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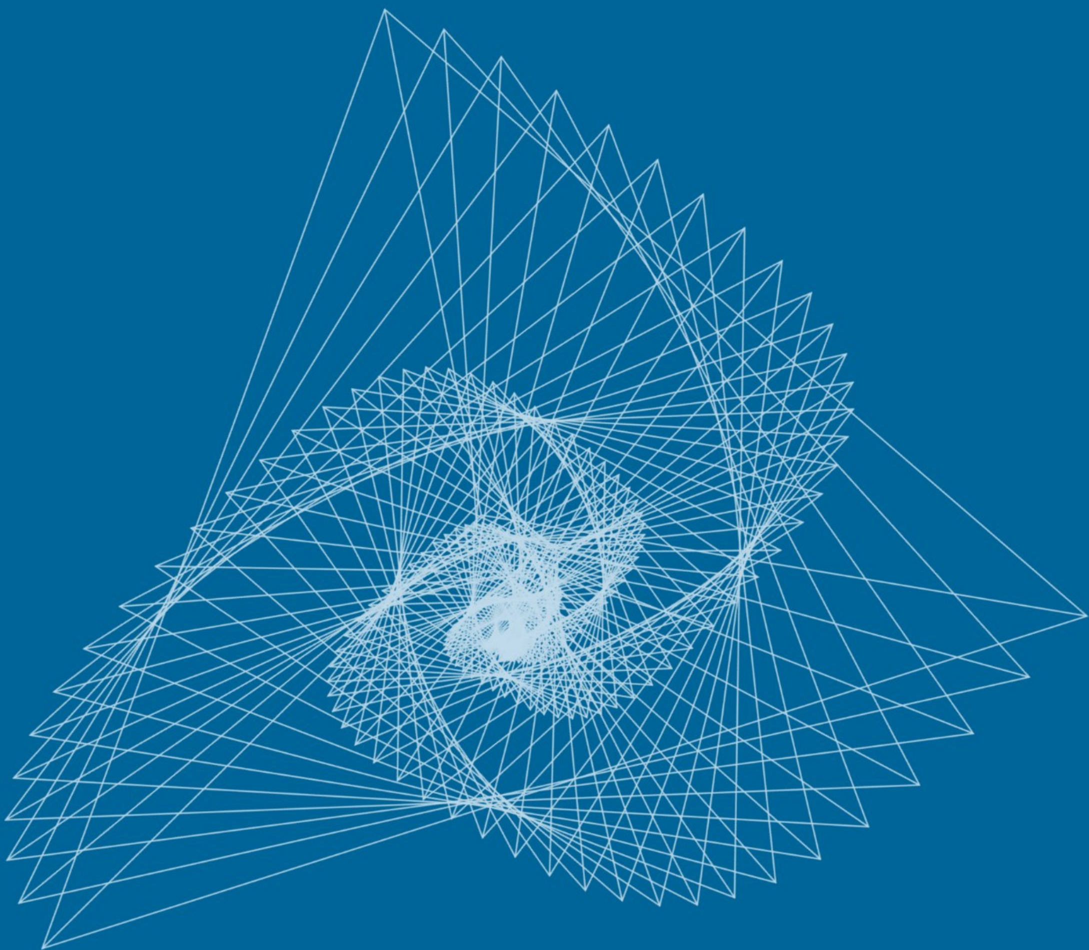

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BESSEL POLYNOMIALS AND SOME CONNECTION FORMULAS IN TERMS OF THE ACTION OF LINEAR DIFFERENTIAL OPERATORS

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Abstract: In this paper, we introduce the concept of the $\mathbb{B}_\alpha$-classical orthogonal polynomials, where $\mathbb{B}_\alpha$ is the raising operator $\mathbb{B}_\alpha := x^2 \cdot d/dx + \big(2(\alpha-1)x+1\big)\mathbb{I}$, with nonzero complex number α and $\mathbb{I}$ representing the identity operator. We show that the Bessel polynomials $B_n^{(\alpha)}(x),\ n\geq 0$, where $\alpha\neq -m/2,\ m\geq -2,\ m\in\mathbb{Z}$, are the only $\mathbb{B}_\alpha$-classical orthogonal polynomials. As an application, we present some new formulas for polynomial solution.

Keywords: Classical orthogonal polynomials, Linear functionals, Bessel polynomials, Raising operators, Connection formulas.

1. Introduction

Let $\{B_n^{(\alpha)}\}_{n\geq 0}$ be the monic Bessel polynomial sequence. It satisfies the following explicit expression [10, 23]

$$B_n^{(\alpha)}(x) = \sum_{\nu=0}^{n} \binom{n}{\nu} \frac{2^{n-\nu}\Gamma(n+2\alpha+\nu-1)}{\Gamma(2n+2\alpha-1)} x^{\nu}, \quad n\geq 0, \tag{1.1}$$

for $\alpha\neq -m/2,\ m\in\mathbb{N}$. To complete the definition, $B_n^{(\alpha)}(0)$ is set equal to

$$B_n^{(\alpha)}(0) = 2^n \frac{\Gamma(n+2\alpha-1)}{\Gamma(2n+2\alpha-1)}, \quad n\geq 0. \tag{1.2}$$

It is well known that the monic Bessel polynomial sequence is classical and satisfies the following relations [8, 10, 16, 23]:

–*The Second-Order Differential Equation* (SODE)

$$x^2 B_n^{(\alpha)''}(x) + 2\big(\alpha x+1\big) B_n^{(\alpha)'}(x) = n(n+2\alpha-1) B_n^{(\alpha)}(x), \quad n\geq 0. \tag{1.3}$$

–*The Lowering Relation* (LR)

$$D B_n^{(\alpha)}(x) = n B_{n-1}^{(\alpha+1)}(x), \quad n\geq 1, \tag{1.4}$$

where $D := d/dx$ is the standard derivate operator.

After a simple calculation, the SODE can be written for $n \geq 0$ as follows

$$\Big(x^2 B_n^{(\alpha)'}(x)\Big)' + \Big(2\big((\alpha-1)x+1\big)B_n^{(\alpha)}(x)\Big)' = (n+1)(n+2\alpha-2)B_n^{(\alpha)}(x). \tag{1.5}$$

Using the LR (1.4), the equation (1.5) becomes for $n \geq 0$

$$\Big(x^2 B_n^{(\alpha)'}(x) + 2\big((\alpha-1)x+1\big)B_n^{(\alpha)}(x)\Big)' = (n+2\alpha-2)B_{n+1}^{(\alpha-1)'}(x).$$

Using the primitive of the last equation, we get

$$x^2 B_n^{(\alpha)'}(x) + 2\big((\alpha-1)x+1\big)B_n^{(\alpha)}(x) = (n+2\alpha-2)B_{n+1}^{(\alpha-1)}(x) + K,$$

with $(\alpha \neq -m/2,\ m \geq -2,\ m \in \mathbb{Z})$, and where, using (1.2), we have

$$K = 2B_n^{(\alpha)}(0) - (n+2\alpha-2)B_{n+1}^{(\alpha-1)}(0) = 0.$$

Then we finally obtain the following *Raising Relation* (RR) satisfied by the monic Bessel polynomials

$$\mathbb{B}_\alpha B_n^{(\alpha)}(x) = (n+2\alpha-2)B_{n+1}^{(\alpha-1)}(x), \tag{1.6}$$

where $\mathbb{B}_\alpha := x^2 D + 2\big((\alpha-1)x+1\big)\mathbb{I}$ is called the degree raising shift operator for the Bessel polynomials with $\mathbb{I}$ representing the identity operator. For more details see also the degree raising shift operator for the family of classical orthogonal polynomials [13].

In view of (1.6), we can say that $\{B_n^{(\alpha)}\}_{n\geq 0}$ is an $\mathbb{B}_\alpha$-classical polynomial sequence, since it satisfies the Hahn's property with respect to the operators $\mathbb{B}_\alpha$, i.e., it is an orthogonal polynomial sequence whose sequence of $\mathbb{B}_\alpha$-derivatives is also orthogonal. Note that an orthogonal polynomial sequence $\{p_n\}_{n\geq 0}$ is called classical, if $\{p_n'\}_{n\geq 0}$ is also orthogonal (see [16–19]). This characterization is essentially the Hahn–Sonine characterization (see [11, 21]) of the classical orthogonal polynomials.

In the same context, a natural question arises about the characterization of $\mathbb{B}_\alpha$-classical orthogonal polynomials. The purpose of this paper is to introduce the concept of the $\mathbb{B}_\alpha$-classical polynomial sequence and to give a complete description of this family of orthogonal polynomials. Note that many researches have been devoted to these topics where lowering, transfer and raising operators have been used (see for example [1–7, 9, 11, 12, 20]).

The paper is organized as follows: Section 2 gives the basic notations and tools that will be used throughout the paper. Section 3 deals with $\mathbb{B}_\alpha$-classical orthogonal polynomial sequence. In Section 4, we put in evidence some differential relations satisfied by the polynomials solution of our problem. In Section 5, we give a conclusion.

2. Preliminaries

Let $\mathcal{P}$ be linear space of polynomials in one variable with complex coefficients and $\mathcal{P}'$ be its dual space, whose elements are linear functionals. We write $\langle u, p\rangle := u(p)$ $(u \in \mathcal{P}',\ p \in \mathcal{P})$. In particular, we denote by $(u)_n := \langle u, x^n\rangle$, $n \geq 0$, the moments of u. Let us define the following operations on $\mathcal{P}'$. For any linear functional u, any polynomial f and any $(a,b) \in \mathbb{C}\backslash\{0\} \times \mathbb{C}$, let $Du := u'$, fu, $h_a u$ and $\tau_b u$ be the linear functionals defined by the duality [15, 16]

$$\langle fu, p\rangle := \langle u, fp\rangle, \quad \langle u', p\rangle := -\langle u, p'\rangle,$$

$$\langle h_a u, p\rangle := \langle u, h_a p\rangle = \langle u, p(ax)\rangle, \quad \langle \tau_b u, p\rangle := \langle u, \tau_{-b} p\rangle = \langle u, p(x+b)\rangle.$$

A linear functional u is called normalized if it satisfies $(u)_0 = 1$. We assume that the linear functionals used in this paper are normalized.

Let $\{p_n\}_{n\geq 0}$ be a sequence of monic polynomials with $\deg p_n = n$, $n \geq 0$ (MPS in short) and let $\{u_n\}_{n\geq 0}$ be its dual sequence, $u_n \in \mathcal{P}'$, defined by $\langle u_n, p_m\rangle = \delta_{n,m}$, $n,\ m \geq 0$. Notice that u_0 is said to be the canonical functional associated with the MPS $\{p_n\}_{n\geq 0}$ (see [16–18]).

Let us recall the following result.

Lemma 1 [16, 17]. *For any $u \in \mathcal{P}'$ and any integer $m \geq 1$, the following statements are equivalent*:

(i) $\langle u, p_{m-1}\rangle \neq 0, \quad \langle u, p_n\rangle = 0, \quad n \geq m,$

(ii) $\exists \lambda_\nu \in \mathbb{C}, \quad 0 \leq \nu \leq m-1, \quad \lambda_{m-1} \neq 0$ *such that* $u = \sum_{\nu=0}^{m-1} \lambda_\nu u_\nu.$

As a consequence, the dual sequence $\{u_n^{[1]}\}_{n\geq 0}$ of $\{p_n^{[1]}\}_{n\geq 0}$ where

$$p_n^{[1]}(x) := (n+1)^{-1} D p_{n+1}(x), \quad n \geq 0,$$

is given by [16, 19] as

$$D u_n^{[1]} = -(n+1) u_{n+1}, \quad n \geq 0.$$

Similarly, the dual sequence $\{\tilde{u}_n\}_{n\geq 0}$ of $\{\tilde{p}_n\}_{n\geq 0}$, where

$$\tilde{p}_n(x) := a^{-n} p_n(ax+b)$$

with $(a,b) \in \mathbb{C}\backslash\{0\} \times \mathbb{C}$, is given by [16, 19]

$$\tilde{u}_n = a^n (h_{a^{-1}} \circ \tau_{-b}) u_n, \quad n \geq 0.$$

A linear functional u is called *regular* if we can associate with it a MPS $\{p_n\}_{n\geq 0}$ such that [16, 19] as

$$\langle u, p_n p_m\rangle = r_n \delta_{n,m}, \quad n,m \geq 0, \quad r_n \neq 0, \quad n \geq 0.$$

The sequence $\{p_n\}_{n\geq 0}$ is then called a monic *orthogonal* polynomial sequence (MOPS in short) with respect to u. Note that $u = (u)_0 u_0 = u_0$, since u is normalized.

