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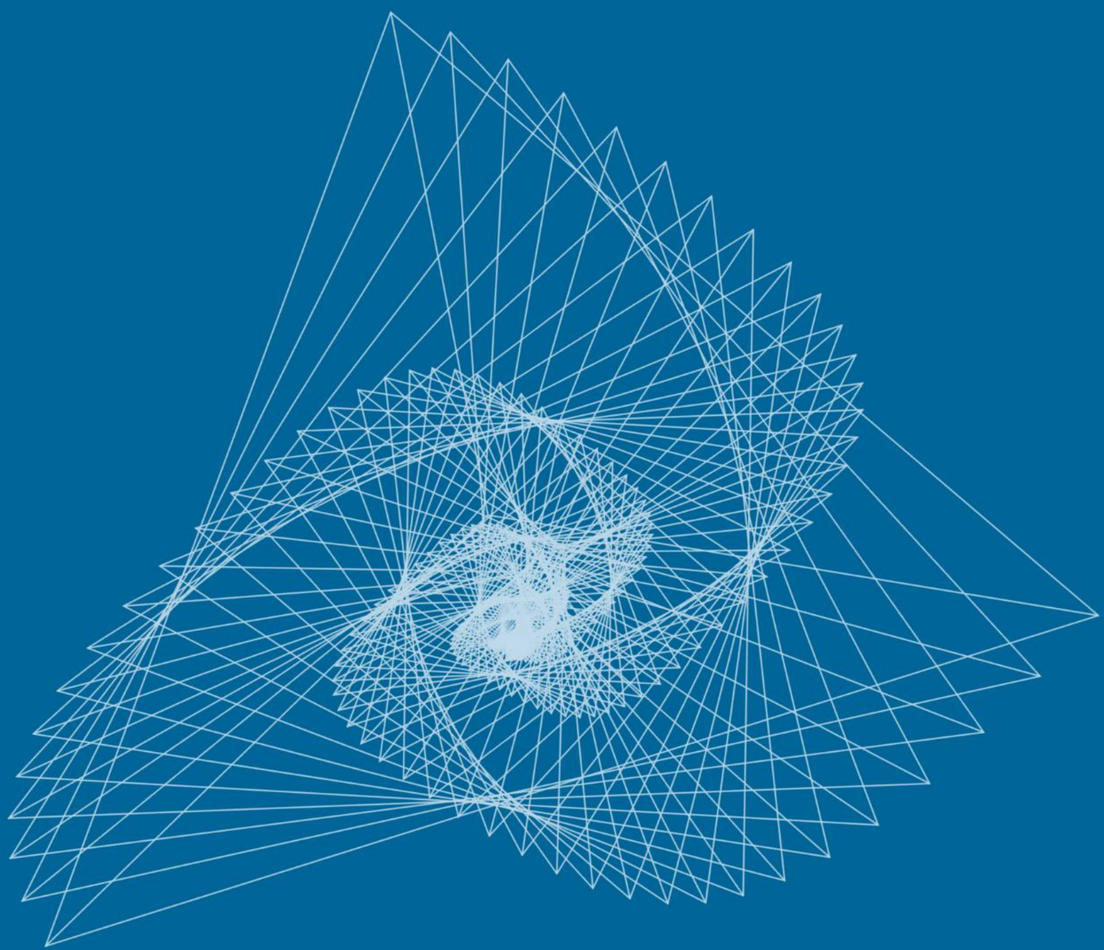

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A MODEL OF AGE–STRUCTURED POPULATION UNDER STOCHASTIC PERTURBATION OF DEATH AND BIRTH RATES[1]

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Abstract: Under consideration is construction of a model of age-structured population reflecting random oscillations of the death and birth rate functions. We arrive at an Itô-type difference equation in a Hilbert space of functions which can not be transformed into a proper Itô equation via passing to the limit procedure due to the properties of the operator coefficients. We suggest overcoming the obstacle by building the model in a space of Hilbert space valued generalized random variables where it has the form of an operator-differential equation with multiplicative noise. The result on existence and uniqueness of the solution to the obtained equation is stated.

Key words: Brownian sheet, Cylindrical Wiener process, Gaussian white noise, Stochastic differential equation, Age-structured population model.

Introduction

A well known model of an age-structured population dynamics is the famous McKendrick–von Foerster equation

$$\frac{\partial u(x,t)}{\partial t}+\frac{\partial u(x,t)}{\partial x}=-m(x)u(x,t), \tag{0.1}$$

where $u(x,t)$ is density of the population at age x at time t (so, that $\int\limits_{x_1}^{x_2} u(s,t)ds$ is the number of individuals with the age belonging to $[x_1;x_2]$ at the time t) and $m(x)$ is the death rate. The usual assumption is that the age of individuals is limited, say $x\in[0;1]$. The process of reproduction is modeled by the boundary condition

$$u(0,t)=\int_0^1 b(x)u(x,t)\,dx. \tag{0.2}$$

Here $b(x)$ is the birth rate which describes the reproductive capacity of the population with respect to age. The model would be more realistic if it reflected random oscillations of the rates of death and birth. Presence of these oscillations can be considered as the result of superposition of multitude of factors connected with different aspects of vital activity of the individuals in the population as well as with unpredictable changes in the environment connected with its physical nature, with food supply, vital activity of competing populations, predators and so on. The assumption of randomness of the oscillations is the way of avoiding unnecessary complication of the model that

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occurs when one tries to reflect the interaction of all these factors which are often hardly subject to formalization.

Stochastically perturbed McKendrick–von Furster equation was for the first time introduced in [10] in a straightforward way by adding a term containing Gaussian white noise and having form $g(t,u)\dot{W}(t)$ (or equivalently $g(t,u)dW(t)$ in the corresponding Itô equation), where $W(t)$ is a Hilbert space valued Wiener process and $g(t,\cdot)$ maps the Hilbert space H, where u considered as a function of t takes values, onto the space of linear bounded operators acting from the separable Hilbert space K, where the values of $W(t)$ lie, to H. In an analogous fashion in [7] was introduced the McKendrick–von Furster equation perturbed with the Levy noise. However both works do not consider the question of choice of appropriate mapping $g(t,\cdot)$.

The aim of our work is clarification of this question in order consistent with the desired properties of the noisy influence on the population.

Since both of the rates are described by functions m and b of age $x \in [0;1]$, it seems natural to model these oscillations by appropriate random processes taking values in spaces of functions of x and to build a model having form of a stochastic equation in such a space. In the present work we discuss problems that arise in building such a model.

We start with a difference equation for the increment of the number of individuals belonging to a small segment of length Δx of the age scale during a small period of time Δt. In section 1 we show that a Brownian sheet naturally arises in modelling the random fluctuations of the death rate. Crucial assumption here is independence between fluctuations of per capita amounts of dead individuals at disjoint segments of the age scale or the time line.

In section 2 we consider passage to limit in the obtained difference equation when Δx tends to zero. We show that the obstacle connected with non-differentiability of the Brownian sheet can be overcome with the help of the concept of a cylindrical random variable on a Hilbert space. Thus, we obtain a difference equation for the increments of the density of the population in a Hilbert space H of functions of $x \in [0;1]$. We show that the random fluctuations of the death rate can be modeled by increments of a cylindrical Wiener process. We also show how this idea can be implemented in modeling the random fluctuations of the rate of birth.

In section 3 we discuss difficulties that arise when we attempt to convert the difference equation into a stochastic differential equation in the Hilbert space H. We show that the use of the theory of Itô-type stochastic differential equations in infinite dimensional Hilbert spaces (see the review of the theory in [5, 6]) is limited due to the properties of the operator coefficients in the difference equation obtained on the previous step. The necessary requirement for the operator-valued integrand of a well defined Itô integral with respect to a cylindrical Wiener process is the condition of being a Hilbert–Schmidt operator, which is not the case here. The way out can be found in setting the equation in the space $(\mathcal{S})_{-\rho}(H)$ of H-valued generalized random variables introduced and studied in [2, 8, 9]. Cylindrical Wiener process $W(t)$ considered a function of t with values in this space happens to be differentiable with the derivative $W'(t) = \mathbb{W}(t)$ being the cylindrical H-valued white noise. We use the established in [3] connection between the Itô integral with respect to a cylindrical Wiener process and the Hitsuda–Skorohod integral. Thus, we finally arrive at a model having form of an operator-differential equation in $(\mathcal{S})_{-\rho}(H)$ and formulate the existence and uniqueness result for the Cauchy problem for this equation.

1. Difference equation

Consider evolution of the population density $u(x,t)$ of an age-structured population, where $x \in [0;1]$ is age, $t \geq 0$ is time. Given $x \in [0;1]$ and $t \geq 0$ we consider the change of the number of individuals that belong to a small segment $[x; x+\Delta x]$ at the moment t during a small time interval

$[t; t + \Delta t]$:

$$\int\limits_{x+\Delta t}^{x+\Delta x+\Delta t} u(s, t + \Delta t)ds - \int\limits_{x}^{x+\Delta x} u(s, t)ds = u(x + \Delta x, t + \Delta t)\Delta x - u(x, t)\Delta x + o(\Delta x).$$

Suppose the change is due to death of individuals and $m(x)$ is the expected rate of death at age x, i.e. $m(x)\Delta x + o(\Delta x)$ is the mean number of dead in the age segment $[x; x + \Delta x]$ in a unit time under constant unit density with respect to age. Suppose also that the population replenishment is due to reproduction which is characterized by the birth rate function $b(x)$ and is described by the boundary condition (0.2). Now let the death rate be subject to random fluctuations, so that omitting the $o(\Delta x)$'s we arrive at the following equation:

$$u(x + \Delta t, t + \Delta t)\Delta x - u(x, t)\Delta x = -u(x, t)m(x)\Delta x\Delta t + u(x, t)\Delta\eta, \tag{1.1}$$

where $\Delta\eta = \Delta\eta^{x,t}_{\Delta x,\Delta t}$ is the random increment of the number of dead individuals in an arbitrary age segment $[x; x + \Delta x]$ during the time $[t; t + \Delta t]$ under constant unit density of population.

The individuals belonging to the age segment $[x; x + \Delta x]$ at the moment t move along the age scale as the time goes and get into the segment $[x + \Delta t; x + \Delta t + \Delta x]$ at the time $t + \Delta t$. This suggests a natural parametrization of the introduced family of random variables by means of parallelograms $\Pi^{x,t}_{\Delta x,\Delta t}$ (the upper and the right sides are supposed to be excluded, see figure 1):

$$\Delta\eta = \Delta\eta^{x,t}_{\Delta x,\Delta t} = \Delta\eta\left(\Pi^{x,t}_{\Delta x,\Delta t}\right). \tag{1.2}$$

We will suppose that the following hypothesis holds.

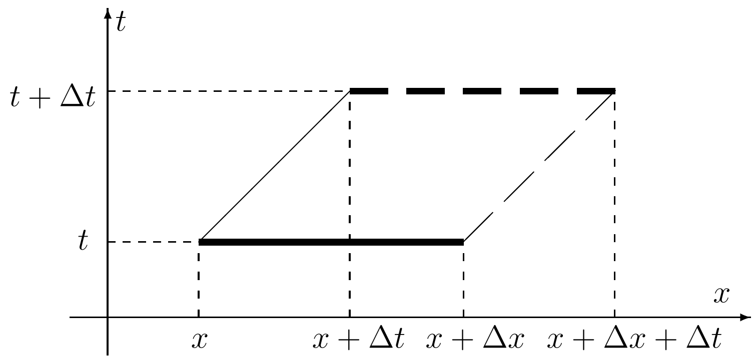

Figure 1. Parallelogram $\Pi^{x,t}_{\Delta x,\Delta t}$.

Hypothesis 1. $\Delta\eta\left(\Pi^{x_k,t_k}_{\Delta x_k,\Delta t_k}\right)$, $k = 1, \ldots, n$, $n \in \mathbb{N}$, *are independent if the parallelograms* $\Pi^{x_k,t_k}_{\Delta x_k,\Delta t_k}$ *are pairwise disjoint.*

Given arbitrary segments $[x; x + \Delta x]$ and $[t; t + \Delta t]$ consider the uniform partition $\{x_k\}$ of $[x; x + \Delta x]$, where $x_k = x + k\Delta x/n$, $k = 0, 1, \ldots, n$ and the corresponding decomposition $\Pi^{x,t}_{\Delta x,\Delta t} = \bigcup_{k=0}^{n-1} \Pi^{x_k,t}_{\Delta x/n,\Delta t}$. The definition of $\Delta\eta$'s implies the following "additivity" for them:

$$\Delta\eta\left(\Pi^{x,t}_{\Delta x,\Delta t}\right) = \sum_{k=0}^{n-1} \Delta\eta\left(\Pi^{x_k,t}_{\Delta x/n,\Delta t}\right).$$

Due to the Hypothesis 1 it follows

$$\mathrm{Var}\left[\Delta\eta\left(\Pi^{x,t}_{\Delta x,\Delta t}\right)\right]=\sum_{k=0}^{n-1}\mathrm{Var}\left[\Delta\eta\left(\Pi^{x_k,t}_{\Delta x/n,\Delta t}\right)\right].$$

This condition will be fulfilled if we let

$$\Delta\eta\left(\Pi^{x,t}_{\Delta x,\Delta t}\right)=\begin{cases}\gamma\sqrt{\Delta x}, \text{ with probability } \lambda\Delta t,\\ 0, \text{ with probability } 1-2\lambda\Delta t,\\ -\gamma\sqrt{\Delta x}, \text{ with probability } \lambda\Delta t,\end{cases} \tag{1.3}$$

for any $x, t, \Delta x, \Delta t$. Here γ and λ are some proportionality factors. This is true since we have

$$\Delta\eta\left(\Pi^{x_k,t}_{\Delta x/n,\Delta t}\right)=\begin{cases}\gamma\sqrt{\dfrac{\Delta x}{n}}, \text{ with probability } \lambda\Delta t,\\ 0, \text{ with probability } 1-2\lambda\Delta t,\\ -\gamma\sqrt{\dfrac{\Delta x}{n}}, \text{ with probability } \lambda\Delta t,\end{cases}$$

and therefore

$$\mathrm{Var}\left[\Delta\eta\left(\Pi^{x_k,t}_{\Delta x/n,\Delta t}\right)\right]=\gamma^2\frac{\Delta x}{n}2\lambda\Delta t. \tag{1.4}$$

Note that the Central Limit Theorem holds for the sequence of series of random variables $\{\xi_k^{(n)}\}_{k=1}^n$, $n=1,2,\dots$, where $\xi_k^{(n)}=\Delta\eta\left(\Pi^{x_k,t}_{\Delta x/n,\Delta t}\right)$, since $\xi_k^{(n)}$ are independent and identically distributed with $E\xi_k^{(n)}=0$ and $\mathrm{Var}\,\xi_k^{(n)}$ given by (1.4). By the Central Limit Theorem we conclude that the distribution of $\dfrac{1}{\gamma\sqrt{2\lambda\Delta x\Delta t}}\sum_{i=1}^n\xi_i^{(n)}$ converges to standard Gaussian when $n\to\infty$. So, the Hypothesis 1 together with the additivity property (1) makes it natural to impose the following hypothesis.

Hypothesis 2. $\Delta\eta\left(\Pi^{x,t}_{\Delta x,\Delta t}\right)\sim N\left(0,2\lambda\gamma^2\Delta x\Delta t\right)$.

Definition 1. *The collection of random variables $\{\Theta(B), B\in\mathcal{B}(\mathbb{R}^2)\}$ is called a Gaussian orthogonal measure on the Borel σ-field $\mathcal{B}(\mathbb{R}^2)$ if the following holds:*

1. *$\Theta(B)\sim N(0,\mu_L(B))$ for all $B\in\mathcal{B}(\mathbb{R}^2)$, where μ_L is the Lebesque measure of B;*
2. *$B_1\cap B_2=\varnothing$ implies $\Theta(B_1)$ and $\Theta(B_2)$ are independent for all $B_1,B_2\in\mathcal{B}(\mathbb{R}^2)$;*
3. *$\Theta(\cup_{k=1}^\infty B_k)=\sum_{k=1}^\infty\Theta(B_k)$ (the series is mean square convergent) for any sequence $\{B_k\}\subset\mathcal{B}(\mathbb{R}^2)$ of pairwise disjoint sets.*

Hypotheses 1 and 2 imply that $\Delta\eta=\gamma\sqrt{2\lambda}\Theta$, where Θ is a Gaussian orthogonal measure on $\mathcal{B}(\mathbb{R}^2)$. Thus, equation (1.1) turns into

$$u(x+\Delta t,t+\Delta t)\Delta x-u(x,t)\Delta x=-u(x,t)m(x)\Delta x\Delta t+\alpha_0u(x,t)\Theta\left(\Pi^{x_k,t}_{\Delta x,\Delta t}\right), \tag{1.5}$$

where $\alpha_0\in\mathbb{R}$ is a constant.

Definition 2. [1, p. 649] *A two-parameter Gaussian random process* $\{\mathbb{B}(x,t)\,, x \geq 0, t \geq 0\}$ *is called a Brownian sheet if it satisfies the following conditions:*

1. $E\left[\mathbb{B}(x,t)\right] = 0\,,$ *for all* $x, t \geq 0$*;*

2. $\mathrm{Cov}\ (\mathbb{B}(x_1,t_1), \mathbb{B}(x_2,t_2)) = \min\{x_1; x_2\} \cdot \min\{t_1; t_2\}$ *for all* $x_1, x_2, t_1, t_2 \geq 0.$

In [4, Definition 12, p. 107] a random process, satisfying the conditions of Definition 2 is called a Wiener–Chentsov random field.

It is easy to see that the random process defined by

$$\mathbb{B}(x,t) := \Theta\left(\Pi^{0,0}_{x,t}\right), \quad x,t \geq 0 \tag{1.6}$$

is a Brownian sheet.

Note that a Brownian sheet on $[0;1] \times [0;T]$ admits the following decomposition

$$\mathbb{B}(x,t) = \sum_{n,k=0}^{n} \theta_{n,k} \frac{8\sqrt{T}}{\pi^2(2n+1)(2k+1)} \sin\frac{\pi(2n+1)t}{2T} \sin\frac{\pi(2k+1)x}{2}, \tag{1.7}$$

where $\theta_{n,k}$ are independent standard Gaussian random variables, defined on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$. Replacing $\Theta\left(\Pi^{x_k,t}_{\Delta x,\Delta t}\right)$ in (1.5) by the increment of the Brownian sheet, defined by (1.6) we obtain

$$\begin{gathered} u(x+\Delta t, t+\Delta t)\Delta x - u(x,t)\Delta x = -u(x,t)\Delta x \mu(x)\Delta t + \\ +\alpha_0 u(x,t)\left[\mathbb{B}(x+\Delta x, t+\Delta t) - \mathbb{B}(x, t+\Delta t) - \mathbb{B}(x+\Delta x, t) + \mathbb{B}(x,t)\right]. \end{gathered}$$

Let $u(x,t)$ be continuously differentiable with respect to x. Then we have

$$\begin{aligned} u(x+\Delta t, t+\Delta t) - u(x,t) &= u(x, t+\Delta t) + \frac{\partial u}{\partial x}(x, t+\Delta t)\Delta t + o(\Delta t) - u(x,t) = \\ &= u(x,t+\Delta t) - u(x,t) + \left[\frac{\partial u}{\partial x}(x,t) + o(1)\right]\Delta t + o(\Delta t) = \\ &= u(x,t+\Delta t) - u(x,t) + \frac{\partial u}{\partial x}(x,t)\Delta t + o(\Delta t). \end{aligned}$$

Omitting the $o(\Delta t)$'s and dividing both sides of the equation by Δx, we obtain the equation

$$\begin{gathered} u(x,t+\Delta t) - u(x,t) = \left(-\frac{\partial u}{\partial x}(x,t) - \mu(x)u(x,t)\right)\Delta t + \\ +\alpha_0 u(x,t)\left[\frac{\mathbb{B}(x+\Delta x, t+\Delta t) - \mathbb{B}(x, t+\Delta t)}{\Delta x} - \frac{\mathbb{B}(x+\Delta x, t) - \mathbb{B}(x,t)}{\Delta x}\right]. \end{gathered} \tag{1.8}$$

Brownian sheet is nowhere differentiable in both variables. Therefore we can not pass to limit in this difference equation letting $\Delta x \to 0$. In the next section we consider this equation in a Hilbert space of functions of $x \in [0;1]$ and justify this passage to limit with the help of the concept of cylindrical random variable.

2. Difference equation in a Hilbert space

The set of functions $e_k(x) = \sqrt{2}\sin\dfrac{x}{\lambda_k}$, $k = 0, 1, \dots,$ used in expansion (1.7), where $\lambda_k = \dfrac{2}{\pi(2k+1)}$ is an orthonormal basis in the space $H = L^2[0;1]$. Note that the random processes $\beta_k(t)$, defined by the series

$$\beta_k(t) = \sum_{n=0}^{\infty} \theta_{n,k} \frac{2\sqrt{2T}}{\pi(2n+1)} \sin\frac{\pi(2n+1)t}{2T}, \quad t \in [0;T], \tag{2.1}$$

are independent Brownian motions (here, as in (1.7), $\theta_{n,k}$ are independent standard Gaussian random variables). Thus, we can rewrite the expansion (1.7) as

$$\mathbb{B}(x,t) = \sum_{k=0}^{\infty} \lambda_k \beta_k(t) e_k(x) \tag{2.2}$$

and consider $\mathbb{B}(t) = \mathbb{B}(\cdot, t)$ as a random process in H. It is easy to see that the series (2.2) is convergent in $L^2(\Omega, \mathcal{F}, \mathrm{P}; H)$ for any t.

Let us introduce the shift operator $\tau_{\Delta x} : H \to H$, defining it on the elements of the basis $\{e_k\}$ by

$$\tau_{\Delta x} e_k = \sin\frac{\Delta x}{\lambda_k}\tilde{e}_k + \cos\frac{\Delta x}{\lambda_k} e_k\,, \tag{2.3}$$

where $\tilde{e}_k(x) := \lambda_k e_k'(x) = \sqrt{2}\cos\frac{x}{\lambda_k}$, $k = 0, 1, \dots.$ Note that the set $\{\tilde{e}_k\}$ is also an orthonormal basis in $L^2[0;1]$. Equation (1.8) can be written as the following difference equation in H:

$$\begin{aligned} u(t+\Delta t) - u(t) &= \left(-\frac{\partial}{\partial x}u(t) - mu(t)\right)\Delta t + \\ &+ \alpha_0 u(t)\left[\frac{\tau_{\Delta x}\mathbb{B}(t+\Delta t) - \mathbb{B}(t+\Delta t)}{\Delta x} - \frac{\tau_{\Delta x}\mathbb{B}(t) - \mathbb{B}(t)}{\Delta x}\right], \end{aligned}$$

where $u(t) = u(\cdot, t)$.

Definition 3. [6, p. 17] *Let H be a Hilbert space. A linear operator $X : H \to L^2(\Omega, \mathcal{F}, \mathrm{P})$ with the properties:*

1. *$X[h] \sim N(0, \|h\|^2)$ for any $h \in H$,*
2. *$X(h_1)$ and $X(h_2)$ are independent if $(h_1, h_2)_H = 0$,*

is called a cylindrical standard Gaussian random variable on H.

It follows from the definition that any cylinder standard Gaussian random variable X is a bounded operator: $X \in \mathcal{L}\left(H; L^2(\Omega, \mathcal{F}, \mathrm{P})\right)$ with $\|X\| = 1$.

Definition 4. [6, p. 19] *A family $\{W(t), t \in \mathbb{R}\}$ is called a cylindrical Wiener process if*

1. *$W(t) : H \to L^2(\Omega, \mathcal{F}, \mathrm{P})$ is a linear operator;*
2. *$W(t)[h]$ is a brownian motion for any $h \in H$;*
3. *$E\left(W(t)[h_1]W(t)[h_2]\right) = t(h_1, h_2)_H$ for any $h_1, h_2 \in H$.*

Let $W(t)$ be a cylindrical Wiener process on a Hilbert space H. It follows from the definition that $\frac{1}{\sqrt{t}}W(t)$ is a cylindrical standard Gaussian random variable on H for any $t>0$. We also have that for any orthonormal basis $\{g_k\}_{k=0}^{\infty}$ in H $\beta_k(t):=W(t)[g_k]$ are independent Brownian motions, therefore one can identify $W(t)$ with the expansion

$$W(t)=\sum_{k=0}^{\infty}\beta_k(t)g_k \tag{2.4}$$

by letting

$$W(t)[h]:=\sum_{k=0}^{\infty}h_k\beta_k(t)\,,\quad h=\sum_{k=0}^{\infty}h_kg_k\in H\,. \tag{2.5}$$

Although the series (2.4) is divergent in $L^2(\Omega,\mathcal{F},\mathrm{P};H)$, the right hand side of the equality (2.5) defines a random variable belonging to $L^2(\Omega,\mathcal{F},\mathrm{P})$ which can be thought of as a scalar product $(W(t),h)_H$. Conversely, any sequence of independent Brownian motions $\{\beta_k(t)\}_{k=0}^{\infty}$ and an orthonormal basis $\{g_k\}_{k=0}^{\infty}$ in H generate a cylindrical Wiener process on H, defined by (2.5).

The next proposition states that when $\Delta x\to 0$, the difference quotients $\dfrac{\tau_{\Delta x}\mathbb{B}(\cdot,t)-\mathbb{B}(\cdot,t)}{\Delta x}$ converge to a cylindrical Wiener process as cylindrical random variables on the Hilbert space $H=L^2[0;1]$.

Proposition 1. *For any $h\in H$*

$$\lim_{\Delta x\to 0}E\left(\frac{\tau_{\Delta x}\mathbb{B}(t)-\mathbb{B}(t)}{\Delta x}-W_0(t),h\right)_H^2=0\,, \tag{2.6}$$

where $W_0(t)$ is the cylindrical Wiener process, defined by the expansion

$$W_0(t)=\sum_{k=0}^{\infty}\beta_k(t)\tilde{e}_k.$$

P r o o f. Let $h=\sum_{k=1}^{\infty}h_ke_k=\sum_{k=1}^{\infty}\tilde{h}_k\tilde{e}_k\in H$. Using the expansion (2.2) and the equality (2.3), we obtain

$$\left(\frac{\tau_{\Delta x}\mathbb{B}(t)-\mathbb{B}(t)}{\Delta x}-W_0(t),h\right)_H=\sum_{k=0}^{\infty}\beta_k(t)\left[\zeta_k(\Delta x)\tilde{h}_k+\gamma_k(\Delta x)h_k\right],$$

where

$$\zeta_k(\Delta x)=\frac{\sin\Delta x/\lambda_k}{\Delta x/\lambda_k}-1,\quad \gamma_k(\Delta x)=\frac{\cos\Delta x/\lambda_k-1}{\Delta x/\lambda_k}.$$

We have

$$E\left(\frac{\tau_{\Delta x}\mathbb{B}(t)-\mathbb{B}(t)}{\Delta x}-W_0(t),h\right)_H^2=t\sum_{k=0}^{\infty}\left[\zeta_k(\Delta x)\tilde{h}_k+\gamma_k(\Delta x)h_k\right]^2 \tag{2.7}$$

and due to the estimate

$$\left[\zeta_k(\Delta x)\tilde{h}_k+\gamma_k(\Delta x)h_k\right]^2\le 2\left[\zeta_k^2(\Delta x)\tilde{h}_k^2+\gamma_k^2(\Delta x)h_k^2\right]\le 4\left[\tilde{h}_k^2+h_k^2\right],$$

we conclude that the series in the right hand side of (2.7) is uniformly convergent with respect to $\Delta x\in\mathbb{R}$. Since $\lim\limits_{\Delta x\to 0}\zeta_k(\Delta x)=\lim\limits_{\Delta x\to 0}\gamma_k(\Delta x)=0$ for any k, it follows (2.6). $\square$

Thus, letting $\Delta x \to 0$ in (1.8), we arrive at the following difference equation in H:

$$u(t+\Delta t) - u(t) = \left(-\frac{\partial}{\partial x}u(t) - mu(t)\right)\Delta t + \alpha_0 u(t)\left(W_0(t+\Delta t) - W_0(t)\right).$$

Note, that the last term in the right hand side of this equation can not be thought of as a product of functions of x. This is due to the fact that the increments of the cylindrical Wiener process are cylindrical Gaussian random variables on H and do not belong to $H = L^2[0;1]$ with probability one. In order to give meaning to the product we rewrite the equation in the following form:

$$u(t+\Delta t) - u(t) = Au(t)\Delta t + \alpha_0 B_0(u(t))\left(W_0(t+\Delta t) - W_0(t)\right), \tag{2.8}$$

where $A = -\dfrac{d}{dx} - m(x) : H \to H$ with the domain

$$D(A) = \left\{ u \in H^1[0;1] \;\middle|\; u(0) = \int_0^1 b(x)u(x)\,dx \right\},$$

and $B_0 : H \to L(H)$ is the operator, defined by $B_0 : u \mapsto B_0(u)$, where $B_0(u)$ is the operator of multiplication by u.

Since for any $u \in H$ we have

$$\|B_0(u)e_k\|_H^2 = \int_0^1 |u(x)e_k(x)|^2 dx \le 2\|u\|_H^2$$

and $h = \sum_{k=0}^{\infty} h_k e_k \in H^1[0,1]$ iff $\|h\|_1^2 := \displaystyle\sum_{k=0}^{\infty}\left(\frac{h_k}{\lambda_k}\right)^2 < \infty$, the following estimate holds:

$$\|B_0(u)h\|_H^2 = \left\|\sum_{k=0}^{\infty} h_k B_0(u)e_k\right\|_H^2 \le \left(\sum_{k=0}^{\infty}|h_k|\|B_0(u)e_k\|\right)^2 \le$$
$$\le \sum_{k=0}^{\infty}\left(\frac{h_k}{\lambda_k}\right)^2 \sum_{k=0}^{\infty}\lambda_k^2\|B_0(u)e_k\|^2 \le \|h\|_1^2\|u\|_H^2 \cdot 2\sum_{k=0}^{\infty}\lambda_k^2.$$

Since $\sum_{k=0}^{\infty}\lambda_k^2 < \infty$, it follows $B_0(u)h \in H$. Therefore the equation (2.8) can be understood in the following weak sense:

$$(u(t+\Delta t) - u(t), h)_H = (u(t), A^*h)_H\Delta t + \alpha_0(W_0(t+\Delta t) - W_0(t))[B_0(u(t))h] \tag{2.9}$$

for any $h \in D(A^*)$. Here $A^*h(x) = h'(x) - m(x)h(x) + b(x)h(0)$ with the domain

$$D(A^*) = \left\{h \in H^1[0;1] \mid h(1) = 0\right\}.$$

Consider the first term in the right hand side of (2.9). We have

$$\begin{aligned}(u(t), A^*h)_H &= (u(t), h')_H - (u(t), mh)_H + (u(t), \langle h, \delta\rangle b)_H = \\ &= (u(t), h')_H - (m, B_0(u(t))h)_H + (b, B_1(u(t))h)_H,\end{aligned} \tag{2.10}$$

where δ is the Dirac delta-function, considered as an element of the space$H^{-1}[0;1]$, the operator $B_1 : H \to \mathcal{L}(H^1[0;1]; H)$ is defined by $B_1(u)h = \langle h, \delta\rangle u$, $u \in H$, $h \in H^1[0;1]$. The second term in the right hand side of (2.9) has appeared as the result of stochastic perturbation of the operator of multiplication by $m(x)$ (the mean rate of death). This operator is represented by the second term in the right hand side of (2.10). Since the third term there corresponds to the operator of multiplication by $b(x)$ (the mean birth rate) initially contained in the boundary condition, it is natural to introduce an analogous stochastic perturbation of this factor by the term $\alpha_1(W_1(t+\Delta t)-W_1(t))[B_1(u(t))h]$, where $W_1(t)$ is a cylindrical Wiener process independent with $W_0(t)$ and α_1 is a constant.

Thus, we arrive at the following equation:

$$u(t+\Delta t)-u(t)=Au(t)\Delta t+\alpha_0 B_0(u(t))\left(W_0(t+\Delta t)-W_0(t)\right)+ \\ +\alpha_1 B_1(u(t))\left(W_1(t+\Delta t)-W_1(t)\right), \tag{2.11}$$

which is understood in the weak sense, namely:

$$(u(t+\Delta t)-u(t),h)_H=(u(t),A^*h)_H\Delta t+\alpha_0(W_0(t+\Delta t)-W_0(t))[B_0(u(t))h] \\ +\alpha_1(W_1(t+\Delta t)-W_1(t))[B_1(u(t))h]$$

for any $h\in D(A^*)$.

3. Differential equation

For any $t>0$ let $\{t_k\}_{k=0}^N$ be a partition of the segment $[0;t]$, where $t_k=k\Delta t$, $\Delta t=t/N$. Summing up the equality (2.11) written for the points t_k we obtain

$$u(t)-u(0)=\sum_{k=0}^{N-1}Au(t_k)\Delta t+\sum_{k=0}^{N-1}B_0(u(t_k))\left(W_0(t_{k+1})-W_0(t_k)\right)+ \\ +\sum_{k=0}^{N-1}B_1(u(t_k))\left(W_1(t_{k+1})-W_1(t_k)\right).$$

Letting $N\to\infty$ we arrive at the following integral Itô equation

$$u(t)-u(0)=\int_0^t Au(s)\,ds+\int_0^t B_0(u(s))\,dW_0(s)+\int_0^t B_1(u(s))\,dW_1(s), \tag{3.1}$$

if the integrals in the right hand side exist. The equation is usually written in the following differential form:

$$du(t)=Au(t)\,dt+B_0(u(t))\,dW_0(t)+B_1(u(t))\,dW_1(t)\,,\quad u(0)=u_0. \tag{3.2}$$

The necessary condition of existence of the integrals in (3.1) is $B_0(u),B_1(u)\in\mathcal{L}_2(H;H)$ (the space of Hilbert–Schmidt operators acting in H) for any $u\in H$. It is not the case here, therefore it is impossible to obtain theorems on existence and uniqueness of solution (weak, or mild) for the problem (3.3) (see, for example, Theorem 6.7, p. 164 in [5], Theorem 3.3, p. 97 in [6]).

The way out can be found in setting the problem in the space of generalized Hilbert-space-valued random variables $(\mathcal{S})_{-\rho}(H)\supset L^2(\Omega,\mathcal{F},P;H)$, $\rho\in[0;1]$ (see the definition and properties of this space in [9]). It turns out that a cylindrical Wiener process $W(t)$ on H is a differentiable $(\mathcal{S})_{-\rho}(H)$-valued function of t. Denote its derivative by $\mathbb{W}(t)$. It is called a cylindrical singular white noise. It was proved in [3] that for any predictable $\mathcal{L}_2(H,H)$-valued process $\Psi(t)$ it holds

$$\int_0^t\Psi(s)\,dW(s)=\int_0^t\Psi(s)\diamond\mathbb{W}(s)ds,$$

where $\diamond$ is the Wick product, if the Itô integral in the left hand side exists. The integral in the right hand side is often called the Hitsuda–Skorohod integral. It can be considered an extension of the Itô integral onto a wider class of integrands.

Thus, the problem (3.3) takes the form

$$\frac{du(t)}{dt} = Au(t) + B_0(u(t)) \diamond \mathbb{W}_0(t) + B_1(u(t)) \diamond \mathbb{W}_1(t), \quad u(0) = u_0. \tag{3.3}$$

in the space $(\mathcal{S})_{-\rho}(H)$. The following theorem is stated here without proof as it is a straightforward generalization of the Theorem 3 in [9].