Proposition 1. [16]. *Let $\{p_n\}_{n\geq 0}$ be a MPS and let $\{u_n\}_{n\geq 0}$ be its dual sequence. The following statements are equivalent*:

(i) $\{p_n\}_{n\geq 0}$ *is orthogonal with respect to* u_0,

(ii) $\{p_n\}_{n\geq 0}$ *satisfies the linear recurrence relation of order two*

$$\begin{cases} p_0(x) = 1, \quad p_1(x) = x - \beta_0, \\ p_{n+2}(x) = (x - \beta_{n+1}) p_{n+1}(x) - \gamma_{n+1} p_n(x), \quad n \geq 0, \end{cases}$$

where

$$\beta_n = \langle u_0, x p_n^2\rangle \langle u_0, p_n^2\rangle^{-1}, \quad n \geq 0,$$

and

$$\gamma_{n+1} = \langle u_0, p_{n+1}^2\rangle \langle u_0, p_n^2\rangle^{-1} \neq 0, \quad n \geq 0,$$

(iii) *the dual sequence* $\{u_n\}_{n\geq 0}$ *satisfies*:

$$u_n = \langle u_0, p_n^2\rangle^{-1} p_n u_0, \quad n \geq 0.$$

A MOPS $\{p_n\}_{n\geq 0}$ is called D-classical, if $\{Dp_n\}_{n\geq 0}$ is also orthogonal (*Hermite, Laguerre, Bessel or Jacobi*) [19]. Moreover, if $\{p_n\}_{n\geq 0}$ is orthogonal with respect to u_0, then there exists a monic polynomial ϕ with $\deg\phi \leq 2$ and a polynomial ψ with $\deg\psi = 1$ such that u_0 satisfies the *Pearson's equation* (PE) [19]

$$D(\phi u_0) + \psi u_0 = 0.$$

A second characterization of these polynomials is that they are the only polynomial solutions of the SODE [8, 19],

$$\phi(x)p''_{n+1}(x) - \psi(x)p'_{n+1}(x) = \lambda_n p_{n+1}(x), \quad n \geq 0,$$

where

$$\lambda_n = (n+1)\Big(\frac{1}{2}\phi''(0)n - \psi'(0)\Big) \neq 0, \quad n \geq 0.$$

Note that if $p_n(x) = B_n^{(\alpha)}(x),\ n \geq 0,\ (\alpha \neq -n/2,\ n \geq 0)$ is the monic Bessel polynomial and we write $\mathcal{B}^{(\alpha)}$ for u_0, then the regular form $\mathcal{B}^{(\alpha)}$ satisfies the following PE [16, 19]

$$D\big(x^2\mathcal{B}^{(\alpha)}\big) - 2(\alpha x + 1)\mathcal{B}^{(\alpha)} = 0, \tag{2.1}$$

and $B_n^{(\alpha)}(x),\ n \geq 0$ satisfies the SODE (1.3).

3. The $\mathbb{B}_\alpha$-classical polynomials

Recall the operator

$$\begin{aligned} \mathbb{B}_\alpha : \mathcal{P} &\longrightarrow \mathcal{P}, \\ f &\longmapsto \mathbb{B}_\alpha(f) := x^2 f' + 2\big((\alpha-1)x + 1\big)f, \end{aligned}$$

with $\alpha \neq -m/2,\ m \geq -2,\ m \in \mathbb{Z}$.

Clearly, the operator $\mathbb{B}_\alpha$ raises the degree of any polynomial. Such an operator is called *raising operator* [14, 22].

Definition 1. *We call a sequence $\{P_n\}_{n\geq 0}$ of orthogonal polynomials $\mathbb{B}_\alpha$-classical if $\{\mathbb{B}_\alpha P_n\}_{n\geq 0}$ is also orthogonal.*

For any MPS $\{P_n\}_{n\geq 0}$ we define

$$Q_{n+1}(x;\alpha) := \frac{1}{n+2\alpha-2}\,\mathbb{B}_\alpha P_n(x), \quad n \geq 0,$$

or equivalently

$$(n+2\alpha-2)Q_{n+1}(x;\alpha) := x^2 P'_n(x) + 2\big((\alpha-1)x+1\big)P_n(x), \quad n \geq 0, \tag{3.1}$$

with initial value $Q_0(x;\alpha) = 1$.

Clearly, $\{Q_{n+1}(.;\alpha)\}_{n\geq 0}$ is a MPS and

$$\deg Q_{n+1}(x;\alpha) = n+1.$$

In the sequel, we write

$$Q_n(x) := Q_n(x;\alpha), \quad n \geq 0,$$

if there is no ambiguity. Our next goal is to describe all the $\mathbb{B}_\alpha$-classical polynomial sequences. Assume that $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ are MOPS satisfying

$$P_{n+2}(x) = (x - \varpi_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \quad n \geq 0, \tag{3.2}$$

with initial values $P_0(x) = 1,\ P_1(x) = x - \varpi_0$, and

$$Q_{n+2}(x) = (x - \theta_{n+1})Q_{n+1}(x) - \zeta_{n+1}Q_n(x), \quad n \geq 0, \tag{3.3}$$

with initial values $Q_0(x) = 1,\ Q_1(x) = x - \theta_0$.

Next, a first result will be deduced as a consequence of relations (3.1), (3.2) and (3.3).

Proposition 2. *The sequences $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ satisfy the following finite type relation*

$$x^2 P_n(x) = Q_{n+2}(x) + s_n Q_{n+1}(x) + t_n Q_n(x), \quad n \geq 0,$$

where

$$s_n = (n + 2\alpha - 2)\big(\varpi_n - \theta_{n+1}\big), \quad n \geq 0,$$
$$t_n = (n + 2\alpha - 3)\gamma_n - (n + 2\alpha - 2)\zeta_{n+1}, \quad n \geq 0,$$

with the convention $\gamma_0 = 0$.

P r o o f. Differentiating (3.2), we obtain

$$P'_{n+2}(x) = (x - \varpi_{n+1})P'_{n+1}(x) - \gamma_{n+1}P'_n(x) + P_{n+1}(x), \quad n \geq 0.$$

We multiply the last equation by x^2 and the relation (3.2) by $2\big((\alpha-1)x+1\big)$, take the sum of the two resulting equations, and substitute (3.1). Then, we get

$$(n + 2\alpha)Q_{n+3}(x) = (n + 2\alpha - 1)(x - \varpi_{n+1})Q_{n+2}(x)$$
$$-(n + 2\alpha - 2)\gamma_{n+1}Q_{n+1}(x) + x^2 P_{n+1}(x), \quad n \geq 0.$$

Using the relation (3.3), we get

$$x^2 P_{n+1}(x) = Q_{n+3}(x) + (n + 2\alpha - 1)\big(\varpi_{n+1} - \theta_{n+2}\big)Q_{n+2}(x)$$
$$+\big((n + 2\alpha - 2)\gamma_{n+1} - (n + 2\alpha - 1)\zeta_{n+2}\big)Q_{n+1}(x), \quad n \geq 0.$$

In fact, this result is valid if $n+1$ is replaced by n with the convention $\gamma_0 = 0$. Hence we got the desired result. □

Note that, for $n = 0$, the Proposition 2 gives

$$x^2 = Q_2(x) + (2\alpha - 2)(\varpi_0 - \theta_1)Q_1(x) - (2\alpha - 2)\zeta_1 Q_0(x), \tag{3.4}$$

and using the fact that

$$Q_1(x) = x - \theta_0 = x + \frac{1}{\alpha - 1},$$

we obtain

$$Q_2(x) = x^2 + (2\alpha - 2)(\theta_1 - \varpi_0)x + (2\alpha - 2)\zeta_1 + 2(\theta_1 - \varpi_0).$$

It gives by comparing with (3.3) for $n = 0$

$$\theta_1 = \frac{-\theta_0 + 2(\alpha - 1)\varpi_0}{2\alpha - 1} = \frac{1}{(\alpha - 1)(2\alpha - 1)} + \frac{2(\alpha - 1)}{2\alpha - 1}\,\varpi_0,$$
$$\zeta_1 = \frac{\theta_0\theta_1 + 2(\varpi_0 - \theta_1)}{2\alpha - 1} = \frac{-1}{(\alpha - 1)^2}.$$

Denote by u_0 and v_0 the regular forms (linear functionals) in $\mathcal{P}'$ corresponding to $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ respectively. Then we can state the following result.

Lemma 2. *The following algebraic relation between the regular forms u_0 and v_0 holds*

$$x^2 v_0 = \frac{2}{(\alpha-1)}\, u_0.$$

P r o o f. According to Proposition 2, we obtain

$$\left\langle x^2 v_0, P_n(x)\right\rangle = 0, \quad n \geq 1. \tag{3.5}$$

On the other hand, by (3.4) we have

$$\begin{aligned}\left\langle x^2 v_0, P_0(x)\right\rangle &= \langle v_0, Q_2(x)\rangle + 2(\alpha-1)(\varpi_0-\theta_1)\langle v_0, Q_1(x)\rangle - 2(\alpha-1)\zeta_1\langle v_0, Q_0(x)\rangle r\\ &= -2(\alpha-1)\zeta_1 = \frac{2}{(\alpha-1)},\end{aligned} \tag{3.6}$$

since $\{Q_n\}_{n\geq 0}$ is orthogonal with respect to the normalized form v_0. According to Lemma 1 and using (3.5) and (3.6), we obtain the desired result. □

Based on PE satisfied by the linear functional of $\mathcal{B}^{(\alpha)}$, we can state the following theorem.

Theorem 1. *The sequence of Bessel polynomials $\{B_n^{(\alpha)}\}_{n\geq 0}$, with $\alpha \neq -m/2,\ m \geq -2,$ $m \in \mathbb{Z}$, is the only $\mathbb{B}_\alpha$-classical orthogonal sequence. More precisely, $P_n(x) = B_n^{(\alpha)}(x)$ and $Q_n(x) = B_n^{(\alpha-1)}(x)$, $n \geq 0$.*

P r o o f. If we apply v_0 in (3.1), we get for $n \geq 0$

$$\left\langle v_0, (n+2\alpha-2)Q_{n+1}(x)\right\rangle = \left\langle v_0, x^2 P_n'(x) + 2\big((\alpha-1)x+1\big)P_n(x)\right\rangle = 0.$$

But the right hand side may be read as

$$\left\langle -D\big(x^2 v_0\big) + 2\big((\alpha-1)x+1\big)v_0, P_n(x)\right\rangle = 0, \quad n \geq 0.$$

Hence we have for all polynomials P, expanding P in the basis $\{P_n\}_{n\geq 0}$, the following relation

$$\left\langle -D\big(x^2 v_0\big) + 2\big((\alpha-1)x+1\big)v_0, P(x)\right\rangle = 0.$$

In other words we have

$$\big(x^2 v_0\big)' - 2\big((\alpha-1)x+1\big)v_0 = 0. \tag{3.7}$$

This implies that v_0 is the Bessel functional $\mathcal{B}^{(\alpha-1)}$ according to the corresponding PE (2.1), i.e.,

$$Q_n(x) = B_n^{(\alpha-1)}(x), \quad n \geq 0,$$

with $\alpha \neq -m/2,\ m \geq -2,\ m \in \mathbb{Z}$.

Multiplying (3.7) by x^2 and using Lemma 2, we obtain

$$\big(x^2 u_0\big)' - 2\big(\alpha x+1\big)u_0 = 0. \tag{3.8}$$

Essentially (3.8) corresponds to the PE of linear functional $\mathcal{B}^{(\alpha)}$ of the sequence of Bessel polynomials $\{B_n^{(\alpha)}\}_{n\geq 0}$. Hence, $P_n(x) = B_n^{(\alpha)}(x),\ n \geq 0$. □

In conclusion, we give the following relation, which is satisfied by Bessel polynomials

$$x^2 B_n^{(\alpha)\prime}(x) + 2\big((\alpha-1)x+1\big)B_n^{(\alpha)}(x) = (n+2\alpha-2)B_{n+1}^{(\alpha-1)}(x), \quad n \geq 0$$

with $\alpha \neq -m/2,\ m \geq -2,\ m \in \mathbb{Z}$.

4. Representations of Bessel polynomials in terms of the action of linear differential operators

In this section, we prove some higher order differential relations between the Bessel polynomials (solution of our problem). First, we need the following fundamental relation

$$\big(xD + (n+\alpha-1)\mathbb{I}\big)B_n^{(\alpha/2)}(x) = (2n+\alpha-1)B_n^{((\alpha+1)/2)}(x), \tag{4.1}$$

which is obtained after a simple calculation from (1.1).

Theorem 2. *The representation of Bessel polynomials $B_n^{((\alpha+m)/2)}(x)$ in terms of action of linear differential operators on the Bessel polynomials $B_n^{(\alpha/2)}(x)$ is given by*

$$B_n^{((\alpha+m)/2)}(x) = \frac{\Gamma(2n+\alpha-1)}{\Gamma(2n+\alpha+m-1)} \sum_{k=0}^{m} \binom{m}{k} \frac{\Gamma(n+\alpha+m-1)}{\Gamma(n+\alpha+m-k-1)} x^{m-k}D^{m-k}B_n^{(\alpha/2)}(x), \quad n \geq 0, \quad m \geq 0. \tag{4.2}$$

P r o o f. We prove this by induction on $m \in \mathbb{N}$. For $m = 0$ this is obvious. Now, suppose (4.2) holds and prove the same for $m+1$ instead of m. Indeed, by differentiating both sides of (4.2) and using (1.4), we get, for all $n \geq 1$,

$$B_{n-1}^{((\alpha+m+2)/2)}(x) = \frac{\Gamma(2n+\alpha-1)}{\Gamma(2n+\alpha+m-1)} \sum_{k=0}^{m} \binom{m}{k} \frac{\Gamma(n+\alpha+m-1)}{\Gamma(n+\alpha+m-k-1)} \times \big[(m-k)x^{m-k-1}D^{m-k-1} + x^{m-k}D^{m-k}\big]B_{n-1}^{((\alpha+2)/2)}(x), \quad n \geq 1.$$

Replacing $\alpha+1$ by α, $n-1$ by n and using the identity (4.1) we obtain for all $n \geq 0$

$$B_n^{((\alpha+m+1)/2)}(x) = \frac{\Gamma(2n+\alpha-1)}{\Gamma(2n+\alpha+m)} \sum_{k=0}^{m} \binom{m}{k} \frac{\Gamma(n+\alpha+m-1)}{\Gamma(n+\alpha+m-k-1)} \times \big[(m-k)x^{m-k-1}D^{m-k-1} + x^{m-k}D^{m-k}\big]\big(xD + (n+\alpha-1)\mathbb{I}\big)B_n^{(\alpha/2)}(x), \quad n \geq 0.$$

Equivalently

$$B_n^{((\alpha+m+1)/2)}(x) = \frac{\Gamma(2n+\alpha-1)}{\Gamma(2n+\alpha+m)} \sum_{k=0}^{m} \binom{m}{k} \frac{\Gamma(n+\alpha+m-1)}{\Gamma(n+\alpha+m-k-1)} \times \Big[(m-k)(n+\alpha+m-k-2)x^{m-k-1}D^{m-k-1} + (n+\alpha+2m-2k-1)x^{m-k}D^{m-k} + x^{m+1-k}D^{m+1-k}\Big]B_n^{(\alpha/2)}(x), \quad n \geq 0.$$

After some calculations, we finally obtain for all $n \geq 0$

$$B_n^{((\alpha+m+1)/2)}(x) = \frac{\Gamma(2n+\alpha-1)}{\Gamma(2n+\alpha+m)} \sum_{k=0}^{m+1} \binom{m+1}{k} \frac{\Gamma(n+\alpha+m)}{\Gamma(n+\alpha+m-k)} \times x^{m+1-k}D^{m+1-k}B_n^{(\alpha/2)}(x), \quad m \geq 0.$$

Hence the desired result is proved. □

5. Conclusion

We have described the $\mathbb{B}_\alpha$-classical orthogonal polynomials using the Pearson's equation that the corresponding linear functionals satisfy. More precisely, we have proved that the Bessel polynomial sequence $\{B_n^{(\alpha)}(x)\}_{n\geq 0}$, where $\alpha \neq -m/2$, $m \geq -2$, $m \in \mathbb{Z}$, is the only $\mathbb{B}_\alpha$-classical sequence. As a consequence, some connection formulas between the corresponding polynomials are deduced.

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Appendix

Table A. Bessel polynomials.

$$\{\mathbf{B_n}\}_{\mathbf{n\geq 0}} \perp \mathcal{B}(\alpha)$$

$$\Phi(x) = x^2, \quad \Psi(x) = -2(\alpha x + 1),$$

$$\beta_0 = -\frac{1}{\alpha}, \quad \beta_{n+1} = \frac{1-\alpha}{(n+\alpha)(n+\alpha+1)}, \quad n \geq 0,$$

$$\gamma_{n+1} = -\frac{(n+1)(n+2\alpha-1)}{(2n+2\alpha-1)(n+\alpha)^2(2n+2\alpha+1)}, \quad n \geq 0,$$

$$x^2 B''_{n+1}(x) + 2(\alpha x + 1)B'_{n+1}(x) - (n+1)(n+2\alpha)B_{n+1}(x) = 0,$$

$$x^2 B'_{n+1}(x) = (n+1)\left(x - \frac{1}{n+\alpha}\right) B_{n+1}(x) - (2n+2\alpha+1)\gamma_{n+1} B_n(x),$$

$$\langle \mathcal{B}(\alpha), f \rangle = J(\alpha)^{-1} \int_0^{+\infty} x^{2\alpha-2} e^{-2/x} \left(\int_x^{+\infty} \xi^{-2\alpha} e^{2/\xi} s(\xi) d\xi \right) f(x) dx,$$

$$J(\alpha) := 4 \int_0^{+\infty} t^{3-8\alpha} e^{2/t^4} e^{-t} \sin(t) \left(\int_0^{t^4} x^{2\alpha-2} e^{-2/x} dx \right) dt,$$

$$s(x) = \begin{cases} 0, & x \leq 0, \\ e^{-x^{1/4}} \sin x^{1/4}, & x > 0. \end{cases}$$

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OUTPUT CONTROLLABILITY OF DELAYED CONTROL SYSTEMS IN A LONG TIME HORIZON[1]

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Abstract: In this paper, we consider the output controllability of finite-dimensional control systems governed by a distributed delayed control. For systems with ordinary controls, this problem was investigated earlier. Nevertheless, in many practical and technical problems the control acts with some delay. We give the necessary and sufficient condition for the output controllability. The main goal of our control is to govern the output of the system to some position on a subspace in a given instant, and then keep this output fixed for the remaining times. This property is called the long-time output controllability. For this, sufficient conditions are given. The introduced notions are applied for the investigation of averaged controllability of systems with delayed controls. The general approach for that is to approximate the system by the ordinary one. Some examples are considered.

Keywords: Output and averaged controllability, Delayed control, Approximation.

1. Introduction

In this paper, we deal with the output controllability of finite-dimensional control systems governed by a distributed delayed control. For systems with ordinary controls, this problem is investigated in [2]. It is known that in many practical and technical problems the controlling actions take place with some delays. The main goal of our control is to govern the output of the system to some position on a subspace in a given time $T > 0$, and then keep this output fixed for the remaining times $t > T$.

Consider a linear autonomous system with delayed controls and observation:

$$\dot{x}(t) = Ax(t) + \int_{-h}^{0} dB(s)u(t+s), \quad t \geq 0, \quad x(t) \in \mathbb{R}^n, \quad u(t) \in \mathbb{R}^m, \tag{1.1}$$

$$y(t) = Cx(t), \quad C \in \mathbb{R}^{p\times n}, \tag{1.2}$$

where elements of the matrix function $B(s)$ belong to $BV[-h, 0]$ (the space of functions of bounded variation) and they are left continuous on $(-h, 0]$, $B(s) = 0$ for $\forall s > 0$, and $B(s) = B(-h)$ for $\forall s \leq -h$. Since the matrix $B(s)$ generates a Borelian measure, any bounded Borelian m-vector function $u(t)$ can be used as a control. The notion of *output controllability* is as follows.

Definition 1. *We say that the system* (1.1) *is* C-output controllable, *if for every* $x_0 \in \mathbb{R}^n$ *and every* $\bar{y} \in \operatorname{im} C = \{y \mid y = Cx,\, x \in \mathbb{R}^n\}$ *there exist an instant* $T > 0$ *and a bounded Borelian control* u *on* $[-h, T]$ *such that the solution* $x(t)$ *of* (1.1) *with initial condition* $x(0) = x_0$ *satisfies* $y(T) = Cx(T) = \bar{y}$.

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Let us recall that the ordinary (state) controllability definition of the system (1.1) follows from Definition 1 when $C = I_n$. The symbol $I_n \in \mathbb{R}^{n\times n}$ means the unity matrix.

In this paper, we are interested in the conditions for having *long-time output controllability*. This controllability notion means that the output of the system enters the subspace and then remains on it for later times. This is defined as follows.

Definition 2. *Given $\bar{y} \in \operatorname{im} C$, the system* (1.1) *is said to be* C-long-time output controllable *(briefly C-LTOC on $\bar{y}$), if for every x_0 there exist a time $T > 0$ and a control u such that the solution of* (1.1)*, with initial condition $x(0) = x_0$ satisfies $y(t) = \bar{y}$ for every $t \in [T, \infty)$.*

It is obvious that state controllability implies C-output controllability for any matrix C. But in order to save the property $y(t) = \bar{y}$ for every $t \geq T$ we need extra conditions on a delayed control. In this paper we only assume the C-output controllability of the system and give a criterion for this. Our main attention is directed to conditions of C-LTOC (or simply LTOC) for systems of the form (1.1), (1.2) and their applications.

The notions of output controllability and C-LTOC can be applied to averaged controllability property of finite-dimensional, parameter dependent systems with delayed controls. The averaged controllability has been introduced in the paper [4]. More precisely, let us consider d realizations of control systems,

$$\dot{x}_i(t) = A_i x(t) + \int_{-h}^{0} dB_i(s)u(t+s), \quad t \geq 0, \quad x_i(t) \in \mathbb{R}^n, \quad u(t) \in \mathbb{R}^m, \quad i \in 1:d, \tag{1.3}$$

and d parameters $p_i > 0$, $\sum_{i=1}^{d} p_i = 1$. Here the matrices $B_i(s)$ have the same properties as $B(s)$ in (1.1).