Theorem 1. *Let A be the generator of a C_0-semigroup of operators in a Hilbert space H, $B_0(\cdot), B_1(\cdot) \in \mathcal{L}\big(H, \mathcal{L}_2(H_Q, H)\big)$, where Q is a positive trace class operator in H with the set of eigenvalues $\{\sigma_j^2\}_{j=1}^{\infty}$ satisfying the condition*

$$\sum_j \sigma_j^{-2} j^{-2p} < \infty\,, \quad \textit{for some } p \in \mathbb{N},$$

and $H_Q = Q^{1/2}(H)$ with the norm $\|x\|_Q = \|Q^{-1/2}x\|_H$. Then the problem (3.3) has a unique solution $u(t) \in (\mathcal{S})_{-0}(H)$ for any $u_0 \in (D(A))$, where $(D(A))$ denotes the domain of A in $(\mathcal{S})_{-0}(H)$.

Remark 1. Conditions of the theorem hold true for the operators, B and B_1 introduced in our model. To show this, note that the functions $\sigma_j \tilde{e}_j(x)$, $j = 0, 1, 2, \ldots$ form an orthonormal basis in H_Q and for any $u \in H = L^2[0;1]$ we have:

$$\|B(u)\|^2_{\mathcal{L}_2(H_Q,H)} = \sum_j \|B(u)\sigma_j \tilde{e}_j\|^2 = \sum_j \sigma_j^2 \int_0^1 u^2(x)\tilde{e}_j(x)\,dx \le 2\sum_j \sigma_j^2 \|u\|^2,$$

$$\|B_1(u)\|^2_{\mathcal{L}_2(H_Q,H)} = \sum_j \|B_1(u)\sigma_j \tilde{e}_j\|^2 = \sum_j \sigma_j^2 \langle \tilde{e}_j, \delta\rangle^2 \|u\|^2 = 2\sum_j \sigma_j^2 \|u\|^2.$$

4. Conclusion

Introduction of stochastic perturbation into McKendrick–von Foerster model of an age-structured population requires taking into account certain properties of the oscillations of rates of death and birth. We have shown that the assumption of independence between the random fluctuations of per capita amounts of dead individuals in disjoint segments of the age scale or the time line together with the analogous assumption on the random fluctuations concerning the process of reproduction in the population lead to a difference equation in the Hilbert space $L^2[0;1]$ with a cylindrical Wiener process. Due to nonregularity of the latter, we finally obtain a model which has the form of an operator-differential equation with cylindrical white noises in the space of generalized Hilbert space-valued random variables satisfying the conditions of the theorem on existence and uniqueness of solutions.

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OPTIMIZATION OF THE ALGORITHM FOR DETERMINING THE HAUSDORFF DISTANCE FOR CONVEX POLYGONS

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Abstract: The paper provides a brief historical analysis of problems that use the Hausdorff distance; provides an analysis of the existing Hausdorff distance optimization elements for convex polygons; and demonstrates an optimization approach. The existing algorithm served as the basis to propose low-level optimization with super-operative memory, ensuring the finding a precise solution by a full search of the corresponding pairs of vertices and sides of polygons with exclusion of certain pairs of vertices and sides of polygons. This approach allows a significant acceleration of the process of solving the set problem.

Key words: Hausdorff distance, Polygon, Optimization, Optimal control theory, Differential games, Theory of image recognition.

1. Introduction

Recognition of images is not a new problem in its essence and arises in the most diverse lines of research, ranging from applied tasks in the field of analog signal security and digitization to the problems of theories of optimal control and differential games. The most intensive development and improvement of methods for solving such problems is observed in the current period. This is due, above all, to the need to release people from the arising huge information loads and to use both thinking and perception characteristic of recognition. All these problems are of pronounced interdisciplinary nature and are the basis for the development of a new generation of specialized and applied technical recognition systems used in various fields, including medicine [21] and artificial intelligence development. One of the earliest lines of research was optical character recognition (OCR).

The recognition and comparison of images [4, 5], including recognition and localization [4, 5] is a relevant problem of the era of digital information processing. As is known, the most important information about the shape is in the outlines of objects. Many real-world objects can be recognized from the images of their outlines, and there is no need to use the original gray-scale images. Due of this, recognition algorithms are most often designed to operate binary, outline, or close to outline images.

One of the known methods for detecting and analyzing objects in binary outline images distinguishable from the surrounding context due to their geometric properties is the geometric measurement of the distance between the image points. One of approaches to solving this problem is to modify the Hausdorff metrics to identify objects geometrically close to arbitrary reference ones specified by bit masks. In this approach, the image is considered as a set of complex elements or a set of points in a two-dimensional Euclidean space. For these sets, the measure of mutual proximity is calculated; in the case of complex elements, the Hausdorff metric is used. Modifications of the Hausdorff metric in image recognition have been used since 1993 [6]. They have an intuitive operation principle, an explicit connection to the object geometry, and do not require any training sample. However, this approach is little known compared with neural networks, although an increasing number of related publications have been released in recent years.

2. Main Results

The biggest disadvantage of algorithms that use modifications of the Hausdorff metric is a rather high computational complexity, on the average $2 \sim 3$ times higher than that of the simplest correlation algorithms. Non-invariance to rotation and scale, which, in the absence of a priori information on the orientation and size of the objects to be recognized, forces to use the scanning of a multitude of versions of the standard sample at different turning angles and scales; therefore, one of the relevant problems is the development of a calculation optimization technology in these algorithms.

Among the large number of publications dealing with this subject, several review works should be singled out [2, 4, 5]. These works touch upon both theoretical and applied aspects in terms of the algorithm development. Back in the 1990s, the works of P. Gruber [4, 5] covering various faces of approximating convex bodies were released, and the agenda adjacent to this topic was further developed and studied in [2]. The author emphasizes that along with the development of the perfect form of classical approaches, outstanding results on attractive sets have been obtained.

The Hausdorff metric [1, 14] denotes distance $h : D \to \mathbb{R}$ on certain given set D between its subsets X, Y, where

$$h(X, Y) = \max \left[\max_{X} \min_{Y} d(x, y),\ \max_{Y} \min_{Y} d(x, y) \right],$$

where $d(x, y)$ is the distance between elements of subsets of the given set.

The definition of the metric space in [20] was formulated for convex unbounded closed subsets of a Banach space using the Hausdorff metric, which establishes the differences of the properties of convex ones with this metric from the properties of the metric space. The research in [20] led to the important assertion that not every object in the metric space can be approximated by generalized polyhedra and, therefore, the generalized polyhedron concept was introduced, and approximation criteria were proposed. It was shown that the uniform continuity of the support function is a necessary and sufficient condition for approximation.

Another no less important problem of minimizing the Hausdorff metric between two convex polygons was addressed in [18]. The authors consider two polygons: one fixed and the other changing its location on the plane (rotation or parallel transfer). Ushakov and Lebedev et al. [11–13, 18] developed and tested the iterative step-by-step shift and rotation algorithms that ensure a reduction in the Hausdorff distance between a moving and a fixed object, using the differential properties of the function of the Euclidean distance to a convex set and the geometric properties of the Chebyshev center of a compact set, and proved the theorems on the correctness of the developed algorithms for a wide range of cases. A multiple start-over of the algorithm allows choosing the best option.

The works of A.B. Kurzhansky [10] and F.L. Chernousko [3] use the approximation of sets of attainability of differential games with ellipsoids and parallelepipeds in solving problems of the optimal control of dynamical systems. In this case, the Hausdorff distance is the criterion of optimality.

Also, the work of A.S. Lakhtin [18] considers the algorithm for obtaining a precise analytic solution of the problem of minimizing the Hausdorff distance through a full search among a finite number of options depending on the number of vertices of the given polygons. The options are pairs or triples of vectors connecting the vertices of two polygons or the vertex of one and the side of the other polygon.

Having analyzed all the presented ideas and methods, it was decided to take the ideas underlying the analytical algorithm from publication [18] and the numerical subgradient method from publication [18] as the basis for the new proposed algorithm.

Of all the algorithms already considered, the most suitable for improvement is the analytical algorithm of the step-by-step displacement between a moving and a fixed object from publication [18]. For the convenience of the material perception, the ideas that underlie this method are provided below.

Suppose that two convex polyhedra $A, B \in \mathbb{R}^n$ are given. It is required to move them so as to minimize the Hausdorff distance between them, which, as is known, is calculated by formula

$$d(A, B) = \max\big\{ \max_{a\in A} \min_{b\in B} \|a-b\|,\ \max_{b\in B} \min_{a\in A} \|a-b\| \big\}.$$

Assuming that polyhedron A is fixed, and B moves by parallel transfer for vector $x \in \mathbb{R}^n$, we have convex function $F(x) = d(A, B + x)$, the minimum point of which x^* is sought for, i.e. $F^* = F(x^*) = \min_{x\in\mathbb{R}^n} F(X)$.

For convenience, a plane case is considered, i.e. $A, B \in \mathbb{R}^2$ are convex polygons, but the idea of the proposed method can be used in spaces of larger dimensions.

It was proven in [18] that $\partial F(x) = \mathrm{co}\,\{L_A(x) \bigcup L_B(x)\}$, where

$$L_A(x) = \{-l : \exists i \in I_A(x) : \langle l, a_i - x\rangle - \rho_B(l) = F(x),\ \|l\| = 1\}$$

and

$$I_A(x) = \{i : \mathrm{dist}\,(a_i, B + x) = F(x)\}.$$

Set $L_B(x)$ is defined similarly.

Define sets $L^*_A(x^*)$ and $L^*_B(x^*)$ as sets of vectors co-directed to single vectors from sets $L_A(x^*)$ and $L_B(x^*)$, respectively, with length $F^* = F(x^*)$.

Type V vectors are called vectors from set $L^*_A(x^*) \bigcup L^*_B(x^*)$, connecting the vertices of different polygons. Type W vectors are called vectors from set $L^*_A(x^*) \bigcup L^*_B(x^*)$, connecting the vertex of one polygon with the side of the other, where the vector is perpendicular to this side. Note that any vector from $L^*_A(x^*) \bigcup L^*_B(x^*)$ is of either type V or type W.

By definition, any type V vector has the form $l_k = b_{j_k} + x^* - a_{i_k}$ and, therefore, complies with equality $F^* = \|l_k\| = \|b_{j_k} + x^* - a_{i_k}\| = \|x^* - (a_{i_k} - b_{j_k})\|$. Geometrically, this corresponds to the distance from the point with coordinates x^* to the point with coordinates $(a_{i_k} - b_{j_k})$.

Let the type W vector be a vector from vertex a_{i_k} to side (b_k+x^*), $(b_{k+1}+x^*)$ then the following equality is met:

$$F^* = \|l_k\| = \frac{(x^* - (a_{i_k} - b_{j_k}) \times (b_{j_k+1} - b_{j_k}))}{(b_{j_k+1} - b_{j_k})}.$$

Geometrically, this corresponds to the distance from the point with coordinates x^* to the straight line passing through points with coordinates $(a_{i_k} - b_{j_k})$, $(a_{i_k} - b_{j_k+1})$.

Thus, the problem of finding the minimum point x^* reduces to a full search among a finite number of options. The pairs of vertices of different polygons give type V vectors, and the pairwise consideration of the vertices of the same polygon with the sides of the other gives type W vectors. In the auxiliary space, it is required each time to solve the problem of optimal placement of point x^* providing the shortest distances to the corresponding points and straight lines. In other words, it is required in each case to find the center of a circle passing through given points and tangent to the given straight lines.

Based on the described work, the algorithm was implemented, which, through a full search among a finite number of options depending on the number of vertices of the given polygons, finds a precise analytical solution of the optimization problem posed.

One of the important drawbacks of the described algorithm is the need for a full search, which leads to a very high computational complexity. But, despite these shortcomings, the algorithm ensures finding a precise solution for a fixed time.

The first stage of the work was the implementation of the algorithm itself without any optimization. C++ was chosen as the programming language for the implementation because of the execution rate, absence of unnecessary calls, similarity with Assembler, as well as abundance of optimization tools and parallelization of the algorithm execution.

The implementation of this algorithm was divided into several logical parts. The first part was the creation of data structures, both to store polyhedra, and to optimize the results at each step of the algorithm, as well as to enable storing and reading data structures from the file. The second part includes all the auxiliary algorithms that perform the following operations: finding vectors from the vertex of one polygon to the other, checking algorithms, whether this vertex-vertex and vertex-side pair of vectors (in both directions) ensures the best optimization. Similarly, the checks for a triple of vectors from the set of vertex-vertex and vertex-side pairs (in both directions) are performed. These algorithms include finding the center of a circle using three points, tangent and two points, two tangents and one point, and three tangents. The third part includes the algorithm for a full search among all possible pairs and triples of vectors found in other parts of the implementation.

The result obtained is new. Prior to this, there has been only a theoretical justification for the analytical method, but no ways for its implementation that could be officially referred to could be found. As a result, this algorithm was implemented. The resulting implementation was tested on a large number of pairs of convex polygons of various types. This result is of independent value, both for subsequent testing of any approaches to optimization, and for testing any subgradient methods.

During the operation of the algorithm, which performs a complete search among all possible pairs and triples of vectors, statistical data about which vectors influence the formation of the final optimal result were accumulated. Based on the processing results of these data, *Hypothesis 1* was formulated. Its idea is that some groups of vectors can be excluded from the search, since they do not participate in the formation of the final optimal result. It was suggested that such vectors are those that go from one polygon to another but intersect any side of the other polygon.

Hypothesis 1 was tested on a set of polygons, based on which the hypothesis was formulated, see Fig. 1 and Fig. 2. But when the set of polygons was expanded, counter examples were found. In Fig. 3 the type V vector from vertex 3 to vertex 5 intersects the side beginning at vertex 6 and ending at vertex 0. In Figure 4, the type V vector from vertex 0 to vertex 3 intersects the side beginning at vertex 1 and ending at vertex 2. Thus, Hypothesis 1 was not confirmed. Nevertheless, this heuristic idea is viable, since in quite a large number of cases for polygons without peculiar features, Hypothesis 1 is fair and provides a substantial reduction in the search options.

Due to finding a counter example for Hypothesis 1, a new hypothesis was required. This hypothesis was based on the idea of using a support function. To begin with, recall the definition of the Hausdorff distance through support functions [19]

$$H(A,B) = \max\Big\{\max_{a\in A}\max_{\|l\|=1}(\langle l,a\rangle - \rho_B(l)),\ \max_{b\in B}\max_{\|l\|=1}(\langle l,b\rangle - \rho_A(l))\Big\}.$$

Also, within the framework of the Hausdorff distance determination through support functions, a minimum is used. Argmin are the elements, to which the minimum is reached. Assume that these will be the required elements.

Further, the notion of the visible part of a polygon is introduced. Fixing the direction vectors and looking in this direction at each polygon from the given pair separately, as, for example, is shown in Fig. 2, shows that only few sides and vertices of the polygons are visible. After this operation, a set of “visible”vertices and sides is obtained, to which boundary sides are added. The boundary sides are those sides of the polygon that were not included in the original sample, but one of the vertices of this side was added. Based on the resulting set of vertices and polygon sides, a set of vertex-vertex and vertex-side vectors that are involved in the full search algorithm are built. The first stage of the algorithm has been completed; the output is the set of vectors for the algorithm under consideration.

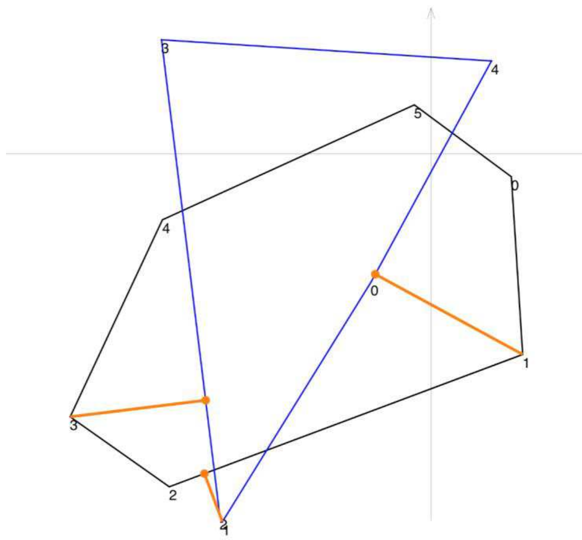

Figure 1. The triple of SSS type vectors compliant with Hypothesis 1

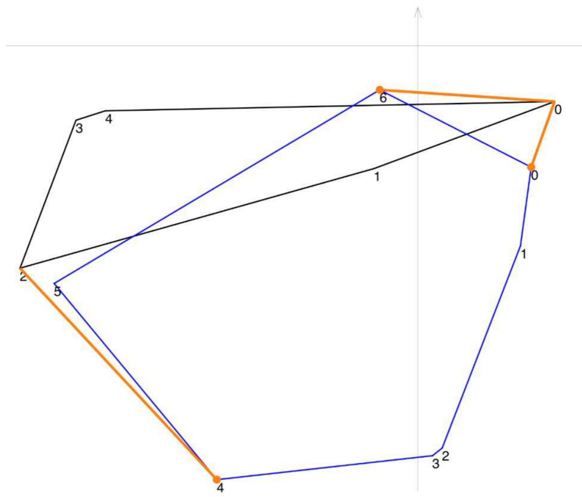

Figure 2. The triple of VVV type vectors compliant with Hypothesis 1

Further, this set of vectors should be reduced to a more limited set. This occurs by crossing the sets of vectors obtained using the algorithm described in the first step, but with a modified direction vector. It is necessary to perform the first stage four times, each time turning the direction vector by 10 degrees relatively to its axis. All the resulting vectors must be crossed to obtain the first part of the set for the search algorithm, see Fig. 5. To obtain the second part of the set, it is necessary to execute the algorithm of intersection of sets of pairs, obtained with the algorithm described in the first step, with direction vectors rotated for 180 degrees, see Fig. 6. The first and second sets obtained are combined and transferred to the full search algorithm.

To test each of the described algorithms, a testing system was developed that included the following components. The main component of the algorithms is the generation of convex polygons. For this purpose, various algorithms that are considered below in more detail were implemented.

The first algorithm is the following one. On a plane, N points are arranged randomly in the following way: $N/4$ points with positive abscissas and ordinates, $N/4$ points with negative abscissas

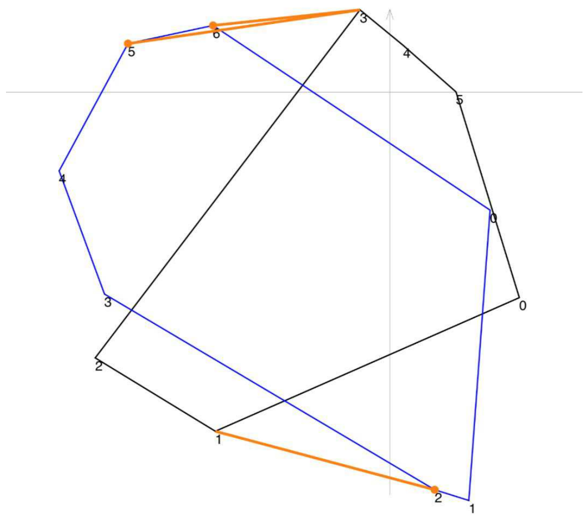

Figure 3. The triple of VVV type vectors non-compliant with Hypothesis 1

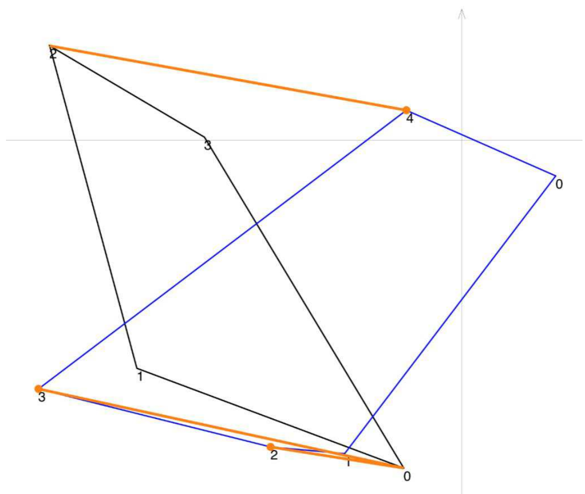

Figure 4. The triple of VVV type vectors non-compliant with Hypothesis 1

and positive ordinates, $N/4$ points with positive abscissas and negative ordinates, $N/4$ points with negative abscissas and ordinates. The next step is to select the point with the lowest abscissa and ordinate values. Relatively to this point, the convex hull of the given set is built using the following algorithm: take the point chosen at the previous stage and choose the next one at the minimum positive turning angle. This algorithm is repeated until the starting point is reached. As a result, a convex polygon is obtained. The downside of this algorithm is that the number of vertices of the polygon obtained at the output cannot be controlled.

The second algorithm is the algorithm for constructing a polygon based on a triangle. At the first stage, the triangle is constructed by placing three arbitrary points on a plane. The input data of the algorithm is the number of vertices for the polygon, which should be provided at the output. To achieve the necessary number of vertices “a point is added to any side of the polygon” iteratively as follows: firstly, an arbitrary side of the polygon is selected, and then a point is chosen between the ends of the segment of this side, so that the polygon remains convex with the added

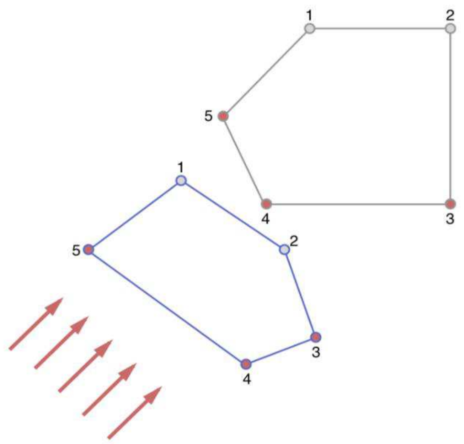

Figure 5. Selection of the vector vertices to perform a search under Hypothesis 2

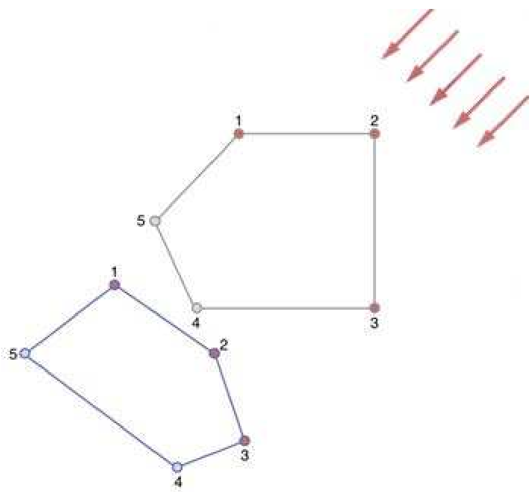

Figure 6. Selection of the vector vertices to perform a search under Hypothesis 2

point. This algorithm is iterated until the required number of vertices is reached.

The third method is manual testing. It was decided to take a set of $20 \sim 30$ polygons different in their construction. The algorithm arbitrarily selects one of them, provided that the second selected polygon (if any) is not similar to the given one.

The described algorithms are used to generate polygons used in the testing of the Hausdorff distance minimization algorithm. At each stage, a pair of polygons is generated, which are sent to the input of one of the algorithms.

To automate the work, the storage systems for the following objects were also created: generated polygons, results of the optimization algorithm, which include the following parameters: the type of algorithm used (full search or some other option), the vector of the second polygon displacement relatively to the first, the type of the set of vectors used to obtain the given displacement vector (V is the vector to the vertex, S is the vector perpendicular to the side of the polygon; the following sets of vectors were considered: $VV, VS, SS, VVV, VVS, VSS, SSS$), the Hausdorff distance obtained after minimization, the number of sets of vectors considered before the desired pair or triple of vectors was found, the list of vectors used in the pair or triple described before. Writing and reading algorithms were developed for the storage system.

Based on the previously described algorithms and storage systems, automated tests were developed that could perform the following functions: generating polygon pairs automatically; saving them to a file for further use; performing a full search algorithm to calculate the costs necessary to minimize the Hausdorff distance; performing one or more of the optimized algorithms; comparing the results to verify the validity of the optimized algorithm; and saving the optimization results.

This testing was performed for all possible combinations of pairs of polygons with 3 to 10 vertices inclusive. The test results for Hypothesis 2 are shown in Table 1. Also, the testing was selectively carried out for polygons with more than 10 vertices. Based on the testing results, the statistical data described in Table 1 were collected, including the number of vertices of polygons A and B, as well as the information about how fewer steps were taken, in percentage of the number of steps in the analytical algorithm, **was deleted** to find the position, at which the minimum Hausdorff distance is reached.

n/m	3	4	5	6	7	8	9	10
3	0%	0-15 %	0-15%	5-15 %	5-15%	10-30%	10-30%	10-30%
4	0-15%	0-20 %	0-20%	0-20 %	0-20%	10-30%	10-30%	15-30%
5	0-15%	10-20 %	10-20%	10-20 %	10-25%	10-30%	10-30%	15-30%
6	5-15%	10-20 %	10-20%	10-20 %	10-25%	10-30%	10-30%	15-30%
7	5-15%	10-20 %	10-20%	10-20 %	10-25%	10-30%	10-30%	15-30%
8	10-30%	10-30 %	10-30%	10-30 %	10-30%	10-30%	10-30%	15-30%
9	10-30%	10-30 %	10-30%	10-30 %	10-30%	10-30%	10-30%	15-30%
10	10-30%	15-30 %	15-30%	15-30 %	15-30%	15-30%	15-30%	15-30%

Table 1. Statistical data obtained by testing Hypothesis 2

The results show that the improvement degree depends on both the geometric features of the polygons and their location. Therefore, the degree of reduction in the number of search steps can vary with the same number of vertices. When selecting polygons, the number of vertices of which does not exceed 10, the reduction in the number of the algorithm steps reaches 30%. The percentage by the average value of which pair of the number of vertices of the polygon A and B grows monotonically. The overall result is a significant acceleration of the algorithm with a number of vertices equal to six or more.

3. Conclusion

As a result, the analytical algorithm was implemented. This result is of independent value, both for subsequent testing of any approaches to optimization, and for any subgradient methods. Two hypotheses were tested. The test of first hypothesis resulted in finding a counter example. As a consequence, the second hypothesis was implemented, for which no counter examples were found on a large and diverse sampling of polygon pairs.

As a result, the algorithm was developed, which ensures finding the precise optimal mutual arrangement of polygons in all the cases tested, despite a significant reduction in the search scope. The advantages achieved are as follows: the ability to solve a large number of practical problems not only accurately, but also quickly; the implemented algorithm combines speed and quality. The only drawback of the algorithm is the absence of a rigorous proof of the fact that vectors determining the optimal position of the polygons will not be ignored in the process of the search reduction. The research can be continued in this direction.

The results of this work can be applied in comparing images [6, 7], recognizing images, recognizing and localizing human faces [8, 9] and emotions on faces [16], as well as in one of the methods of medical imaging based on wave transformation using the Hausdorff distance [21].

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EVALUATION OF SOME NON-ELEMENTARY INTEGRALS INVOLVING SINE, COSINE, EXPONENTIAL AND LOGARITHMIC INTEGRALS: PART I

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Abstract: The non-elementary integrals $\mathrm{Si}_{\beta,\alpha} = \int [\sin{(\lambda x^\beta)}/(\lambda x^\alpha)] dx$, $\beta \geq 1$, $\alpha \leq \beta + 1$ and $\mathrm{Ci}_{\beta,\alpha} = \int [\cos{(\lambda x^\beta)}/(\lambda x^\alpha)] dx$, $\beta \geq 1$, $\alpha \leq 2\beta + 1$, where $\{\beta, \alpha\} \in \mathbb{R}$, are evaluated in terms of the hypergeometric functions ${}_1F_2$ and ${}_2F_3$, and their asymptotic expressions for $|x| \gg 1$ are also derived. The integrals of the form $\int [\sin^n{(\lambda x^\beta)}/(\lambda x^\alpha)] dx$ and $\int [\cos^n{(\lambda x^\beta)}/(\lambda x^\alpha)] dx$, where n is a positive integer, are expressed in terms $\mathrm{Si}_{\beta,\alpha}$ and $\mathrm{Ci}_{\beta,\alpha}$, and then evaluated. $\mathrm{Si}_{\beta,\alpha}$ and $\mathrm{Ci}_{\beta,\alpha}$ are also evaluated in terms of the hypergeometric function ${}_2F_2$. And so, the hypergeometric functions, ${}_1F_2$ and ${}_2F_3$, are expressed in terms of ${}_2F_2$. The exponential integral $\mathrm{Ei}_{\beta,\alpha} = \int (e^{\lambda x^\beta}/x^\alpha) dx$ where $\beta \geq 1$ and $\alpha \leq \beta + 1$ and the logarithmic integral $\mathrm{Li} = \int_\mu^x dt/\ln t$, $\mu > 1$, are also expressed in terms of ${}_2F_2$, and their asymptotic expressions are investigated. For instance, it is found that for $x \gg 2$, $\mathrm{Li} \sim x/\ln x + \ln{(\ln x/\ln 2)} - 2 - \ln 2 \, {}_2F_2(1,1;2,2;\ln 2)$, where the term $\ln{(\ln x/\ln 2)} - 2 - \ln 2 \, {}_2F_2(1,1;2,2;\ln 2)$ is added to the known expression in mathematical literature $\mathrm{Li} \sim x/\ln x$. The method used in this paper consists of expanding the integrand as a Taylor and integrating the series term by term, and can be used to evaluate the other cases which are not considered here. This work is motivated by the applications of sine, cosine exponential and logarithmic integrals in Science and Engineering, and some applications are given.

Key words: Non-elementary integrals, Sine integral, Cosine integral, Exponential integral, Logarithmic integral, Hyperbolic sine integral, Hyperbolic cosine integral, Hypergeometric functions, Asymptotic evaluation, Fundamental theorem of calculus.

1. Introduction

Definition 1. *An elementary function is a function of one variable constructed using that variable and constants, and by performing a finite number of repeated algebraic operations involving exponentials and logarithms. An indefinite integral which can be expressed in terms of elementary functions is an elementary integral. And if, on the other hand, it cannot be evaluated in terms of elementary functions, then it is non-elementary* [6, 10].

Liouville 1938's Theorem gives conditions to determine whether a given integral is elementary or non-elementary [6, 10]. For instance, it was shown in [6, 10], using Liouville 1938's Theorem, that the integral $\mathrm{Si}_{1,1} = \int (\sin x/x) dx$ is non-elementary. With similar arguments as in [6, 10], One can show that $\mathrm{Ci}_{1,1} = \int (\cos x/x) dx$ is also non-elementary. Using the Euler formulas $e^{\pm ix} = \cos x \pm i \sin x$, and noticing that if the integral of a function $g(x)$ is elementary, then both its real and imaginary parts are elementary [6], one can, for instance, prove that the integrals $\mathrm{Si}_{\beta,\alpha} = \int [\sin{(\lambda x^\beta)}/(\lambda x^\alpha)] dx, \beta \geq 1, \alpha \geq 1$, and $\mathrm{Ci}_{\beta,\alpha} = \int [\cos{(\lambda x^\beta)}/(\lambda x^\alpha)] dx$, where $\beta \geq 1$ and $\alpha \geq 1$, are non-elementary by using the fact that their real and imaginary parts are non-elementary. The integrals $\int [\sin^n{(\lambda x^\beta)}/(\lambda x^\alpha)] dx$ and $\int [\cos^n{(\lambda x^\beta)}/(\lambda x^\alpha)] dx$, where n is a positive integer, are also non-elementary since they can be expressed in terms of $\mathrm{Si}_{\beta,\alpha}$ and $\mathrm{Ci}_{\beta,\alpha}$.

To my knowledge, no one has evaluated these integrals before. To this end, in this paper, formulas for these non-elementary integrals are expressed in terms of the hypergeometric functions ${}_1F_2$ and ${}_2F_3$ whose properties, for example, the asymptotic expansions for large argument ($|\lambda x| \gg 1$), are known [9]. We do so by expanding the integrand in terms of its Taylor series and by integrating the series term by term as in [7]. And therefore, their corresponding definite integrals can be evaluated using the Fundamental Theorem of Calculus (FTC). For example, the sine integral

$$\mathrm{Si}_{\beta,\alpha} = \int_A^B \frac{\sin(\lambda x^\beta)}{(\lambda x^\alpha)} dx, \quad \beta \geq 1, \quad \alpha \leq \beta + 1,$$

is evaluated for any A and B using the FTC.

On the other hand, the integrals $\mathrm{Ei}_{\beta,\alpha} = \int (e^{\lambda x^\beta}/x^\alpha) dx$ and $\int dx/\ln x$, are expressed in terms of the hypergeometric function ${}_2F_2$. This is quite important since one may re-investigate the asymptotic behavior of the exponential (Ei) and logarithmic (Li) integrals [3] using the asymptotic expressions of the hypergeometric function ${}_2F_2$ which are known [9].

Some other non-elementary integrals which can be written in terms of $\mathrm{Ei}_{\beta,\alpha}$ or $\int dx/\ln x$ are also evaluated. For instance, as a result of substitution, the integral $\int e^{\lambda e^{\beta x}} dx$ is written in terms of $\mathrm{Ei}_{\beta,1} = \int (e^{\lambda x^\beta}/x) dx$ and then evaluated in terms of ${}_2F_2$, and using integration by parts, the integral $\int \ln(\ln x) dx$ is written in terms of $\int dx/\ln x$ and then evaluated in terms of ${}_2F_2$ as well.

Using the Euler identity $e^{\pm ix} = \cos(x) \pm i \sin(x)$ or the hyperbolic identity $e^{\pm x} = \cosh(x) \pm \sinh(x)$, $\mathrm{Si}_{\beta,\alpha}$ and $\mathrm{Ci}_{\beta,\alpha}$ are evaluated in terms $\mathrm{Ei}_{\beta,\alpha}$. And hence, the hypergeometric functions ${}_1F_2$ and ${}_2F_3$ are expressed in terms of the hypergeometric ${}_2F_2$.

This type of integrals find applications in many fields in Science and Engineering. For instance, in wireless telecommunications, the random attenuation capacity of a channel, known as fading capacity, is calculated as [11]

$$C_{\text{fading}} = E[\log_2(1 + P|H|^2)] = \int_0^\infty \log_2(1 + P\xi) e^{-\xi} d\xi = \frac{1}{\ln 2} e^{1/P} \left[E_{1,1}(\infty) - E_{1,1}\left(\frac{1}{P}\right)\right],$$

where the fading coefficient H is a complex Gaussian random variable, and $E\left(|X|^2 \leq P\right)$ is the maximum average transmitted power of a complex-valued channel input X. In number theory, the prime number theorem states that [3]

$$\pi(x) \sim \mathrm{Li}(x) = \int_\mu^x \frac{dx}{\ln x}, \quad \mu > 1,$$

where $\pi(x)$ denotes the number of primes small than or equal to x. Moreover, there are applications of sine and cosine integrals in electromagnetic theory, see for example Lebedev [5]. Therefore, it is quite important to adequately evaluate these integrals.

For that reason, the main goal of this paper is to evaluate non-elementary integrals of sine, cosine, exponential and logarithmic integrals type in terms of elementary and special functions with well known properties so that the fundamental theorem of calculus can be used so that we can avoid to use numerical integration.