Definition 3. *We say that the flock of systems* (1.3) *is* controllable in average for the weights *$p_i > 0$ if for all initial states $x_{10}, \dots, x_{d0}$ and every $\bar{y} \in \mathbb{R}^n$ there exist an instant $T > 0$ and a bounded Borelian control u on $[-h, T]$ such that the solutions of* (1.3) *satisfy the equality $\sum_{i=1}^{d} p_i x_i(T) = \bar{y}$.*

Let us use the Matlab notation for matrices and vectors. We can see that the averaged controllability notion is exactly the C-output controllability of (1.1)–(1.2) with matrices:

$$A = \operatorname{diag}[A_1, \dots, A_d], \quad B(s) = [B_1(s); \dots; B_d(s)], \quad C = [p_1 I_n, \dots, p_d I_n],$$

where $x = [x_1; \dots; x_d] \in \mathbb{R}^{nd}$. The flock of systems (1.3) is called *simultaneously controllable* if corresponding system (1.1) is state controllable. Of course, the simultaneous controllability of (1.3) implies the averaged controllability. We can also define the notion of *long-time averaged controllability* (briefly LTAC on $\bar{y}$). We say that systems (1.3) are LTAC on $\bar{y}$ for the weights $p_i > 0$ if for every initial states $x_{10}, \dots, x_{d0}$ there exist an instant $T > 0$ and an admissible control u such that the corresponding mean value is the following

$$\sum_{i=1}^{d} p_i x_i(t) = \bar{y}$$

for every $t \in [T, \infty)$.

In this paper, we obtain conditions of C-output controllability and C-LTOC for general systems (1.1)–(1.2) and apply them for the LTAC property of (1.3). Besides, we get the algorithm for constructing of necessary control in special cases. Some examples are considered.

2. Output controllability of the system

First, note that the initial condition x_0 does not play any role in Definition 1. System (1.1), (1.2) is C-output controllable iff for every $\bar{y} \in \operatorname{im} C$ there exist an instant $T > 0$ and an admissible control u on $[-h, T]$ such that

$$C \int_0^T e^{A(T-\theta)} \int_{-h}^0 dB(s)u(\theta+s)d\theta = \bar{y}.$$

Setting $\alpha = \theta + s$ we have

$$\int_{-h}^0 dB(s)u(\theta+s) = \int_{\theta-h}^{\theta} dB(\alpha-\theta)u(\alpha),$$

and by Fubini's theorem we get the equivalent equality

$$\int_{-h}^T \mathcal{B}(T,\alpha)u(\alpha)d\alpha = \bar{y}, \quad \text{where} \quad \mathcal{B}(T,\alpha) = C\int_{\alpha\vee 0}^{(\alpha+h)\wedge T} e^{A(T-\theta)}dB(\alpha-\theta). \tag{2.1}$$

The $n\times m$-matrix function $\mathcal{B}(T,\alpha)$ is of bounded variation with respect to α and, therefore, belongs to the space $L_2^{p\times m}[-h,T]$ (the space of square integrable matrices or vectors). We can prove the following lemma.

Lemma 1. *System (1.1), (1.2) is C-output controllable iff there is a segment $[a,b]$, $-h \le a < b \le T$, such that*

$$\operatorname{rank}\left(\int_a^b \mathcal{B}(T,\alpha)\mathcal{B}'(T,\alpha)d\alpha\right) = \operatorname{rank} C. \tag{2.2}$$

P r o o f. Let condition (2.2) be satisfied. Since

$$\operatorname{im}\int_a^b \mathcal{B}(T,\alpha)\mathcal{B}'(T,\alpha)d\alpha \subset \operatorname{im} C,$$

we obtain the equality of subspaces in this inclusion. For every $\bar{y} \in \operatorname{im} C$ there exists a vector $v \in \mathbb{R}^p$ such that

$$\int_a^b \mathcal{B}(T,\alpha)\mathcal{B}'(T,\alpha)d\alpha v = \bar{y}.$$

Then $u(\alpha) = \mathcal{B}'(T,\alpha)v$ is a bounded Borelian control on $[a,b]$. We can take $u(\alpha) = 0$, $\alpha \notin [a,b]$, and satisfy (2.1) for any $T \ge b$. On the contrary, let condition (2.1) be valid, but there is a vector $\bar{y} \in \operatorname{im} C$ such that

$$\bar{y} \notin \operatorname{im}\int_{-h}^T \mathcal{B}(T,\alpha)\mathcal{B}'(T,\alpha)d\alpha.$$

Then we have a contradiction with (2.1) as

$$L_2^m[-h,T] = \{u(\alpha) : u(\alpha) = \mathcal{B}'(T,\alpha)v,\ v \in \mathbb{R}^p\} \bigoplus \left\{u(\alpha) : \int_{-h}^T \mathcal{B}(T,\alpha)u(\alpha)d\alpha = 0\right\}.$$

Therefore, there are no functions $u \in L_2^m[-h,T]$ satisfying (2.1). □

Corollary 1. *Condition (2.2) holds iff the equality*

$$l'\mathcal{B}(T,\alpha)=0 \quad \textit{a.e. on } [a,b] \quad \textit{implies that} \quad l\in\ker C'. \tag{2.3}$$

P r o o f. Condition (2.2) is equivalent to the equality

$$\ker C'=\ker\int_a^b \mathcal{B}(T,\alpha)\mathcal{B}'(T,\alpha)d\alpha,$$

or, in other words, the implication (2.3) is fulfilled. □

Corollary 2. *The function $\mathcal{B}(T,\alpha)$ from (2.1) can be expressed in the form*

$$\mathcal{B}(T,\alpha)=Ce^{A(T-\alpha)}\boldsymbol{b}(T,\alpha), \quad \textit{where} \quad \boldsymbol{b}(T,\alpha)=\int_{(\alpha-T)\vee(-h)}^{\alpha\wedge 0} e^{As}dB(s). \tag{2.4}$$

If $T>h$ and $a=0$, $b=T-h$, then

$$\boldsymbol{b}(T,\alpha)=\int_{-h}^{0} e^{As}dB(s)=\text{const}$$

on $[a,b]$. Hence, the implication (2.3) is equivalent to the rank condition

$$\operatorname{rank} C\left[\int_{-h}^{0} e^{A(s+h)}dB(s), A\int_{-h}^{0} e^{A(s+h)}dB(s),\dots,A^{n-1}\int_{-h}^{0} e^{A(s+h)}dB(s)\right]=\operatorname{rank} C. \tag{2.5}$$

P r o o f. Setting $\alpha-\theta=s$ in (2.1) we get (2.4). As $\mathbf{b}(T,\alpha)=$const on segment $[a,b]$, the relation $l'\mathcal{B}(T,\alpha)=0$ can be differentiated with respect to α many times. So, we come to the equivalence of implication (2.3) and rank condition (2.5) by the theorem of Cayley–Hamilton [5, Theorem 7.2.4]. □

Let us discuss the Lemma 1 and its Corollaries. If condition (2.2) does not hold for some segment $[a,b]$, it can be hold for grater ones. The rank condition for C-output controllability is possible if the matrix function $B(s)$ is piecewise-constant as in the case of lumped delays. The simplest case of lumped delays is given by

$$B(s)=-B_0\chi_{(-\infty,0]}(s)-B_1\chi_{(-\infty,-1]}(s), \tag{2.6}$$

where the indicator function $\chi_{(a,b]}(s)=1$ if $s\in(a,b]$, and $\chi_{(a,b]}(s)=0$, elsewhere. Let $T>1$. We can divide the segment $[-1,T]=(T-1,T]\cup[0,T-1]\cup[-1,0)$ into three parts. On the first semi-interval the implication

$$l'Ce^{A(T-\alpha)}B_0\equiv 0\Rightarrow l\in\ker C'$$

is equivalent to the condition

$$\operatorname{rank} C[B_0,AB_0,\dots,A^{n-1}B_0]=\operatorname{rank} C \tag{2.7}$$

by the theorem of Cayley–Hamilton. On $[0,T-1]$ we have

$$\mathbf{b}(T,\alpha)=B_0+e^{-A}B_1,$$

and we are in the conditions of Corollary 2. On the remaining semi-interval the implication

$$l'Ce^{A(T-\alpha-1)}B_1\equiv 0\Rightarrow l\in\ker C'$$

is equivalent to condition

$$\operatorname{rank} Ce^{A(T-1)}[B_1, AB_1, \ldots, A^{n-1}B_1] = \operatorname{rank} C. \tag{2.8}$$

If $0 \le T < 1$, we have only two segments $[0, T]$ and $[-1, T-1]$. Therefore, we get two conditions in (2.7) and (2.8), where the matrix of $e^{A(T-1)}$ is absent. Conditions (2.5), (2.7) do not depend on T, but condition in (2.8) nevertheless depends. It distinguishes case of delayed controls from the ordinary one.

Remark 1. Of course, we can also consider partly different Definitions 1–3 with null initial controls, i.e. when $u(t) = 0$ for $t < 0$. Then the integral in (2.1) is considered on $[0, T]$, the parameter $a \ge 0$ in Lemma 1, and condition (2.8) is not necessary. In addition, we may demand that $u(t) = 0$ when $t \in [T-h, T]$, $T > h$. Then we have $0 \le a < b \le T - h$ in Lemma 1.

3. The property of LTOC

Suppose further that system (1.1)–(1.2) is C-output controllable at some instant $T > h$ and $u(t) = 0$ if $t \notin [0, T-h]$ as in the Remark 1. Then it is easily seen that the C-output controllability on $[0, T]$ is equivalent to C-output controllability on $[a, T+a]$ for all $a \ge 0$ when $u(t) = 0$ if $t \notin [a, T+a-h]$. Therefore, in order to get $y(t) \equiv \bar{y}$, $\forall t \ge T$, we need to obtain the conditions for the property

$$Cx_0 = Cx(t) \quad \forall t \ge 0. \tag{3.1}$$

By derivation with respect to t in (3.1), we have:

$$CAx(t) + C\int_{-h}^{0} dB(s)u(t+s) = 0 \quad \forall t \ge 0. \tag{3.2}$$

Introduce the subspace

$$\mathcal{U} = \left\{ x \in \mathbb{R}^n : \exists \text{ an admissible function } u(\cdot) \ s.t. \ x = \int_{-h}^{0} dB(s)u(s) \right\}. \tag{3.3}$$

To satisfy (3.2), one needs to have the inclusion $CAx(t) \in C\mathcal{U}$ for $\forall t \ge 0$. Since $\mathbb{R}^p = C\mathcal{U} \oplus C\mathcal{U}^{\perp}$, we can take an orthonormal basis $\{h_1, \ldots, h_q\}$ in $C\mathcal{U}^{\perp}$, where $q = p - \dim(C\mathcal{U})$, and the corresponding matrix $H = [h_1, \ldots, h_q] \in \mathbb{R}^{p\times q}$. As a result, we get a projector $P_0 = HH' \in \mathbb{R}^{p\times p}$ on the subspace $C\mathcal{U}^{\perp}$.

Consequently, condition (3.2) is $L_0x(t) = 0$ $\forall t \ge 0$, where $L_0 = P_0CA$. Therefore, if we introduce the matrix $C_1 = [C; L_0] \in \mathbb{R}^{2p\times n}$, then $C_1x_0 = C_1x(t)$ $\forall t \ge 0$, as in (3.1).

We can iterate the process similar to ordinary case with no delays as in [2] to define $C_2 = [C, L_1]$, $L_1 = P_1C_1A$, and so on. After k steps we get

$$C_{k+1} = [C; L_k], \quad L_k = P_kC_kA \in \mathbb{R}^{(k+1)p\times n}, \tag{3.4}$$

where $P_k \in \mathbb{R}^{(k+1)p\times(k+1)p}$ is the orthogonal projector on $C_k\mathcal{U}^{\perp}$. The process stops when $\ker C_{k+1} = \ker C_k$. The condition (3.1) can be fulfilled iff $L_kx_0 = 0$. To be more exact, the following assertion holds.

Lemma 2. *We have* $\ker C_{k+1} \subset \ker C_k \subset \mathbb{R}^n$ *and* $\ker L_{k+1} \subset \ker L_k \subset \mathbb{R}^n$ *for every* $k \in \mathbb{N}\cup\{0\}$. *There exists a number* $K \in 0 : n$ *such that* $\ker C_{K+1} = \ker C_K$. *Here* $C_0 = C$. *For every* $i \in \mathbb{N}$ *we have* $\ker C_{K+i} = \ker C_K$.

P r o o f. We will argue by induction. As $\ker C_1 = \ker C_0 \cap \ker L_0$, we trivially obtain $\ker C_1 \subset \ker C_0$. Suppose that $\ker C_k \subset \ker C_{k-1}$ for some $k \in \mathbb{N}$. Then we notice that

$$L_k x = 0 \;\Leftrightarrow\; \exists\, u \in \mathcal{U} \quad s.t. \quad C_k(Ax - u) = 0$$
$$\Rightarrow\; C_{k-1}(Ax - u) = 0 \;\Leftrightarrow\; L_{k-1}x = 0.$$

This means that $\ker L_k \subset \ker L_{k-1}$. Therefore, $\ker C_{k+1} \subset \ker C_k$. It is obvious that there exists a number $K \in 0 : n$ such that $\ker C_{K+1} = \ker C_K \subset \mathbb{R}^n$. It follows that $\ker L_{K+1} = \ker L_K \subset \mathbb{R}^n$. Indeed,

$$L_K x = 0 \;\Leftrightarrow\; \exists\, u \in \mathcal{U} \quad s.t. \quad C_K(Ax - u) = 0$$
$$\Rightarrow\; C_{K+1}(Ax - u) = 0 \;\Leftrightarrow\; L_{K+1}x = 0.$$

Hence, by induction, we obtain the final assertion. □

Note also that $\operatorname{im} L_k \subset \operatorname{im} C_k$ for every $k \in \mathbb{N} \cup \{0\}$. This is equivalent to the inclusion $\ker A'C_k'P_k \supset \ker C_k'$. Indeed, if $C_k'z = 0$, then $z \perp C_k\mathcal{U} \Rightarrow\; z \in C_k\mathcal{U}^{\perp} \;\Rightarrow\; P_k z = z$.

The problem of control with delays to ensure equality (3.1) is more difficult than for ordinary controls. Let us prove the lemma.

Lemma 3. *Let C_k, $k \in \mathbb{N} \cup \{0\}$, be the sequence defined by* (3.4)*, and let $K \in 0 : n$ such that $\ker C_{K+1} = \ker C_K$. Then there exists a function $v(t) \in \mathcal{U}$ such that* (3.1) *holds where*

$$\dot{x}(t) = Ax(t) + v(t), \quad x(0) = x_0, \tag{3.5}$$

if and only if $L_K x_0 = 0$ with L_K defined by (3.4)*.*

P r o o f. It follows from (3.1) that $C_K x_0 = C_K x(t)$ and $L_K x_0 = 0$. On the contrary, assume that $L_K x_0 = 0$. We need

$$C_K x_0 = C_K x(t) \quad \forall t \geq 0.$$

After derivation we get

$$C_K \dot{x}(t) = C_K(Ax(t) + v(t)), \quad v(t) \in \mathcal{U}. \tag{3.6}$$

If we find $v(t)$ with $C_K \dot{x}(t) = 0$, then the lemma is proved. We can write

$$C_K \dot{x}(t) = L_K x(t) + (I_{(K+1)p} - P_K)C_K Ax(t) + C_K v(t).$$

Here $I_{(K+1)p} - P_K$ is a projector on $C_K\mathcal{U}$. Hence, there exists a continuous closed-loop control $v(x)$ such that

$$(I_{(K+1)p} - P_K)C_K Ax + C_K v(x) = 0.$$

Relation (3.6) under such a control reduces to

$$C_K \dot{x}(t) = L_K x(t).$$

Let us write the orthogonal expansion for $x(t)$:

$$x(t) - x_0 = x_0(t) + x_1(t), \tag{3.7}$$

where $x_0(t) \in \ker C_K$ and $x_1(t) \in \operatorname{im} C_K'$. Then

$$C_k x(t) = C_K(x_0 + x_1(t))$$

and

$$L_K x(t) = L_K x_1(t)$$

as

$$\ker C_K = \ker C_{K+1} = \ker C_0 \cap \ker L_K.$$

Thus, we get that

$$C_K \dot{x}_1(t) = L_K x_1(t).$$

The matrix C_K is invertible on the subspace $\operatorname{im} C'_K$ and $x_1(0) = 0$ from (3.7). Therefore, $x_1(t) = 0$ $\forall t \geq 0$. The lemma is proved. □

It follows from Lemma 3 that conditions

$$L_K x_0 = 0 \quad \text{and} \quad \left(I_{(K+1)p} - P_K\right) C_K A x + C_K v(x) = 0$$

are necessary and sufficient for the solution $x(t)$ of equation (3.5) to satisfy (3.1). They define the function $\bar{v}(t) = v(x(t))$, but for our purposes we need a function $u(t)$ such that

$$\int_{(-h)\vee(-t)}^{0} dB(s)u(t+s) = \bar{v}(t), \quad t \geq 0. \tag{3.8}$$

This is an integral equation. It can have no solutions. Therefore, in next sections we consider the approximation scheme to exclude equations like (3.8). Now, we formulate the general result.

Theorem 1. *Let be given $\bar{y} \in \operatorname{im} C$. For every $x_0 \in \mathbb{R}^n$ there exists an admissible control for the system (1.1) such that the solution satisfies $Cx(t) = \bar{y}$ $\forall t \geq T$ if and only if*

$$[\bar{y}; 0] \in \operatorname{im} C_{K+1}$$

and system (1.1), (1.2) is C_{K+1}-output controllable in the sense of Remark 1, i.e. the condition like (2.2) holds for some $0 \leq a < b \leq T - h$ with

$$\mathcal{B}(T, \alpha) = C_{K+1} e^{A(T-\alpha)} \boldsymbol{b}(T, \alpha)$$

and $\operatorname{rank} C_{K+1}$. *Here C_k is the sequence defined by (3.4) and the number K is defined by Lemma 2. Besides, equation (3.8) has to be resolved for the function $\bar{v}(t)$ defined in Lemma 3.*

P r o o f. According to the Lemma 3 the control exists iff the system is transferred to the state $x(T)$ such that $C_{K+1}x(T) = [\bar{y}; 0]$. This is possible for every $\bar{y} \in \operatorname{im} C$ and every $x_0 \in \mathbb{R}^n$ iff the system is C_{K+1}-output controllable. After that we solve the problem as in Lemma 3 which does not depend of initial instant T. □

We do not give any sufficient conditions for the existence of a solution of integral equation (3.8). This is considered in some special cases. For example, in simplest case (2.6) we have the difference equation

$$B_0 u(t) + B_1 u(t-1) = \bar{v}(t), \quad u(t) = 0 \quad \text{if} \quad t < 0$$

which can be resolved step-by-step on segments $[i-1, i]$:

$$B_0 u_i(t) = \bar{v}(t) - B_1 u_{i-1}(t-1), \quad t \in [i-1, i], \quad i \in \mathbb{N}, \tag{3.9}$$

where $u_0(t) = 0$.

Example 1. Consider the flock of two systems of the form:

$$\dot{x}_1^1 = -x_1^2 + \sum_{i,j=1}^2 b_{1ij} u_i(t-j+1), \quad \dot{x}_1^2 = x_1^1; \quad \text{first system},$$
$$\dot{x}_2^1 = x_2^1 + 2x_2^2 + \sum_{i,j=1}^2 b_{2ij} u_i(t-j+1), \quad \dot{x}_2^2 = x_2^2; \quad \text{second system},$$

where we have the case with $p_1 = p_2 = 1/2$ and $C = [I_2, I_2]/2$. Condition (3.1) reduces to the requirement: $Cx_0 = 0$ implies $Cx(t) = 0$ if $t \geq 0$. Here $x(t) \in \mathbb{R}^4$ is the composed vector. Below we study the example in detail for various coefficients b_{lij}.

4. The system in the infinite-dimensional space

Let us now rewrite the system (1.1)–(1.2) in the infinite-dimensional space following [1]. We can write

$$\int_{-h}^0 dB(s)u(s) = B_0 u(0) + \mathbb{B}u, \quad \text{where} \quad B_0 = -B(0), \quad \mathbb{B}u = \int_{[-h,0)} dB(s)u(s). \tag{4.1}$$

Formula (4.1) is true for continuous vector-functions u, but we want to use functions $\{u \in \mathcal{H} = L_2^m[-h, 0]\}$. In this case we consider the operator $\mathbb{B}$ as unbounded with dense domain $\mathbf{b}(\mathbb{B}) = W = W_{1,2}^m[-h, 0]$ (the Sobolev space). If $u \in W$, the function $\phi(t, s) = u(t + s)$ satisfies the equation in partial derivatives:

$$\dot{\phi}(t, s) = D\phi(t, s), \quad \phi(0, s) = u_0(s), \quad \phi(t, 0) = u(t), \tag{4.2}$$

with the operator $D = d/ds$. Equation (4.2) is considered in $\mathcal{H}$ with unbounded D. The left-shift C_0-semigroup S_t on $\mathcal{H}$ is defined by

$$(S_t u)(s) = \begin{cases} u(t+s), & s \in [-h, -t] \\ 0, & s \in (-t, 0] \end{cases} \quad \text{if} \quad t \leq h, \quad \text{and} \quad (S_t u)(s) = 0 \quad \text{if} \quad t > h.$$

The infinitesimal generator for S_t is D with dense domain

$$\mathbf{b}(D) = W^0 = \{u \in W : u(0) = 0\} \subset \mathcal{H}.$$

As shown in [1, Lemma 1.1], the solution

$$\phi(t, s) = \begin{cases} u_0(t+s), & s \in [-h, -t] \\ u(t+s), & s \in (-t, 0] \end{cases} \quad \text{if} \quad t \leq h, \quad \text{and} \quad \phi(t, s) = u(t+s) \quad \text{if} \quad t > h,$$

of equation (4.2), $\phi(t, \cdot) \in \mathcal{H}$, can be represented by

$$\phi(t) = S_t \phi_0 + \int_0^t S_{t-r} \Delta u(r) dr, \tag{4.3}$$

where the operator $\Delta \in \mathcal{L}(\mathbb{R}^m, W^*)$ (the space of linear operators) is given by the relation $(\Delta u, w) = u'w(0)$ for all $w \in W$. So, in spite of the fact that equality (4.3) is considered in $W^* \supset \mathcal{H} \supset W$ and the integration is also fulfilled in W^*, we have $\phi(t, \cdot) \in \mathcal{H}$ for every $u \in L_{2,loc}^m[0, \infty)$.

Introducing the operators $\mathbf{A} = [A, \mathbb{B}; 0, D]$, $\mathbf{B} = [B_0; \Delta]$, and C_0-semigroup $\mathbb{T}$ by

$$\mathbb{T}_{t-r} z = \left[e^{A(t-r)} x + \int_r^t e^{A(t-\alpha)} \mathbb{B} S_{\alpha-r} \phi d\alpha; S_{\alpha-r} \phi \right],$$

where $z = [x; \phi] \in Z = \mathbb{R}^n \times \mathcal{H}$, we can write the mild solution

$$z(t) = \mathbb{T}_t z_0 + \int_0^t \mathbb{T}_{t-r}\mathbf{B}u(r)dr, \quad \text{for the equation} \tag{4.4}$$

$$\dot{z}(t) = \mathbf{A}z(t) + \mathbf{B}u(r), \quad z(0) = z_0. \tag{4.5}$$

Here the operator $\mathbf{A}$ is unbounded on Z with domains $\mathbf{b}(\mathbf{A}) = \mathbb{R}^n \times W^0$. For the operator B we have $\mathbf{B} \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n \times W^*)$.