Part I is indeed devoted to the cases $\mathrm{Si}_{\beta,\alpha} = \int [\sin(\lambda x^\beta)/(\lambda x^\alpha)] dx$, $\beta \geq 1$, $\alpha \leq \beta + 1$, $\mathrm{Ci}_{\beta,\alpha} = \int [\cos(\lambda x^\beta)/(\lambda x^\alpha)] dx$, $\beta \geq 1$, $\alpha \leq 2\beta + 1$ and $\mathrm{Ei}_{\beta,\alpha} = \int (e^{\lambda x^\beta}/x^\alpha) dx$ where $\beta \geq 1$, $\alpha \leq \beta + 1$, where $\{\beta, \alpha\} \in \mathbb{R}$. The other cases $\mathrm{Si}_{\beta,\alpha} = \int [\sin(\lambda x^\beta)/(\lambda x^\alpha)] dx$, $\beta \geq 1$, $\alpha > \beta + 1$, $\mathrm{Ci}_{\beta,\alpha} = \int [\cos(\lambda x^\beta)/(\lambda x^\alpha)] dx$, $\beta \geq 1$, $\alpha > 2\beta + 1$ and $\mathrm{Ei}_{\beta,\alpha} = \int (e^{\lambda x^\beta}/x^\alpha) dx$ where $\beta \geq 1$, $\alpha > \beta + 1$, where $\{\beta, \alpha\} \in \mathbb{R}$, which may involve series whose properties are not necessary known will be considered in Part 2 [8].

Before we proceed to the objectives of this paper (see sections 2, 3, 4 and 5), we first define the generalized hypogeometric function as it is an important mathematical that we are going to use throughout the paper.

Definition 2. *The generalized hypergeometric function, denoted as ${}_pF_q$, is a special function given by the series* [1, 9]

$${}_pF_q(a_1, a_2, \cdots, a_p; b_1, b_2, \cdots, b_q; x) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \frac{x^n}{n!},$$

where $a_1, a_2, \cdots, a_p$ and $;b_1, b_2, \cdots, b_q$ are arbitrary constants, $(\vartheta)_n = \Gamma(\vartheta + n)/\Gamma(\vartheta)$ (Pochhammer's notation [1]*) for any complex ϑ, with $(\vartheta)_0 = 1$, and Γ is the standard gamma function* [1, 9].

2. Evaluation of the sine integral and related integrals

Proposition 1. *The function $G(x) = x\, {}_1F_2\left(\frac{1}{2}; \frac{3}{2}, \frac{3}{2}; -\frac{\lambda^2 x^2}{4}\right)$, where ${}_1F_2$ is a hypergeometric function* [1] *and λ is an arbitrarily constant, is the antiderivative of the function $g(x) = \frac{\sin(\lambda x)}{\lambda x}$. Thus,*

$$\int \frac{\sin(\lambda x)}{\lambda x} dx = x\, {}_1F_2\left(\frac{1}{2}; \frac{3}{2}, \frac{3}{2}; -\frac{\lambda^2 x^2}{4}\right) + C.$$

P r o o f. To prove Proposition 1, we expand $g(x)$ as Taylor series and integrate the series term by term. We also use the gamma duplication formula [1]

$$\Gamma(2\alpha) = (2\pi)^{-\frac{1}{2}} 2^{2\alpha - \frac{1}{2}} \Gamma(\alpha) \Gamma\left(\alpha + \frac{1}{2}\right), \quad \alpha \in \mathbb{C},$$

the Pochhammer's notation for the gamma function [1],

$$(\alpha)_n = \alpha(\alpha + 1) \cdots (\alpha + n - 1) = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)}, \quad \alpha \in \mathbb{C},$$

and the property of the gamma function $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ (eg., $\Gamma(n + 3/2) = (n + 1/2)\Gamma(n + 1/2)$ for any real n). We then obtain

$$\begin{aligned}
\int g(x)dx &= \int \frac{\sin(\lambda x)}{\lambda x} dx = \int \frac{1}{\lambda x} \sum_{n=0}^{\infty} (-1)^n \frac{(\lambda x)^{2n+1}}{(2n+1)!} dx = \lambda \sum_{n=0}^{\infty} (-1)^n \frac{\lambda^{2n}}{(2n+1)!} \frac{x^{2n+1}}{2n+1} + C \\
&= \frac{x}{2} \sum_{n=0}^{\infty} (-1)^n \frac{\lambda^{2n}}{(2n+1)!} \frac{x^{2n}}{n + 1/2} + C = \frac{x}{2} \sum_{n=0}^{\infty} \frac{\Gamma(n + 1/2)}{\Gamma(2n+2)\Gamma(n + 3/2)} (-\lambda^2 x^2)^n + C \\
&= x \sum_{n=0}^{\infty} \frac{(1/2)_n}{(3/2)_n (3/2)_n} \frac{\left(-\lambda^2 x^2/4\right)^n}{n!} + C = x\, {}_1F_2\left(\frac{1}{2}; \frac{3}{2}, \frac{3}{2}; -\frac{\lambda^2 x^2}{4}\right) + C = G(x) + C.
\end{aligned}$$

□

In the following lemma, we assume that the function $G(x)$ is unknown and therefore we establish its properties such as the inflection points and its behaviour as $x \to \pm\infty$.

Lemma 1. *Let $G(x)$ be the antiderivative for $g(x) = \frac{\sin x}{x}$ $(\lambda = 1)$, and $G(0) = 0$.*

1. *Then $G(x)$ is linear around $x = 0$ and the point $(0, G(0)) = (0, 0)$ is an inflection point of the curve $Y = G(x)$, $x \in \mathbb{R}$.*

2. *And $\lim_{x\to-\infty} G(x) = -\theta$ while $\lim_{x\to+\infty} G(x) = \theta$, where θ is a positive finite constant.*

P r o o f.

1. The series
$$g(x) = \frac{\sin x}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{(\lambda x)^{2n}}{(2n+1)!}$$
gives $G'(0) = g(0) = 1$. Then, around $x = 0$, $G(x) \sim x$ since $G'(0) = g(0) = 1$ and $G(0) = 0$. Moreover,
$$G''(x) = g'(x) = \left(\frac{\sin x}{x}\right)' = -\lambda^2 x \sum_{n=0}^{\infty} (-1)^n \frac{(2n+2)(\lambda x)^{2n}}{(2n+3)!},$$
and so $G''(0) = g'(0) = 0$. Hence, by the second derivative test, the point $(0, G(0)) = (0, 0)$ is an inflection point of the curve $Y = G(x)$.

2. It is straight forward, using Squeeze theorem, to obtain $\lim_{x\to-\infty} g(x) = \lim_{x\to+\infty} g(x) = 0$. And since both $g(x)$ and $G(x)$ are analytic on $\mathbb{R}$, then $G(x)$ has to be constant as $x \to \pm\infty$ by Liouville Theorem (section 3.1.3 in [4]) since if a complex function is entire on $\mathbb{C}$ then both its imaginary and real parts are analytic on the real line $\mathbb{R}$ including at $x \to \pm\infty$. Also, there exists some numbers $\delta > 0$ and ϵ such that if $|x| > \delta$ then $||\sin x|/x - 1/x| < \epsilon$, and $\lim_{x\to-\infty} (|\sin x|/x)/(1/x) = \lim_{x\to+\infty} (|\sin x|/x)/(1/x) = \pm 1$. This makes the function $g_1(x) = -1/x$ an envelop of $g(x)$ away from $x = 0$ if $\sin x < 0$ and $g_2(x) = 1/x$ an envelop of $g(x)$ away from $x = 0$ if $\sin x > 0$. Moreover, on one hand, $g_2' \le G'' \le g_1'$ if $x < -\delta$, and g_1' and g_2' do not change signs. While on another hand, $g_1' \le G'' \le g_2'$ if $x > \delta$, and also g_1' and g_2' do not change signs. Therefore there exists some number $\theta > 0$ such $G(x)$ oscillates about θ if $x > \delta$ and $G(x)$ oscillates about $-\theta$ if $x < -\delta$. And $|G(x)| \le \theta$ if $|x| \le \delta$. □

Example 1. For instance, if $\lambda = 1$, then
$$\int \frac{\sin x}{x} dx = x\,{}_1F_2\left(\frac{1}{2}; \frac{3}{2}, \frac{3}{2}; -\frac{x^2}{4}\right) + C. \tag{2.1}$$

By Proposition 1, the antiderivative of $g(x) = \dfrac{\sin x}{x}$ is $G(x) = x\,{}_1F_2\left(\frac{1}{2}; \frac{3}{2}, \frac{3}{2}; -\frac{x^2}{4}\right)$, and the graph of $G(x)$ is shown in Figure 1. It is in agreement with Lemma 1. It is seen in Figure 1 that $(0, G(0)) = (0, 0)$ is an inflection point and that G attains some constants as $x \to \pm\infty$ as predicted by Lemma 1.

In the following lemma, we obtain the values of $G(x)$, the antiderivative of the function $g(x) = \sin(\lambda x)/(\lambda x)$, as $x \to \pm\infty$ using the asymptotic expansion of the hypergeometric function ${}_1F_2$.

Lemma 2. *Consider $G(x)$ in Proposition 1,and preferably assume that $\lambda > 0$.*

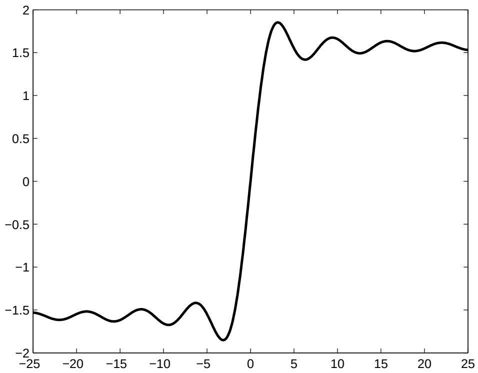

Figure 1. $G(x)$ is the antiderivative of $\sin(x)/x$ given in (2.1).

1. *Then,*

$$G(-\infty) = \lim_{x\to-\infty} G(x) = \lim_{x\to\infty} x\ {}_1F_2\left(\frac{1}{2}; \frac{3}{2}, \frac{3}{2}; -\frac{\lambda^2 x^2}{4}\right) = -\frac{\pi}{2\lambda}, \tag{2.2}$$

and

$$G(+\infty) = \lim_{x\to+\infty} G(x) = \lim_{x\to\infty} x\ {}_1F_2\left(\frac{1}{2}; \frac{3}{2}, \frac{3}{2}; -\frac{\lambda^2 x^2}{4}\right) = \frac{\pi}{2\lambda}. \tag{2.3}$$

2. *And by the FTC,*

$$\int\limits_{-\infty}^{\infty} \frac{\sin(\lambda x)}{\lambda x} dx = G(+\infty) - G(-\infty) = \frac{\pi}{2\lambda} - \left(-\frac{\sqrt{\pi}}{2\lambda}\right) = \frac{\pi}{\lambda}. \tag{2.4}$$

Proof.

1. To prove (2.2) and (2.3), we use the asymptotic formula for the hypergeometric function ${}_1F_2$ which is valid for $|z| \gg 1$ and $-\pi \leq \arg z \leq \pi$, where $\arg z$ is the argument of z in the complex plane. It can be derived using formulas (16.11.1), (16.11.2) and (16.11.8) in [9] and is given by

$$\begin{aligned}
&{}_1F_2\left(a_1; b_1, b_2; -z\right) = \\
&\Gamma(b_1)\Gamma(b_2) z^{-a_1}\left\{\sum_{n=0}^{R-1} \frac{(a_1)_n}{\Gamma(b_1 - a_1 - n)\Gamma(b_2 - a_1 - n)} \frac{(-z)^{-n}}{n!} + O(|z|^{-R})\right\} \\
&+ \frac{\Gamma(b_1)\Gamma(b_2)}{\Gamma(a_1)} + \frac{e^{2z^{1/2} e^{-i\pi/2}} (ze^{-i\pi})^{(a_1 - b_1 - b_2 + 1/2)/2}}{\sqrt{\pi}} \left\{\sum_{n=0}^{S-1} \frac{\mu_n}{2^{n+1}} (ze^{-i\pi})^{-n} + O(|z|^{-S})\right\} \\
&+ \frac{\Gamma(b_1)\Gamma(b_2)}{\Gamma(a_1)} \frac{e^{2z^{1/2} e^{i\pi/2}} (ze^{i\pi})^{(a_1 - b_1 - b_2 + 1/2)/2}}{\sqrt{\pi}} \left\{\sum_{n=0}^{S-1} \frac{\mu_n}{2^{n+1}} (ze^{i\pi})^{-n} + O(|z|^{-S})\right\},
\end{aligned} \tag{2.5}$$

where a_1, b_1 and b_2 are constants and the coefficient μ_n is given by formula (16.11.4) in [9]. We then set $z = \lambda^2 x^2/4$, $a_1 = 1/2$, $b_1 = 3/2$ and $b_2 = 3/2$, and we obtain

$${}_1F_2\left(\frac{1}{2}; \frac{3}{2}, \frac{3}{2}; -\frac{\lambda^2 x^2}{4}\right) = \frac{\pi}{2}\left(\lambda^2 x^2\right)^{-1/2}\left\{\sum_{n=0}^{R-1} \frac{(1/2)_n}{n!}\left(i\frac{\lambda x}{2}\right)^{-2n} + O\left(\left|\frac{\lambda x}{2}\right|^{-2R}\right)\right\}$$

$$-\frac{\sqrt{\pi}}{\lambda^2x^2}\frac{e^{-i\lambda x}}{2}\left\{\sum_{n=0}^{S-1}\frac{\mu_n}{2^n}\left(-i\frac{\lambda x}{2}\right)^{-2n}+O\left(\left|\frac{\lambda x}{2}\right|^{-2S}\right)\right\}$$
$$-\frac{\sqrt{\pi}}{\lambda^2x^2}\frac{e^{i\lambda x}}{2}\left\{\sum_{n=0}^{S-1}\frac{\mu_n}{2^n}\left(i\frac{\lambda x}{2}\right)^{-2n}+O\left(\left|\frac{\lambda x}{2}\right|^{-2S}\right)\right\}.$$

Then, for $|x|\gg 1$,

$$\frac{\pi}{2}\left(\lambda^2x^2\right)^{-1/2}\left\{\sum_{n=0}^{R-1}\frac{(1/2)_n}{n!}\left(i\frac{\lambda x}{2}\right)^{-2n}+O\left(\left|\frac{\lambda x}{2}\right|^{-2R}\right)\right\}\sim\frac{\pi}{2\lambda|x|},$$

while

$$-\frac{\sqrt{\pi}}{\lambda^2x^2}\frac{e^{i\lambda x}}{2}\left\{\sum_{n=0}^{S-1}\frac{\mu_n}{2^n}\left(-i\frac{\lambda x}{2}\right)^{-2n}-\sum_{n=0}^{S-1}\frac{\mu_n}{2^n}\left(i\frac{\lambda x}{2}\right)^{-2n}+O\left(\left|\frac{\lambda x}{2}\right|^{-2S}\right)\right\}$$
$$\sim\frac{\sqrt{\pi}}{(\lambda x)^2}\frac{e^{i\lambda x}+e^{-i\lambda x}}{2}=\sqrt{\pi}\frac{\cos(\lambda x)}{(\lambda x)^2}.$$

We then obtain,

$$x\,{}_1F_2\left(\frac{1}{2};\frac{3}{2},\frac{3}{2};-\frac{\lambda^2x^2}{4}\right)\sim\frac{\pi}{2\lambda}\frac{x}{|x|}-\frac{\sqrt{\pi}}{\lambda}\frac{\cos(\lambda x)}{\lambda x},\quad |x|\to\infty.$$

Hence,

$$G(-\infty)=\lim_{x\to-\infty}x\,{}_1F_2\left(\frac{1}{2};\frac{3}{2},\frac{3}{2};-\frac{\lambda^2x^2}{4}\right)=\lim_{x\to-\infty}\left(\frac{\pi}{2\lambda}\frac{x}{|x|}-\frac{\sqrt{\pi}}{\lambda}\frac{\cos(\lambda x)}{\lambda x}\right)=-\frac{\pi}{2\lambda}$$

and

$$G(+\infty)=\lim_{x\to+\infty}x\,{}_1F_2\left(\frac{1}{2};\frac{3}{2},\frac{3}{2};-\frac{\lambda^2x^2}{4}\right)=\lim_{x\to+\infty}\left(\frac{\pi}{2\lambda}\frac{x}{|x|}-\frac{\sqrt{\pi}}{\lambda}\frac{\cos(\lambda x)}{\lambda x}\right)=\frac{\pi}{2\lambda}.$$

2. By the FTC,

$$\int_{-\infty}^{+\infty}\frac{\sin(\lambda x)}{\lambda x}dx=\lim_{y\to-\infty}\int_{y}^{0}\frac{\sin(\lambda x)}{\lambda x}dx+\lim_{y\to+\infty}\int_{0}^{y}\frac{\sin(\lambda x)}{\lambda x}dx=G(+\infty)-G(-\infty)$$
$$=\lim_{y\to+\infty}y\,{}_1F_2\left(\frac{1}{2};\frac{3}{2},\frac{3}{2};-\frac{\lambda^2y^2}{4}\right)-\lim_{y\to-\infty}y\,{}_1F_2\left(\frac{1}{2};\frac{3}{2},\frac{3}{2};-\frac{\lambda^2y^2}{4}\right)=\frac{\pi}{\lambda}.$$

We now verify whether this is correct or not using Fubini's Theorem [2]. We first observe that

$$\int_{-\infty}^{+\infty}\frac{\sin(\lambda x)}{\lambda x}dx=2\int_{0}^{+\infty}\frac{\sin(\lambda x)}{\lambda x}dx$$

since the integrand is an even function. We have in terms of double integrals and using Fubini's Theorem that

$$\int_{0}^{+\infty}\frac{\sin(\lambda x)}{\lambda x}dx=\frac{1}{\lambda}\int_{0}^{+\infty}\int_{0}^{+\infty}e^{-sx}\sin(\lambda x)dsdx=\int_{0}^{+\infty}\int_{0}^{+\infty}e^{-sx}\sin(\lambda x)dxds.\tag{2.6}$$

Now using the fact that the inside integral in (2.6) is the Laplace transform of $\sin(\lambda x)$ [1] yields

$$\int_0^{+\infty}\int_0^{+\infty} e^{-sx}\sin(\lambda x)dxds = \int_0^{+\infty} \frac{\lambda}{s^2+\lambda^2}ds = \arctan(+\infty) - \arctan 0 = \frac{\pi}{2}.$$

Hence,

$$\int_{-\infty}^{+\infty} \frac{\sin(\lambda x)}{\lambda x}dx = 2\int_0^{+\infty} \frac{\sin(\lambda x)}{\lambda x}dx = \frac{2}{\lambda}\int_0^{+\infty}\int_0^{+\infty} e^{-sx}\sin(\lambda x)dsdx = 2\frac{\pi}{2\lambda} = \frac{\pi}{\lambda}$$

as before. □

Setting $\lambda = 1$ as in Example 1, Lemma 2 gives $\lim_{x\to-\infty} G(x) = -\theta = -\pi/2$ while $\lim_{x\to+\infty} G(x) = \theta = \pi/2$. And these are the exact values of $G(x)$ as $x \to \pm\infty$ in Figure 1.

Theorem 1. *If $\beta \geq 1$ and $\alpha < \beta + 1$, then the function*

$$G(x) = \frac{x^{\beta-\alpha+1}}{\beta-\alpha+1}\,{}_1F_2\left(-\frac{\alpha}{2\beta}+\frac{1}{2\beta}+\frac{1}{2};-\frac{\alpha}{2\beta}+\frac{1}{2\beta}+\frac{3}{2},\frac{3}{2};-\frac{\lambda^2x^{2\beta}}{4}\right),$$

where ${}_1F_2$ is a hypergeometric function [1] and λ is an arbitrarily constant, is the antiderivative of the function $g(x) = \dfrac{\sin(\lambda x^\beta)}{\lambda x^\alpha}$. Thus,

$$Si_{\beta,\alpha} = \int \frac{\sin(\lambda x^\beta)}{\lambda x^\alpha}dx = \frac{x^{\beta-\alpha+1}}{\beta-\alpha+1}\,{}_1F_2\left(-\frac{\alpha}{2\beta}+\frac{1}{2\beta}+\frac{1}{2};-\frac{\alpha}{2\beta}+\frac{1}{2\beta}+\frac{3}{2},\frac{3}{2};-\frac{\lambda^2x^{2\beta}}{4}\right)+C. \tag{2.7}$$

And for $|x| \gg 1$,

$$\begin{aligned}&\frac{x^{\beta-\alpha+1}}{\beta-\alpha+1}\,{}_1F_2\left(-\frac{\alpha}{2\beta}+\frac{1}{2\beta}+\frac{1}{2};-\frac{\alpha}{2\beta}+\frac{1}{2\beta}+\frac{3}{2},\frac{3}{2};-\frac{\lambda^2x^{2\beta}}{4}\right)\\ &\sim \frac{(2/\lambda)^{1+1/\beta-\alpha/\beta}}{\beta-\alpha+1}\frac{\Gamma(-\alpha/(2\beta)+1/(2\beta)+3/2)}{\Gamma(1+\alpha/(2\beta)-1/(2\beta))}\frac{\sqrt{\pi}}{2}\frac{x^{\beta-\alpha+1}}{|x|^{\beta-\alpha+1}} - \frac{\beta-\alpha+1}{\beta}\frac{\sqrt{\pi}}{\lambda^2}\frac{\cos(\lambda x^\beta)}{x^{\beta+\alpha-1}}.\end{aligned} \tag{2.8}$$

P r o o f.

$$\begin{aligned}\mathrm{Si}_{\beta,\alpha} &= \int g(x)dx = \int \frac{\sin(\lambda x^\beta)}{\lambda x^\alpha}dx = \int \frac{1}{\lambda x^\alpha}\sum_{n=0}^{\infty}(-1)^n\frac{(\lambda x^\beta)^{2n+1}}{(2n+1)!}dx\\ &= \sum_{n=0}^{\infty}(-1)^n\frac{\lambda^{2n}}{(2n+1)!}\int x^{2\beta n+\beta-\alpha}dx = \lambda\sum_{n=0}^{\infty}(-1)^n\frac{\lambda^{2n}}{(2n+1)!}\int x^{2\beta n+\beta-\alpha}dx\\ &= \sum_{n=0}^{\infty}(-1)^n\frac{\lambda^{2n}}{(2n+1)!}\frac{x^{2\beta n+\beta-\alpha+1}}{2\beta n+\beta-\alpha+1}+C\\ &= \frac{x^{\beta-\alpha+1}}{2\beta}\sum_{n=0}^{\infty}(-1)^n\frac{\lambda^{2n}}{(2n+1)!}\frac{x^{2\beta n}}{n-\alpha/(2\beta)+1/(2\beta)+1/2}+C\end{aligned}$$

$$= \frac{x^{\beta-\alpha+1}}{2\beta} \sum_{n=0}^{\infty} \frac{\Gamma\left(n - \alpha/(2\beta) + 1/(2\beta) + 1/2\right)}{\Gamma(2n+2)\Gamma\left(n - \alpha/(2\beta) + 1/(2\beta) + 3/2\right)} (-\lambda^2 x^{2\beta})^n + C$$

$$= \frac{x^{\beta-\alpha+1}}{\beta-\alpha+1} \sum_{n=0}^{\infty} \frac{\left(-\alpha/(2\beta) + 1/(2\beta) + 1/2\right)_n}{(3/2)_n \left(-\alpha/(2\beta) + 1/(2\beta) + 3/2\right)_n} \frac{\left(-\lambda^2 x^{2\beta}/4\right)^n}{n!} + C$$

$$= \frac{x^{\beta-\alpha+1}}{\beta-\alpha+1} \, {}_1F_2\left(-\frac{\alpha}{2\beta} + \frac{1}{2\beta} + \frac{1}{2}; -\frac{\alpha}{2\beta} + \frac{1}{2\beta} + \frac{3}{2}, \frac{3}{2}; -\frac{\lambda^2 x^{2\beta}}{4}\right) + C = G(x) + C.$$

To prove (2.8), we use the asymptotic formula for the hypergeometric function ${}_1F_2$ in equation (2.5), and proceed as in Lemma 2. □

Beside, we can show as above that if $\beta \geq 1$ and $\alpha < \beta + 1$, then

$$\int \frac{\sinh(\lambda x^{\beta})}{\lambda x^{\alpha}} dx = \frac{x^{\beta-\alpha+1}}{\beta-\alpha+1} \, {}_1F_2\left(-\frac{\alpha}{2\beta} + \frac{1}{2\beta} + \frac{1}{2}; -\frac{\alpha}{2\beta} + \frac{1}{2\beta} + \frac{3}{2}, \frac{3}{2}; \frac{\lambda^2 x^{2\beta}}{4}\right) + C. \tag{2.9}$$

Corollary 1. *Let $\beta = \alpha$. If $\alpha \geq 1$, then*

$$\int_{-\infty}^{0} \frac{\sin(\lambda x^{\alpha})}{\lambda x^{\alpha}} dx = G(0) - G(-\infty) = \left(\frac{2}{\lambda}\right)^{1/\alpha} \frac{\Gamma\left(1/(2\alpha) + 1\right)}{\Gamma\left(3/2 - 1/(2\alpha)\right)} \frac{\sqrt{\pi}}{2}, \tag{2.10}$$

$$\int_{0}^{+\infty} \frac{\sin(\lambda x^{\alpha})}{\lambda x^{\alpha}} dx = G(+\infty) - G(0) = \left(\frac{2}{\lambda}\right)^{1/\alpha} \frac{\Gamma\left(1/(2\alpha) + 1\right)}{\Gamma\left(3/2 - 1/(2\alpha)\right)} \frac{\sqrt{\pi}}{2} \tag{2.11}$$

and

$$\int_{-\infty}^{+\infty} \frac{\sin(\lambda x^{\alpha})}{\lambda x^{\alpha}} dx = G(+\infty) - G(-\infty) = \left(\frac{2}{\lambda}\right)^{1/\alpha} \frac{\Gamma\left(1/(2\alpha) + 1\right)}{\Gamma\left(3/2 - 1/(2\alpha)\right)} \sqrt{\pi}. \tag{2.12}$$

P r o o f. If $\beta = \alpha$, Theorem 1 gives

$$G(-\infty) = \lim_{x \to -\infty} x \, {}_1F_2\left(\frac{1}{2\alpha}; \frac{1}{2\alpha} + 1, \frac{3}{2}; -\frac{\lambda^2 x^{2\alpha}}{4}\right)$$

$$= \lim_{x \to -\infty} \left(\left(\frac{2}{\lambda}\right)^{1/\alpha} \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(1/(2\alpha) + 1\right)}{\Gamma\left(3/2 - 1/(2\alpha)\right)} \frac{x}{|x|} - \frac{\sqrt{\pi}}{\alpha\lambda^2} \frac{\cos(\lambda x^{\alpha})}{x^{2\alpha-1}}\right) = -\left(\frac{2}{\lambda}\right)^{1/\alpha} \frac{\Gamma\left(1/(2\alpha) + 1\right)}{\Gamma\left(3/2 - 1/(2\alpha)\right)} \frac{\sqrt{\pi}}{2}$$

and

$$G(+\infty) = \lim_{x \to +\infty} x \, {}_1F_2\left(\frac{1}{2\alpha}; \frac{1}{2\alpha} + 1, \frac{3}{2}; -\frac{\lambda^2 x^{2\alpha}}{4}\right)$$

$$= \lim_{x \to +\infty} \left(\left(\frac{2}{\lambda}\right)^{1/\alpha} \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(1/(2\alpha) + 1\right)}{\Gamma\left(3/2 - 1/(2\alpha)\right)} \frac{x}{|x|} - \frac{\sqrt{\pi}}{\alpha\lambda^2} \frac{\cos(\lambda x^{\alpha})}{x^{2\alpha-1}}\right) = \left(\frac{2}{\lambda}\right)^{1/\alpha} \frac{\Gamma\left(1/(2\alpha) + 1\right)}{\Gamma\left(3/2 - 1/(2\alpha)\right)} \frac{\sqrt{\pi}}{2}.$$

Hence, by the FTC,

$$\int_{-\infty}^{0} \frac{\sin(\lambda x^\alpha)}{\lambda x^\alpha} dx = G(0) - G(-\infty) = 0 - \left(- \left(\frac{2}{\lambda}\right)^{1/\alpha} \frac{\Gamma(1/(2\alpha)+1)}{\Gamma(3/2 - 1/(2\alpha))} \frac{\sqrt{\pi}}{2} \right)$$

$$= \left(\frac{2}{\lambda}\right)^{1/\alpha} \frac{\Gamma(1/(2\alpha)+1)}{\Gamma(3/2 - 1/(2\alpha))} \frac{\sqrt{\pi}}{2}, \tag{2.13}$$

$$\int_{0}^{+\infty} \frac{\sin(\lambda x^\alpha)}{\lambda x^\alpha} dx = G(+\infty) - G(0) = \left(\frac{2}{\lambda}\right)^{1/\alpha} \frac{\Gamma(1/(2\alpha)+1)}{\Gamma(3/2 - 1/(2\alpha))} \frac{\sqrt{\pi}}{2} - 0$$

$$= \left(\frac{2}{\lambda}\right)^{1/\alpha} \frac{\Gamma(1/(2\alpha)+1)}{\Gamma(3/2 - 1/(2\alpha))} \frac{\sqrt{\pi}}{2}. \tag{2.14}$$

And combining (2.13) and (2.14) gives (2.12). □

Theorem 2. *If $\beta \geq 1$ and $\alpha < \beta + 1$, then the FTC gives*

$$\int_{A}^{B} \frac{\sin(\lambda x^\beta)}{\lambda x^\alpha} dx = G(B) - G(A), \tag{2.15}$$

for any A and any B, and where G is given in Theorem 1.

P r o o f. Equation (2.15) holds by Theorem 1, Corollary 1 and Lemma 2. Since the FTC works for $A = -\infty$ and $B = 0$ in (2.10), $A = 0$ and $B = +\infty$ in (2.11) and $A = -\infty$ and $B = +\infty$ in (2.12) by Corollary 1 for any $\beta = \alpha \geq 1$ and by Lemma 2 for $\beta = \alpha = 1$, then it has to work for other values of $A, B \in \mathbb{R}$ and for $\beta \geq 1$ and $\alpha < \beta + 1$ since the case with $\beta = \alpha \geq 1$ is derived from the case with $\beta \geq 1$ and $\alpha < \beta + 1$. □

Theorem 3. *Let $\beta \geq 1$, then the function*

$$G(x) = \ln|x| - \frac{\left(\lambda x^\beta/2\right)^2}{6\beta} {}_2F_3\left(1,1;2,2,\frac{5}{2}; -\frac{\lambda^2 x^{2\beta}}{4}\right),$$

where ${}_2F_3$ is a hypergeometric function [1] and λ is an arbitrarily constant, is the antiderivative of the function $g(x) = \dfrac{\sin(\lambda x^\beta)}{\lambda x^{\beta+1}}$. Thus,

$$Si_{\beta,\beta+1} = \int \frac{\sin(\lambda x^\beta)}{\lambda x^{\beta+1}} dx = \ln|x| - \frac{(\lambda x^\beta/2)^2}{6\beta} {}_2F_3\left(1,1;2,2,\frac{5}{2}; -\frac{\lambda^2 x^{2\beta}}{4}\right) + C.$$

P r o o f.

$$\mathrm{Si}_{\beta,\beta+1} = \int g(x)dx = \int \frac{\sin(\lambda x^\beta)}{\lambda x^{\beta+1}} dx = \int \frac{1}{\lambda x^{\beta+1}} \sum_{n=0}^{\infty} (-1)^n \frac{(\lambda x^\beta)^{2n+1}}{(2n+1)!} dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{\lambda^{2n}}{(2n+1)!} \int x^{2\beta n - 1} dx = \int \frac{dx}{x} + \sum_{n=1}^{\infty} (-1)^n \frac{\lambda^{2n}}{(2n+1)!} \int x^{2\beta n-1} dx$$

$$= \ln|x| + \sum_{n=0}^{\infty} (-1)^{n+1} \frac{\lambda^{2n+2}}{(2n+3)!} \frac{x^{2\beta n + 2\beta}}{2\beta n + 2\beta} + C$$

$$= \ln|x| - \frac{\lambda^2 x^{2\beta}}{2\beta} \sum_{n=0}^{\infty} (-1)^n \frac{\lambda^{2n}}{(2n+3)!} \frac{x^{2\beta n}}{n+1} + C$$

$$= \ln|x| - \frac{\lambda^2 x^{2\beta}}{2\beta} \sum_{n=0}^{\infty} \frac{(\Gamma(n+1))^2}{\Gamma(2n+4)\Gamma(n+2)} \frac{\left(-\lambda^2 x^{2\beta}\right)^n}{n!} + C$$

$$= \ln|x| - \frac{\left(\lambda x^{\beta}/2\right)^2}{6\beta} \sum_{n=0}^{\infty} \frac{(1)_n (1)_n}{(2)_n (2)_n (5/2)_n} \frac{\left(-\lambda^2 x^{2\beta}/4\right)^n}{n!} + C$$

$$= \ln|x| - \frac{\left(\lambda x^{\beta}/2\right)^2}{6\beta} \, {}_2F_3\left(1,1;2,2,\frac{5}{2};-\frac{\lambda^2 x^{2\beta}}{4}\right) + C = G(x) + C.$$

□

3. Evaluation of the cosine integral and related integrals

Theorem 4. *If $\beta \geq 1$ and $\alpha < 2\beta + 1$, then the function*

$$G(x) = \frac{1}{\lambda} \frac{x^{1-\alpha}}{1-\alpha} - \frac{\lambda x^{2\beta-\alpha+1}}{2\beta-\alpha+1} \, {}_2F_3\left(1, -\frac{\alpha}{2\beta} + \frac{1}{2\beta} + 1; -\frac{\alpha}{2\beta} + \frac{1}{2\beta} + 2, \frac{3}{2}, 2; -\frac{\lambda^2 x^{2\beta}}{4}\right),$$

where ${}_2F_3$ is a hypergeometric function [1] and λ is an arbitrarily constant, is the antiderivative of the function $g(x) = \dfrac{\cos(\lambda x^{\beta})}{\lambda x^{\alpha}}$. Thus,

$$\int \frac{\cos(\lambda x^{\beta})}{\lambda x^{\alpha}} dx = \frac{1}{\lambda} \frac{x^{1-\alpha}}{1-\alpha} - \frac{1}{2} \frac{\lambda x^{2\beta-\alpha+1}}{2\beta-\alpha+1} \, {}_2F_3\left(1, -\frac{\alpha}{2\beta} + \frac{1}{2\beta} + 1; -\frac{\alpha}{2\beta} + \frac{1}{2\beta} + 2, \frac{3}{2}, 2; -\frac{\lambda^2 x^{2\beta}}{4}\right) + C, \tag{3.16}$$

and for $|x| \gg 1$,

$$\begin{aligned} &\frac{\lambda x^{2\beta-\alpha+1}}{2\beta-\alpha+1} \, {}_2F_3\left(1, -\frac{\alpha}{2\beta} + \frac{1}{2\beta} + 1; -\frac{\alpha}{2\beta} + \frac{1}{2\beta} + 2, \frac{3}{2}, 2; -\frac{\lambda^2 x^{2\beta}}{4}\right) \\ &\sim \frac{\sqrt{\pi}\lambda}{2\beta} \Gamma\left(-\frac{\alpha}{2\beta} + \frac{1}{2\beta} + 1\right) \left(\frac{2}{\lambda}\right)^{-\alpha/\beta + 1/\beta + 2} + \frac{\sqrt{\pi}}{\lambda\beta} x^{-\alpha+1} + \frac{2}{\lambda^2\beta} \frac{\cos(\lambda x^{\beta})}{x^{\beta+\alpha-1}}. \end{aligned} \tag{3.17}$$