Equation (4.5) has no delays in control, but a recurrent procedure like in (3.4) is, unfortunately, impossible for infinite-dimensional system (4.5) to find a C-LTOC control. Therefore, we pass to finite-dimensional approximations of the obtained system.

5. Finite-dimensional approximation

We use the averaging approximation of the delayed system following [3]. For every positive integer N, we define the finite-dimensional linear subspace $\mathcal{H}^N$ of $\mathcal{H}$ by

$$\mathcal{H}^N = \Big\{u \in \mathcal{H} : u = \sum_{i=1}^N v_i\chi_i, \quad v_i \in \mathbb{R}^m\Big\},$$

where χ_i denote the characteristic function of $[t_i, t_{i-1})$ for $i \in 1:N$ and $t_i = -ih/N$, $i \in 0:N$. The subspace $\mathcal{H}^N$ is isometrically isomorphic to $\mathbb{R}^{mN}$ by means of the embedding $\gamma^N : \mathbb{R}^{mN} \to \mathcal{H}^N$ such that $(\gamma^N g)(s) = v_i$, $s \in [t_i, t_{i-1})$, $i \in 1:N$, where $g = [v_1; \dots; v_N]$. On $\mathbb{R}^{mN}$, we define the induced inner product

$$\langle f, g\rangle_N = f'Q^N g, \quad f, g \in \mathbb{R}^{mN},$$

where

$$Q^N = \operatorname{diag}[I_m, \dots, I_m]h/N \in \mathbb{R}^{mN\times mN}.$$

The corresponding vector and matrix norms will be denoted by $\|\cdot\|_N$. The dual mapping $\gamma^{N*} : \mathcal{H}^N \to \mathbb{R}^{mN}$ has the natural extension $\pi^N : \mathcal{H} \to \mathbb{R}^{mN}$ defined by

$$\pi^N u = [v_1; \dots; v_N], \quad v_i = \int_{t_i}^{t_{i-1}} u(s)dsN/h, \quad i \in 1:N.$$

We have that $P^N = \gamma^N\pi^N$ is an self-adjoint orthogonal projector onto $\mathcal{H}^N$ and $\pi^N\gamma^N = I_{mN}$. Introduce the the following matrices:

$$B_i^N = \lim_{s\uparrow t_i}(B(s + h/N) - B(s)) = B(t_{i-1}) - B(t_i), \quad i \in 1:N.$$

Note that the matrix $B(s)$ is left-continuous. For $\phi \in \mathcal{H}$, let $\pi^N\phi = g = [v_1; \dots; v_N] \in \mathbb{R}^{mN}$. Then we can approximate the infinite-dimensional operators as follows:

$$\mathbb{B}\phi \approx \mathbb{B}P^N\phi = \sum_{i=1}^N B_i^N v_i; \quad D\phi \approx \nabla P^N\phi = \sum_{i=1}^N N(v_{i-1} - v_i)\chi_i/h, \quad v_0 = 0;$$

$$\mathbf{B}u \approx [B_0u; N\chi_1 u/h].$$

Denote by Z^N the space $\mathbb{R}^n\times\mathcal{H}^N$. Introduce the approximating operators $\mathbf{A}^N = [A, \mathbb{B}P^N; 0, \nabla P^N] : Z^N \to Z^N$ and $\mathbf{B}^N = [B_0; N\chi_1/h] : \mathbb{R}^m \to Z^N$. Let $\mathbb{T}_t^N$ denote the C_0-semigroup generated by $\mathbf{A}^N$ on Z^N and let $\bar{\pi}^N = [I_n, 0; 0, \pi^N]$, $\bar{\gamma}^N = [I_n, 0; 0, \gamma^N]$ be the operators on Z and on $\mathbb{R}^{n+mN}$, respectively. The following theorem is true.

Theorem 2 [3, Theorem 3.1]. *Let the matrix $B(s)$ have the form*

$$B(s) = -\sum_{i=0}^{q} B_i\chi_{(-\infty,-h_i]}(s) - \int_s^0 B_{01}(r)dr, \quad 0 = h_0 < \cdots < h_q = h, \tag{5.1}$$

where $B_{01}(\cdot) \in L_2^{n\times m}[-h,0]$. Then there exist constants M and ω independent of N such that

$$\|e^{\bar\pi^N \mathbf{A}^N \bar\gamma^N t}\|_N \le Me^{\omega t}.$$

It follows from definitions of operators that

$$\bar\pi^N \mathbf{A}^N \bar\gamma^N = [A, \mathbb{B}\gamma^N; 0, \pi^N\nabla\gamma^N] \in \mathbb{R}^{(n+mN)\times(n+mN)}.$$

Therefore, the finite-dimensional approximation for (4.4), (4.5) is written as

$$\begin{aligned} \dot x(t) &= Ax(t) + \mathbb{B}\gamma^N g(t) + B_0 u(t),\\ \dot g(t) &= \pi^N\nabla\gamma^N g(t) + N[u(t); 0; \ldots; 0]/h. \end{aligned} \tag{5.2}$$

Since $g(t) = [v_1(t); \ldots; v_N(t)]$ we can write the matrices of system (5.2), where the state vector is $[x(t); g(t)]$, in the following form

$$\begin{aligned} \mathbf{A}^N &= \left[A, B_1^N, \ldots, B_N^N; 0_{mN\times n}, (Q^N)^{-1}V\right],\\ V &= \begin{bmatrix} -I_m, & 0, & \ldots & ,0, & 0;\\ I_m, & -I_m, & \ldots & ,0, & 0;\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0, & 0, & \ldots & ,I_m, & -I_m \end{bmatrix},\\ \mathbf{B}^N &= [B_0; I_m N/h; \ldots; 0; 0]. \end{aligned} \tag{5.3}$$

By Trotter–Kato theorem and Theorem 2 the following estimates are true [3, Theorems 4.4 and 4.10].

(i) If $z \in \mathbf{b}(\mathbf{A})$, then

$$\left\|[I_n; P^N]\mathbb{T}_t z - \mathbb{T}_t^N[I_n; P^N]z\right\| \le \alpha_1 e^{\alpha t}(h/N)\|z\|_{\mathbb{R}^n\times W}, \quad \forall N \in \mathbb{N}, \quad t > 4h.$$

(ii) For $t > 5h$ and $\forall N > N_0$,

$$\left\|[I_n; P^N]\mathbb{T}_t - \mathbb{T}_t^N[I_n; P^N]\right\| \le \alpha_2 e^{\alpha t}(h/N).$$

(iii) There exists a positive constant α_3, dependent on t but independent on N, such that for every $u(\cdot) \in L_2^m[0,t]$ and all $N \in \mathbb{N}$, we have

$$\left\|\int_0^t \mathbb{T}^N(t-r)\mathbf{B}^N u(r)dr\right\|_{Z^N} \le \alpha_3\|u\|_{L_2^m[0,t]}.$$

From (iii) it follows that

$$\lim_{N\to\infty}\int_0^t \mathbb{T}^N(t-r)\mathbf{B}^N u(r)dr = \int_0^t \mathbb{T}(t-r)\mathbf{B}u(r)dr.$$

It is unknown whether estimates in Theorem 2 and in (i)–(iii) without an assumption (5.1) are true.

6. Application to averaged controllability and examples

For the flock (1.3), Lemma 1 can be reformulated in the following way.

Lemma 4. *The flock of systems* (1.3) *is controllable in average for the weights* $p_i > 0$ *iff there is a segment* $[a, b]$, $-h \le a < b \le T$, *such that*

$$\operatorname{rank}\left(\int_a^b \mathcal{B}(T,\alpha)\mathcal{B}'(T,\alpha)d\alpha\right) = n, \tag{6.1}$$

where

$$\mathcal{B}(T,\alpha) = \sum_{i=1}^{d} p_i \int_{\alpha\vee 0}^{(\alpha+h)\wedge T} e^{A_i(T-\theta)} dB_i(\alpha-\theta).$$

Of course, the condition (6.1) holds iff the equality

$$l'\mathcal{B}(T,\alpha) = 0 \quad \text{a.e. on } [a,b] \quad \text{implies that} \quad l = 0. \tag{6.2}$$

Corollary 2 has the form.

Corollary 3. *The function* $\mathcal{B}(T,\alpha)$ *from* (6.1) *can be expressed in the form*

$$\mathcal{B}(T,\alpha) = \sum_{i=1}^{d} p_i e^{A_i(T-\alpha)} \boldsymbol{b}_i(T,\alpha), \quad \text{where} \quad \boldsymbol{b}_i(T,\alpha) = \int_{(\alpha-T)\vee(-h)}^{\alpha\wedge 0} e^{A_i s} dB_i(s).$$

If $T > h$ *and* $a = 0$, $b = T - h$, *then*

$$\boldsymbol{b}_i(T,\alpha) = \int_{-h}^{0} e^{A_i s} dB_i(s) = \text{const}$$

on $[a, b]$. *Hence, the implication* (6.2) *is equivalent to the rank condition*

$$\begin{aligned}\operatorname{rank}\Big[\sum_{i=1}^{d} p_i \int_{-h}^{0} e^{A_i(s+h)} dB_i(s),\ \sum_{i=1}^{d} p_i A_i \int_{-h}^{0} e^{A_i(s+h)} dB_i(s),\\ \dots, \sum_{i=1}^{d} p_i A_i^{nd-1} \int_{-h}^{0} e^{A_i(s+h)} dB_i(s)\Big] = n.\end{aligned} \tag{6.3}$$

Let us pass to the property of LTAC. For the sake of example, we restrict the analysis to the case of the null control, i.e. the goal is to steer and keep the average equal to zero. We also consider the case with $d = 2$ components, and we chose $p_1 = p_2 = 1/2$, and $B_1(s) = B_2(s) = B(s)$. We do the remark.

Remark 2. All the statements in Section 3 are still valid, if at step $k+1$ in (3.4) we consider any matrix $C_{k+1} = [RC; \tilde{L}_k]$, with $R \in \mathbb{R}^{p\times p}$, $\det R \neq 0$, and $\tilde{L}_k$ is a matrix of n columns such that $\ker C \cap \ker L_k = \ker C \cap \ker \tilde{L}_k$, where L_k is defined by (3.4). With this modification, $\bar{y}$ has to be modified in $R\bar{y}$.

Let $\mathcal{U}$ be the subspace defined by (3.3). In what follows, P denotes the orthogonal projector of $\mathbb{R}^n$ on $\mathcal{U}^\perp$, and we set $E = (A_1 - A_2)/2$, $F = (A_1 + A_2)/2$.

Instead of the sequence C_k introduced in (3.4), we use the sequence Ξ_k defined by

$$\Xi_k = \begin{bmatrix} I_n, & I_n; \\ PE, & -PE; \\ PEF, & -PEF; \\ \vdots & \vdots \\ PEF^{k-1}, & -PEF^{k-1} \end{bmatrix} \in \mathbb{R}^{(k+1)n\times 2n}. \tag{6.4}$$

We can note the following.

- For $k = 0$ $\quad \Xi_0 = 2C = [I_n, I_n]$.
- For $k = 1$, let P_0 be the orthogonal projector of $\mathbb{R}^n$ on $\Xi_0[\mathcal{U};\mathcal{U}]^\perp = \mathcal{U}^\perp$. We see $P_0 = P$. Then we set $\tilde{L}_1 = [\Xi_0; P\Xi_0 A] = [I_n, I_n; PA_1, PA_2]$. Since $\ker \tilde{L}_1 = \ker \Xi_1$, matrix Ξ_1 is suitable, according to Remark 2.
- Assume that at step k the matrix Ξ_k given by (6.4) is suitable. We define P_k, the orthogonal projector of $\mathbb{R}^{(k+1)n}$ on $\Xi_k[\mathcal{U};\mathcal{U}]^\perp = \operatorname{diag}[P, I_n, \dots, I_n]$. Then we set

$$\tilde{L}_{k+1} = [\Xi_0; P_k\Xi_k A] = \begin{bmatrix} I_n, & I_n; \\ PA_1, & PA_2; \\ PEA_1, & -PEA_2; \\ \vdots & \vdots \\ PEF^{k-1}A_1, & -PEF^{k-1}A_2 \end{bmatrix}.$$

  It is obvious that $\ker \tilde{L}_{k+1} = \ker \Xi_{k+1}$. So, Ξ_{k+1} is suitable.