P r o o f. If $\beta \geq 1$ and $\alpha < 2\beta + 1$,

$$\int g(x)dx = \int \frac{\cos(\lambda x^{\beta})}{\lambda x^{\alpha}} dx = \int \frac{1}{\lambda x^{\alpha}} \sum_{n=0}^{\infty} (-1)^n \frac{(\lambda x^{\beta})^{2n}}{(2n)!} dx$$

$$= \int \frac{1}{\lambda x^{\alpha}} dx + \frac{1}{\lambda} \int \sum_{n=1}^{\infty} (-1)^n \frac{\lambda^{2n}}{(2n)!} x^{2\beta n - \alpha} dx$$

$$= \frac{1}{\lambda} \frac{x^{1-\alpha}}{1-\alpha} - \frac{1}{\lambda} \sum_{n=0}^{\infty} (-1)^n \frac{\lambda^{2n+2}}{(2n+2)!} \int x^{2\beta n + 2\beta - \alpha} dx$$

$$= \frac{1}{\lambda} \frac{x^{1-\alpha}}{1-\alpha} - \lambda \sum_{n=0}^{\infty} (-1)^n \frac{\lambda^{2n}}{(2n+2)!} \frac{x^{2\beta n + 2\beta - \alpha + 1}}{2\beta n + 2\beta - \alpha + 1} + C$$

$$= \frac{1}{\lambda} \frac{x^{1-\alpha}}{1-\alpha} - \frac{\lambda x^{2\beta-\alpha+1}}{2\beta} \sum_{n=0}^{\infty} \frac{\Gamma(n - \alpha/(2\beta) + 1/(2\beta) + 1)}{\Gamma(2n+3)\Gamma(n - \alpha/(2\beta) + 1/(2\beta) + 2)} (-\lambda^2 x^{2\beta})^n + C$$

$$= \frac{1}{\lambda} \frac{x^{1-\alpha}}{1-\alpha} - \frac{1}{2} \frac{\lambda x^{2\beta-\alpha+1}}{2\beta-\alpha+1} - \sum_{n=0}^{\infty} \frac{(1)_n \left(-\alpha/(2\beta) + 1/(2\beta) + 1\right)_n}{(3/2)_n (2)_n \left(-\alpha/(2\beta) + 1/(2\beta) + 2\right)_n} \frac{\left(-\lambda^2 x^{2\beta}/4\right)^n}{n!} + C$$

$$= \frac{1}{\lambda} \frac{x^{1-\alpha}}{1-\alpha} - \frac{1}{2} \frac{\lambda x^{2\beta-\alpha+1}}{2\beta-\alpha+1} \, {}_2F_3\left(1, -\frac{\alpha}{2\beta} + \frac{1}{2\beta} + 1; -\frac{\alpha}{2\beta} + \frac{1}{2\beta} + 2, \frac{3}{2}, 2; -\frac{\lambda^2 x^{2\beta}}{4}\right) + C.$$

To prove (3.17), we use the asymptotic expression of ${}_2F_3\,(a_1, a_2; b_1, b_2, b_3; -z)$ for $|z| \gg 1$, where a_1, a_2, b_1, b_2 and b_3 are constants, and $-\pi \le \arg z \le \pi$. It can be obtained using formulas 16.11.1, 16.11.2 and 16.11.8 in [9] and is given by

$$
\begin{aligned}
&{}_2F_3\,(a_1, a_2; b_1, b_2, b_3; -z) = \\
&\frac{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)}{\Gamma(a_2)} z^{-a_1} \left\{ \sum_{n=0}^{R-1} \frac{(a_1)_n \Gamma(a_1 - a_2 - n)}{\Gamma(b_1 - a_1 - n)\Gamma(b_2 - a_1 - n)\Gamma(b_3 - a_1 - n)} \frac{(-z)^{-n}}{n!} + O(|z|^{-R}) \right\} \\
&+ \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)}{\Gamma(a_1)} z^{-a_2} \left\{ \sum_{n=0}^{R-1} \frac{(a_2)_n \Gamma(a_2 - a_1 - n)}{\Gamma(b_1 - a_2 - n)\Gamma(b_2 - a_2 - n)\Gamma(b_3 - a_2 - n)} \frac{(-z)^{-n}}{n!} + O(|z|^{-R}) \right\} \\
&+ \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)}{\Gamma(a_1)\Gamma(a_2)} \frac{e^{2z^{1/2} e^{-i\pi/2}} (z e^{-i\pi})^{(a_1 + a_2 - b_1 - b_2 - b_3 + 1/2)/2}}{\sqrt{\pi}} \left\{ \sum_{n=0}^{S-1} \frac{\mu_n}{2^{n+1}} (z e^{-i\pi})^{-n} + O(|z|^{-S}) \right\} \\
&+ \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)}{\Gamma(a_1)\Gamma(a_2)} \frac{e^{2z^{1/2} e^{i\pi/2}} (z e^{i\pi})^{(a_1 + a_2 - b_1 - b_2 - b_3 + 1/2)/2}}{\sqrt{\pi}} \left\{ \sum_{n=0}^{S-1} \frac{\mu_n}{2^{n+1}} (z e^{i\pi})^{-n} + O(|z|^{-S}) \right\},
\end{aligned}
\tag{3.18}
$$

where the coefficient μ_n is given by formula 16.11.4 in [9].

We now set $z = \frac{\lambda^2 x^{2\beta}}{4}$, $a_1 = 1$, $a_2 = -\frac{\alpha}{2\beta} + \frac{1}{2\beta} + 1$, $b_1 = -\frac{\alpha}{2\beta} + \frac{1}{2\beta} + 2$, $b_2 = \frac{3}{2}$ and $b_3 = 2$ in (3.18) to obtain

$$
\begin{aligned}
&{}_2F_3 \left(1, -\frac{\alpha}{2\beta} + \frac{1}{2\beta} + 1; -\frac{\alpha}{2\beta} + \frac{1}{2\beta} + 2, \frac{3}{2}, 2; -\frac{\lambda^2 x^{2\beta}}{4} \right) \\
&\sim \frac{\sqrt{\pi}}{\lambda^2} \left(-\frac{\alpha}{\beta} + \frac{1}{\beta} + 2 \right) \frac{1}{x^{2\beta}} + \frac{\sqrt{\pi}}{2} \Gamma \left(-\frac{\alpha}{2\beta} + \frac{1}{2\beta} + 2 \right) \left(\frac{2}{\lambda x^\beta} \right)^{-\alpha/\beta + 1/\beta + 2} \\
&\quad + \frac{2}{\lambda^3} \left(-\frac{\alpha}{\beta} + \frac{1}{\beta} + 2 \right) \frac{\cos(\lambda x^\beta)}{x^{3\beta}}.
\end{aligned}
\tag{3.19}
$$

Hence, multiplying (3.19) with $\lambda x^{2\beta - \alpha + 1}/(2\beta - \alpha + 1)$ gives (3.17). □

On the other hand, we can show as above that if $\beta \ge 1$ and $\alpha < 2\beta + 1$, then

$$
\int \frac{\cosh(\lambda x^\beta)}{\lambda x^\alpha} dx = \frac{1}{\lambda} \frac{x^{1-\alpha}}{1-\alpha} + \frac{1}{2} \frac{\lambda x^{2\beta - \alpha + 1}}{2\beta - \alpha + 1} {}_2F_3 \left(1, -\frac{\alpha}{2\beta} + \frac{1}{2\beta} + 1; -\frac{\alpha}{2\beta} + \frac{1}{2\beta} + 2, \frac{3}{2}, 2; \frac{\lambda^2 x^{2\beta}}{4} \right) + C.
$$

Theorem 5. *Let $\beta \ge 1$, then the function*

$$
G(x) = -\frac{1}{2\lambda\beta x^{2\beta}} - \frac{\lambda}{2} \ln|x| + \frac{\lambda}{6\beta} \left(\frac{\lambda x^\beta}{4} \right)^2 {}_2F_3 \left(1, 1; 2, \frac{5}{2}, 3; -\frac{\lambda^2 x^{2\beta}}{4} \right),
$$

where ${}_2F_3$ is a hypergeometric function [1] and λ is an arbitrarily constant, is the antiderivative of the function $g(x) = \frac{\cos(\lambda x^\beta)}{\lambda x^{2\beta+1}}$. Thus,

$$
Ci_{\beta, 2\beta+1} = \int \frac{\cos(\lambda x^\beta)}{\lambda x^{2\beta+1}} dx = -\frac{1}{2\lambda\beta x^{2\beta}} - \frac{\lambda}{2} \ln|x| + \frac{\lambda}{6\beta} \left(\frac{\lambda x^\beta}{4} \right)^2 {}_2F_3 \left(1, 1; 2, \frac{5}{2}, 3; -\frac{\lambda^2 x^{2\beta}}{4} \right) + C.
\tag{3.20}
$$

We also have,

$$
Ci_{\beta, 1} = \int \frac{\cos(\lambda x^\beta)}{\lambda x} dx = \frac{1}{\lambda} \ln|x| - \frac{\lambda x^{2\beta}}{4\beta} {}_2F_3 \left(1, 1; \frac{3}{2}, 2, 2; -\frac{\lambda^2 x^{2\beta}}{4} \right) + C.
\tag{3.21}
$$

P r o o f.

$$
\begin{aligned}
\mathrm{Ci}_{\beta,2\beta+1} &= \int g(x)dx = \int \frac{\cos\left(\lambda x^{\beta}\right)}{\lambda x^{2\beta+1}}dx = \int \frac{1}{\lambda x^{2\beta+1}}\sum_{n=0}^{\infty}(-1)^n\frac{(\lambda x^{\beta})^{2n}}{(2n)!}dx \\
&= \int \frac{1}{\lambda x^{2\beta+1}}dx + \frac{1}{\lambda}\int\sum_{n=1}^{\infty}(-1)^n\frac{\lambda^{2n}}{(2n)!}x^{2\beta n-2\beta-1}dx \\
&= -\frac{1}{2\lambda\beta x^{2\beta}} - \frac{1}{\lambda}\sum_{n=0}^{\infty}(-1)^n\frac{\lambda^{2n+2}}{(2n+2)!}\int x^{2\beta n-1}dx \\
&= -\frac{1}{2\lambda\beta x^{2\beta}} - \frac{\lambda}{2}\int\frac{dx}{x} - \lambda\sum_{n=1}^{\infty}(-1)^n\frac{\lambda^{2n}}{(2n+2)!}\int x^{2\beta n-1}dx \\
&= -\frac{1}{2\lambda\beta x^{2\beta}} - \frac{\lambda}{2}\ln|x| + \lambda^3\sum_{n=0}^{\infty}(-1)^n\frac{\lambda^{2n}}{(2n+4)!}\int x^{2\beta n+2\beta-1}dx \\
&= -\frac{1}{2\lambda\beta x^{2\beta}} - \frac{\lambda}{2}\ln|x| + \lambda^3\sum_{n=0}^{\infty}(-1)^n\frac{\lambda^{2n}}{(2n+4)!}\frac{x^{2\beta n+2\beta}}{2\beta n+2\beta} + C \\
&= -\frac{1}{2\lambda\beta x^{2\beta}} - \frac{\lambda}{2}\ln|x| - \frac{\lambda^3 x^{2\beta}}{2\beta}\sum_{n=0}^{\infty}\frac{(\Gamma(n+1))^2}{\Gamma(2n+5)\Gamma(n+2)}\frac{(-\lambda^2x^{2\beta})^n}{n!} + C \\
&= -\frac{1}{2\lambda\beta x^{2\beta}} - \frac{\lambda}{2}\ln|x| + \frac{\lambda}{6\beta}\left(\frac{\lambda x^{\beta}}{4}\right)^2\sum_{n=0}^{\infty}\frac{(1)_n(1)_n}{(2)_n(5/2)_n(3)_n}\frac{\left(-\lambda^2x^{2\beta}/4\right)^n}{n!} + C \\
&= -\frac{1}{2\lambda\beta x^{2\beta}} - \frac{\lambda}{2}\ln|x| + \frac{\lambda}{6\beta}\left(\frac{\lambda x^{\beta}}{4}\right)^2 {}_2F_3\left(1,1;2,\frac{5}{2},3;-\frac{\lambda^2x^{2\beta}}{4}\right) + C.
\end{aligned}
$$

The proof of (3.21) is similar, we do not show it here. □

4. Evaluation of some integrals involving $\mathbf{Si}_{\alpha,\beta}$ and $\mathbf{Ci}_{\alpha,\beta}$

The integral $\int \frac{\cos^n\left(\lambda x^{\beta}\right)}{\lambda x^{\alpha}}dx$, where n is a positive integer and $\beta \geq 1, \alpha < 2\beta+1$, can be written in terms of (3.16) and then evaluated.

Example 2. In this example, the integral $\int \frac{\cos^4\left(\lambda x^{\beta}\right)}{\lambda x^{\alpha}}dx$ is evaluated by linearizing $\cos^4\left(\lambda x^{\beta}\right)$.

$$
\begin{aligned}
&\int \frac{\cos^4\left(\lambda x^{\beta}\right)}{\lambda x^{\alpha}}dx = \frac{1}{8}\int\frac{\cos\left(4\lambda x^{\beta}\right)}{\lambda x^{\alpha}}dx + \frac{1}{2}\int\frac{\cos\left(2\lambda x^{\beta}\right)}{\lambda x^{\alpha}}dx + \frac{3}{8}\int dx = \\
&\frac{1}{8\lambda}\frac{x^{1-\alpha}}{1-\alpha} - \frac{1}{4}\frac{\lambda x^{2\beta-\alpha+1}}{2\beta-\alpha+1}\,{}_2F_3\left(1,-\frac{\alpha}{2\beta}+\frac{1}{2\beta}+1;-\frac{\alpha}{2\beta}+\frac{1}{2\beta}+2,\frac{3}{2},2;-4\lambda^2x^{2\beta}\right) \\
&+\frac{1}{2\lambda}\frac{x^{1-\alpha}}{1-\alpha} - \frac{1}{2}\frac{\lambda x^{2\beta-\alpha+1}}{2\beta-\alpha+1}\,{}_2F_3\left(1,-\frac{\alpha}{2\beta}+\frac{1}{2\beta}+1;-\frac{\alpha}{2\beta}+\frac{1}{2\beta}+2,\frac{3}{2},2;-\lambda^2x^{2\beta}\right) + \frac{3x}{8} + C.
\end{aligned}
$$

If $\beta \geq 1$ and $\alpha < \beta+1$, the integral $\int \frac{\sin^n\left(\lambda x^{\beta}\right)}{\lambda x^{\alpha}}dx$, where n is a positive integer, can be written either in terms of (2.7) if n odd, and then evaluated.

Example 3. In this example, the integral $\int \frac{\sin^3(\lambda x^\beta)}{\lambda x^\alpha}dx$ is evaluated by linearizing $\sin^3(\lambda x^\beta)$.

$$\int \frac{\sin^3(\lambda x^\beta)}{\lambda x^\alpha}dx = -\frac{1}{4}\int \frac{\sin(3\lambda x^\beta)}{\lambda x^\alpha}dx + \frac{3}{4}\int \frac{\sin(\lambda x^\beta)}{\lambda x^\alpha}dx$$
$$= -\frac{1}{4}\frac{x^{\beta-\alpha+1}}{\beta-\alpha+1}\,{}_1F_2\left(-\frac{\alpha}{2\beta}+\frac{1}{2\beta}+\frac{1}{2};-\frac{\alpha}{2\beta}+\frac{1}{2\beta}+\frac{3}{2},\frac{3}{2};-\frac{9\lambda^2x^{2\beta}}{4}\right)$$
$$+\frac{3}{4}\frac{x^{\beta-\alpha+1}}{\beta-\alpha+1}\,{}_1F_2\left(-\frac{\alpha}{2\beta}+\frac{1}{2\beta}+\frac{1}{2};-\frac{\alpha}{2\beta}+\frac{1}{2\beta}+\frac{3}{2},\frac{3}{2};-\frac{\lambda^2x^{2\beta}}{4}\right)+C.$$

Example 4. Let us now evaluate the integrals $\int \sin(\lambda/x^\mu)dx$ and $\int \cos(\lambda/x^\mu)dx$.

1. The integral $\int \sin(\lambda/x^\mu)dx$ is evaluated using the substitution $u = 1/x$ and Theorem 1 if $\mu > 1$. Then, we have

$$\int \sin\left(\frac{\lambda}{x^\mu}\right)dx = -\int \frac{\sin(\lambda u^\mu)}{u^2}du = -\frac{\lambda u^{\mu-1}}{\mu-1}\,{}_1F_2\left(-\frac{1}{2\mu}+\frac{1}{2};-\frac{1}{2\mu}+\frac{3}{2},\frac{3}{2};-\frac{\lambda^2u^{2\mu}}{4}\right)$$
$$= -\frac{\lambda(1/x)^{\mu-1}}{\mu-1}\,{}_1F_2\left(-\frac{1}{2\mu}+\frac{1}{2};-\frac{1}{2\mu}+\frac{3}{2},\frac{3}{2};-\frac{\lambda^2}{4x^{2\mu}}\right)+C,\quad \mu>1. \tag{4.22}$$

The integral $\int \sin(\lambda/x^\mu)dx$ is evaluated using the substitution $u = 1/x$ and Theorem 3 if $\mu = 1$. Then, we have

$$\int \sin\left(\frac{\lambda}{x}\right)dx = -\int \frac{\sin(\lambda u)}{u^2}du = -\ln|u| + \frac{(\lambda u/2)^2}{6}\,{}_2F_3\left(1,1;2,2,\frac{5}{2};-\frac{\lambda^2u^2}{4}\right)$$
$$= \ln|x| + \frac{(\lambda/(2x))^2}{6}\,{}_2F_3\left(1,1;2,2,\frac{5}{2};-\frac{\lambda^2}{4x^2}\right)+C.$$

2. Making the substitution $u = 1/x$ and applying Theorem 4 gives

$$\int \cos\left(\frac{\lambda}{x^\mu}\right)dx = -\int \frac{\cos(\lambda u^\mu)}{u^2}du = \frac{1}{u}+\frac{\lambda u^{2\mu-1}}{2\mu-1}\,{}_2F_3\left(1,-\frac{1}{2\mu}+1;-\frac{1}{2\mu}+2,\frac{3}{2},2;-\frac{\lambda^2u^{2\mu}}{4}\right)$$
$$= x+\frac{\lambda(1/x)^{2\mu-1}}{2\mu-1}\,{}_2F_3\left(1,-\frac{1}{2\mu}+1;-\frac{1}{2\mu}+2,\frac{3}{2},2;-\frac{\lambda^2}{4x^{2\mu}}\right)+C,\quad \mu>1. \tag{4.23}$$

Making the substitution $u = 1/x$ and applying Theorem 5, then for $\mu = 1$, we have

$$\int \cos\left(\frac{\lambda}{x}\right)dx = -\int \frac{\cos(\lambda u)}{u^2}du = \frac{1}{2\lambda u^2}+\frac{\lambda}{2}\ln|u| - \frac{\lambda}{6}\left(\frac{\lambda u}{4}\right)^2\,{}_2F_3\left(1,1;2,\frac{5}{2},3;-\frac{\lambda^2x^2}{4}\right)$$
$$= \frac{x^2}{2\lambda}-\frac{\lambda}{2}\ln|x|-\frac{\lambda}{6}\left(\frac{\lambda}{4x}\right)^2\,{}_2F_3\left(1,1;2,\frac{5}{2},3;-\frac{\lambda^2}{4x^2}\right)+C.$$

5. Evaluation of exponential (Ei) and logarithmic (Li) integrals

Theorem 6. *If $\beta \geq 1$, then for any constant λ,*

$$\int \frac{e^{\lambda x^\beta}}{x}dx = \ln|x| + \frac{\lambda x^\beta}{\beta}\,{}_2F_2(1,1;2,2;\lambda x^\beta)+C,$$

and

$$\lambda x^\beta\,{}_2F_2(1,1;2,2;\lambda x^\beta) \sim -2 + \frac{e^{\lambda x^\beta}}{\lambda x^\beta}, \quad |x| \gg 1. \tag{5.24}$$

P r o o f.

$$\int \frac{e^{\lambda x^\beta}}{x}dx = \int \frac{1}{x}\sum_{n=0}^{\infty}\frac{(\lambda x^\beta)^n}{n!}dx = \int \frac{dx}{x} + \int \sum_{n=1}^{\infty}\frac{\lambda^n x^{\beta n-1}}{n!}dx = \ln|x| + \sum_{n=1}^{\infty}\frac{\lambda^n}{n!}\int x^{\beta n-1}dx$$
$$= \ln|x| + \sum_{n=1}^{\infty}\frac{\lambda^n}{n!}\frac{x^{\beta n}}{\beta n} = \ln|x| + \sum_{n=0}^{\infty}\frac{\lambda^{n+1}}{(n+1)!}\frac{x^{\beta n+\beta}}{\beta n+\beta}$$
$$= \ln|x| + \frac{\lambda x^\beta}{\beta}\sum_{n=0}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+2)\Gamma(n+2)}(\lambda x^\beta)^n + C$$
$$= \ln|x| + \frac{\lambda x^\beta}{\beta}\sum_{n=0}^{\infty}\frac{(1)_n(1)_n}{(2)_n(2)_n}\frac{(\lambda x^\beta)^n}{n!} + C = \ln|x| + \frac{\lambda x^\beta}{\beta}\,{}_2F_2(1,1;2,2;\lambda x^\beta) + C.$$

To derive the asymptotic expression of $\lambda x^\beta\,{}_2F_2(1,1;2,2;\lambda x^\beta)$, $|x| \gg 1$, we use the asymptotic expression of the hypergeometric function ${}_2F_2(a_1,a_2;b_1,b_2;z)$ for $|z| \gg 1$, where $z \in \mathbb{C}$, and a_1, a_2, b_1 and b_2 are constants. It can be obtained using formulas 16.11.1, 16.11.2 and 16.11.7 in [9] and is given by

$$\begin{aligned} &{}_2F_2(a_1,a_2;b_1,b_2;z) = \\ &= \frac{\Gamma(b_1)\Gamma(b_2)}{\Gamma(a_2)}(ze^{\pm i\pi})^{-a_1}\left\{\sum_{n=0}^{R-1}\frac{(a_1)_n\Gamma(a_1-a_2-n)}{\Gamma(b_1-a_1-n)\Gamma(b_2-a_1-n)_n}\frac{(ze^{\pm i\pi})^{-n}}{n!} + O(|z|^{-R})\right\} \\ &+\frac{\Gamma(b_1)\Gamma(b_2)}{\Gamma(a_1)}(ze^{\pm i\pi})^{-a_2}\left\{\sum_{n=0}^{R-1}\frac{(a_2)_n\Gamma(a_2-a_1-n)}{\Gamma(b_1-a_2-n)\Gamma(b_2-a_2-n)_n}\frac{(ze^{\pm i\pi})^{-n}}{n!} + O(|z|^{-R})\right\} \\ &+\frac{\Gamma(b_1)\Gamma(b_2)}{\Gamma(a_1)\Gamma(a_2)}e^z z^{a_1+a_2-b_1-b_2}\left\{\sum_{n=0}^{S-1}\frac{\mu_n}{2^n}z^{-n} + O(|z|^{-S})\right\}, \end{aligned} \tag{5.25}$$

where the coefficient μ_n is given by formula 16.11.4. And the upper or lower signs are chosen according as z lies in the upper (above the real axis) or lower half-plane (below the real axis).

Setting $z = \lambda x^\beta, a_1 = 1, a_2 = 1, b_1 = 2$ and $b_2 = 2$ in (5.25) yields

$${}_2F_2(1,1;2,2;\lambda x^\beta) \sim \frac{-2}{\lambda x^\beta} + \frac{e^{\lambda x^\beta}}{\lambda^2 x^{2\beta}}, \quad |x| \gg 1.$$

Hence,

$$\lambda x^\beta\,{}_2F_2(1,1;2,2;\lambda x) \sim -2 + \frac{e^{\lambda x^\beta}}{\lambda x^\beta}, \quad |x| \gg 1.$$

This ends the proof. □

Example 5. The random attenuation capacity of a channel or fading capacity [11] can now be evaluated in terms of the natural logarithm ln and the hypergeometric function ${}_2F_2$ as

$$\begin{aligned} C_{\text{fading}} = E[\log_2(1+P|H|^2)] &= \frac{1}{\ln 2}e^{1/P}\left[E_{1,1}(\infty) - E_{1,1}\left(\frac{1}{P}\right)\right] \\ &= \frac{1}{\ln 2}e^{1/P}\left[\ln P + \frac{1}{P}\,{}_2F_2\left(1,1;2,2;-\frac{1}{P}\right)\right] \end{aligned}$$

Example 6. One can now evaluate $\int e^{\lambda e^{\beta x}}dx$ in terms of ${}_2F_2$ using the substitution $u = e^x$, and obtain

$$\int e^{\lambda e^{\beta x}}dx = \int \frac{e^{\lambda u^\beta}}{u}du = \ln u + \frac{\lambda u^\beta}{\beta}\,{}_2F_2(1,1;2,2;\lambda u^\beta) + C = x + \frac{\lambda e^{\beta x}}{\beta}\,{}_2F_2(1,1;2,2;\lambda e^{\beta x}) + C.$$

Theorem 7. *The logarithmic integral is given by*

$$Li = \int_\mu^x \frac{dt}{\ln t} = \ln\left(\frac{\ln x}{\ln \mu}\right) + \ln x\,{}_2F_2(1,1;2,2;\ln x) - \ln \mu\,{}_2F_2(1,1;2,2;\ln \mu), \quad \mu > 1.$$

And for $x \gg \mu$,

$$Li = \int_\mu^x \frac{dt}{\ln t} \sim \frac{x}{\ln x} + \ln\left(\frac{\ln x}{\ln \mu}\right) - 2 - \ln \mu\,{}_2F_2(1,1;2,2;\ln \mu). \tag{5.26}$$

P r o o f. Making the substitution $u = \ln x$ and using (4.22) gives

$$\int_\mu^x \frac{dx}{\ln x} = \int_{\ln \mu}^{\ln x} \frac{e^u}{u}du = [\ln u + u\,{}_2F_2(1,1;2,2;u)]_{\ln \mu}^{\ln x}$$
$$= \ln\left(\frac{\ln x}{\ln \mu}\right) + \ln x\,{}_2F_2(1,1;2,2;\ln x) - \ln \mu\,{}_2F_2(1,1;2,2;\ln \mu).$$

Now setting $z = \ln x, a_1 = 1,\ a_2 = 1,\ b_1 = 2$ and $b_2 = 2$ in (5.24) or in (4.23) yields

$${}_2F_2(1,1;2,2;\ln x) \sim \frac{-2}{\ln x} + \frac{x}{(\ln x)^2}, \quad x \gg 1.$$

This gives

$$\ln x\,{}_2F_2(1,1;2,2;\ln x) \sim -2 + \frac{x}{\ln x}, \quad x \gg 1.$$

Hence for $x \gg \mu$,

$$\mathrm{Li} = \int_\mu^x \frac{dt}{\ln t} \sim \frac{x}{\ln x} + \ln\left(\frac{\ln x}{\ln \mu}\right) - 2 - \ln \mu\,{}_2F_2(1,1;2,2;\ln \mu).$$

□

We importantly note that Theorem 7 adds the term $\ln(\ln x/\ln \mu) - 2 - \ln \mu\,{}_2F_2(1,1;2,2;\ln \mu)$ to the known asymptotic expression of the logarithmic integral in mathematical literature, $\mathrm{Li} \sim x/\ln x$ [1, 9]. And this term is negligible if $x \sim O(10^6)$ or higher.

We can now slightly improve the prime number Theorem [3] as following,

Corollary 2. *Let $\pi(x)$ denotes the number of primes small than or equal to x and $\mu > 1$. Then for $x \gg \mu$,*

$$\pi(x) - \frac{x}{\ln x} \sim \ln\left(\frac{\ln x}{\ln \mu}\right) - 2 - \ln \mu\,{}_2F_2(1,1;2,2;\ln \mu).$$

The proof follows directly from equation (5.26) in Theorem 7.

Example 7. One can now evaluate $\int \ln(\ln x)dx$ using integration by parts.

$$\int \ln(\ln x)dx = x\ln(\ln x) - \int \frac{1}{\ln x}dx$$
$$= x\ln(\ln x) - \ln(\ln x) - \ln x\, {}_2F_2(1,1;2,2;\ln x) + C.$$

Theorem 8. *For $\beta \geq 1$ and $\alpha < \beta+1$, we have*

$$Ei_{\beta,\alpha} = \int \frac{e^{\lambda x^{\beta}}}{\lambda x^{\alpha}}dx = \frac{1}{\lambda}\frac{x^{1-\alpha}}{1-\alpha} + \frac{x^{\beta-\alpha+1}}{\beta-\alpha+1}\, {}_2F_2\left(1, -\frac{\alpha}{\beta}+\frac{1}{\beta}+1; 2, -\frac{\alpha}{\beta}+\frac{1}{\beta}+2; \lambda x^{\beta}\right) + C,$$

and for $|x| \gg 1$,

$$\begin{aligned} &\frac{\lambda x^{\beta-\alpha+1}}{\beta-\alpha+1}\, {}_2F_2\left(1, -\frac{\alpha}{\beta}+\frac{1}{\beta}+1; 2, -\frac{\alpha}{\beta}+\frac{1}{\beta}+2; \lambda x\right) \\ &\sim \frac{\lambda}{\beta}\Gamma\left(-\frac{\alpha}{\beta}+\frac{1}{\beta}+1\right)\left(-\frac{1}{\lambda}\right)^{-\alpha/\beta+1/\beta+1} - \frac{x^{-\alpha+1}}{\beta} + \frac{1}{\lambda\beta}\frac{e^{\lambda x^{\beta}}}{x^{\beta+\alpha-1}}. \end{aligned} \tag{5.27}$$

We also have,

$$Ei_{\beta,\beta+1} = \int \frac{e^{\lambda x^{\beta}}}{\lambda x^{\beta+1}}dx = -\frac{1}{\beta x^{\beta}} + \ln(|x|) + \frac{\lambda x^{\beta}}{2\beta}\, {}_2F_2\left(1,1;2,2;\lambda x^{\beta}\right) + C. \tag{5.28}$$

P r o o f.

$$\begin{aligned} \mathrm{Ei}_{\beta,\alpha} &= \int \frac{e^{\lambda x^{\beta}}}{\lambda x^{\alpha}}dx = \int \frac{1}{\lambda x^{\alpha}}\sum_{n=0}^{\infty}\frac{(\lambda x^{\beta})^n}{n!}dx = \frac{1}{\lambda}\int\frac{dx}{x^{\alpha}} + = \frac{1}{\lambda x^{\alpha}}\int\sum_{n=1}^{\infty}\frac{(\lambda x^{\beta})^n}{n!}dx \\ &= \frac{1}{\lambda}\frac{x^{1-\alpha}}{1-\alpha} + \frac{1}{\lambda}\sum_{n=1}^{\infty}\frac{\lambda^n}{n!}\int x^{\beta n-\alpha}dx = \frac{1}{\lambda}\frac{x^{1-\alpha}}{1-\alpha} + \sum_{n=0}^{\infty}\frac{\lambda^n}{(n+1)!}\frac{x^{\beta n+\beta-\alpha+1}}{\beta n+\beta-\alpha+1} + C \\ &= \frac{1}{\lambda}\frac{x^{1-\alpha}}{1-\alpha} + \frac{x^{\beta-\alpha+1}}{\beta}\sum_{n=0}^{\infty}\frac{\Gamma\left(n-\frac{\alpha}{\beta}+\frac{1}{\beta}+1\right)}{\Gamma(n+2)\Gamma\left(n-\frac{\alpha}{\beta}+\frac{1}{\beta}+2\right)}\left(\lambda x^{\beta}\right)^n + C \\ &= \frac{1}{\lambda}\frac{x^{1-\alpha}}{1-\alpha} + \frac{x^{\beta-\alpha+1}}{\beta-\alpha+1}\sum_{n=0}^{\infty}\frac{(1)_n\left(-\frac{\alpha}{\beta}+\frac{1}{\beta}+1\right)_n}{(2)_n\left(-\frac{\alpha}{\beta}+\frac{1}{\beta}+2\right)_n}\frac{(x^{\beta})^n}{n!} + C \\ &= \frac{1}{\lambda}\frac{x^{1-\alpha}}{1-\alpha} + \frac{x^{\beta-\alpha+1}}{\beta-\alpha+1}\, {}_2F_2\left(1, -\frac{\alpha}{\beta}+\frac{1}{\beta}+1; 2, -\frac{\alpha}{\beta}+\frac{1}{\beta}+2; \lambda x^{\beta}\right) + C. \end{aligned}$$

Now setting $a_1 = 1$, $a_2 = -\frac{\alpha}{\beta}+\frac{1}{\beta}+1$, $b_1 = 2$, $b_2 = -\frac{\alpha}{\beta}+\frac{1}{\beta}+2$ and $z = \lambda x^{\beta}$ in (5.25) gives,

$$\begin{aligned} &{}_2F_2\left(1, -\frac{\alpha}{\beta}+\frac{1}{\beta}+1; 2, -\frac{\alpha}{\beta}+\frac{1}{\beta}+2; \lambda x^{\beta}\right) \\ &\sim -\left(-\frac{\alpha}{\beta}+\frac{1}{\beta}+1\right)\frac{1}{\lambda x^{\beta}} + \Gamma\left(-\frac{\alpha}{\beta}+\frac{1}{\beta}+2\right)\left(\frac{1}{\lambda x^{\beta}}\right)^{-\alpha/\beta+1/\beta+1} + \frac{e^{\lambda x^{\beta}}}{\lambda^2 x^{2\beta}}. \end{aligned} \tag{5.29}$$

Hence, multiplying (5.29) with $\dfrac{\lambda x^{\beta-\alpha+1}}{\beta-\alpha+1}$ gives (5.27). The proof of (5.28) is similar to that of (3.20). □

Theorem 9. *For any constants α, β and λ,*

$$ {}_1F_2\left(-\frac{\alpha}{2\beta}+\frac{1}{2\beta}+\frac{1}{2};-\frac{\alpha}{2\beta}+\frac{1}{2\beta}+\frac{3}{2},\frac{3}{2};-\frac{\lambda^2x^{2\beta}}{4}\right)= $$
$$ \frac{1}{2}\Big[{}_2F_2\left(1,-\frac{\alpha}{\beta}+\frac{1}{\beta}+1;2,-\frac{\alpha}{\beta}+\frac{1}{\beta}+2;i\lambda x^{\beta}\right)+{}_2F_2\left(1,-\frac{\alpha}{\beta}+\frac{1}{\beta}+1;2,-\frac{\alpha}{\beta}+\frac{1}{\beta}+2;-i\lambda x^{\beta}\right)\Big], \tag{5.30} $$

or

$$ {}_1F_2\left(-\frac{\alpha}{2\beta}+\frac{1}{2\beta}+\frac{1}{2};-\frac{\alpha}{2\beta}+\frac{1}{2\beta}+\frac{3}{2},\frac{3}{2};\frac{\lambda^2x^{2\beta}}{4}\right)= $$
$$ \frac{1}{2}\Big[{}_2F_2\left(1,-\frac{\alpha}{\beta}+\frac{1}{\beta}+1;2,-\frac{\alpha}{\beta}+\frac{1}{\beta}+2;\lambda x^{\beta}\right)+{}_2F_2\left(1,-\frac{\alpha}{\beta}+\frac{1}{\beta}+1;2,-\frac{\alpha}{\beta}+\frac{1}{\beta}+2;-\lambda x^{\beta}\right)\Big]. \tag{5.31} $$

P r o o f. Using Theorem 8, we obtain

$$ \begin{aligned} &\int\frac{\sin(\lambda x^{\beta})}{x^{\alpha}}dx=\frac{1}{2i}\int\frac{e^{i\lambda x^{\beta}}-e^{-i\lambda x^{\beta}}}{x^{\alpha}}dx\\ &=\frac{1}{2}\frac{\lambda x^{\beta-\alpha+1}}{\beta-\alpha+1}\Big[{}_2F_2\left(1,-\frac{\alpha}{\beta}+\frac{1}{\beta}+1;2,-\frac{\alpha}{\beta}+\frac{1}{\beta}+2;i\lambda x^{\beta}\right)\\ &\quad+{}_2F_2\left(1,-\frac{\alpha}{\beta}+\frac{1}{\beta}+1;2,-\frac{\alpha}{\beta}+\frac{1}{\beta}+2;-i\lambda x^{\beta}\right)\Big]+C. \end{aligned} \tag{5.32} $$

Hence, comparing (2.7) with (5.32) gives (5.30). Or on the other hand,

$$ 2\int\frac{\sinh(\lambda x^{\beta})}{x^{\alpha}}dx=\int\frac{e^{\lambda x^{\beta}}-e^{-\lambda x^{\beta}}}{x^{\alpha}}dx=\frac{\lambda x^{\beta-\alpha+1}}{\beta-\alpha+1}\times $$
$$ \Big[{}_2F_2\left(1,-\frac{\alpha}{\beta}+\frac{1}{\beta}+1;2,-\frac{\alpha}{\beta}+\frac{1}{\beta}+2;\lambda x^{\beta}\right)+{}_2F_2\left(1,-\frac{\alpha}{\beta}+\frac{1}{\beta}+1;2,-\frac{\alpha}{\beta}+\frac{1}{\beta}+2;-\lambda x^{\beta}\right)\Big]+C. \tag{5.33} $$

Hence, comparing (2.9) with (5.33) gives (5.31). □

Theorem 10. *For any constants α, β and λ,*

$$ \frac{ix^{2\beta-\alpha+1}}{2\beta-\alpha+1}\,{}_2F_3\left(1,-\frac{\alpha}{2\beta}+\frac{1}{2\beta}+1;-\frac{\alpha}{2\beta}+\frac{1}{2\beta}+2,\frac{3}{2},2;-\frac{\lambda^2x^{2\beta}}{4}\right)=\frac{x^{\beta-\alpha+1}}{\beta-\alpha+1}\times $$
$$ \Big[{}_2F_2\left(1,-\frac{\alpha}{\beta}-\frac{1}{\beta}+1;2,-\frac{\alpha}{\beta}+\frac{1}{\beta}+2;i\lambda x^{\beta}\right)-{}_2F_2\left(1,-\frac{\alpha}{\beta}+\frac{1}{\beta}+1;2,-\frac{\alpha}{\beta}+\frac{1}{\beta}+2;-i\lambda x^{\beta}\right)\Big]. \tag{5.34} $$

Or,

$$\begin{aligned}
&\frac{x^{2\beta-\alpha+1}}{2\beta-\alpha+1}\,{}_2F_3\left(1,-\frac{\alpha}{2\beta}+\frac{1}{2\beta}+1;-\frac{\alpha}{2\beta}+\frac{1}{2\beta}+2,\frac{3}{2},2;\frac{\lambda^2x^{2\beta}}{4}\right)\\
&=\frac{x^{\beta-\alpha+1}}{\beta-\alpha+1}\Big[\,{}_2F_2\left(1,-\frac{\alpha}{\beta}-\frac{1}{\beta}+1;2,-\frac{\alpha}{\beta}+\frac{1}{\beta}+2;\lambda x^{\beta}\right)\\
&\quad+{}_2F_2\left(1,-\frac{\alpha}{\beta}+\frac{1}{\beta}+1;2,-\frac{\alpha}{\beta}+\frac{1}{\beta}+2;-\lambda x^{\beta}\right)\Big].
\end{aligned}\tag{5.35}$$

We prove Theorem 10 as Theorem 9 using Theorems 4 and 8.