As in Lemma 2, we have $\ker \Xi_{k+1} \subset \ker \Xi_k \subset \ker \Xi_0 \subset \mathbb{R}^{2n}$. Since $\dim(\ker \Xi_0) = n$ there exists $K \in 0:n$ such that $\ker \Xi_{K+1} = \ker \Xi_K$, and we have $\ker \Xi_K = \ker \Xi_n$ (see Lemma 2).

As a consequence of Theorem 1 and the above considerations, we obtain the following result.

Corollary 4. *Let $d = 2$ and let $A_1, A_2 \in \mathbb{R}^{n\times n}$, and $B_1(s) = B_2(s)$. Then for every $x_{10}, x_{20} \in \mathbb{R}^n$ the flock of systems (1.3) is LTAC to 0 for $p_1 = p_2 = 1/2$ if and only if the condition like (2.2) holds for some $0 \le a < b \le T - h$ with*

$$\mathcal{B}(T,\alpha) = \Xi_n \operatorname{diag}\left[\int_{\alpha\vee 0}^{(\alpha+h)\wedge T} e^{A_1(T-\theta)} dB(\alpha-\theta),\ \int_{\alpha\vee 0}^{(\alpha+h)\wedge T} e^{A_2(T-\theta)} dB(\alpha-\theta)\right]$$

and $\operatorname{rank}\Xi_n$*, where the matrix Ξ_n is given by (6.4) for $k = n$.*

Remark 3. The Corollary 4 ensures that the solutions $x_1(t)$ and $x_2(t)$ of (1.3) (with $d = 2$ and $B_1(s) = B_2(s) = B(s)$) can be steered to some $[x_1(T); x_2(T)] \in \ker \Xi_n$. This condition can be equivalently rewritten as

$$x_1(T) + x_2(T) = 0,$$
$$x_1(T) - x_2(T) \in \left\{ g \in \mathbb{R}^N : EF^k g = 0 \quad \forall k \in 0:n-1 \right\}.$$

Let $g = (x_1 - x_2)/2$ and $f = (x_1 + x_2)/2$. Then for every control $v(t) \in \mathcal{U}$ we have

$$\begin{cases} \dot{f} = Ff + Eg + v(t), \\ \dot{g} = Ef + Fg, \end{cases} \quad \Leftrightarrow \quad \begin{cases} \dot{x}_1 = A_1x_1 + v(t), \\ \dot{x}_2 = A_2x_2 + v(t). \end{cases}$$

Now it becomes obvious that $f(t) = 0$ for $t \geq T$ if and only if $v(t) = -Eg(t)$ and $g(t)=e^{F(t-T)}g(T)$ such that $Eg(t) = 0$ for $t \geq T$. Note that $v(t) \in \mathcal{U}$.

Of course, we need a control $u(t)$, $t \geq T$, such that

$$\int_{(-h)\vee(T-t)}^{0} dB(s)u(t+s) = v(t), \quad t \geq T, \tag{6.5}$$

similarly to (3.8).

Example 2. Let us return to the flock in the Example 1. We have $A_1 = [0, -1; 1, 0]$, $A_2 = [1, 2; 0, 1]$. The system has 8 parameters. Let $b_{111} = b_{112} = b_{211} = b_{212} = 1$. Other parameters are equal to zero. It corresponds to one control with one delay in the form $[u(t) + u(t-1); 0]$. The flock is controllable in average for every $T > 1$ in the sense of the Remark 1, as condition (6.3) is fulfilled. Here we have the projector $P = [0, 0; 0, 1]$. It was shown in [2] that the systems with one scalar ordinary control:

$$\dot{x}_1 = A_1x_1 + [v(t); 0], \quad \dot{x}_2 = A_2x_2 + [v(t); 0],$$

are controllable in average, but not simultaneously controllable. Moreover, this system has the long-time averaged controllability property. Hence, there is a control $v(t)$, $t \geq T$, owing to the Remark 3. We can find a control $u(t)$, $t \in [0, T-1]$, such that $x_1(T) + x_2(T) = 0$ due to controllability. Equation (6.5) is $u(t) + u(t-1) = v(t)$, $t \geq T$. As in (3.9), it can be resolved step-by-step on segments $[T+i-1, T+i]$:

$$u_i(t) = v(t) - u_{i-1}(t-1), \quad t \in [T+i-1, T+i], \quad i \in \mathbb{N},$$

where $u_0(t) = 0$.

We can also analyze the property of LTAC for the case $B_1(s) \neq B_2(s)$ when $d = 2$. Then we use the general considerations of Section 3. Note that equation (3.9) can be easily resolved only if the matrix B_0 is square and $\det B_0 \neq 0$. For our examples, it corresponds to the condition $\det [b_{111}, b_{121}; b_{211}, b_{221}] \neq 0$. This determinant equals zero in Example 2, but, nevertheless, we found the $u(t)$.

Example 3. Let $b_{112} = b_{211} = 1$ and others parameters equal zero. It corresponds to one control with one delay in the forms $[u(t-1); 0]$ for the first system and $[u(t); 0]$ for the second one. The average controllability for every $T > 1$ is easily verified due to condition (6.3). Introduce $B_1 = [0, 1; 0, 0]$, $B_2 = [1, 0; 0, 0]$, and $v(t) = [v_1(t); v_2(t)]$. The corresponding systems with ordinary controls have the form:

$$\dot{x}_1 = A_1x_1 + B_1v(t), \quad \dot{x}_2 = A_2x_2 + B_2v(t).$$

This system has the LTAC property with $v_1(t) \neq v_2(t)$. We cannot solve the equation $u(t-1) = v_2(t)$, $u(t) = v_1(t)$. It may be solved only if $v_1(t-1) = v_2(t)$. Let us pass to the approximation from Section 5. Let b=[1; 0], then our flock of systems is written as

$$\dot{x}_1(t) = A_1x_1(t) + bu(t-1), \quad \dot{x}_2(t) = A_2x_2(t) + bu(t),$$
$$y(t) = (x_1(t) + x_2(t))/2.$$

We need to approximate only the first system. Here $m = 1$ and $t_i = -i/N$, $i \in 0 : N$. As $B(s) = -b\chi_{(-\infty,-1]}(s)$, the matrix $B_N^N = b$ and $B_i^N = 0$, $i \in 1 : N-1$. Therefore, matrices (5.3) have the form

$$\mathbf{A}_1^N = \left[A_1, 0_{2\times(N-1)}, b; 0_{N\times 2}, (Q^N)^{-1}V\right],$$

$$V = \left[\begin{array}{ccccc} -1, & 0, & \dots & ,0, & 0; \\ 1, & -1, & \dots & ,0, & 0; \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0, & 0, & \dots & ,1, & -1 \end{array}\right],$$

$$\mathbf{b}^N = [0_{2\times 1}; N; \dots; 0; 0],$$

where $\mathbf{A}_1^N \in \mathbb{R}^{(2+N)\times(2+N)}$, $V \in \mathbb{R}^{N\times N}$, and $\mathbf{b}^N \in \mathbb{R}^{2+N}$. We compose the matrices $A = \mathrm{diag}\left[\left[\mathbf{A}_1^N\right], A_2\right]$, $B = \left[\left[\mathbf{b}^N\right]; b\right]$, and $C = [I_2, 0_{2\times N}, I_2]/2$. For the obtained system $\dot{x} = Ax + Bu$, we verify the property of C-LTOC. It does not hold for any N. The flock has not the property of LTAC for the case $b_{112} = b_{211} = 1$ and others equal zero.

It can be verified that the approximating system in Example 2 has the LTAC property.

7. Conclusion and open problems

In this paper, we considered the notion of output controllability for ordinary systems with retarded controls and gave the necessary and sufficient condition for that. For the notion of long-time output controllability, we obtained only sufficient conditions. This notions were applied for the investigation of averaged controllability of mentioned systems. The general approach for that is to approximate the systems by the ordinary ones. In connection with the results obtained, a number of interesting open questions arise.

- Assume that there exists a number $N_0 \in \mathbb{N}$ such that for every $N \geq N_0$ the approximating system has the C-LTOC property. Is it sufficient for C-LTOC property of the original system? And vice versa, if the original system has C-LTOC property, whether it is sufficient for C-LTOC of the approximating system?
- How to obtain any rank conditions for output controllability of systems with delays in the state and control? The same question about the C-LTOC property of such a systems.
- We considered the LTAC property for flocks with finite number of members. Can the results be extended for flocks with infinite members?
- Does output controllability imply output feedback stabilisation? Suppose that the system is output controllable, does it exist a feedback control $u(t) = Ky(t)$ such that y(t) goes to zero as t goes to ∞?

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ON ONE INEQUALITY OF DIFFERENT METRICS FOR TRIGONOMETRIC POLYNOMIALS

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Abstract: We study the sharp inequality between the uniform norm and $L^p(0,\pi/2)$-norm of polynomials in the system $\mathscr{C}=\{\cos(2k+1)x\}_{k=0}^{\infty}$ of cosines with odd harmonics. We investigate the limit behavior of the best constant in this inequality with respect to the order n of polynomials as $n\to\infty$ and provide a characterization of the extremal polynomial in the inequality for a fixed order of polynomials.

Keywords: Trigonometric cosine polynomial in odd harmonics, Nikol'skii different metrics inequality.

1. Problem statement, backgroung, and some preliminaries

1.1. Some notation

This paper considers classical spaces of complex-valued functions of one variable; in fact, we use not the spaces themselves but the norms of these spaces on some subspaces of polynomials. Let $I=[a,b]$ be an interval on the real axis, and let υ be a nonnegative, integrable function on I called a weight. For $0<p<\infty$, the space $L^p_\upsilon=L^p_\upsilon(I)$ consists of complex-valued, Lebesgue measurable on I functions f such that the function $\upsilon|f|^p$ is integrable on I. The functional

$$\|f\|_p=\|f\|_{L^p_\upsilon(I)}=\left(\frac{1}{b-a}\int_a^b|f(x)|^p\upsilon(x)\,dx\right)^{1/p},\quad f\in L^p_\upsilon, \tag{1.1}$$

is a norm in the space $L^p_\upsilon=L^p_\upsilon(I)$ for $1\le p<\infty$, but not for $0<p<1$. Nevertheless, for all $0<p<\infty$, we will refer to (1.1) as a norm or, more precisely, as a p-norm. The space $L^2_\upsilon=L^2_\upsilon(I)$ (here $p=2$) is a Hilbert space with the inner product

$$\langle f,g\rangle=\frac{1}{b-a}\int_a^b f(x)\overline{g(x)}\upsilon(x)\,dx,\quad f,g\in L^2_\upsilon.$$

In the case of unit weight $v(x) \equiv 1$, the weight symbol is omitted in the notation of spaces and their norms. By $L^\infty = L^\infty(I)$ we mean the space $C = C(I)$ of functions continuous (bounded) on the interval I with the uniform norm

$$\|f\|_\infty = \|f\|_{C(I)} = \max\big\{|f(x)|\colon x \in (I)\big\}.$$

The space $C = C(I)$ contains the subspace $C_0 = C(I)_0$ of functions f vanishing at the right end point of the interval: $f(b) = 0$. In what follows, the parameter a is equal to zero: $a = 0$, and b is 1, $\pi/2$, or π depending on the situation.