6. Conclusion

$\mathrm{Si}_{\beta,\alpha}=\int[\sin(\lambda x^\beta)/(\lambda x^\alpha)]dx$, $\beta\geq 1$, $\alpha\leq\beta+1$, and $\mathrm{Ci}_{\beta,\alpha}=\int[\cos(\lambda x^\beta)/(\lambda x^\alpha)]dx$, $\beta\geq 1$, $\alpha\leq 2\beta+1$, were expressed in terms of the hypergeometric functions ${}_1F_2$ and ${}_2F_3$ respectively, and their asymptotic expressions for $|x|\gg 1$ were obtained (see Theorems 1,2, 3, 4 and 5). Once derived, formulas for the hyperbolic sine and hyperbolic cosine integrals were readily deduced from those of the sine and cosine integrals.

On the other hand, the exponential integral $\mathrm{Ei}_{\beta,\alpha}=\int(e^{\lambda x^\beta}/x^\alpha)dx$, $\beta\geq 1$, $\alpha\leq\beta+1$, and the logarithmic integral $\int dx/\ln x$ were expressed in terms of the hypergeometric function ${}_2F_2$, and their asymptotic expressions for $|x|\gg 1$ were also obtained (see Theorems 6, 7 and 8). Therefore, their corresponding definite integrals can now be evaluated using the FTC rather than using numerical integration.

Using the Euler and hyperbolic identities $\mathrm{Si}_{\beta,\alpha}$ and $\mathrm{Ci}_{\beta,\alpha}$ were expressed in terms of $\mathrm{Ei}_{\beta,\alpha}$. And hence, some expressions of the hypergeometric functions ${}_1F_2$ and ${}_2F_3$ in terms of ${}_2F_2$ were derived (see Theorems 9 and 10).

The evaluation of the logarithmic integral $\int dx/\ln x$ in terms of the function ${}_2F_2$ and its asymptotic expression ${}_2F_2$ for $|x|\gg 1$ allowed us to add the term $\ln(\ln x/\ln\mu)-2-\ln\mu\,{}_2F_2(1,1;2,2;\ln\mu)$, $\mu>1$, to the known asymptotic expression of the logarithmic integral, which is $\mathrm{Li}=\int_2^x dt/\ln t\sim x/\ln x$ [1, 9], so that it is given by $\mathrm{Li}=\int_\mu^x dt/\ln t\sim x/\ln x+\ln(\ln x/\ln\mu)-2-\ln\mu\,{}_2F_2(1,1;2,2;\ln\mu)$ in Theorem 7. Beside, this leads to Corollary 2 which is an improvement of the prime number Theorem [3].

In addition, other non-elementary integrals which can be written in terms of $\mathrm{Ei}_{\beta,1}$ and $\int dx/\ln x$ and then evaluated were given as examples. For instance, using substitution, the $\int e^{\lambda e^{\beta x}}dx$ was written in terms of $\mathrm{Ei}_{\beta,1}$ and therefore evaluated in terms of ${}_2F_2$, and using integration by parts, the non-elementary integral $\int\ln(\ln x)dx$ was written in terms of $\int dx/\ln x$ and therefore evaluated in terms of ${}_2F_2$.

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EVALUATION OF SOME NON-ELEMENTARY INTEGRALS INVOLVING SINE, COSINE, EXPONENTIAL AND LOGARITHMIC INTEGRALS: PART II

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Abstract: The non-elementary integrals $\mathrm{Si}_{\beta,\alpha} = \int[\sin(\lambda x^\beta)/(\lambda x^\alpha)]dx$, $\beta \geq 1$, $\alpha > \beta+1$ and $\mathrm{Ci}_{\beta,\alpha} = \int[\cos(\lambda x^\beta)/(\lambda x^\alpha)]dx$, $\beta \geq 1$, $\alpha > 2\beta+1$, where $\{\beta,\alpha\} \in \mathbb{R}$, are evaluated in terms of the hypergeometric function ${}_2F_3$. On the other hand, the exponential integral $\mathrm{Ei}_{\beta,\alpha} = \int(e^{\lambda x^\beta}/x^\alpha)dx$, $\beta \geq 1$, $\alpha > \beta+1$ is expressed in terms of ${}_2F_2$. The method used to evaluate these integrals consists of expanding the integrand as a Taylor series and integrating the series term by term.

Key words: Non-elementary integrals, Sine integral, Cosine integral, Exponential integral, Hyperbolic sine integral, Hyperbolic cosine integral, Hypergeometric functions.

1. Introduction

Let us first give the definition of the non-elementary integral. This definition is also given in Part I [6], we repeat it here for reference.

Definition 1. *An elementary function is a function of one variable constructed using that variable and constants, and by performing a finite number of repeated algebraic operations involving exponentials and logarithms. An indefinite integral which can be expressed in terms of elementary functions is an elementary integral. And if, on the other hand, it cannot be evaluated in terms of elementary functions, then it is non-elementary* [4, 9].

The cases consisting of the non-elementary integrals $\mathrm{Si}_{\beta,\alpha} = \int[\sin(\lambda x^\beta)/(\lambda x^\alpha)]dx$, $\beta \geq 1$, $\alpha \leq \beta+1$ and $\mathrm{Ci}_{\beta,\alpha} = \int[\cos(\lambda x^\beta)/(\lambda x^\alpha)]dx$, $\beta \geq 1$, $\alpha \leq 2\beta+1$, where $\{\beta,\alpha\} \in \mathbb{R}$, were considered and evaluated in terms of the hypergeometric functions ${}_1F_2$ and ${}_2F_3$ in Part I [6], and their asymptotic expressions for $|x| \gg 1$ were derived too in Part I [6]. The exponential integral $\mathrm{Ei}_{\beta,\alpha} = \int(e^{\lambda x^\beta}/x^\alpha)dx$ where $\beta \geq 1$ and $\alpha \leq \beta+1$ was expressed in terms of ${}_2F_2$, and its asymptotic expression for $|x| \gg 1$ was derived as well in Part I [6].

Here, we investigate other cases which were not treated neither in Part I [6] nor elsewhere. We evaluate $\mathrm{Si}_{\beta,\alpha} = \int[\sin(\lambda x^\beta)/(\lambda x^\alpha)]dx$, $\beta \geq 1$, $\alpha > \beta+1$ and $\mathrm{Ci}_{\beta,\alpha} = \int[\cos(\lambda x^\beta)/(\lambda x^\alpha)]dx$, $\beta \geq 1$, $\alpha > 2\beta+1$ and $\mathrm{Ei}_{\beta,\alpha} = \int(e^{\lambda x^\beta}/x^\alpha)dx$, $\beta \geq 1$, $\alpha > \beta+1$. In order to take into account all possibilities, we write these integrals as $\mathrm{Si}_{\beta,\beta+\alpha} = \int[\sin(\lambda x^\beta)/(\lambda x^{\beta+\alpha})]dx$, $\beta \geq 1$, $\alpha > 1$, $\mathrm{Ci}_{\beta,2\beta+\alpha} = \int[\cos(\lambda x^\beta)/(\lambda x^{2\beta+\alpha})]dx$, $\beta \geq 1, \alpha > 1$, and $\mathrm{Ei}_{\beta,\beta+\alpha} = \int(e^{\lambda x^\beta}/x^{\beta+\alpha})dx$, $\beta \geq 1$, $\alpha > 1$ where $\{\beta,\alpha\} \in \mathbb{R}$. On one hand, $\mathrm{Si}_{\beta,\beta+\alpha}$ and $\mathrm{Ci}_{\beta,2\beta+\alpha}$ are expressed in terms of the hypergeometric function ${}_2F_3$, while on another hand, $\mathrm{Ei}_{\beta,\beta+\alpha}$ is expressed in terms of the hypergeometric function ${}_2F_2$.

These integrals involving a power function x^β in the argument of the numerator are the generalizations of the exponential, sine and cosine integrals in [7] (see sections 8.19 and 8.21 respectively), which have applications in different fields in science, applied sciences and engineering including physics, nuclear technology, mathematics, probability, statistics, and so on. For instance, the generalized exponential integral $E_{1,1+\alpha}$ is used in fluidodynamics and transport theory, where it is applied to the solution of Milne's integral equations [2], there are also used in modeling radiative transfer processes in the atmosphere and in nuclear reactors [10], etc. Exponential asymptotics involving generalized exponential integrals are used in probability theory, see for example [3]. On the hand, generalized sine and cosine integrals are frequently utilized in Fourier analysis and related domains [8]. Therefore, we are justified to further generalize these functions and their connections to hypergeometric functions.

Before we proceed to the main objectives of this paper consisting of evaluating the above interesting cases of non-elementary integrals (see sections 2, 3 and 4), we first define the generalized hypergeometric function as it is an important tool that we are going to use in the paper.

Definition 2. *The generalized hypergeometric function, denoted as ${}_pF_q$, is a special function given by the series* [1, 7]

$$ {}_pF_q(a_1, a_2, \cdots, a_p; b_1, b_2, \cdots, b_q; x) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \frac{x^n}{n!}, $$

where $a_1, a_2, \cdots, a_p$ and $; b_1, b_2, \cdots, b_q$ are arbitrary constants, $(\vartheta)_n = \Gamma(\vartheta+n)/\Gamma(\vartheta)$ (Pochhammer's notation [1, 7]*) for any complex ϑ, with $(\vartheta)_0 = 1$, and Γ is the standard gamma function* [1].

2. Evaluation of the sine integral $\mathbf{Si}_{\beta,\beta+\alpha}$, $\beta \geq 1$, $\alpha > 1$

Theorem 1. *Let $\beta \geq 1$ and $\alpha > 1$, and let $\alpha = m\beta + \epsilon$, where m is an integer ($m \in \mathbb{N}$) and $-\beta < \epsilon < \beta$.*

1. If $\epsilon = 0$, then

$$ Si_{\beta,\beta+\alpha} = \int \frac{\sin(\lambda x^\beta)}{\lambda x^{\beta+\alpha}} dx = \sum_{n=-m}^{-1} (-1)^{n+m} \frac{\lambda^{2n+2m}}{\Gamma(2n+2m+2)} \frac{x^{2\beta n+1}}{2\beta n+1} + \frac{(-1)^m \lambda^{2m} x}{2^{m+1}\sqrt{\pi}\Gamma(m+1)\Gamma(m+3/2)(2\beta+1)} \, {}_2F_3\left(1, 1+\frac{1}{2\beta}; m+1, m+\frac{3}{2}, 2+\frac{1}{2\beta}; -\frac{\lambda^2 x^{2\beta}}{4}\right) + C, \tag{2.1} $$

where $m = \alpha/\beta$.

2. If $\epsilon = 1$, then

$$ Si_{\beta,\beta+\alpha} = \int \frac{\sin(\lambda x^\beta)}{\lambda x^{\beta+\alpha}} dx = (-1)^m \frac{\lambda^{2m}}{\Gamma(2m+2)} \ln|x| + \sum_{n=-m}^{-1} (-1)^{n+m} \frac{\lambda^{2n+2m}}{\Gamma(2n+2m+2)} \frac{x^{2\beta n}}{2\beta n} + \frac{(-1)^{m+1} \lambda^{2m+2} x^{2\beta}}{2^{2m+4}\sqrt{\pi}\Gamma(m+2)\Gamma(m+5/2)\beta} \, {}_2F_3\left(1, 1; m+2, m+\frac{5}{2}, 2; -\frac{\lambda^2 x^{2\beta}}{4}\right) + C, \tag{2.2} $$

where $m = (\alpha-1)/\beta$.

3. *Finally, if* $\epsilon \in (-\beta, 0) \cup (0,1) \cup (1,\beta)$, *we have*

$$\begin{aligned}\int \frac{\sin(\lambda x^{\beta})}{\lambda x^{\beta+\alpha}} dx = (-1)^m \frac{\lambda^{2m}}{\Gamma(2m+2)} \frac{x^{1-\epsilon}}{1-\epsilon} + \sum_{n=-m}^{-1} (-1)^{n+m} \frac{\lambda^{2n+2m}}{\Gamma(2n+2m+2)} \frac{x^{2\beta n-\epsilon+1}}{2\beta n - \epsilon + 1} \\ + \frac{(-1)^{m+1}\lambda^{2m+2} x^{2\beta-\epsilon+1}}{2^{2m+3}\sqrt{\pi}\Gamma(m+2)\Gamma(m+5/2)(2\beta-\epsilon+1)} \times \\ \times {}_2F_3\left(1, 1+\frac{1-\epsilon}{2\beta}; m+2, m+\frac{5}{2}, 2+\frac{1-\epsilon}{2\beta}; -\frac{\lambda^2 x^{2\beta}}{4}\right) + C,\end{aligned} \tag{2.3}$$

where $m = (\alpha-\epsilon)/\beta$.

P r o o f. We proceed as in [5, 6]. We expand $g(x)$ as Taylor series and integrate the series term by term. We use the gamma duplication formula[1], the gamma property $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ and Pochhammer's notation (see Definition 2). We also set $\alpha = m\beta + \epsilon$, and then we obtain

$$\begin{aligned}\int \frac{\sin(\lambda x^{\beta})}{\lambda x^{\beta+\alpha}} dx &= \int \frac{1}{\lambda x^{\beta} x^{\alpha}} \sum_{n=0}^{\infty} (-1)^n \frac{(\lambda x^{\beta})^{2n+1}}{(2n+1)!} dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{\lambda^{2n}}{(2n+1)!} x^{2\beta n-\alpha} dx \\ &= \int \sum_{n=0}^{m-1} (-1)^n \frac{\lambda^{2n}}{(2n+1)!} x^{2\beta n - 2\beta m - \epsilon} dx + \int \sum_{n=m}^{\infty} (-1)^n \frac{\lambda^{2n}}{(2n+1)!} x^{2\beta n-2\beta m-\epsilon} dx \\ &= \int \sum_{n=0}^{m-1} (-1)^n \frac{\lambda^{2n}}{(2n+1)!} x^{2\beta(n-m)-\epsilon} dx + \int \sum_{n=m}^{\infty} (-1)^n \frac{\lambda^{2n}}{(2n+1)!} x^{2\beta(n-m)-\epsilon} dx \\ &= \int \sum_{n=-m}^{-1} (-1)^{n+m} \frac{\lambda^{2n+2m}}{(2n+2m+1)!} x^{2\beta n-\epsilon} dx + \int \sum_{n=0}^{\infty} (-1)^{n+m} \frac{\lambda^{2n+2m}}{(2n+2m+1)!} x^{2\beta n-\epsilon} dx \\ &= \int \sum_{n=-m}^{-1} (-1)^{n+m} \frac{\lambda^{2n+2m}}{\Gamma(2n+2m+2)} x^{2\beta n-\epsilon} dx + \int \sum_{n=0}^{\infty} (-1)^{n+m} \frac{\lambda^{2n+2m}}{\Gamma(2n+2m+2)} x^{2\beta n-\epsilon} dx \\ &= (-1)^m \frac{\lambda^{2m}}{\Gamma(2m+2)} \int \frac{dx}{x^{\epsilon}} + \int \sum_{n=-m}^{-1} (-1)^{n+m} \frac{\lambda^{2n+2m}}{\Gamma(2n+2m+2)} x^{2\beta n-\epsilon} dx \\ &\quad + \int \sum_{n=1}^{\infty} (-1)^{n+m} \frac{\lambda^{2n+2m}}{\Gamma(2n+2m+2)} x^{2\beta n-\epsilon} dx \\ &= (-1)^m \frac{\lambda^{2m}}{\Gamma(2m+2)} \int \frac{dx}{x^{\epsilon}} + \int \sum_{n=-m}^{-1} (-1)^{n+m} \frac{\lambda^{2n+2m}}{\Gamma(2n+2m+2)} x^{2\beta n-\epsilon} dx \\ &\quad + \int \sum_{n=0}^{\infty} (-1)^{n+m+1} \frac{\lambda^{2n+2m+2}}{\Gamma(2n+2m+4)} x^{2\beta n+2\beta-\epsilon} dx \\ &= (-1)^m \frac{\lambda^{2m}}{\Gamma(2m+2)} \int \frac{dx}{x^{\epsilon}} + \sum_{n=-m}^{-1} (-1)^{n+m} \frac{\lambda^{2n+2m}}{\Gamma(2n+2m+2)} \frac{x^{2\beta n-\epsilon+1}}{2\beta n-\epsilon+1} \\ &\quad + \sum_{n=0}^{\infty} (-1)^{n+m+1} \frac{\lambda^{2n+2m+2}}{\Gamma(2n+2m+4)} \frac{x^{2\beta n+2\beta-\epsilon+1}}{2\beta n+2\beta-\epsilon+1} + C_1 \\ &= (-1)^m \frac{\lambda^{2m}}{\Gamma(2m+2)} \int \frac{dx}{x^{\epsilon}} + \sum_{n=-m}^{-1} (-1)^{n+m} \frac{\lambda^{2n+2m}}{\Gamma(2n+2m+2)} \frac{x^{2\beta n-\epsilon+1}}{2\beta n-\epsilon+1}\end{aligned} \tag{2.4}$$

$$+\frac{(-1)^{m+1}\lambda^{2m+2}x^{2\beta-\epsilon+1}}{2^{2m+3}\sqrt{\pi}\Gamma(m+2)\Gamma(m+5/2)(2\beta-\epsilon+1)}\sum_{n=0}^{\infty}\frac{(1)_n(1+(1-\epsilon)/(2\beta))_n}{(m+2)_n(m+5/2)_n(2+(1-\epsilon)/(2\beta))_n}\frac{\left(-\lambda^2x^{2\beta}/4\right)^n}{n!}+C_1$$

$$=(-1)^m\frac{\lambda^{2m}}{\Gamma(2m+2)}\int\frac{dx}{x^\epsilon}+\sum_{n=-m}^{-1}(-1)^{n+m}\frac{\lambda^{2n+2m}}{\Gamma(2n+2m+2)}\frac{x^{2\beta n-\epsilon+1}}{2\beta n-\epsilon+1}$$

$$+\frac{(-1)^{m+1}\lambda^{2m+2}x^{2\beta-\epsilon+1}}{2^{2m+3}\sqrt{\pi}\Gamma(m+2)\Gamma(m+5/2)(2\beta-\epsilon+1)}\times$$

$$\times{}_2F_3\left(1,1+\frac{1-\epsilon}{2\beta};m+2,m+\frac{5}{2},2+\frac{1-\epsilon}{2\beta};-\frac{\lambda^2x^{2\beta}}{4}\right)+C_1.$$

1. For $\epsilon=0$, we substitute $\epsilon=0$ in (2.4), and hence, we obtain

$$\int\frac{\sin(\lambda x^\beta)}{\lambda x^{\beta+\alpha}}dx=\int\frac{1}{\lambda x^\beta x^\alpha}\sum_{n=0}^{\infty}(-1)^n\frac{(\lambda x^\beta)^{2n+1}}{(2n+1)!}dx$$

$$=\int\sum_{n=-m}^{-1}(-1)^{n+m}\frac{\lambda^{2n+2m}}{\Gamma(2n+2m+2)}x^{2\beta n}dx+\int\sum_{n=0}^{\infty}(-1)^{n+m}\frac{\lambda^{2n+2m}}{\Gamma(2n+2m+2)}x^{2\beta n}dx$$

$$=\sum_{n=-m}^{-1}(-1)^{n+m}\frac{\lambda^{2n+2m}}{\Gamma(2n+2m+2)}\frac{x^{2\beta n+1}}{2\beta n+1}+\sum_{n=0}^{\infty}(-1)^{n+m}\frac{\lambda^{2n+2m}}{\Gamma(2n+2m+2)}\frac{x^{2\beta n+1}}{2\beta n+1}$$

$$=\sum_{n=-m}^{-1}(-1)^{n+m}\frac{\lambda^{2n+2m}}{\Gamma(2n+2m+2)}\frac{x^{2\beta n+1}}{2\beta n+1}$$

$$+\frac{(-1)^m\lambda^{2m}x}{2^{m+1}\sqrt{\pi}(2\beta+1)\Gamma(m+1)\Gamma(m+3/2)}\sum_{n=0}^{\infty}\frac{(1)_n(1+1/(2\beta))_n}{(m+1)_n(m+3/2)_n(2+1/(2\beta))_n}\frac{\left(-\lambda^2x^{2\beta}/4\right)^n}{n!}$$

$$=\sum_{n=-m}^{-1}(-1)^{n+m}\frac{\lambda^{2n+2m}}{\Gamma(2n+2m+2)}\frac{x^{2\beta n+1}}{2\beta n+1}$$

$$+\frac{(-1)^m\lambda^{2m}x}{2^{m+1}\sqrt{\pi}\Gamma(m+1)\Gamma(m+3/2)(2\beta+1)}\,{}_2F_3\left(1,1+\frac{1}{2\beta};m+1,m+\frac{3}{2},2+\frac{1}{2\beta};-\frac{\lambda^2x^{2\beta}}{4}\right)+C,$$

which is (2.1), and where $m=\alpha/\beta$.

2. For $\epsilon=1$, we set $\epsilon=1$ in (2) and obtain

$$\int\frac{\sin(\lambda x^\beta)}{\lambda x^{\beta+\alpha}}dx=(-1)^m\frac{\lambda^{2m}}{\Gamma(2m+2)}\ln|x|+\sum_{n=-m}^{-1}(-1)^{n+m}\frac{\lambda^{2n+2m}}{\Gamma(2n+2m+2)}\frac{x^{2\beta n}}{2\beta n}$$

$$+\frac{(-1)^{m+1}\lambda^{2m+2}x^{2\beta}}{2^{2m+4}\sqrt{\pi}\Gamma(m+2)\Gamma(m+5/2)\beta}\,{}_2F_3\left(1,1;m+2,m+\frac{5}{2},2;-\frac{\lambda^2x^{2\beta}}{4}\right)+C,$$

which is (2.2), and where $m=(\alpha-1)/\beta$.

3. For $\epsilon\in(-\beta,0)\cup(0,1)\cup(1,\beta)$, (2) gives

$$\int\frac{\sin(\lambda x^\beta)}{\lambda x^{\beta+\alpha}}dx=(-1)^m\frac{\lambda^{2m}}{\Gamma(2m+2)}\frac{x^{1-\epsilon}}{1-\epsilon}+\sum_{n=-m}^{-1}(-1)^{n+m}\frac{\lambda^{2n+2m}}{\Gamma(2n+2m+2)}\frac{x^{2\beta n-\epsilon+1}}{2\beta n-\epsilon+1}$$

$$+\frac{(-1)^{m+1}\lambda^{2m+2}x^{2\beta-\epsilon+1}}{2^{2m+3}\sqrt{\pi}\Gamma(m+2)\Gamma(m+5/2)(2\beta-\epsilon+1)}\times$$

$$\times{}_2F_3\left(1,1+\frac{1-\epsilon}{2\beta};m+2,m+\frac{5}{2},2+\frac{1-\epsilon}{2\beta};-\frac{\lambda^2x^{2\beta}}{4}\right)+C,$$

which is (2.3), and where $m = (\alpha - \epsilon)/\beta$. □

Example 1. In this example, we evaluate $\int \left[\sin(x^2)/x^{3.5}\right] dx$. We first observe that $\lambda = 1$ and $\beta = 2$. We also have $3.5 = \beta + \alpha = 2 + 1.5 = 2 + (1)2 - 0.5 = \beta + m\beta + \epsilon$, and so $m = 1$ and $\epsilon = -0.5$. Substituting $\lambda = 1, \beta = 2, m = 1$ and $\epsilon = -0.5$ in (2.3) gives

$$\int \frac{\sin(x^2)}{x^{3.5}} dx = -\frac{x^{1.5}}{9} - \frac{x^{-2.5}}{2.5} + \frac{x^{5.5}}{540\pi} \, {}_2F_3\left(1, \frac{9}{8}; 3, \frac{7}{2}, \frac{17}{8}; -\frac{x^4}{4}\right) + C.$$

We can use the same procedure for the hyperbolic sine integral, the results are stated in the following theorem. Its proof is similar to that of Theorem 1, we will omit it.

Theorem 2. *Let $\beta \geq 1$ and $\alpha > 1$, and let $\alpha = m\beta + \epsilon$, where m is an integer ($m \in \mathbb{N}$) and $-\beta < \epsilon < \beta$.*

1. *If $\epsilon = 0$, then*

$$\int \frac{\sinh(\lambda x^\beta)}{\lambda x^{\beta+\alpha}} dx = \sum_{n=-m}^{-1} \frac{\lambda^{2n+2m}}{\Gamma(2n+2m+2)} \frac{x^{2\beta n+1}}{2\beta n+1}$$
$$+ \frac{\lambda^{2m} x}{2^{m+1}\sqrt{\pi}\Gamma(m+1)\Gamma(m+3/2)(2\beta+1)} \, {}_2F_3\left(1, 1 + \frac{1}{2\beta}; m+1, m+\frac{3}{2}, 2+\frac{1}{2\beta}; \frac{\lambda^2 x^{2\beta}}{4}\right) + C,$$

where $m = \alpha/\beta$.

2. *If $\epsilon = 1$, then*

$$\int \frac{\sinh(\lambda x^\beta)}{\lambda x^{\beta+\alpha}} dx = \frac{\lambda^{2m}}{\Gamma(2m+2)} \ln|x| + \sum_{n=-m}^{-1} \frac{\lambda^{2n+2m}}{\Gamma(2n+2m+2)} \frac{x^{2\beta n}}{2\beta n}$$
$$+ \frac{\lambda^{2m+2} x^{2\beta}}{2^{2m+4}\sqrt{\pi}\Gamma(m+2)\Gamma(m+5/2)\beta} \, {}_2F_3\left(1, 1; m+2, m+\frac{5}{2}, 2; \frac{\lambda^2 x^{2\beta}}{4}\right) + C,$$

where $m = (\alpha - 1)/\beta$.

3. *Finally, if $\epsilon \in (-\beta, 0) \cup (0, 1) \cup (1, \beta)$, we have*

$$\int \frac{\sinh(\lambda x^\beta)}{\lambda x^{\beta+\alpha}} dx = \frac{\lambda^{2m}}{\Gamma(2m+2)} \frac{x^{1-\epsilon}}{1-\epsilon} + \sum_{n=-m}^{-1} \frac{\lambda^{2n+2m}}{\Gamma(2n+2m+2)} \frac{x^{2\beta n-\epsilon+1}}{2\beta n-\epsilon+1}$$
$$+ \frac{\lambda^{2m+2} x^{2\beta-\epsilon+1}}{2^{2m+3}\sqrt{\pi}\Gamma(m+2)\Gamma(m+5/2)(2\beta-\epsilon+1)} \times$$
$$\times {}_2F_3\left(1, 1 + \frac{1-\epsilon}{2\beta}; m+2, m+\frac{5}{2}, 2+\frac{1-\epsilon}{2\beta}; \frac{\lambda^2 x^{2\beta}}{4}\right) + C,$$

where $m = (\alpha - \epsilon)/\beta$.

3. Evaluation of the cosine integral $\mathbf{Ci}_{\beta,2\beta+\alpha}$, $\beta \geq 1$, $\alpha > 1$

Theorem 3. *Let $\beta \geq 1$ and $\alpha > 1$, and let $\alpha = 2\beta m + \epsilon$, where m is an integer ($m \in \mathbb{N}$) and $-2\beta < \epsilon < 2\beta$.*

1. If $\epsilon = 0$, then

$$Ci_{\beta,2\beta+\alpha} = \int \frac{\cos(\lambda x^\beta)}{\lambda x^{2\beta+\alpha}} dx = \frac{1}{\lambda}\frac{x^{1-2\beta-\alpha}}{1-2\beta-\alpha} + \sum_{n=-m}^{-1} (-1)^{n+m+1} \frac{\lambda^{2n+2m+1}}{\Gamma(2n+2m+3)} \frac{x^{2\beta n+1}}{2\beta n+1}$$
$$+ \frac{(-1)^m \lambda^{2m} x}{2^{m+2}\sqrt{\pi}\Gamma(m+3/2)\Gamma(m+2)(2\beta+1)} \, {}_2F_3\left(1, 1+\frac{1}{2\beta}; m+\frac{3}{2}, m+2, 2+\frac{1}{2\beta}; -\frac{\lambda^2 x^{2\beta}}{4}\right) + C, \tag{3.5}$$

where $m = \alpha/(2\beta)$.

2. If $\epsilon = 1$, then

$$\int \frac{\cos(\lambda x^\beta)}{\lambda x^{2\beta+\alpha}} dx = \frac{1}{\lambda}\frac{x^{1-2\beta-\alpha}}{1-2\beta-\alpha} + \frac{(-1)^m \lambda^{2m+1}}{\Gamma(2m+3)} \ln|x| + \sum_{n=-m}^{-1} (-1)^{n+m+1} \frac{\lambda^{2n+2m+1}}{\Gamma(2n+2m+3)} \frac{x^{2\beta n}}{2\beta n}$$
$$+ \frac{(-1)^{m+1} \lambda^{2m+3} x^{2\beta}}{2^{2m+5}\sqrt{\pi}\Gamma(m+5/2)\Gamma(m+3)\beta} \, {}_2F_3\left(1, 1; m+\frac{5}{2}, m+3, 2; -\frac{\lambda^2 x^{2\beta}}{4}\right) + C, \tag{3.6}$$

where $m = (\alpha-1)/(2\beta)$.

3. Finally, if $\epsilon \in (-2\beta, 0) \cup (0,1) \cup (1, 2\beta)$, we have

$$\begin{aligned} \int \frac{\cos(\lambda x^\beta)}{\lambda x^{2\beta+\alpha}} dx &= \frac{1}{\lambda}\frac{x^{1-2\beta-\alpha}}{1-2\beta-\alpha} + \frac{(-1)^m \lambda^{2m+1}}{\Gamma(2m+3)} \frac{x^{1-\epsilon}}{1-\epsilon} \\ &+ \sum_{n=-m}^{-1} (-1)^{n+m+1} \frac{\lambda^{2n+2m+1}}{\Gamma(2n+2m+3)} \frac{x^{2\beta n-\epsilon+1}}{2\beta n-\epsilon+1} \\ &+ \frac{(-1)^{m+1} \lambda^{2m+3} x^{2\beta-\epsilon+1}}{2^{2m+4}\sqrt{\pi}\Gamma(m+5/2)\Gamma(m+3)(2\beta-\epsilon+1)} \times \\ \times {}_2F_3&\left(1, 1+\frac{1-\epsilon}{2\beta}; m+\frac{5}{2}, m+3, 2+\frac{1-\epsilon}{2\beta}; -\frac{\lambda^2 x^{2\beta}}{4}\right) + C, \end{aligned} \tag{3.7}$$

where $m = (\alpha-\epsilon)/(2\beta)$.

P r o o f. We proceed as in Theorem 1. We have

$$\begin{aligned} \int \frac{\cos(\lambda x^\beta)}{\lambda x^{2\beta+\alpha}} dx &= \int \frac{1}{\lambda x^{2\beta+\alpha}} \sum_{n=0}^{\infty} (-1)^n \frac{(\lambda x^\beta)^{2n}}{(2n)!} dx \\ &= \int \frac{1}{\lambda x^{2\beta+\alpha}} dx + \frac{1}{\lambda} \int \sum_{n=1}^{\infty} (-1)^n \frac{\lambda^{2n}}{(2n)!} x^{2\beta n-2\beta-\alpha} dx \\ &= \int \frac{1}{\lambda x^{2\beta+\alpha}} dx + \frac{1}{\lambda} \int \sum_{n=0}^{\infty} (-1)^{n+1} \frac{\lambda^{2n+2}}{(2n+2)!} x^{2\beta n-\alpha} dx \end{aligned}$$

$$= \int \frac{1}{\lambda x^{2\beta+\alpha}} dx + \int \sum_{n=0}^{m-1} (-1)^{n+1} \frac{\lambda^{2n+1}}{(2n+2)!} x^{2\beta n-2\beta m-\epsilon} dx + \int \sum_{n=m}^{\infty} (-1)^{n+1} \frac{\lambda^{2n+1}}{(2n+2)!} x^{2\beta n-2\beta m-\epsilon} dx$$

$$
\begin{aligned}
&= \int \frac{1}{\lambda x^{2\beta+\alpha}}dx + \int \sum_{n=0}^{m-1}(-1)^{n+1}\frac{\lambda^{2n+1}}{(2n+2)!}x^{2\beta(n-m)-\epsilon}dx + \int \sum_{n=m}^{\infty}(-1)^{n+1}\frac{\lambda^{2n+1}}{(2n+2)!}x^{2\beta(n-m)-\epsilon}dx \\
&= \int \frac{1}{\lambda x^{2\beta+\alpha}}dx + \int \sum_{n=-m}^{-1}(-1)^{n+m+1}\frac{\lambda^{2n+2m+1}}{(2n+2m+2)!}x^{2\beta n-\epsilon}dx \\
&\quad + \int \sum_{n=0}^{\infty}(-1)^{n+m+1}\frac{\lambda^{2n+2m+1}}{(2n+2m+2)!}x^{2\beta n-\epsilon}dx \\
&= \int \frac{1}{\lambda x^{2\beta+\alpha}}dx + \int \sum_{n=-m}^{-1}(-1)^{n+m+1}\frac{\lambda^{2n+2m+1}}{\Gamma(2n+2m+3)}x^{2\beta n-\epsilon}dx \\
&\quad + \int \sum_{n=0}^{\infty}(-1)^{n+m+1}\frac{\lambda^{2n+2m+1}}{\Gamma(2n+2m+3)}x^{2\beta n-\epsilon}dx \\
&= \int \frac{1}{\lambda x^{2\beta+\alpha}}dx + (-1)^{m+1}\frac{\lambda^{2m+1}}{\Gamma(2m+3)}\int\frac{dx}{x^{\epsilon}} + \int \sum_{n=-m}^{-1}(-1)^{n+m+1}\frac{\lambda^{2n+2m+1}}{\Gamma(2n+2m+3)}x^{2\beta n-\epsilon}dx \\
&\quad + \int \sum_{n=1}^{\infty}(-1)^{n+m+1}\frac{\lambda^{2n+2m+1}}{\Gamma(2n+2m+3)}x^{2\beta n-\epsilon}dx \\
&= \int \frac{1}{\lambda x^{2\beta+\alpha}}dx + (-1)^{m+1}\frac{\lambda^{2m+1}}{\Gamma(2m+3)}\int\frac{dx}{x^{\epsilon}} + \int \sum_{n=-m}^{-1}(-1)^{n+m+1}\frac{\lambda^{2n+2m+1}}{\Gamma(2n+2m+3)}x^{2\beta n-\epsilon}dx \\
&\quad + \int \sum_{n=0}^{\infty}(-1)^{n+m}\frac{\lambda^{2n+2m+3}}{\Gamma(2n+2m+5)}x^{2\beta n+2\beta-\epsilon}dx \\
&= \frac{1}{\lambda}\frac{x^{1-2\beta-\alpha}}{1-2\beta-\alpha} + (-1)^{m+1}\frac{\lambda^{2m+1}}{\Gamma(2m+3)}\int\frac{dx}{x^{\epsilon}} + \sum_{n=-m}^{-1}(-1)^{n+m+1}\frac{\lambda^{2n+2m+1}}{\Gamma(2n+2m+3)}\frac{x^{2\beta n-\epsilon+1}}{2\beta n-\epsilon+1} \\
&\quad + \sum_{n=0}^{\infty}(-1)^{n+m}\frac{\lambda^{2n+2m+3}}{\Gamma(2n+2m+5)}\frac{x^{2\beta n+2\beta-\epsilon+1}}{2\beta n+2\beta-\epsilon+1} + C_1 \\
&= \frac{1}{\lambda}\frac{x^{1-2\beta-\alpha}}{1-2\beta-\alpha} + (-1)^{m+1}\frac{\lambda^{2m+1}}{\Gamma(2m+3)}\int\frac{dx}{x^{\epsilon}} + \sum_{n=-m}^{-1}(-1)^{n+m+1}\frac{\lambda^{2n+2m+1}}{\Gamma(2n+2m+3)}\frac{x^{2\beta n-\epsilon+1}}{2\beta n-\epsilon+1} \\
&+ \frac{(-1)^{m+1}\lambda^{2m+3}x^{2\beta-\epsilon+1}}{2^{2m+4}\sqrt{\pi}\Gamma(m+5/2)\Gamma(m+3)(2\beta-\epsilon+1)}\sum_{n=0}^{\infty}\frac{(1)_n\,(1+(1-\epsilon)/(2\beta))_n}{(m+5/2)_n\,(m+3)_n\,(2+(1-\epsilon)/(2\beta))_n}\frac{\left(-\lambda^2x^{2\beta}/4\right)^n}{n!} + C_1 \\
&= \frac{1}{\lambda}\frac{x^{1-2\beta-\alpha}}{1-2\beta-\alpha} + (-1)^{m}\frac{\lambda^{2m+1}}{\Gamma(2m+3)}\int\frac{dx}{x^{\epsilon}} + \sum_{n=-m}^{-1}(-1)^{n+m+1}\frac{\lambda^{2n+2m+1}}{\Gamma(2n+2m+3)}\frac{x^{2\beta n-\epsilon+1}}{2\beta n-\epsilon+1} \\
&+ \frac{(-1)^{m+1}\lambda^{2m+3}x^{2\beta-\epsilon+1}}{2^{2m+4}\sqrt{\pi}\Gamma(m+5/2)\Gamma(m+3)(2\beta-\epsilon+1)}\;{}_2F_3\left(1,1+\frac{1-\epsilon}{2\beta};m+\frac{5}{2},m+3,2+\frac{1-\epsilon}{2\beta};-\frac{\lambda^2x^{2\beta}}{4}\right) + C_1.
\end{aligned}
\tag{3.8}
$$

1. For $\epsilon = 0$, we substitute $\epsilon = 0$ in (3.8), and hence, we obtain

$$
\begin{aligned}
\int \frac{\cos(\lambda x^{\beta})}{\lambda x^{2\beta+\alpha}}dx &= \int\frac{dx}{\lambda x^{2\beta+\alpha}} + \int\sum_{n=-m}^{-1}(-1)^{n+m+1}\frac{\lambda^{2n+2m+1}}{\Gamma(2n+2m+3)}x^{2\beta n}dx \\
&+ \int\sum_{n=0}^{\infty}(-1)^{n+m+1}\frac{\lambda^{2n+2m+1}}{\Gamma(2n+2m+3)}x^{2\beta n}dx = \frac{1}{\lambda}\frac{x^{1-2\beta-\alpha}}{1-2\beta-\alpha}
\end{aligned}
$$

$$+\sum_{n=-m}^{-1}(-1)^{n+m+1}\frac{\lambda^{2n+2m+1}}{\Gamma(2n+2m+3)}\frac{x^{2\beta n+1}}{2\beta n+1}+\sum_{n=0}^{\infty}(-1)^{n+m+1}\frac{\lambda^{2n+2m+1}}{\Gamma(2n+2m+3)}\frac{x^{2\beta n+1}}{2\beta n+1}$$

$$=\frac{1}{\lambda}\frac{x^{1-2\beta-\alpha}}{1-2\beta-\alpha}+\sum_{n=-m}^{-1}(-1)^{n+m+1}\frac{\lambda^{2n+2m+1}}{\Gamma(2n+2m+3)}\frac{x^{2\beta n+1}}{2\beta n+1}$$

$$+\frac{(-1)^m\lambda^{2m}x}{2^{m+2}\sqrt{\pi}(2\beta+1)\Gamma(m+3/2)\Gamma(m+2)}\sum_{n=0}^{\infty}\frac{(1)_n\left(1+1/(2\beta)\right)_n}{(m+3/2)_n(m+2)_n\left(2+1/(2\beta)\right)_n}\frac{\left(-\lambda^2x^{2\beta}/4\right)^n}{n!}$$

$$=\frac{1}{\lambda}\frac{x^{1-2\beta-\alpha}}{1-2\beta-\alpha}+\sum_{n=-m}^{-1}(-1)^{n+m+1}\frac{\lambda^{2n+2m+1}}{\Gamma(2n+2m+3)}\frac{x^{2\beta n+1}}{2\beta n+1}$$

$$+\frac{(-1)^m\lambda^{2m}x}{2^{m+2}\sqrt{\pi}\Gamma(m+3/2)\Gamma(m+2)(2\beta+1)}\,{}_2F_3\left(1,1+\frac{1}{2\beta};m+\frac{3}{2},m+2,2+\frac{1}{2\beta};-\frac{\lambda^2x^{2\beta}}{4}\right)+C,$$

which is (3.5), and where $m=\alpha/(2\beta)$.

2. For $\epsilon=1$, we set $\epsilon=1$ in (3.8) and obtain

$$\int\frac{\cos\left(\lambda x^{\beta}\right)}{\lambda x^{2\beta+\alpha}}dx=\frac{1}{\lambda}\frac{x^{1-2\beta-\alpha}}{1-2\beta-\alpha}+\frac{(-1)^m\lambda^{2m+1}}{\Gamma(2m+3)}\ln|x|+\sum_{n=-m}^{-1}(-1)^{n+m+1}\frac{\lambda^{2n+2m+1}}{\Gamma(2n+2m+3)}\frac{x^{2\beta n}}{2\beta n}$$

$$+\frac{(-1)^{m+1}\lambda^{2m+3}x^{2\beta}}{2^{2m+5}\sqrt{\pi}\Gamma(m+5/2)\Gamma(m+3)\beta}\,{}_2F_3\left(1,1;m+\frac{5}{2},m+3,2;-\frac{\lambda^2x^{2\beta}}{4}\right)+C,$$

which is (3.6), and where $m=(\alpha-1)/(2\beta)$.

3. For $\epsilon\in(-2\beta,0)\cup(0,1)\cup(1,2\beta)$, (3.8) gives

$$\int\frac{\cos\left(\lambda x^{\beta}\right)}{\lambda x^{2\beta+\alpha}}dx=\frac{1}{\lambda}\frac{x^{1-2\beta-\alpha}}{1-2\beta-\alpha}+\frac{(-1)^m\lambda^{2m+1}}{\Gamma(2m+3)}\frac{x^{1-\epsilon}}{1-\epsilon}$$

$$+\sum_{n=-m}^{-1}(-1)^{n+m+1}\frac{\lambda^{2n+2m+1}}{\Gamma(2n+2m+3)}\frac{x^{2\beta n-\epsilon+1}}{2\beta n-\epsilon+1}$$

$$+\frac{(-1)^{m+1}\lambda^{2m+3}x^{2\beta-\epsilon+1}}{2^{2m+4}\sqrt{\pi}\Gamma(m+5/2)\Gamma(m+3)(2\beta-\epsilon+1)}\times$$

$$\times{}_2F_3\left(1,1+\frac{1-\epsilon}{2\beta};m+\frac{5}{2},m+3,2+\frac{1-\epsilon}{2\beta};-\frac{\lambda^2x^{2\beta}}{4}\right)+C,$$

which is (3.7), and where $m=(\alpha-\epsilon)/(2\beta)$. □

Example 2. In this example, we evaluate $\int\left[\cos(x)/x^5\right]dx$. We first observe that $\lambda=1$ and $\beta=1$. We also have $5=2\beta+\alpha=2+3=2+2(1)(1)+1=\beta+2\beta m+\epsilon$, and so $m=1$ and $\epsilon=1$. Substituting $\lambda=1$, $\beta=1$, $m=1$ and $\epsilon=1$ in (3.6) gives

$$\int\frac{\cos(x)}{x^5}dx=-\frac{x^{-4}}{4}-\frac{x^{-2}}{4}+\frac{\ln|x|}{24}+\frac{x^2}{720\pi}\,{}_2F_3\left(1,1;\frac{7}{2},4,2;-\frac{x^2}{4}\right)+C.$$

We can use the same procedure for the hyperbolic cosine integral, the results are stated in the next theorem. Its proof is similar to Theorem 3's proof, we will omit it.

Theorem 4. *Let $\beta\geq 1$ and $\alpha>1$, and let $\alpha=2\beta m+\epsilon$, where m is an integer $(m\in\mathbb{N})$ and $-2\beta<\epsilon<2\beta$.*

1. If $\epsilon = 0$, then

$$\int \frac{\cosh(\lambda x^{\beta})}{\lambda x^{2\beta+\alpha}}dx = \frac{1}{\lambda}\frac{x^{1-2\beta-\alpha}}{1-2\beta-\alpha} + \sum_{n=-m}^{-1}\frac{\lambda^{2n+2m+1}}{\Gamma(2n+2m+3)}\frac{x^{2\beta n+1}}{2\beta n+1}$$
$$+\frac{\lambda^{2m}x}{2^{m+2}\sqrt{\pi}\Gamma(m+3/2)\Gamma(m+2)(2\beta+1)}\ {}_2F_3\left(1,1+\frac{1}{2\beta};m+\frac{3}{2},m+2,2+\frac{1}{2\beta};\frac{\lambda^2x^{2\beta}}{4}\right)+C,$$

where $m = \alpha/(2\beta)$.

2. If $\epsilon = 1$, then

$$\int \frac{\cosh(\lambda x^{\beta})}{\lambda x^{2\beta+\alpha}}dx = \frac{1}{\lambda}\frac{x^{1-2\beta-\alpha}}{1-2\beta-\alpha} + \frac{\lambda^{2m+1}}{\Gamma(2m+3)}\ln|x| + \sum_{n=-m}^{-1}\frac{\lambda^{2n+2m+1}}{\Gamma(2n+2m+3)}\frac{x^{2\beta n}}{2\beta n}$$
$$+\frac{\lambda^{2m+3}x^{2\beta}}{2^{2m+5}\sqrt{\pi}\Gamma(m+5/2)\Gamma(m+3)\beta}\ {}_2F_3\left(1,1;m+\frac{5}{2},m+3,2;\frac{\lambda^2x^{2\beta}}{4}\right)+C,$$

where $m = (\alpha-1)/(2\beta)$.

3. Finally, if $\epsilon \in (-2\beta,0)\cup(0,1)\cup(1,2\beta)$, we have

$$\int \frac{\cos(\lambda x^{\beta})}{\lambda x^{2\beta+\alpha}}dx = \frac{1}{\lambda}\frac{x^{1-2\beta-\alpha}}{1-2\beta-\alpha} + \frac{(-1)^m\lambda^{2m+1}}{\Gamma(2m+3)}\frac{x^{1-\epsilon}}{1-\epsilon} + \sum_{n=-m}^{-1}\frac{\lambda^{2n+2m+1}}{\Gamma(2n+2m+3)}\frac{x^{2\beta n-\epsilon+1}}{2\beta n-\epsilon+1}$$
$$+\frac{\lambda^{2m+3}x^{2\beta-\epsilon+1}}{2^{2m+4}\sqrt{\pi}\Gamma(m+5/2)\Gamma(m+3)(2\beta-\epsilon+1)}\times$$
$$\times{}_2F_3\left(1,1+\frac{1-\epsilon}{2\beta};m+\frac{5}{2},m+3,2+\frac{1-\epsilon}{2\beta};\frac{\lambda^2x^{2\beta}}{4}\right)+C,$$

where $m = (\alpha-\epsilon)/(2\beta)$.

4. Evaluation of the exponential integral $\mathbf{Ei}_{\beta,\beta+\alpha}$, $\beta \geq 1$, $\alpha > 1$

Theorem 5. *Let $\beta \geq 1$ and $\alpha > 1$, and let $\alpha = \beta m + \epsilon$, where m is an integer ($m \in \mathbb{N}$) and $-\beta < \epsilon < \beta$.*

1. If $\epsilon = 0$, then

$$Ei_{\beta,\beta+\alpha} = \int \frac{e^{\lambda x^{\beta}}}{\lambda x^{\beta+\alpha}}dx = \frac{1}{\lambda}\frac{x^{1-\beta-\alpha}}{1-\beta-\alpha} + \sum_{n=-m}^{-1}\frac{\lambda^{n+m}}{\Gamma(n+m+2)}\frac{x^{\beta n+1}}{\beta n+1}$$
$$+\frac{\lambda^m x}{\Gamma(m+2)(\beta+1)}\ {}_2F_2\left(1,1+\frac{1}{\beta};m+2,2+\frac{1}{\beta};\lambda x^{\beta}\right)+C, \tag{4.9}$$

where $m = \alpha/(\beta)$.

2. If $\epsilon = 1$, then

$$Ei_{\beta,\beta+\alpha} = \int \frac{e^{\lambda x^{\beta}}}{\lambda x^{\beta+\alpha}}dx = \frac{1}{\lambda}\frac{x^{1-\beta-\alpha}}{1-\beta-\alpha} + \frac{\lambda^m}{\Gamma(m+2)}\ln|x| + \sum_{n=-m}^{-1}\frac{\lambda^{n+m}}{\Gamma(n+m+2)}\frac{x^{\beta n}}{\beta n}$$
$$+\frac{\lambda^{m+1}x^{\beta}}{\Gamma(m+3)\beta}\ {}_2F_2\left(1,1;m+3,2;\lambda x^{\beta}\right)+C, \tag{4.10}$$

where $m = (\alpha-1)/(\beta)$.

3. Finally, if $\epsilon \in (-\beta, 0) \cup (0, 1) \cup (1, \beta)$, we have

$$\begin{aligned}\int \frac{e^{\lambda x^\beta}}{\lambda x^{\beta+\alpha}} dx = \frac{1}{\lambda} \frac{x^{1-\beta-\alpha}}{1-\beta-\alpha} + \frac{\lambda^m}{\Gamma(m+2)} \frac{x^{1-\epsilon}}{1-\epsilon} + \sum_{n=-m}^{-1} \frac{\lambda^{n+m}}{\Gamma(n+m+2)} \frac{x^{\beta n-\epsilon+1}}{\beta n-\epsilon+1} \\ + \frac{\lambda^{m+1} x^{\beta-\epsilon+1}}{\Gamma(m+3)(\beta-\epsilon+1)} \, {}_2F_2\left(1, 1+\frac{1-\epsilon}{\beta}; m+3, 2+\frac{1-\epsilon}{\beta}; \lambda x^\beta\right) + C,\end{aligned} \tag{4.11}$$

where $m = (\alpha - \epsilon)/(\beta)$.

P r o o f. We proceed as before. Then, we have

$$\begin{aligned}
\int \frac{e^{\lambda x^\beta}}{\lambda x^{\beta+\alpha}} dx &= \int \frac{1}{\lambda x^{\beta+\alpha}} \sum_{n=0}^{\infty} \frac{(\lambda x^\beta)^n}{n!} dx = \int \frac{1}{\lambda x^{\beta+\alpha}} dx + \frac{1}{\lambda} \int \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} x^{\beta n-\beta-\alpha} dx \\
&= \int \frac{1}{\lambda x^{\beta+\alpha}} dx + \frac{1}{\lambda} \int \sum_{n=0}^{\infty} \frac{\lambda^{n+1}}{(n+1)!} x^{\beta n-\alpha} dx \\
&= \int \frac{1}{\lambda x^{\beta+\alpha}} dx + \int \sum_{n=0}^{m-1} \frac{\lambda^n}{(n+1)!} x^{\beta n-\beta m-\epsilon} dx + \int \sum_{n=m}^{\infty} \frac{\lambda^n}{(n+1)!} x^{\beta n-\beta m-\epsilon} dx \\
&= \int \frac{1}{\lambda x^{\beta+\alpha}} dx + \int \sum_{n=0}^{m-1} \frac{\lambda^n}{(n+1)!} x^{\beta(n-m)-\epsilon} dx + \int \sum_{n=m}^{\infty} \frac{\lambda^n}{(n+1)!} x^{\beta(n-m)-\epsilon} dx \\
&= \int \frac{1}{\lambda x^{\beta+\alpha}} dx + \int \sum_{n=-m}^{-1} \frac{\lambda^{n+m}}{(n+m+1)!} x^{\beta n-\epsilon} dx + \int \sum_{n=0}^{\infty} \frac{\lambda^{n+m}}{(n+m+1)!} x^{\beta n-\epsilon} dx \\
&= \int \frac{1}{\lambda x^{\beta+\alpha}} dx + \int \sum_{n=-m}^{-1} \frac{\lambda^{n+m}}{\Gamma(n+m+2)} x^{\beta n-\epsilon} dx + \int \sum_{n=0}^{\infty} \frac{\lambda^{n+m}}{\Gamma(n+m+2)} x^{\beta n-\epsilon} dx \\
&= \int \frac{1}{\lambda x^{\beta+\alpha}} dx + \frac{\lambda^m}{\Gamma(m+2)} \int \frac{dx}{x^\epsilon} + \int \sum_{n=-m}^{-1} \frac{\lambda^{n+m}}{\Gamma(n+m+2)} x^{\beta n-\epsilon} dx \\
&\quad + \int \sum_{n=1}^{\infty} \frac{\lambda^{n+m}}{\Gamma(n+m+2)} x^{\beta n-\epsilon} dx = \int \frac{1}{\lambda x^{\beta+\alpha}} dx + \frac{\lambda^m}{\Gamma(m+2)} \int \frac{dx}{x^\epsilon} \\
&\quad + \int \sum_{n=-m}^{-1} \frac{\lambda^{n+m}}{\Gamma(n+m+2)} x^{\beta n-\epsilon} dx + \int \sum_{n=0}^{\infty} \frac{\lambda^{n+m+1}}{\Gamma(n+m+3)} x^{\beta n+\beta-\epsilon} dx \\
&= \frac{1}{\lambda} \frac{x^{1-\beta-\alpha}}{1-\beta-\alpha} + \frac{\lambda^m}{\Gamma(m+2)} \int \frac{dx}{x^\epsilon} + \sum_{n=-m}^{-1} \frac{\lambda^{n+m}}{\Gamma(n+m+2)} \frac{x^{\beta n-\epsilon+1}}{\beta n-\epsilon+1} \\
&\quad + \sum_{n=0}^{\infty} \frac{\lambda^{n+m+1}}{\Gamma(n+m+3)} \frac{x^{\beta n+\beta-\epsilon+1}}{\beta n+\beta-\epsilon+1} + C_1 \\
&= \frac{1}{\lambda} \frac{x^{1-\beta-\alpha}}{1-\beta-\alpha} + \frac{\lambda^m}{\Gamma(m+2)} \int \frac{dx}{x^\epsilon} + \sum_{n=-m}^{-1} \frac{\lambda^{n+m}}{\Gamma(n+m+2)} \frac{x^{\beta n-\epsilon+1}}{\beta n-\epsilon+1} \\
&\quad + \frac{\lambda^{m+1} x^{\beta-\epsilon+1}}{\Gamma(m+3)(\beta-\epsilon+1)} \sum_{n=0}^{\infty} \frac{(1)_n \left(1+(1-\epsilon)/\beta\right)_n}{(m+3)_n \left(2+(1-\epsilon)/\beta\right)_n} \frac{\left(\lambda x^\beta\right)^n}{n!} + C_1 \\
&= \frac{1}{\lambda} \frac{x^{1-\beta-\alpha}}{1-\beta-\alpha} + \frac{\lambda^m}{\Gamma(m+2)} \int \frac{dx}{x^\epsilon} + \sum_{n=-m}^{-1} \frac{\lambda^{n+m}}{\Gamma(n+m+2)} \frac{x^{\beta n-\epsilon+1}}{\beta n-\epsilon+1}
\end{aligned} \tag{4.12}$$

$$+\frac{\lambda^{m+1}x^{\beta-\epsilon+1}}{\Gamma(m+3)(\beta-\epsilon+1)}\ {}_2F_2\left(1,1+\frac{1-\epsilon}{\beta};m+3,2+\frac{1-\epsilon}{\beta};\lambda x^\beta\right)+C_1.$$

1. For $\epsilon=0$, we substitute $\epsilon=0$ in (4.12), and hence, we obtain

$$\int\frac{e^{\lambda x^\beta}}{\lambda x^{\beta+\alpha}}dx=\int\frac{dx}{\lambda x^{\beta+\alpha}}+\int\sum_{n=-m}^{-1}\frac{\lambda^{n+m}}{\Gamma(n+m+2)}x^{\beta n}dx+\int\sum_{n=0}^{\infty}\frac{\lambda^{n+m}}{\Gamma(n+m+2)}x^{\beta n}dx$$
$$=\frac{1}{\lambda}\frac{x^{1-\beta-\alpha}}{1-\beta-\alpha}+\sum_{n=-m}^{-1}\frac{\lambda^{n+m}}{\Gamma(n+m+2)}\frac{x^{\beta n+1}}{\beta n+1}+\sum_{n=0}^{\infty}\frac{\lambda^{n+m}}{\Gamma(n+m+2)}\frac{x^{\beta n+1}}{\beta n+1}$$
$$=\frac{1}{\lambda}\frac{x^{1-\beta-\alpha}}{1-\beta-\alpha}+\sum_{n=-m}^{-1}\frac{\lambda^{n+m}}{\Gamma(n+m+2)}\frac{x^{\beta n+1}}{\beta n+1}$$
$$+\frac{\lambda^m x}{\Gamma(m+2)(\beta+1)}\sum_{n=0}^{\infty}\frac{(1)_n\,(1+1/\beta)_n}{(m+2)_n\,(2+1/\beta)_n}\frac{\left(\lambda x^\beta\right)^n}{n!}=\frac{1}{\lambda}\frac{x^{1-\beta-\alpha}}{1-\beta-\alpha}$$
$$+\sum_{n=-m}^{-1}\frac{\lambda^{n+m}}{\Gamma(n+m+2)}\frac{x^{\beta n+1}}{\beta n+1}+\frac{\lambda^m x}{\Gamma(m+2)(\beta+1)}\ {}_2F_2\left(1,1+\frac{1}{\beta};m+2,2+\frac{1}{\beta};\lambda x^\beta\right)+C,$$

which is (4.9), and where $m=\alpha/\beta$.

2. For $\epsilon=1$, we set $\epsilon=1$ in (4.12) and obtain

$$\int\frac{e^{\lambda x^\beta}}{\lambda x^{\beta+\alpha}}dx=\frac{1}{\lambda}\frac{x^{1-\beta-\alpha}}{1-\beta-\alpha}+\frac{\lambda^m}{\Gamma(m+2)}\ln|x|+\sum_{n=-m}^{-1}\frac{\lambda^{n+m}}{\Gamma(n+m+2)}\frac{x^{\beta n}}{\beta n}$$
$$+\frac{\lambda^{m+1}x^\beta}{\Gamma(m+3)\beta}\ {}_2F_2\left(1,1;m+3,2;\lambda x^\beta\right)+C,$$

which is (4.10), and where $m=(\alpha-1)/\beta$.

3. For $\epsilon\in(-\beta,0)\cup(0,1)\cup(1,\beta)$, (4.12) gives

$$\int\frac{e^{\lambda x^\beta}}{\lambda x^{\beta+\alpha}}dx=\frac{1}{\lambda}\frac{x^{1-\beta-\alpha}}{1-\beta-\alpha}+\frac{\lambda^m}{\Gamma(m+2)}\frac{x^{1-\epsilon}}{1-\epsilon}+\sum_{n=-m}^{-1}\frac{\lambda^{n+m}}{\Gamma(n+m+2)}\frac{x^{\beta n-\epsilon+1}}{\beta n-\epsilon+1}$$
$$+\frac{\lambda^{m+1}x^{\beta-\epsilon+1}}{\Gamma(m+3)(\beta-\epsilon+1)}\ {}_2F_2\left(1,1+\frac{1-\epsilon}{\beta};m+3,2+\frac{1-\epsilon}{\beta};\lambda x^\beta\right)+C,$$

which is (4.11), and where $m=(\alpha-\epsilon)/\beta$. □

Example 3. In this example, we evaluate $\int(e^{-x^2}/x^4)dx$. We first observe that $\lambda=-1$ and $\beta=2$. We also have $4=\beta+\alpha=2+2=2+2(1)+0=\beta+\beta m+\epsilon$, and so $m=1$ and $\epsilon=0$. Substituting $\lambda=1$, $\beta=1$, $m=1$ and $\epsilon=0$ in (4.9) gives

$$\int\frac{e^{-x^2}}{x^4}dx=\frac{x^{-3}}{3}-\frac{1}{x}-\frac{x}{4}\ {}_2F_2\left(1,2;3,3;-x^2\right)+C.$$

Corollary 1. *Let $\alpha>1$ and let $\alpha=m+\epsilon$, where m is an integer $(m\in\mathbb{N})$ and $-1<\epsilon\leq 1$.*

1. *If $\epsilon = 0$ or 1, then*

$$Ei_{1,1+\alpha} = \int \frac{e^{\lambda x}}{\lambda x^{1+\alpha}} dx = -\frac{1}{\lambda \alpha x^{\alpha}} + \frac{\lambda^m}{\Gamma(m+2)} \ln|x| + \sum_{n=-m}^{-1} \frac{\lambda^{n+m}}{\Gamma(n+m+2)} \frac{x^n}{n}$$
$$+ \frac{\lambda^{m+1} x}{\Gamma(m+3)\beta} \, {}_2F_2\left(1,1;m+3,2;\lambda x\right) + C,$$

where $m = \alpha - 1$.

2. *And if $\epsilon \in (-1,0) \cup (0,1)$, we have*

$$\int \frac{e^{\lambda x}}{\lambda x^{1+\alpha}} dx = -\frac{1}{\lambda \alpha x^{\alpha}} + \frac{\lambda^m}{\Gamma(m+2)} \frac{x^{1-\epsilon}}{1-\epsilon} + \sum_{n=-m}^{-1} \frac{\lambda^{n+m}}{\Gamma(n+m+2)} \frac{x^{n-\epsilon+1}}{n-\epsilon+1}$$
$$+ \frac{\lambda^{m+1} x^{2-\epsilon}}{\Gamma(m+3)(2-\epsilon)} \, {}_2F_2\left(1,2-\epsilon;m+3,3-\epsilon;\lambda x\right) + C,$$

where $m = \alpha - \epsilon$.

P r o o f.

1. If $\epsilon = 0$ or 1 implies $\alpha = m + \epsilon$ is an integer ($\alpha \in \mathbb{N}$) since ($m \in \mathbb{N}$). Morever, $\alpha = m + \epsilon$ implies $\beta = 1$ in Theorem 5. Therefore, we obtain (1) by setting $\beta = 1$ in (4.10).

2. For $\epsilon \in (-1,0) \cup (0,1)$, we set $\beta = 1$ in (4.11) and obtain (2). □

Example 4. In this example, we evaluate $\int \left(e^{-x}/x^{3.7}\right) dx$. We first observe that $\lambda = -1$. We also have $3.7 = 1 + \alpha = 1 + 2.7 = 1 + 2 + 0.7 = 1 + m + \epsilon$, and so $m = 2$ and $\epsilon = 0.7$. Substituting $\lambda = -1, m = 2$ and $\epsilon = 0.7$ in (2) gives

$$\int \frac{e^{-x}}{x^{3.7}} dx = \frac{x^{-2.7}}{2.7} - \frac{x^{0.3}}{1.8} - \frac{x^{-1.7}}{1.7} + \frac{x^{-0.7}}{1.4} - \frac{x^{1.3}}{31.2} \, {}_2F_2\left(1,1.3;5,2.3;-x\right) + C.$$

5. Conclusion

Formulas for the non-elementary integrals $\mathrm{Si}_{\beta,\alpha} = \int[\sin(\lambda x^{\beta})/(\lambda x^{\alpha})]dx$, $\beta \geq 1$, $\alpha > \beta + 1$, and $\mathrm{Ci}_{\beta,\alpha} = \int[\cos(\lambda x^{\beta})/(\lambda x^{\alpha})]dx$, $\beta \geq 1$, $\alpha > 2\beta + 1$, were explicitly derived in terms of the hypergeometric function ${}_2F_3$ (see Theorems 1 and 2). Once derived, formulas for the hyperbolic sine and hyperbolic cosine integrals were deduced from those of the sine and cosine integrals (see Theorems 2 and 4). On the other hand, the exponential integral $\mathrm{Ei}_{\beta,\alpha} = \int(e^{\lambda x^{\beta}}/x^{\alpha})dx$, $\beta \geq 1$, $\alpha > \beta + 1$ was expressed in terms of the hypergeometric function ${}_2F_2$ (see Theorem 5 and Corollary 1).

Beside, illustrative examples were given. Therefore, their corresponding definite integrals can now be evaluated using the FTC rather than using numerical integration.

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A NUMERICAL TECHNIQUE FOR THE SOLUTION OF GENERAL EIGHTH ORDER BOUNDARY VALUE PROBLEMS: A FINITE DIFFERENCE METHOD

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Abstract: In this article, we present a novel finite difference method for the numerical solution of the eighth order boundary value problems in ordinary differential equations. We have discretized the problem by using the boundary conditions in a natural way to obtain a system of equations. Then we have solved system of equations to obtain a numerical solution of the problem. Also we obtained numerical values of derivatives of solution as a byproduct of the method. The numerical experiments show that proposed method is efficient and fourth order accurate.

Key words: Boundary value problem, Eighth order equation, Finite difference method, Fourth order method.

1. Introduction

In the present article we have considered general eighth order boundary value problem of the following form:

$$u^{(8)}(x) = f(x, u, u', u'', u^{(3)}, u^{(4)}, u^{(5)}, u^{(6)}, u^{(7)}), \quad a < x < b \tag{1.1}$$

and the boundary conditions are

$$\begin{gathered} u(a) = \alpha_1, \quad u''(a) = \alpha_2, \quad u^{(4)}(a) = \alpha_3, \quad u^{(6)}(a) = \alpha_4, \\ u(b) = \beta_1, \quad u''(b) = \beta_2, \quad u^{(4)}(b) = \beta_3 \quad \text{and} \quad u^{(6)}(b) = \beta_4, \end{gathered}$$

where $u(x)$ and forcing function $f(x, u, u', u'', u^{(3)}, u^{(4)}, u^{(5)}, u^{(6)}, u^{(7)})$ are real and smooth function in [a,b] and α_1, α_2, α_3, α_4, β_1, β_2, β_3 and β_4 are constant.

The above eight order boundary value problem arises in physics such as fluid dynamics, vibrations and so on [1, 2]. For the detail discussion on the existence and uniqueness of the solution of higher order differential equations and corresponding BVPs, reader can refer [3]. So we have assumed that there exists a unique solution to boundary value problem (1.1).

In general it is difficult to obtain analytical solution of the (1.1) for the arbitrary forcing function f. Hence we desire some numerical technique for its numerical solution. We have some numerical methods for either same or different source function as in problem (1.1), for examples Galerkin Method [4, 5], variational iterational technique [6], finite difference method [7], Adomian decomposition method [8] and references there in.

In this article, we have developed numerical method to obtain numerical solution of general eighth order boundary problem (1.1) using finite difference method which involves discretizing the eighth order equation using values of the $u(x), u''(x), u^{(4)}(x)$ and $u^{(6)}(x)$ at discrete points. At each discrete point problem (1.1) reduced into a system of equations. Finally we have solved a well structured system of equations for the numerical solution of problem (1.1) and some other by-products.

We have presented our work in this article as follows. In Section 2 we have proposed our finite difference method and in Section 3 the derivation of the proposed finite difference method. In Section 4 we have tested proposed method on model problems and short discussion on numerical results. A summary on development and performance of the proposed method are presented in Section 5.

2. The Difference Method

Let us assume problem (1.1) posses solution and it will be $u(x)$ such that

$$u^{(8)}(x) = f(x, u, u', u'', u^{(3)}, u^{(4)}, u^{(5)}, u^{(6)}, u^{(7)}), \quad a < x < b \tag{2.1}$$

and the boundary conditions are

$$\begin{gathered} u(a) = \alpha_1, \quad u''(a) = \alpha_2, \quad u^{(4)}(a) = \alpha_3, \quad u^{(6)}(a) = \alpha_4, \\ u(b) = \beta_1, \quad u''(b) = \beta_2, \quad u^{(4)}(b) = \beta_3 \quad \text{and} \quad u^{(6)}(b) = \beta_4, \end{gathered}$$

where source function f is regular and differentiable in $[a, b]$. To derive and develop a numerical method for the solution of the problem we need following definitions and approximations.

To introduce finite number of discrete mesh points we partition the interval $[a, b]$ in which the solution of problem (1.1) is desired. In these subintervals discrete mesh points $a \le x_0 < x_1 < x_2 < \cdots < x_{N+1} \le b$ are generated by using uniform step length h such that $x_i = a + ih$, $i = 0, 1, 2, \ldots, N+1$. We wish to determine the numerical solution of the problem (1.1) at these discrete mesh points x_i. We denote the numerical approximation of $u(x)$ and f respectively by u_i and f_i. Hence, the boundary value problem (1.1) may be written as

$$u_i^{(8)} = F_i, \tag{2.2}$$

where $F_i = f(x_i, u_i, u_i', u_i'', u_i^{(3)}, u_i^{(4)}, u_i^{(5)}, u_i^{(6)}, u_i^{(7)})$ at the discrete mesh point $x = x_i$, $i = 1, 2, \ldots, N$. Let

$$\overline{\overline{u_i'}} = \frac{1}{2h}(u_{i+1} - u_{i-1}) - \frac{h}{12}(u''_{i+1} - u''_{i-1}), \tag{2.3}$$

$$\overline{\overline{u_i^{(3)}}} = \frac{1}{2h}(u''_{i+1} - u''_{i-1}) - \frac{h}{12}(u^{(4)}_{i+1} - u^{(4)}_{i-1}), \tag{2.4}$$

$$\overline{\overline{u_i^{(5)}}} = \frac{1}{2h}(u^{(4)}_{i+1} - u^{(4)}_{i-1}) - \frac{h}{12}(u^{(6)}_{i+1} - u^{(6)}_{i-1}), \tag{2.5}$$

$$\overline{u_i^{(7)}} = \frac{1}{2h}(u^{(6)}_{i+1} - u^{(6)}_{i-1}), \tag{2.6}$$

$$\overline{\overline{u'_{i+1}}} = \frac{1}{2h}(u_{i+1} - u_{i-1}) + \frac{h}{3}(u''_{i+1} + 2u''_i), \tag{2.7}$$

$$\overline{\overline{u'_{i-1}}} = \frac{1}{2h}(u_{i+1} - u_{i-1}) - \frac{h}{3}(2u''_i + u''_{i-1}), \tag{2.8}$$

$$\overline{\overline{u^{(3)}_{i+1}}} = \frac{1}{2h}(u''_{i+1} - u''_{i-1}) + \frac{h}{3}(u^{(4)}_{i+1} + 2u^{(4)}_i), \tag{2.9}$$

$$\overline{\overline{u^{(3)}_{i-1}}} = \frac{1}{2h}(u''_{i+1} - u''_{i-1}) - \frac{h}{3}(2u^{(4)}_i + u^{(4)}_{i-1}), \tag{2.10}$$

$$\overline{\overline{u^{(5)}_{i+1}}} = \frac{1}{2h}(u^{(4)}_{i+1} - u^{(4)}_{i-1}) + \frac{h}{3}(u^{(6)}_{i+1} + 2u^{(6)}_i), \tag{2.11}$$

$$\overline{\overline{u^{(5)}_{i-1}}} = \frac{1}{2h}(u^{(4)}_{i+1} - u^{(4)}_{i-1}) - \frac{h}{3}(2u^{(6)}_i + u^{(6)}_{i-1}), \tag{2.12}$$

$$\overline{u^{(7)}_{i+1}} = \frac{1}{2h}(3u^{(6)}_{i+1} - 4u^{(6)}_{i} + u^{(6)}_{i-1}), \tag{2.13}$$

$$\overline{u^{(7)}_{i+1}} = \frac{1}{2h}(-u^{(6)}_{i+1} + 4u^{(6)}_{i} - 3u^{(6)}_{i-1}), \tag{2.14}$$

$$\overline{F}_{i+1} = f(x_{i+1}, u_{i+1}, \overline{\overline{u'_{i+1}}}, u''_{i+1}, \overline{\overline{u^{(3)}_{i+1}}}, u^{(4)}_{i+1}, \overline{\overline{u^{(5)}_{i+1}}}, u^{(6)}_{i+1}, \overline{u^{(7)}_{i+1}}), \tag{2.15}$$

$$\overline{F}_{i-1} = f(x_{i-1}, u_{i-1}, \overline{\overline{u'_{i-1}}}, u''_{i-1}, \overline{\overline{u^{(3)}_{i-1}}}, u^{(4)}_{i-1}, \overline{\overline{u^{(5)}_{i-1}}}, u^{(6)}_{i-1}, \overline{u^{(7)}_{i-1}}), \tag{2.16}$$

$$\overline{\overline{u^{(7)}_{i}}} = \overline{u^{(7)}_{i}} - \frac{8971}{202084} h(\overline{F}_{i+1} - \overline{F}_{i-1}), \tag{2.17}$$

$$\widehat{u^{(7)}_{i}} = \overline{u^{(7)}_{i}} - \frac{739}{16620} h(\overline{F}_{i+1} - \overline{F}_{i-1}), \tag{2.18}$$

$$\widehat{\widehat{u^{(7)}_{i}}} = \overline{u^{(7)}_{i}} - \frac{155}{732} h(\overline{F}_{i+1} - \overline{F}_{i-1}), \tag{2.19}$$

$$\widetilde{\widetilde{u^{(7)}_{i}}} = \overline{u^{(7)}_{i}} - \frac{1}{20} h(\overline{F}_{i+1} - \overline{F}_{i-1}), \tag{2.20}$$

$$\overline{\overline{F}}_{i} = f(x_{i}, u_{i}, \overline{\overline{u'_{i}}}, u''_{i}, \overline{\overline{u^{(3)}_{i}}}, u^{(4)}_{i}, \overline{\overline{u^{(5)}_{i}}}, u^{(6)}_{i}, \overline{\overline{u^{(7)}_{i}}}), \tag{2.21}$$

$$\widehat{F}_{i} = f(x_{i}, u_{i}, \overline{\overline{u'_{i}}}, u''_{i}, \overline{\overline{u^{(3)}_{i}}}, u^{(4)}_{i}, \overline{\overline{u^{(5)}_{i}}}, u^{(6)}_{i}, \widehat{u^{(7)}_{i}}), \tag{2.22}$$

$$\widehat{\widehat{F}}_{i} = f(x_{i}, u_{i}, \overline{\overline{u'_{i}}}, u''_{i}, \overline{\overline{u^{(3)}_{i}}}, u^{(4)}_{i}, \overline{\overline{u^{(5)}_{i}}}, u^{(6)}_{i}, \widehat{\widehat{u^{(7)}_{i}}}), \tag{2.23}$$

$$\widetilde{\widetilde{F}}_{i} = f(x_{i}, u_{i}, \overline{\overline{u'_{i}}}, u''_{i}, \overline{\overline{u^{(3)}_{i}}}, u^{(4)}_{i}, \overline{\overline{u^{(5)}_{i}}}, u^{(6)}_{i}, \widetilde{\widetilde{u^{(7)}_{i}}}) \tag{2.24}$$

at these node $x = x_i, i = 1, .., N$. Following the ideas in [9], thus we propose our finite difference method for a numerical solution of problem (2.2),

$$\begin{aligned}
-720(u_{i+1} - 2u_i + u_{i-1}) + 360h^2(u''_{i+1} + u''_{i-1}) - 150h^4(u^{(4)}_{i+1} + u^{(4)}_{i-1}) + 61h^6(u^{(6)}_{i+1} + u^{(6)}_{i-1}) \\
= \frac{h^8}{1260}(5902\overline{F}_{i+1} + 50521\overline{\overline{F}}_{i} + 5902\overline{F}_{i-1}), \\
24(u''_{i+1} - 2u''_i + u''_{i-1}) - 12h^2(u^{(4)}_{i+1} + u^{(4)}_{i-1}) + 5h^4(u^{(6)}_{i+1} + u^{(6)}_{i-1}) \\
= \frac{h^6}{840}(323\overline{F}_{i+1} + 2770\widehat{F}_{i} + 323\overline{F}_{i-1}), \\
-2(u^{(4)}_{i+1} - 2u^{(4)}_{i} + u^{(4)}_{i-1}) + h^2(u^{(6)}_{i+1} + u^{(6)}_{i-1}) = \frac{h^4}{90}(7\overline{F}_{i+1} + 61\widehat{\widehat{F}}_{i} + 7\overline{F}_{i-1}), \\
u^{(6)}_{i+1} - 2u^{(6)}_{i} + u^{(6)}_{i-1} = \frac{h^2}{12}(\overline{F}_{i+1} + 10\widetilde{\widetilde{F}}_{i} + \overline{F}_{i-1}).
\end{aligned} \tag{2.25}$$

If the forcing function f in problem (1.1) is linear then the system of equations (2.25) will be linear otherwise we will obtain system of nonlinear equations.

3. Derivation of the Difference Method

In this section we shall out line the derivation of the proposed method (2.25). Using Taylor series and undetermine coefficients method, it is to verify that the following discretization

$$\begin{aligned}
-720(u_{i+1} - 2u_i + u_{i-1}) + 360h^2(u''_{i+1} + u''_{i-1}) - 150h^4(u^{(4)}_{i+1} + u^{(4)}_{i-1}) + 61h^6(u^{(6)}_{i+1} + u^{(6)}_{i-1}) \\
= \frac{h^8}{1260}(5902F_{i+1} + 50521F_{i} + 5902F_{i-1}),
\end{aligned} \tag{3.1}$$

for the solution of problem (1.1) when source function $F = f(x, u)$ is of $O(h^4)$. To discretize problem (1.1) at discrete points, we need approximations of order four for the source function F. So let outline method to obtain fourth order approximation for the forcing functions F.

Though the approximations (2.3)–(2.5) and (2.7)–(2.12) are fourth order approximation to $u'_i,..$ respectively. But some approximations defined in section 2 are not of order four. From (2.6), let expand each term in right in Taylor series about a point $x = x_i$ and simplify, we have

$$\overline{u_i^{(7)}} = u_i^{(7)} + \frac{h^2}{6} u_i^{(9)} + O(h^4) \tag{3.2}$$

From (2.13) and (2.14) respectively, we have

$$\overline{u_{i+1}^{(7)}} = u_{i+1}^{(7)} - \frac{h^2}{3} u_i^{(9)} - \frac{h^3}{12} u_i^{(10)} + O(h^4) \tag{3.3}$$

$$\overline{u_{i-1}^{(7)}} = u_{i-1}^{(7)} - \frac{h^2}{3} u_i^{(9)} + \frac{h^3}{12} u_i^{(10)} + O(h^4) \tag{3.4}$$

Let us define

$$\overline{\overline{u_i^{(7)}}} = \overline{u_i^{(7)}} + a_1 h(\overline{F}_{i+1} - \overline{F}_{i-1}) \tag{3.5}$$

where a_1 is free parametric constant and to be determined under appropriate condition.

Using (3.3) in (2.15), we will obtain

$$\overline{F}_{i+1} = F_{i+1} + (-\frac{h^2}{3} u_i^{(9)} - \frac{h^3}{12} u_i^{(10)})(\frac{\partial f}{\partial u^{(7)}})_{i+1} + O(h^4) \tag{3.6}$$

and similarly from (2.16) and (3.4) we have

$$\overline{F}_{i-1} = F_{i-1} + (-\frac{h^2}{3} u_i^{(9)} + \frac{h^3}{12} u_i^{(10)})(\frac{\partial f}{\partial u^{(7)}})_{i-1} + O(h^4) \tag{3.7}$$

Thus, using (3.2), (3.6) and (3.7) in (3.5), we have

$$\overline{\overline{u^{(7)}}}_i = u_i^{(7)} + (\frac{h^2}{6} + 2a_1 h^2) u_i^{(9)} + O(h^4) \tag{3.8}$$

Using (3.8) in (2.21) and simplify, we have

$$\overline{\overline{F}}_i = F_i + (\frac{h^2}{6} + 2a_1 h^2) u_i^{(9)} (\frac{\partial f}{\partial u^{(7)}})_i + O(h^4) \tag{3.9}$$

Let us consider the expression, $5902\overline{F}_{i+1} + 50521\overline{\overline{F}}_i + 5902\overline{F}_{i-1}$ and simplify this expression using (3.6), (3.7) and (3.9). We will obtain

$$\begin{aligned} 5902\overline{F}_{i+1} + 50521\overline{\overline{F}}_i + 5902\overline{F}_{i-1} &= 5902F_{i+1} + 50521F_i + 5902F_{i-1} \\ &+ \frac{h^2}{6}(26913 + 606252a_1)(u^{(9)}\frac{\partial f}{\partial u^{(7)}})_i + O(h^4) \end{aligned} \tag{3.10}$$

Thus from (3.10), we conclude that $5902\overline{F}_{i+1} + 50521\overline{\overline{F}}_i + 5902\overline{F}_{i-1}$ will provide fourth order approximation to $5902F_{i+1} + 50521F_i + 5902F_{i-1}$ if

$$a_1 = \frac{-8971}{202084}$$

i.e.

$$5902\overline{F}_{i+1}+50521\overline{\overline{F}}_i+5902\overline{F}_{i-1}=5902F_{i+1}+50521F_i+5902F_{i-1}+O(h^4) \tag{3.11}$$

Similarly, we can find other fourth order approximations of the terms in (2.25)

$$\begin{aligned} 323\overline{F}_{i+1}+2770\widehat{\overline{F}}_i+323\overline{F}_{i-1}&=323F_{i+1}+2770F_i+323F_{i-1}+O(h^4)\\ 7\overline{F}_{i+1}+61\widehat{\widehat{F}}_i+7\overline{F}_{i-1}&=7F_{i+1}+61F_i+7F_{i-1}+O(h^4)\\ \overline{F}_{i+1}+10\tilde{\tilde{\overline{F}}}_i+\overline{F}_{i-1}&=F_{i+1}+10F_i+F_{i-1}+O(h^4) \end{aligned} \tag{3.12}$$

Thus by using (3.11) and (3.12) in (3.1), we will get our proposed fourth order difference method (2.25) for the numerical solution of the problem (1.1). Moreover we are getting the numerical value of the derivative of the solution of the problem (1.1) as a byproduct of the method.

4. Numerical Results

To test the computational efficiency of method (2.25), we have considered three model problems. In each model problem, we took uniform step size h. In Table 1, Table 3 and Table 4, we have shown $MAEU$, $MAEV$, $MAEW$ and $MAES$ the maximum absolute error in the solution $u(x)$, second, fourth and sixth derivatives of solution $u(x)$ of the problems (1.1) respectively for different values of N. We have used the following formulas in computation of $MAEU$, $MAEV$, $MAEW$ and $MAES$:

$$\begin{aligned} MAEU&=\max_{1\le i\le N}|U_i-u(x_i)|\\ MAEV&=\max_{1\le i\le N}\left|U_i''-u''(x_i)\right|\\ MAEW&=\max_{1\le i\le N}\left|U_i^{(4)}-u^{(4)}(x_i)\right|\\ MAES&=\max_{1\le i\le N}\left|U_i^{(6)}-u^{(6)}(x_i)\right| \end{aligned}$$

where $u(x_i)$ and U_i are respectively exact and computed value of the solution of the problem and similarly we have defined others terms in the above expression. The order of the convergence of the proposed method (2.25) is estimated by using following formula,

$$O_N=\log_r\left(\frac{MAEU_N}{MAEU_{rN}}\right)$$

where r is ratio of the uniform step lengths h.

We have used Newton Raphson and Gauss Seidel method to solve system of nonlinear/linear equations (2.25). All computations were performed on a Windows 2007 Ultimate operating system in the GNU FORTRAN environment version 99 compiler (2.95 of gcc) on Intel Core i3-2330M, 2.20 GHz PC. The solutions are computed on N nodes and iteration is continued until either the maximum difference between two successive iterates is less than 10^{-9} or the number of iteration reached 10^3.

Problem 1. The model linear problem in [4] given as:

$$u^{(8)}(x)=-u^{(7)}-2u^{(6)}-2u^{(5)}-2u^{(4)}-2u^{(3)}-2u''-u'-u+14\cos(x)-4(4+x)\sin(x),$$
$$0<x<1,$$

subject to boundary conditions

$$u(1) = 0, \quad u''(1) = 4\cos(1) + 2\sin(1), \quad u^{(4)}(1) = -8\cos(1) - 12\sin(1),$$
$$u^{(6)}(1) = 12\cos(1) + 30\sin(1), \quad u(0) = 0, \quad u''(0) = 0, \quad u^{(4)}(0) = 0 \quad \text{and} \quad u^{(6)}(0) = 0.$$

The analytical solution of the problem is $u(x) = (x^2 - 1)\sin(x)$. The $MAEU$, $MAEV$, $MAEW$ and $MAES$ were computed by method (2.25) for different values of N and presented in Table 1.

Problem 2. The model linear problem in [5] given as:

$$u^{(8)}(x) = -\sin(x)u^{(5)} - (1 - x^2)u^{(4)} - u(x) + (3 + \sin(x) - x^2)\exp(x), \quad 0 < x < 1,$$

subject to boundary conditions

$$u(0) = 1, \quad u''(0) = 1, \quad u^{(4)}(0) = 1, \quad u^{(6)}(0) = 1, \quad u(1) = \exp(1),$$
$$u''(1) = \exp(1), \quad u^{(4)}(1) = \exp(1) \quad \text{and} \quad u^{(6)}(1) = \exp(1).$$

The analytical solution of the problem is $u(x) = \exp(x)$. The $MAEU$, $MAEV$, $MAEW$ and $MAES$ were computed by method (2.25) for different values of N and presented in Table 2.

Problem 3. Consider the following non-linear model problem given as:

$$u^{(8)}(x) = -\sin(u(x))u^{(3)} + f(x), \quad 0 < x < 1,$$

subject to boundary conditions

$$u(0) = 0, \quad u''(0) = -2, \; u^{(4)}(0) = 0, \; u^{(6)}(0) = 8, \; u(1) = (1 - \exp(1))\sin(1),$$
$$u''(1) = -2\exp(1)\cos(1) - \sin(1), \qquad u^{(4)}(1) = (4\exp(1) + 1)\sin(1),$$
$$\text{and} \quad u^{(6)}(1) = 8\exp(1)\cos(1) - \sin(1).$$

where $f(x)$ is calculated so that the analytical solution of the problem is $u(x) = (1-\exp(x))\sin(x)$. The $MAEU$, $MAEV$, $MAEW$ and $MAES$ were computed by method (2.25) for different values of N and presented in Table 3.

Table 1. Maximum absolute error (Problem 1).

N	ERROR			
	MAEU	MAEV	MAEW	MAES
4	.13351440(-4)	.13279915(-3)	.15587807(-2)	.47683716(-3)
8	.11324883(-5)	.12874603(-4)	.15258789(-3)	.34332275(-4)
16	.29802322(-7)	.23841858(-6)	.66757202(-5)	.11444092(-4)

Table 2. Maximum absolute error (Problem 2).

N	ERROR			
	MAEU	MAEV	MAEW	MAES
4	.13853703(-6)	.40756808(-6)	.48722518(-5)	.42238746(-4)
8	.82736904(-7)	.82599129(-7)	.19914191(-6)	.31914194(-5)

Table 3. Maximum absolute error (Problem 3).

N	ERROR			
	MAEU	MAEV	MAEW	MAES
4	.17881393(-6)	.71525574(-6)	.10967255(-4)	.10585785(-3)
8	.59604645(-7)	.23841858(-6)	.11920929(-5)	.57220459(-5)

The numerical results obtained in numerical experiment in considered model problems validate the fourth order accuracy. Also we have fourth order accurate numerical value of the second, fourth and sixth derivative of solution of problem as a byproduct of the proposed method (2.25).

5. Conclusion

In the present article, we have described a novel finite difference method for the numerical solution of the eighth order BVP's in ordinary differential equations. We have transformed the problem into system of algebraic equations at mesh points $x = x_i$, $i = 1, 2, \ldots, N$. Then the system of algebraic equations is solved for the solution of the problem. The proposed method in numerical experiments has shown its efficiency and fourth order accuracy. The advantage of the proposed method is that we also get fourth order accurate numerical value of the derivatives of the solution as byproduct.

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ASYMPTOTIC EXPANSION OF A SOLUTION FOR THE SINGULARLY PERTURBED OPTIMAL CONTROL PROBLEM WITH A CONVEX INTEGRAL QUALITY INDEX AND SMOOTH CONTROL CONSTRAINTS[1]

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Abstract: The paper deals with the problem of optimal control with a convex integral quality index for a linear steady-state control system in the class of piecewise continuous controls with smooth control constraints. In a general case, to solve such a problem, the Pontryagin maximum principle is applied as the necessary and sufficient optimum condition. The main difference from the preceding article [10] is that the terminal part of the convex integral quality index depends not only on slow, but also on fast variables. In a particular case, we derive an equation that is satisfied by an initial vector of the conjugate system. Then this equation is extended to the optimal control problem with the convex integral quality index for a linear system with the fast and slow variables. It is shown that the solution of the corresponding equation as $\varepsilon \to 0$ tends to the solution of an equation corresponding to the limit problem. The results obtained are applied to study a problem which describes the motion of a material point in $\mathbb{R}^n$ for a fixed interval of time. The asymptotics of the initial vector of the conjugate system that defines the type of optimal control is built. It is shown that the asymptotics is a power series of expansion.

Keywords: Optimal control, Singularly perturbed problems, Asymptotic expansion, Small parameter.

Introduction

The paper is devoted to studying the asymptotics of the initial vector of a conjugated state and an optimal value of the quality index in the optimal control problem [1–3] for a linear system with a fast and slow variable (see review [4]), convex integral quality index [3, Chapter 3], and smooth geometrical constraints for control.

Singularly perturbed problems of optimal control have been considered in different settings in [5–7]. The solving of problems with a closed and bounded control area meets certain difficulties. That is why the problems with fast and slow variables and closed constraints for control have been studied to a less extent. A significant contribution to solving these problems was made by Dontchev and Kokotovic. Problems with constraints for control in the form of a polygon are dealt with in [5, 7]. The structure of such optimal control is a relay function with values in the apexes of the polygon. No optimal control with constraints in the form of a sphere, which is a continuous function with a finite and countable number of discontinuity points, has been considered so far.

The asymptotics of solutions of the perturbed control problem was formulated differently in papers [8–10].

The main difference from the preceding article [10] is that the terminal part of the convex integral quality index depends not only on slow, but also on fast variables. In the present work,

[1]The paper is a translation of the paper "Asymptotic expansion of a solution for the singularly perturbed optimal control problem with a convex integral quality index and smooth control constraints" by A.A.Shaburov published in Proceedings of the Institute of Mathematics and Informatics at Udmurt State University, 2017, vol. 50, pp. 110–120.

the basic equation for searching for the asymptotics of the initial vector of the conjugated state of the problem under consideration and optimal control is obtained.

General relationships are applied to the case of the optimal control with a point of a small mass in an n-dimensional space under the action of a bounded force.

1. Construction of complete asymptotic expansion of vector λ_ε for an optimal control problem with fast and slow variables

Let us consider a problem that belongs to the class of piecewise continuous controls optimal control problem for a linear stationary system with a convex integral quality index:

$$\begin{cases} \dot{x}_\varepsilon = y_\varepsilon, \quad t \in [0,T], \quad \|u\| \leqslant 1, \\ \varepsilon \cdot \dot{y}_\varepsilon = -y_\varepsilon + u, \quad x_\varepsilon(0) = x^0, \quad y_\varepsilon(0) = y^0, \\ J(u) = \dfrac{1}{2}\|z_\varepsilon(T)\|^2 + \displaystyle\int_0^T \|u(t)\|^2\,dt \to \min, \quad z_\varepsilon(T) = \left(x_\varepsilon(T) \;\; y_\varepsilon(T)\right)^T, \end{cases} \tag{1.1}$$

where x_ε, y_ε, $u \in \mathbb{R}^n$, $z_\varepsilon \in \mathbb{R}^{2n}$. Henceforward $\|\cdot\|$ is the Euclidean norm in corresponding space.

Problem (1.1) simulates a motion of a material point of small mass $\varepsilon > 0$ with the coefficient of the medium resistance equals to 1 in the space $\mathbb{R}^n$ under action of the constrained control force $u(t)$.

Note that in the considered convex integral quality index J, where the first term can be interpreted as a fine for the control error at a finite time instant T, whereas the second is used to account for the energy costs of the implementation of the control.

Controllable system (1.1) contains fast and slow variables. The terminal part of the convex integral quality index depends not only on slow, but also on fast variables. For each fixed $\varepsilon > 0$ the problem (1.1) takes the form

$$\begin{cases} \dot{z} = \mathcal{A}_\varepsilon z + \mathcal{B}_\varepsilon u, \quad z(0) = z^0, \quad \|u(t)\| \leqslant 1, \quad t \in [0,T], \\ J(u) = \varphi(z(T)) + \displaystyle\int_0^T \|u(t)\|^2\,dt \to \min, \end{cases} \tag{1.2}$$

where $z \in \mathbb{R}^{\widetilde{n}}$, $u \in \mathbb{R}^n$,

$$z_\varepsilon(t) = \begin{pmatrix} x_\varepsilon(t) \\ y_\varepsilon(t) \end{pmatrix}, \quad z_\varepsilon^0 = \begin{pmatrix} x^0 \\ y^0 \end{pmatrix}, \quad \widetilde{n} = 2n, \quad \varphi(z_\varepsilon) = \frac{1}{2}\|z_\varepsilon\|^2,$$

$$\mathcal{A}_\varepsilon = \begin{pmatrix} A_{11} & A_{12} \\ \varepsilon^{-1}A_{21} & \varepsilon^{-1}A_{22} \end{pmatrix}, \quad \mathcal{B}_\varepsilon = \begin{pmatrix} B_1 \\ \varepsilon^{-1}B_2 \end{pmatrix}$$

Here $A_{11} = \mathcal{O}$, $A_{12} = I$, $A_{21} = \mathcal{O}$, $A_{22} = -I$, $B_1 = \mathcal{O}$, $B_2 = I$, and $\mathcal{O}$ and I are the zero and the identity matrices of dimensional $n \times n$ respectively.

Calculating $e^{\mathcal{A}_\varepsilon t}$ and $\nabla(\frac{1}{2}\|z_\varepsilon(T)\|^2)$, we obtain

$$e^{\mathcal{A}_\varepsilon t} = \begin{pmatrix} I & \varepsilon(1 - e^{-t/\varepsilon})I \\ \mathcal{O} & e^{-t/\varepsilon}I \end{pmatrix}, \quad \nabla\left(\frac{1}{2}\|z_\varepsilon(T)\|^2\right) = z_\varepsilon(T). \tag{1.3}$$

Thus, the following conditions are valid:

- for all sufficiently small $\varepsilon > 0$ the pair $(\mathcal{A}_\varepsilon, \mathcal{B}_\varepsilon)$ is completely controllable, that is,

$$\operatorname{rank}(\mathcal{B}_\varepsilon, \mathcal{A}_\varepsilon\mathcal{B}_\varepsilon, \ldots, \mathcal{A}_\varepsilon{}^{2n-1}\mathcal{B}_\varepsilon) = 2n;$$

- all eigenvalues of matrix A_{22} have negative real parts;
- the pair (A_{22}, B_2) is completely controllable.

Under the formulated conditions applied to the problem (1.2), the Pontryagin maximum principle is a necessary and sufficient optimum criterion. In this case, the problem has a unique solution [3, p. 3.5, theorem 14]. As well, the following statement is valid:

Statement 1. *The pair $z_\varepsilon(t)$, $u_\varepsilon(t)$ is a solution of the maximum principle problem if and only if $u_\varepsilon(t)$ is determined with the following formula:*

$$u_\varepsilon(t) = \frac{\mathcal{B}_\varepsilon{}^* e^{\mathcal{A}_\varepsilon{}^* t}\lambda_\varepsilon}{S\left(\|\mathcal{B}_\varepsilon{}^* e^{\mathcal{A}_\varepsilon{}^* t}\lambda_\varepsilon\|\right)}, \quad S(\xi) := \begin{cases} 2, & 0 \leqslant \xi \leqslant 2, \\ \xi, & \xi > 2, \end{cases}$$

and the vector λ_ε is the unique solution of the equation

$$-\lambda_\varepsilon = \nabla\varphi\Bigg(e^{\mathcal{A}_\varepsilon T} z_\varepsilon^0 + \int_0^T e^{\mathcal{A}_\varepsilon \tau}\mathcal{B}_\varepsilon \frac{\mathcal{B}_\varepsilon{}^* e^{\mathcal{A}_\varepsilon{}^* \tau}\lambda_\varepsilon}{S\left(\|\mathcal{B}_\varepsilon{}^* e^{\mathcal{A}_\varepsilon{}^* \tau}\lambda_\varepsilon\|\right)}\, d\tau\Bigg), \tag{1.4}$$

where $\nabla\varphi$ is the subgradient function in the sense of convex analysis. Besides $u_\varepsilon(t)$ is a unique optimal control in the problem (1.2) [10, Statement 1].

Definition 1. *The vector λ_ε, that satisfies the equation (1.4), will be called as a vector determining the optimal control in the problem (1.2). Note that since $\nabla\varphi(z_\varepsilon) = \begin{pmatrix} x_\varepsilon \\ y_\varepsilon \end{pmatrix}$, then the vector λ_ε, which determines the optimal control in the problem (1.2), has the form $\lambda_\varepsilon = \begin{pmatrix} l_\varepsilon \\ \rho_\varepsilon \end{pmatrix}$, $l_\varepsilon \in \mathbb{R}^n, \rho_\varepsilon \in \mathbb{R}^n$.*

Definition 2. *The vectors l_ε, ρ_ε also will be called as* ***a vectors determining the optimal control in the problem*** *(1.2).*

By virtue (1.3) the equation (1.4) transforms into system:

$$\begin{cases} -l_\varepsilon = x^0 + \varepsilon\left(1 - e^{-T/\varepsilon}\right) y^0 + \displaystyle\int_0^T \frac{\left(1 - e^{-t/\varepsilon}\right)\left(l_\varepsilon + e^{-t/\varepsilon}\left(\varepsilon^{-1}\rho_\varepsilon - l_\varepsilon\right)\right)}{S\left(\|l_\varepsilon + e^{-\frac{t}{\varepsilon}}\left(\varepsilon^{-1}\rho_\varepsilon - l_\varepsilon\right)\|\right)}\, dt, \\ -\rho_\varepsilon = e^{-T/\varepsilon} y^0 + \displaystyle\int_0^T \frac{e^{-t/\varepsilon}\left(l_\varepsilon + e^{-t/\varepsilon}\left(\varepsilon^{-1}\rho_\varepsilon - l_\varepsilon\right)\right)}{\varepsilon \cdot S\left(\|l_\varepsilon + e^{-t/\varepsilon}\left(\varepsilon^{-1}\rho_\varepsilon - l_\varepsilon\right)\|\right)}\, dt. \end{cases} \tag{1.5}$$

Let us note that the optimal control $u_\varepsilon^o(\tau)$ in the problem (1.1) by virtue 1 is expressed through the vectors $l_\varepsilon, \rho_\varepsilon$ as follows:

$$u_\varepsilon^o(\tau) = \frac{l_\varepsilon + e^{-\tau/\varepsilon}\left(\varepsilon^{-1}\rho_\varepsilon - l_\varepsilon\right)}{S\left(\|l_\varepsilon + e^{-\tau/\varepsilon}\left(\varepsilon^{-1}\rho_\varepsilon - l_\varepsilon\right)\|\right)}. \tag{1.6}$$

The main problem posed for (1.1) is to determine the complete asymptotic expansion in powers of the small parameter ε of optimal control, optimal values of the quality index and the optimal process. Formula (1.6) shows that if it is possible to obtain the complete asymptotic expansion of the vectors $l_\varepsilon, \rho_\varepsilon$, which determine the optimal control in problem (1.1), then this vectors can also be used for the asymptotic expansions of the above values.

We introduce some notation. If the vector-function $f_\varepsilon(t)$ is such that $f_\varepsilon(t) = O(\varepsilon^\alpha)$ as $\varepsilon \to 0$ for any $\alpha > 0$ uniformly with respect to $t \in [0, T]$, then instead of $f_\varepsilon(t)$ we will write $\mathbb{O}$. In particular, $e^{-\gamma T/\varepsilon} = \mathbb{O}$.

Theorem 1. *Let the vectors $l_\varepsilon, \rho_\varepsilon$ are the unique solutions of the equation (1.5) in the problem (1.1), and the vector l_0 is the unique solution of the equation*

$$-l_0 = x^0 + \frac{l_0}{S(\|l_0\|)} T. \tag{1.7}$$

Then $l_\varepsilon \to l_0$ and $\varepsilon^{-1}\rho_\varepsilon \to -l_0$ as $\varepsilon \to +0$.

P r o o f. It is known that the attainability set for the controllable system under control from (1.1) is uniformly bounded by the time instant T at $\varepsilon \in (0, \varepsilon_0]$ (see., for example, [6, Theorem 3.1]).

Writing the first equation from (1.5):

$$-l_\varepsilon = x^0 + \varepsilon \left(1 - e^{-T/\varepsilon}\right) y^0 + \int_0^T \frac{(1 - e^{-t/\varepsilon}) \left(l_\varepsilon + e^{-t/\varepsilon} \left(\varepsilon^{-1}\rho_\varepsilon - l_\varepsilon\right)\right)}{S\left(\|l_\varepsilon + e^{-t/\varepsilon} \left(\varepsilon^{-1}\rho_\varepsilon - l_\varepsilon\right)\|\right)} \, dt.$$

Taking into account that the expression under integral is uniformly constrained and that $O(e^{-t/\varepsilon}) = = e^{-t/\varepsilon} \left(\varepsilon^{-1}\rho_\varepsilon - l_\varepsilon\right)$ as $\varepsilon \to 0$, a proof of that $l_\varepsilon \to l_0$, is carried out almost literally [10, Theorem 1]. Hence, it is enough to show that $\varepsilon^{-1}\rho_\varepsilon \to -l_0$ for a full proof of this theorem.

Let us show that the vector ρ_ε can be presented in the form of $\rho_\varepsilon = \varepsilon \cdot r_\varepsilon$, where $r_\varepsilon \to r_0 \in \mathbb{R}^n$ as $\varepsilon \to +0$. Writing the second equation from (1.5):

$$-\rho_\varepsilon = e^{-T/\varepsilon} y^0 + \int_0^T \frac{e^{-t/\varepsilon} \left(l_\varepsilon + e^{-t/\varepsilon}(\varepsilon^{-1}\rho_\varepsilon - l_\varepsilon)\right)}{\varepsilon \cdot S\left(\|l_\varepsilon + e^{-t/\varepsilon}(\varepsilon^{-1}\rho_\varepsilon - l_\varepsilon)\|\right)} \, dt. \tag{1.8}$$

Let $\tau := t/\varepsilon$. The equation (1.8) rewriting as

$$-\rho_\varepsilon = \mathbb{O} + \int_0^\infty \frac{e^{-\tau} \left(l_\varepsilon + e^{-\tau}(\varepsilon^{-1}\rho_\varepsilon - l_\varepsilon)\right)}{S\left(\|l_\varepsilon + e^{-\tau}(\varepsilon^{-1}\rho_\varepsilon - l_\varepsilon)\|\right)} \, d\tau, \quad \varepsilon \to 0.$$

Replacing the variable $\xi := e^{-\tau}$, we obtain

$$-\rho_\varepsilon = \mathbb{O} + \int_0^1 \frac{l_\varepsilon + \xi(\varepsilon^{-1}\rho_\varepsilon - l_\varepsilon)}{S\left(\|l_\varepsilon + \xi(\varepsilon^{-1}\rho_\varepsilon - l_\varepsilon)\|\right)} \, d\xi, \quad \varepsilon \to 0.$$

Thus, the vector ρ_ε is bounded. Let us prove that a sequence $\{\varepsilon^{-1}\rho_\varepsilon\}$ is bounded. By contradiction, we find $\varepsilon_n \to 0 : \|\varepsilon^{-1}\rho_\varepsilon\| \to \infty$. For simplicity, the n dependence of ε will be omitted.

Let us divide the integral into two terms by means of introduction of complementary parameter $\alpha(\varepsilon)$:

$$-\rho_\varepsilon = \mathbb{O} + \int_0^{\alpha(\varepsilon)} \frac{l_\varepsilon + \xi(\varepsilon^{-1}\rho_\varepsilon - l_\varepsilon)}{S\left(\|l_\varepsilon + \xi(\varepsilon^{-1}\rho_\varepsilon - l_\varepsilon)\|\right)} \, d\xi + \int_{\alpha(\varepsilon)}^1 \frac{\xi\varepsilon^{-1}\rho_\varepsilon + (1 - \xi)l_\varepsilon}{S\left(\|\xi\varepsilon^{-1}\rho_\varepsilon + (1 - \xi)l_\varepsilon\|\right)} \, d\xi, \quad \varepsilon \to 0, \tag{1.9}$$

where $\alpha(\varepsilon) = O(\varepsilon^\gamma)$ as $\varepsilon \to 0$ and for a certain positive number γ.

So far as $\|\xi\varepsilon^{-1}\rho_\varepsilon\| = \xi\|\varepsilon^{-1}\rho_\varepsilon\| \to \infty$ and the vector l_ε is bounded. Choice of the point of division of an integral depends on the number $\gamma \in (0,1)$ as follows:

$$\alpha(\varepsilon) := \frac{1}{\|\varepsilon^{-1}\rho_\varepsilon\|^\gamma} \leqslant \xi,$$

where, because expression under integral sign is bounded, $\alpha(\varepsilon) = o(1)$ as $\varepsilon \to 0$.

Notice that $\|\xi\varepsilon^{-1}\rho_\varepsilon\| \geqslant \alpha(\varepsilon)\|\varepsilon^{-1}\rho_\varepsilon\| \to \infty$, i. e. at sufficiently small the inequality $\varepsilon : \|\xi\varepsilon^{-1}\rho_\varepsilon - (1-\xi)l_\varepsilon\| > 2$ is satisfied. Dividing and multiplying the function under the second integral sign in (1.9) by a factor $\|\rho_\varepsilon\|$ and having got rid of a factor ε^{-1} at ρ_ε, we obtain

$$-\rho_\varepsilon = \mathbb{O} + o(1) + \int_{\alpha(\varepsilon)}^1 \frac{\xi\dfrac{\rho_\varepsilon}{\|\rho_\varepsilon\|} + o(1)}{\left\|\xi\dfrac{\rho_\varepsilon}{\|\rho_\varepsilon\|} + o(1)\right\|}\, d\xi. \tag{1.10}$$

Let, without loss of generality, $\bar\rho$ be a partial limit of the vectors $\rho_\varepsilon/\|\rho_\varepsilon\|$ as $\varepsilon \to +0$, i. e. $\rho_{\varepsilon_k}/\|\rho_{\varepsilon_k}\| \to \bar\rho$ for a certain $\{\varepsilon_k\}$ so that $\varepsilon_k \to +0$. Moreover, $\|\bar\rho\| = 1$. Passing to the limit as $k \to \infty$ in (1.10), we obtain, that $-\rho_o = \bar\rho$. Consequently, $\|\rho_0\| = 1$ and $-\rho_0 = \rho_0$.

The received contradiction leads to the fact that $\rho_\varepsilon = O(\varepsilon)$, and we can rewrite the vector $\rho_\varepsilon = \varepsilon\cdot r_\varepsilon$, where the sequence $\{r_\varepsilon\}$ is bounded.

Divide the integral into two terms. Taking into account $r_\varepsilon \longrightarrow r_0$ as $\varepsilon \to 0$

$$0 = \int_0^1 \frac{l_0 + \xi(r_0 - l_0)}{S\left(\|l_0 + \xi(r_0 - l_0)\|\right)}\, d\xi = \int_0^1 \frac{l_0}{S\left(\|l_0 + \xi(r_0 - l_0)\|\right)}\, d\xi + \int_0^1 \frac{\xi(r_0 - l_0)}{S\left(\|l_0 + \xi(r_0 - l_0)\|\right)}\, d\xi =$$
$$= \mu_1 l_0 + \mu_2(r_0 - l_0) = \widetilde{\mu} l_0 + \mu_2 r_0,$$

where $\widetilde{\mu} = \mu_1 - \mu_2$.

Positive numbers $\mu_1,\ \mu_2$ are represented by integrals

$$\mu_1 = \int_0^1 \frac{d\xi}{S\left(\|l_0 + \xi(r_0 - l_0)\|\right)}, \qquad \mu_2 = \int_0^1 \frac{\xi}{S\left(\|l_0 + \xi(r_0 - l_0)\|\right)}\, d\xi.$$

We can suppose, that $r_0 = \mu\cdot l_0$, where $\mu := -\widetilde{\mu}/\mu_2$.

Change of variable in integration $\nu := 1 + \xi(\mu - 1)$ allows to rewrite an integral equation as follows

$$\frac{l_0}{\mu - 1}\int_1^\mu \frac{\nu}{S(\|l_0\|\cdot|\nu|)}\, d\nu.$$

Integral is equal to zero at $\mu = 1$. Let $\mu \neq 1$, then the function under integral sign is uneven function on a variable ν. Consequently, the integral is equal to zero at $\mu = -1$. We prove, that $\rho_\varepsilon = \varepsilon r_\varepsilon$, besides a first term $r_0 = -l_0$ is a bounding vector. Theorem 1.1 is proved. □

From (1.5) and (1.7) we obtain two cases:

$$\begin{aligned}
&1)\ \|x^0\| < T + 2 \Longrightarrow l_0 = -\frac{2}{2+T}\,x^0 \qquad \text{and} \qquad \|l_0\| < 2,\\
&2)\ \|x^0\| > T + 2 \Longrightarrow l_0 = -\frac{\|x^0\| - T}{\|x^0\|}\,x^0 \qquad \text{and} \qquad \|l_0\| > 2.
\end{aligned} \tag{1.11}$$

1. Consider the first case $\|x^0\| < T+2$.

By virtue of (1.11) and Theorem 1 the inequality $\|l_\varepsilon\| < 2$ is valid for all sufficiently small ε. Taking into account that $(1-e^{-t/\varepsilon}) \leqslant 1$ at any $t \geqslant 0$ and $\varepsilon > 0$, from (1.5) we obtain for $l_\varepsilon,\ \rho_\varepsilon$ the rewriting system of equations:

$$\begin{cases} -l_\varepsilon = x^0 + \varepsilon\left(1-e^{-T/\varepsilon}\right)y^0 + \displaystyle\int_0^T \frac{(1-e^{-t/\varepsilon})\left(l_\varepsilon + e^{-t/\varepsilon}\left(\varepsilon^{-1}\rho_\varepsilon - l_\varepsilon\right)\right)}{2}\,dt, \\ -\rho_\varepsilon = e^{-T/\varepsilon}y^0 + \displaystyle\int_0^T \frac{e^{-t/\varepsilon}\left(l_\varepsilon + e^{-t/\varepsilon}\left(\varepsilon^{-1}\rho_\varepsilon - l_\varepsilon\right)\right)}{2\varepsilon}\,dt. \end{cases} \tag{1.12}$$

The solution of (1.12) are vectors

$$\rho_\varepsilon = \frac{2\varepsilon(x^0+\varepsilon y^0)}{(T+2)+2\varepsilon(3+2T)-6\varepsilon^2} + \mathbb{O}, \quad l_\varepsilon = \frac{-2(x^0+\varepsilon y^0)(1+4\varepsilon)}{(T+2)+2\varepsilon(3+2T)-6\varepsilon^2} + \mathbb{O}, \quad \varepsilon \to 0.$$

It follows from these representations that λ_ε is expanded as $\varepsilon \to 0$ into the asymptotic power series. Moreover, we can obtain explicit form for the first two coefficients of vectors $l_\varepsilon, r_\varepsilon$.

Theorem 2. *Suppose that $\|x^0\| < T+2$. Then the vectors l_ε, r_ε, which determine the optimal control in problem (1.1), are expanded as $\varepsilon \to 0$ into a power asymptotic series:*

$$l_\varepsilon \overset{as}{=} l_0 + \sum_{k=1}^{\infty} \varepsilon^k l_k, \ \textit{where, in particular, } l_0 = \frac{-2x^0}{T+2}, \quad l_1 = \frac{-8x^0}{T+2} - \frac{2y^0}{T+2} + \frac{4(3+2T)x^0}{(T+2)^2},$$

$$r_\varepsilon \overset{as}{=} r_0 + \sum_{k=1}^{\infty} \varepsilon^k r_k, \ \textit{where, in particular, } r_0 = \frac{2x^0}{T+2}, \quad r_1 = \frac{2y^0}{T+2} - \frac{4(3+2T)x^0}{(T+2)^2}.$$

2. Now consider the case $\|x^0\| > T+2$.

Let $l_\varepsilon = l_0 + l,\ \rho_\varepsilon = -\varepsilon l_0 + \varepsilon r$, where l, r — are infinitesimal numbers.

Rewriting the system of equations (1.12), replacing the variable $\eta := e^{-t/\varepsilon}$:

$$\begin{cases} -l_0 - l = x^0 + \varepsilon y^0 + \mathbb{O} + \varepsilon \displaystyle\int_{e^{-T/\varepsilon}}^1 \frac{(1-\eta)\big(l_0 + l + \eta(r-l-2l_0)\big)}{\eta \cdot S\big(\|l_0 + l + \eta(r-l-2l_0)\|\big)}\,d\eta, \quad \varepsilon \to 0, \\ -\varepsilon\left(-l_0 + r\right) = \mathbb{O} + \displaystyle\int_{e^{-T/\varepsilon}}^1 \frac{l_0 + l + \eta(r-l-2l_0)}{S\big(\|l_0 + l + \eta(r-l-2l_0)\|\big)}\,d\eta, \quad \varepsilon \to 0. \end{cases}$$

For simplicity, we will reduce a condition $\varepsilon \to 0$.

Replacing the variable $\xi := 1 - 2\eta$. Then factor under the integral sign in the rewriting system as a function $\psi_\varepsilon(\eta)$ contains vectors $l_\varepsilon,\ \rho_\varepsilon$, as follows

$$\psi_\varepsilon(\xi) := \xi l_0 + \lambda + \xi\nu,$$

where $\lambda = (l+r)/2$, $\nu = (l-r)/2$. For a small variables l, r we can receive the following expressions

$$l = \lambda + \nu, \quad r = \lambda - \nu.$$

Taking into account that we have a new representations of vectors l, r we rewrite the system of equation as follows

$$\begin{cases} -l_0-\lambda-\nu = x^0+\varepsilon y^0+\mathbb{O}+\dfrac{\varepsilon}{2}\displaystyle\int_{-1}^{\beta(\varepsilon)}\frac{(1+\xi)\,(\xi l_0+\lambda+\xi\nu)}{(1-\xi)S\left(\|\xi l_0+\lambda+\xi\nu\|\right)}\,d\xi, \\ -\varepsilon\left(-l_0+\lambda-\nu\right)=\mathbb{O}+\dfrac{1}{2}\displaystyle\int_{-1}^{\beta(\varepsilon)}\frac{\xi l_0+\lambda+\xi\nu}{S\left(\|\xi l_0+\lambda+\xi\nu\|\right)}\,d\xi, \end{cases} \tag{1.13}$$

where $\beta(\varepsilon) := 1-2e^{-T/\varepsilon}$. Notice that $\beta(\varepsilon)\to 1$ as $\varepsilon\to 0$.

Having transformed a factor

$$\frac{1+\xi}{1-\xi}=1+\frac{2\xi}{1-\xi}$$

under the integral sign and divided the integral from the first equation of system (1.13) into two terms, we find

$$\int_{-1}^{\beta(\varepsilon)}\frac{(1+\xi)}{(1-\xi)}\cdot\frac{\xi l_0+\lambda+\xi\nu}{S\left(\|\xi l_0+\lambda+\xi\nu\|\right)}\,d\xi =$$

$$=\int_{-1}^{\beta(\varepsilon)}\frac{\xi l_0+\lambda+\xi\nu}{S\left(\|\xi l_0+\lambda+\xi\nu\|\right)}\,d\xi+2\int_{-1}^{\beta(\varepsilon)}\frac{\xi}{(1-\xi)}\cdot\frac{\xi l_0+\lambda+\xi\nu}{S\left(\|\xi l_0+\lambda+\xi\nu\|\right)}\,d\xi.$$

Calculating the switching points $\xi_1,\ \xi_2$ from a constraint $\|\xi l_0+\lambda+\xi\nu\|=2$, we set

$$\xi_{1,2}=\frac{-\langle l_0;\lambda\rangle-\langle\nu;\lambda\rangle\pm\sqrt{\left(\langle l_0;\lambda\rangle+\langle\nu;\lambda\rangle\right)^2-(\|\lambda\|^2-4)(\|l_0\|^2+\|\nu\|^2+2\langle l_0;\nu\rangle)}}{\|l_0\|^2+\|\nu\|^2+2\langle l_0;\nu\rangle}.$$

Henceforward $\langle\cdot;\cdot\rangle$ is a scalar product in a corresponding space.

Using a binomial expansion and expansion of quadratic root as a small parameter, we find $\xi_1,\ \xi_2$:

$$\xi_{1,2}=\pm\frac{2}{\|l_0\|}-\frac{\langle l_0;\lambda\rangle}{\|l_0\|^2}\mp\frac{2\langle l_0;\nu\rangle}{\|l_0\|^3}+O\left(\|\lambda\|^2+\|\nu\|^2\right).$$

We can extend the integral from the second equation of system (1.13) at the point $\xi=1$:

$$\int_{-1}^{\beta(\varepsilon)}\frac{\xi l_0+\lambda+\xi\nu}{S\left(\|\xi l_0+\lambda+\xi\nu\|\right)}\,d\xi=\int_{-1}^{1}\frac{\xi l_0+\lambda+\xi\nu}{S\left(\|\xi l_0+\lambda+\xi\nu\|\right)}\,d\xi+\mathbb{O}=2\varepsilon(l_0-\lambda+\nu).$$

Introducing into consideration a vector function $F(\lambda,\nu,\varepsilon):=\begin{pmatrix}F_1(\lambda,\nu,\varepsilon)\\ F_2(\lambda,\nu,\varepsilon)\end{pmatrix}$, we rewrite system (1.13) as follows $F(\lambda,\nu,\varepsilon)=0$, where

$$F_1(\lambda,\nu,\varepsilon):=l_0+\lambda+\nu+x^0+\varepsilon y^0+\mathbb{O}+\varepsilon^2(l_0+\nu-\lambda)+\varepsilon\left(\int_{-1}^{\xi_2}\frac{\xi}{(1-\xi)}\cdot\frac{\xi l_0+\lambda+\xi\nu}{\|\xi l_0+\lambda+\xi\nu\|}\,d\xi\right)+$$

$$+\varepsilon\left(\int_{\xi_2}^{\xi_1}\frac{\xi}{(1-\xi)}\cdot\frac{\xi l_0+\lambda+\xi\nu}{2}\,d\xi+\int_{\xi_1}^{\beta(\varepsilon)}\frac{\xi}{(1-\xi)}\cdot\frac{\xi l_0+\lambda+\xi\nu}{\|\xi l_0+\lambda+\xi\nu\|}\,d\xi\right)=0,\quad\varepsilon\to 0, \tag{1.14}$$

$$F_2(\lambda,\nu,\varepsilon):=\varepsilon(\lambda-l_0-\nu)+\mathbb{O}+$$

$$+\frac{1}{2}\left(\int_{-1}^{\xi_2}\frac{\xi l_0+\lambda+\xi\nu}{\|\xi l_0+\lambda+\xi\nu\|}\,d\xi+\int_{\xi_2}^{\xi_1}\frac{\xi l_0+\lambda+\xi\nu}{2}\,d\xi+\int_{\xi_1}^{1}\frac{\xi l_0+\lambda+\xi\nu}{\|\xi l_0+\lambda+\xi\nu\|}\,d\xi\right)=0,\quad\varepsilon\to 0, \tag{1.15}$$

where $\xi_1,\ \xi_2$ are the switching points of control $u(t)$.

Let us remove a singularity at the point $\xi = 1$, divide the integral from the first equation of the system into two terms:

$$\int_{\xi_1}^{\beta(\varepsilon)} \frac{\xi}{1-\xi} \cdot \frac{\xi l_0 + \lambda + \xi\nu}{\|\xi l_0 + \lambda + \xi\nu\|}\, d\xi =$$

$$= \int_{\xi_1}^{\beta(\varepsilon)} \frac{\xi}{1-\xi} \cdot \left(\frac{\xi l_0 + \lambda + \xi\nu}{\|\xi l_0 + \lambda + \xi\nu\|} - \frac{l_0 + \lambda + \nu}{\|l_0 + \lambda + \nu\|} \right) d\xi + \int_{\xi_1}^{\beta(\varepsilon)} \frac{\xi}{1-\xi} \cdot \frac{l_0 + \lambda + \nu}{\|l_0 + \lambda + \nu\|}\, d\xi.$$

Calculating the second integral:

$$\frac{l_0 + \lambda + \nu}{\|l_0 + \lambda + \nu\|} \int_{\xi_1}^{\beta(\varepsilon)} \frac{\xi}{1-\xi}\, d\xi = \frac{l_0 + \lambda + \nu}{\|l_0 + \lambda + \nu\|} \cdot \left(-\left(1 - 2e^{-T/\varepsilon} - \xi_1\right) - \left(\ln 2 - \frac{T}{\varepsilon} - \ln(1-\xi_1) \right) \right).$$

Let us expand terms $1-\xi_1$ and $\ln(1-\xi_1)$ as a small parameter:

$$1 - \xi_1 = 1 - \frac{2}{\|l_0\|} + \frac{\langle l_0; \lambda \rangle}{\|l_0\|^2} + \frac{2\langle l_0; \nu \rangle}{\|l_0\|^3} + O\left(\|\lambda\|^2 + \|\nu\|^2\right),$$

$$\ln(1-\xi_1) = \ln\left(1 - \frac{2}{\|l_0\|}\right) + \frac{\langle l_0; \lambda \rangle}{\|l_0\|(\|l_0\| - 2)} + \frac{2\langle l_0; \nu \rangle}{\|l_0\|^2(\|l_0\| - 2)} + O\left(\|\lambda\|^2 + \|\nu\|^2\right).$$

Calculating the Gateau derivative of function $\rho/\|\rho\|$, we obtain

$$D\left(\frac{\rho}{\|\rho\|}\right)\bigg|_{\rho=\rho_0\neq 0} (\triangle\rho) = \frac{\triangle\rho\|\rho_0\|^2 - \langle \triangle\rho; \rho_0 \rangle \rho_0}{\|\rho_0\|^3}. \tag{1.16}$$

We can use the formula (1.16) to find a partial derivatives

$$\frac{\partial F_1(\lambda,\nu,\varepsilon)}{\partial \lambda}\bigg|_{\lambda,\nu,\varepsilon=0} (\triangle\lambda), \qquad \frac{\partial F_1(\lambda,\nu,\varepsilon)}{\partial \nu}\bigg|_{\lambda,\nu,\varepsilon=0} (\triangle\nu).$$

Taking into account that the unique term in the right side of equation (1.14) has no order $o(1)$, and according to formula (1.16) we find

$$\frac{\partial F_1(\lambda,\nu,\varepsilon)}{\partial \lambda}\bigg|_{\lambda,\nu,\varepsilon=0} (\triangle\lambda) = \triangle\lambda + T \cdot \frac{\triangle\lambda\|l_0\|^2 - l_0\langle l_0; \triangle\lambda \rangle}{\|l_0\|^3},$$

$$\frac{\partial F_1(\lambda,\nu,\varepsilon)}{\partial \nu}\bigg|_{\lambda,\nu,\varepsilon=0} (\triangle\nu) = \triangle\nu + T \cdot \frac{\triangle\nu\|l_0\|^2 - l_0\langle l_0; \triangle\nu \rangle}{\|l_0\|^3}.$$

Function $F_2(\lambda,\nu,\varepsilon)$ from the second equation from (1.15) transforms to

$$F_2(\lambda,\nu,\varepsilon) = \frac{1}{2}\left(\int_{-1}^{\xi_2^o} \frac{\xi l_0 + \lambda + \xi\nu}{\|\xi l_0 + \lambda + \xi\nu\|}\, d\xi + \int_{\xi_2^o}^{\xi_2} \frac{\xi l_0 + \lambda + \xi\nu}{\|\xi l_0 + \lambda + \xi\nu\|}\, d\xi + \int_{\xi_2}^{\xi_1} \frac{\xi l_0 + \lambda + \xi\nu}{2}\, d\xi \right) +$$

$$+ \frac{1}{2}\left(\int_{\xi_1}^{\xi_1^o} \frac{\xi l_0 + \lambda + \xi\nu}{\|\xi l_0 + \lambda + \xi\nu\|}\, d\xi + \int_{\xi_1^o}^{1} \frac{\xi l_0 + \lambda + \xi\nu}{\|\xi l_0 + \lambda + \xi\nu\|}\, d\xi \right) + \varepsilon(\lambda - \nu - l_0),$$

where $\xi_{1,2}^o = \lim\limits_{\varepsilon\to 0} \xi_{1,2}$.

Calculating the third integral:

$$\int_{\xi_2}^{\xi_1} \frac{\xi l_0 + \lambda + \xi\nu}{2} d\xi = \frac{2\lambda}{\|l_0\|} - \frac{2 l_0 \langle l_0; \lambda \rangle}{\|l_0\|^3},$$

and calculating partial derivatives of the third integral :

$$\frac{1}{2}\frac{\partial}{\partial\lambda}\left(\frac{2\lambda}{\|l_0\|}-\frac{2l_0\langle l_0;\lambda\rangle}{\|l_0\|^3}\right)\Big|_{\lambda,\nu,\varepsilon=0}(\triangle\lambda)=\frac{\triangle\lambda}{\|l_0\|}-\frac{l_0\langle l_0;\lambda\rangle}{\|l_0\|^3},$$

$$\frac{1}{2}\frac{\partial}{\partial\nu}\left(\frac{2\lambda}{\|l_0\|}-\frac{2l_0\langle l_0;\lambda\rangle}{\|l_0\|^3}\right)\Big|_{\lambda,\nu,\varepsilon=0}(\triangle\nu)=0.$$

Calculating derivatives of first and fifth integrals, we use formula (1.16):

$$\frac{\partial}{\partial\lambda}\left(\int_{-1}^{\xi_2^o}\frac{\xi l_0+\lambda+\xi\nu}{\|\xi l_0+\lambda+\xi\nu\|}\,d\xi\right)\Big|_{\lambda,\nu,\varepsilon=0}(\triangle\lambda)=\frac{\partial}{\partial\lambda}\left(\int_{\xi_1^o}^{1}\frac{\xi l_0+\lambda+\xi\nu}{\|\xi l_0+\lambda+\xi\nu\|}\,d\xi\right)\Big|_{\lambda,\nu,\varepsilon=0}(\triangle\lambda)=$$

$$=\frac{\triangle\lambda\|l_0\|^2-\langle l_0;\triangle\lambda\rangle l_0}{\|l_0\|^3}\cdot\left(-\ln\frac{2}{\|l_0\|}\right),$$

$$\frac{\partial}{\partial\nu}\left(\int_{-1}^{\xi_2^o}\frac{\xi l_0+\lambda+\xi\nu}{\|\xi l_0+\lambda+\xi\nu\|}\,d\xi\right)\Big|_{\lambda,\nu,\varepsilon=0}(\triangle\nu)=\frac{\triangle\nu\|l_0\|^2-\langle l_0;\triangle\nu\rangle l_0}{\|l_0\|^3}\cdot\left(\frac{2}{\|l_0\|}-1\right),$$

$$\frac{\partial}{\partial\nu}\left(\int_{\xi_1^o}^{1}\frac{\xi l_0+\lambda+\xi\nu}{\|\xi l_0+\lambda+\xi\nu\|}\,d\xi\right)\Big|_{\lambda,\nu,\varepsilon=0}(\triangle\nu)=\frac{\triangle\nu\|l_0\|^2-\langle l_0;\triangle\nu\rangle l_0}{\|l_0\|^3}\cdot\left(1-\frac{2}{\|l_0\|}\right).$$

Calculating derivatives of second and fourth integrals, we take into account the following formula

$$\left(\frac{\partial}{\partial\lambda}\int_{\alpha(\lambda)}^{\beta(\lambda)}f(t,\lambda)\,dt\right)\Big|_{\lambda=\lambda_0}(\triangle\lambda)=$$

$$=\int_{\alpha(\lambda)}^{\beta(\lambda)}\frac{\partial f}{\partial\lambda}(\triangle\lambda)\,dt+f(\beta(\lambda),\lambda)\cdot\frac{\partial\beta}{\partial\lambda}\Big|_{\lambda=\lambda_0}(\triangle\lambda)-f(\alpha(\lambda),\lambda)\cdot\frac{\partial\alpha}{\partial\lambda}\Big|_{\lambda=\lambda_0}(\triangle\lambda). \tag{1.17}$$

Since each integral contains only one multiple limit and integral from the partial derivative of the expression under the integral sign is equal to zero, and taking into account the formula (1.17) we obtain

$$\frac{\partial}{\partial\lambda}\int_{\xi_2^o}^{\xi_2}\frac{\xi l_0+\lambda+\xi\nu}{\|\xi l_0+\lambda+\xi\nu\|}\,d\xi\Big|_{\lambda,\nu,\varepsilon=0}(\triangle\lambda)=\frac{\partial\xi_2}{\partial\lambda}\Big|_{\lambda,\nu,\varepsilon=0}(\triangle\lambda)\cdot\frac{\xi_2 l_0+\lambda+\xi_2\nu}{\|\xi_2 l_0+\lambda+\xi_2\nu\|}\Big|_{\lambda,\nu,\varepsilon=0}(\triangle\lambda)=\frac{l_0\langle l_0;\triangle\lambda\rangle}{\|l_0\|^3},$$

$$\frac{\partial}{\partial\lambda}\int_{\xi_1}^{\xi_1^o}\frac{\xi l_0+\lambda+\xi\nu}{\|\xi l_0+\lambda+\xi\nu\|}\,d\xi\Big|_{\lambda,\nu,\varepsilon=0}(\triangle\lambda)=-\frac{\partial\xi_1}{\partial\lambda}\Big|_{\lambda,\nu,\varepsilon=0}(\triangle\lambda)\cdot\frac{\xi_1 l_0+\lambda+\xi_1\nu}{\|\xi_1 l_0+\lambda+\xi_1\nu\|}\Big|_{\lambda,\nu,\varepsilon=0}(\triangle\lambda)=\frac{l_0\langle l_0;\triangle\lambda\rangle}{\|l_0\|^3}.$$

Following this line of reasoning, we find

$$\frac{\partial}{\partial\nu}\int_{\xi_2^o}^{\xi_2}\frac{\xi l_0+\lambda+\xi\nu}{\xi l_0+\lambda+\xi\nu}\,d\xi\Big|_{\lambda,\nu,\varepsilon=0}(\triangle\nu)=-\frac{2l_0\langle l_0;\triangle\nu\rangle}{\|l_0\|^4},$$

$$\frac{\partial}{\partial\nu}\int_{\xi_1}^{\xi_1^o}\frac{\xi l_0+\lambda+\xi\nu}{\xi l_0+\lambda+\xi\nu}\,d\xi\Big|_{\lambda,\nu,\varepsilon=0}(\triangle\nu)=\frac{2l_0\langle l_0;\triangle\nu\rangle}{\|l_0\|^4}.$$

Let us write the partial derivatives $\frac{\partial F_2(\lambda,\nu,\varepsilon)}{\partial\lambda}\Big|_{\lambda,\nu,\varepsilon=0}(\triangle\lambda)$, $\frac{\partial F_2(\lambda,\nu,\varepsilon)}{\partial\nu}\Big|_{\lambda,\nu,\varepsilon=0}(\triangle\nu)$:

$$\frac{\partial F_2(\lambda,\nu,\varepsilon)}{\partial\lambda}\Big|_{\lambda,\nu,\varepsilon=0}(\triangle\lambda)=\frac{\triangle\lambda}{\|l_0\|}-\ln\frac{2}{\|l_0\|}\left(\frac{\triangle\lambda\|l_0\|^2-\langle l_0;\triangle\lambda\rangle l_0}{\|l_0\|^3}\right),\quad\frac{\partial F_2(\lambda,\nu,\varepsilon)}{\partial\nu}\Big|_{\lambda,\nu,\varepsilon=0}(\triangle\nu)=0.$$

Then we obtain, that $F_1(0,0,0)=0$, $F_2(0,0,0)=0$ and functions $F_1(\cdot,\cdot,\cdot)$, $F_2(\cdot,\cdot,\cdot)$ are infinitely differentiable in λ, ν, ε in a certain neighborhood of the point $(0;0;0)$.

Show that operator

$$\mathcal{F}(\triangle\lambda, \triangle\nu) := D\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}\Bigg|_{\lambda,\nu,\varepsilon=0} =$$
$$= \begin{pmatrix} \triangle\lambda + T\dfrac{\triangle\lambda\|l_0\|^2 - l_0\langle l_0; \triangle\lambda\rangle}{\|l_0\|^3} + \triangle\nu + T\dfrac{\triangle\nu\|l_0\|^2 - l_0\langle l_0; \triangle\nu\rangle}{\|l_0\|^3} \\ \\ \dfrac{\triangle\lambda}{\|l_0\|} - \ln\dfrac{2}{\|l_0\|}\left(\dfrac{\triangle\lambda\|l_0\|^2 - \langle l_0; \triangle\lambda\rangle l_0}{\|l_0\|^3}\right) \end{pmatrix}, \tag{1.18}$$

is continuously reversible.

Consider the equation $\mathcal{F}(0,0)(\triangle\lambda, \triangle\nu) =: (g_1, g_2)$. Multiplying scalarly the first and second coordinates of vectors (1.18), we find unknown couples of multiply scalarly:

$$\langle l_0; \triangle\lambda\rangle = \|l_0\|\langle l_0; g_2\rangle, \quad \langle l_0; \triangle\nu\rangle = \langle l_0; g_1 - \|l_0\|g_2\rangle.$$

The reversible operator $\mathcal{F}^{-1}(g_1, g_2)$ is equal:

$$\mathcal{F}^{-1}(g_1, g_2) =$$
$$= \begin{pmatrix} \left(g_1 + T\dfrac{l_0\langle l_0; g_2\rangle}{\|l_0\|^2} + T\dfrac{l_0\langle l_0; g_1 - \|l_0\|g_2\rangle}{\|l_0\|^3}\right)\dfrac{\|l_0\|}{\|l_0\| + T} - \left(g_2 - \ln\dfrac{2}{\|l_0\|}\dfrac{l_0\langle l_0; g_2\rangle}{\|l_0\|^2}\right)\dfrac{\|l_0\|}{1 - \ln(2/\|l_0\|)} \\ \left(g_2 - \ln\dfrac{2}{\|l_0\|}\dfrac{l_0\langle l_0; g_2\rangle}{\|l_0\|^2}\right)\dfrac{\|l_0\|}{1 - \ln(2/\|l_0\|)} \end{pmatrix}.$$

Thus, the implicit function theorem is applicable. It means that the vectors $l_\varepsilon, r_\varepsilon$ (as a functions of ε) are infinitely differentiable with respect to ε for all small ε and, therefore, l_ε, r_ε can be expanded into the asymptotic series. The coefficients of this series can be found via the standard procedure: substituting the series into the equation $\mathcal{F}(\lambda, \nu, \varepsilon) = 0$, expanding values dependent on ε into the asymptotic series in power of ε and equating terms of the same order of smallness with respect to ε, we obtain equations of the form $\mathcal{F}(\triangle\lambda_k, \triangle\nu_k) = (g_{1,k}, g_{2,k})$ with the right parts known. Then, by the formula (1) we find l_k, r_k.

Theorem 3. *Suppose that $\|x^0\| > T + 2$. Then the vectors l_ε, r_ε, which determine the optimal control in problem* (1.1) *are expanded as $\varepsilon \to 0$ into the power asymptotic series:*

$$l_\varepsilon \overset{as}{=} l_0 + \sum_{k=1}^{\infty} \varepsilon^k l_k, \quad r_\varepsilon \overset{as}{=} r_0 + \sum_{k=1}^{\infty} \varepsilon^k r_k.$$

2. Conclusion

1. Both in the first and the second cases under consideration, from (1.14), (1.15) and the asymptotic expansion of l_ε the asymptotic expansions of both the quality index and optimal control as well as optimal state of the system are conventionally obtained. With this, the asymptotic expansions of the optimal control and optimal state of the system will be exponentially decreasing boundary layers in the neighborhood of point $t = 0$. Moreover, if $t \geqslant \varepsilon^\beta$ and $\beta \in (0,1)$, then the optimal control $u^o(t)$ is constant plus the asymptotic zero.

2. It follows form the formulas $F_1(\lambda,\nu,\varepsilon)=0$, $F_2(\lambda,\nu,\varepsilon)=0$ that λ_ε lies in the subspace Π, generated by vectors x^0 and y^0. Therefore, for all $t\in[0,T]$ and $u^o_\varepsilon(t)$, and $x_\varepsilon(t)$, and $y_\varepsilon(t)$ lie in the same subspace Π. In this way, the problem (1.1) is equivalent to the corresponding two-dimensional problem.

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