We define the spaces of 2π-periodic functions accordingly: the spaces $\mathcal{L}^p_{2\pi}$, $0 < p < \infty$, with p-norm

$$\|f\|_p = \|f\|_{\mathcal{L}^p_{2\pi}} = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^p dx\right)^{1/p}$$

and the space $C_{2\pi}$ of continuous 2π-periodic functions with the uniform norm

$$\|f\|_\infty = \|f\|_{C_{2\pi}} = \max\{|f(x)|\colon x \in \mathbb{R}\} = \max\{|f(x)|\colon x \in [-\pi, \pi]\}.$$

1.2. Nikol'skii C–L^p inequality: the classical case

Let $\mathcal{F}_n = \mathcal{F}_n(\mathbb{C}),\ n \geq 1$, be the set of trigonometric polynomials

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)$$

of order (at most) n with complex coefficients. In function theory and its applications, inequalities between two different norms of polynomials are of great importance. Such inequalities arose in Jackson's paper [19], but they were thoroughly investigated and applied by Nikol'skii [25]; [26, Ch. 3, Sect. 3.3], in this connection, they are called Nikol'skii inequalities or inequalities of different metrics. Large studies have been devoted to such inequalities, see [25]; [26, Ch. 3, Sect. 3.3]; [21, Ch. 3, Sects. 3.5–3.6]; [11, Ch. 8, Sect. 8.4]; [17, 24] and the bibliography therein. In this paper, the authors will need some information about the Nikol'skii inequalities

$$\|f_n\|_\infty \leq C(n)_p \|f_n\|_p, \quad f_n \in \mathcal{F}_n, \tag{1.2}$$

between the uniform norm and p-norm of polynomials (see, the same sources [21, Ch. 3, Sects. 3.5, 3.6]; [17, 24]). We assume that $C(n)_p$ is the best (the smallest possible) constant in this inequality. Employing harmonic analysis, it is easy to obtain (see, for example, [21, Ch. 3, Sect. 3.5, Theorem 3.5.1]) that if $p = 2$, then

$$C(n)_2 = \sqrt{2n+1} \tag{1.3}$$

and inequality (1.2) becomes an equality at the Dirichlet kernel

$$D_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \cos kx, \tag{1.4}$$

i.e., the Dirichlet kernel is an extremal polynomial. The exact values of $C(n)_p$ for $p \neq 2$ are unknown. There are constructive estimates for $C(n)_p$, $0 < p < \infty$, mostly upper ones; see [11, Ch. 8, Sect. 8.4]; [1]; [12, Sect. 7.2]; [15–18] and the bibliography therein. Note for the future the upper estimate of Badkov [12, Sect. 7.2, Theorem 7.2]

$$C(n)_p \leq 4n^{1/p}, \quad 0 < p < \infty. \tag{1.5}$$

It is not the best at the moment, but sufficient for us in what follows.

Much research has been devoted to inequality (1.2) for $p = 1$; information about the history and results related to this inequality can be found in [10, 15–17, 29, 30]. The following estimates are simple and quite rough:

$$n + 1 \le C(n)_1 \le 2n + 1. \tag{1.6}$$

The upper estimate follows from the representation of polynomials $f_n \in \mathcal{F}_n$ in the form of convolution

$$f_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f_n(t)D_n(x-t)dt, \quad f_n \in \mathcal{F}_n,$$

with the Dirichlet kernel (1.4). The Fejér kernel

$$F_n(t) = \sum_{k=0}^{n} D_k(t) =$$

$$= \frac{n+1}{2} + \sum_{k=1}^{n}(n+1-k)\cos kt = \frac{1}{2}\left(\frac{\sin(n+1)\frac{t}{2}}{\sin\frac{t}{2}}\right)^2, \quad t \ne 2\nu\pi, \quad \nu \in \mathbb{Z},$$

provides the former inequality in (1.6). This kernel is nonnegative (see, for example, [27, Vol. 2, Part 6, Sect. 3, Problem 18]) and such that

$$\|F_n\|_\infty = F_n(0) = \frac{(n+1)^2}{2}, \quad \|F_n\|_1 = \frac{1}{\pi}\int_0^\pi F_n(t)dt = \frac{n+1}{2}.$$

This implies the former inequality in (1.6).

Taikov in [29] gives the result of S.B. Stechkin that (for $p = 1$) there exists a constant $c > 0$ such that

$$C(n)_1 = c\,n + o(n), \quad n \to \infty. \tag{1.7}$$

Stechkin's proof of this result is given in [30]. Estimates (1.6) imply that $1 \le c \le 2$. Taikov [29] obtained substantially closer two-sided bounds for the constant c. Hörmander and Bernhardsson have obtained [14] the best estimates currently:

$$1.081857643 \le c \le 1.081857645.$$

Let $\mathcal{E}(\sigma)$ be the space of entire functions of exponential type (at most) $\sigma > 0$, and let $\mathcal{E}(\sigma)_p$ for $0 < p \le \infty$ be the space of functions $f \in \mathcal{E}(\sigma)$ belonging on the real axis to the spaces $L^p = L^p(\mathbb{R})$ with finite norms

$$\|f\|_p = \|f\|_{L^p(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(x)|^p dx\right)^{1/p}, \quad 0 < p < \infty,$$
$$\|f\|_\infty = \|f\|_{C(\mathbb{R})} = \sup\{|f(x)|\colon x \in \mathbb{R}\}, \quad p = \infty.$$

For any $0 < p < \infty$ on $\mathcal{E}(\sigma)_p$, we have the inequality

$$\|f\|_\infty \le A_p \sigma^{1/p}\|f\|_p, \quad f \in \mathcal{E}(\sigma)_p, \tag{1.8}$$

ascending to Nikol'skii [25]; [26, Ch. 3, Sect. 3.3], in which A_p is a finite constant depending only on the parameter p; see details in [15, 16]. In what follows, we assume that A_p is the least possible, i.e., best constant in (1.8). The exact value of this quantity is currently known only for $p = 2$; namely, $A_2 = 1/\sqrt{\pi}$; see, for example, [31, Ch. IV, Sect. 4.9, Subsect. 4.9.53, (28)].

Gorbachev [16] obtained a significantly more informative assertion in comparison with (1.7). Namely, he proved that the following limit relation is true:

$$\lim_{n\to\infty} \frac{C(n)_1}{2\pi n} = A_1. \tag{1.9}$$

Actuality, Gorbachev obtained [16] an even more precise result; namely, he proved the following two-sided inequality:

$$A_1 \le \frac{C(n)_1}{2\pi n} \le \frac{n+1}{n} A_1,$$

which entails, in particular, (1.9).

The following statement belongs to Levin and Lubinsky [22, Theorem 2.1, (2.1)]; it means that a relation similar to (1.9) holds for all $0 < p < \infty$.

Theorem A. *The following limit relation is valid for* $0 < p < \infty$:

$$\lim_{n\to\infty} \frac{C(n)_p}{(2\pi n)^{1/p}} = A_p. \tag{1.10}$$

This statement is part of a more general result of Ganzburg and Tikhonov [15, Theorem 1.5, (1.19)]. Nikol'skii inequalities and, more generally, Bernstein–Nikol'skii inequalities

$$\|f_n^{(r)}\|_q \le C^{(r)}(n)_{p,q}\|f_n\|_p, \quad f_n \in \mathcal{F}_n, \tag{1.11}$$

for trigonometric polynomials and similar inequalities for entire functions are a broad area of function theory. Ganzburg and Tikhonov, in the already cited paper [15], and Gorbachev and Mart'yanov in [18] studied the relationship between exact constants in Bernstein–Nikol'skii (C, L^p)-inequalities and, more generally, (L^q, L^p)-inequalities for polynomials and entire functions of exponential type. We presented here in Theorem A only some results on this topic, which we will need in what follows; for a rich overview of these studies, see [17].

Over the last century, extensive investigations have been carried out on sharp inequalities, i.e., the study of exact constants and extremal functions in inequalities (1.11) for trigonometric polynomials, as well as for algebraic polynomials and entire functions of exponential type; for specific results and further references, see [1, 2, 10, 12, 13, 15, 17, 21, 23, 24, 28, 31].

1.3. Nikol'skii inequality between the uniform norm and L^p-norm on the interval $[0, \pi/2]$ for polynomials in the cosine system with odd harmonics

1.3.1. Nikol'skii inequality for $\mathscr{C}_n$-polynomials

Let $\mathscr{C}_n = \mathscr{C}_n(\mathbb{C})$, $n \ge 0$, be the set of polynomials

$$\phi_n(x) = \sum_{k=0}^{n} a_k \cos(2k+1)x \tag{1.12}$$

with complex coefficients in the cosine system with odd harmonics

$$\mathscr{C} = \{\cos(2k+1)x\}_{k=0}^{\infty}. \tag{1.13}$$

The functions (1.12) will be called $\mathscr{C}_n$-polynomials or $\mathscr{C}$-polynomials of order n. The functions (1.12) are trigonometric polynomials; as trigonometric polynomials they have order $2n+1$. Note that the functions (1.12) vanish at the point $x = \pi/2$: $\varphi_n(\pi/2) = 0$, so none of them is the identical unity.

The main goal of this paper is to study the sharp inequality

$$\|\phi_n\|_\infty \le M(n)_p\|\phi_n\|_p, \quad \phi_n \in \mathscr{C}_n, \tag{1.14}$$

between the uniform norm

$$\|\phi_n\|_\infty = \|\phi_n\|_{C[0,\pi/2]} = \max\left\{|\phi_n(x)|\colon t \in [0,\pi/2]\right\},$$

and integral p-norm ($0 < p < \infty$)

$$\|\phi_n\|_p = \|\phi_n\|_{L^p(0,\pi/2)} = \left(\frac{2}{\pi}\int_0^{\pi/2} |\phi_n(x)|^p dx\right)^{1/p}$$

of $\mathscr{C}_n$-polynomials on the interval $[0,\pi/2]$. Inequality (1.14) appeared in the authors' paper [8] in connection with the study of a variant of the generalized translation in system (1.13) on an interval.

In connection with inequality (1.14), the question naturally arises of the sharp pointwise inequality

$$|\varphi_n(t)| \le M(n,t)_p \|\varphi_n\|_{L^p(0,\pi/2)}, \quad \varphi_n \in \mathscr{C}_n, \tag{1.15}$$

for points $t \in [0,\pi/2]$. Especially important, as will be seen from what follows, is the inequality (1.15) at the end point $t = 0$:

$$|\varphi_n(0)| \le M(n,0)_p \|\varphi_n\|_{L^p(0,\pi/2)}, \quad \varphi_n \in \mathscr{C}_n. \tag{1.16}$$

The study of inequalities (1.14), (1.15), and (1.16) includes, in particular, the study of the properties of extremal polynomials at which the inequalities turn into equalities. It is clear that if the polynomial ϕ_n^* is extremal in one of these inequalities, then, for any constant c, the polynomial $c\,\phi_n^*$ is also extremal. If any extremal polynomial of this inequality has the form $c\,\phi_n^*$ with some constant c, then ϕ_n^* is said to be the unique extremal polynomial.

The following statement is proved in the authors' paper [8, Theorem 4].

Theorem B. *For $1 \le p < \infty$ and $n \ge 0$, the following statements hold.*

(1) *The best constants in inequalities (1.14) and (1.16) coincide:*

$$M(n)_p = M(n,0)_p. \tag{1.17}$$

(2) *The polynomial φ_n^* extremal in inequality (1.16) attains its uniform norm at the point 0 and is also extremal in inequality (1.14).*

The authors do not know whether equality (1.17) holds for $0 < p < 1$; in this case, we can only state that $M(n,0)_p \le M(n)_p$.

1.3.2. Approximation interpretation of inequalities

The problems of studying inequalities (1.14), (1.15), and (1.16) can be reformulated as approximation problems; we will do this only for inequality (1.16). Consider the set

$$\mathscr{C}_n[0] = \{\phi_n \in \mathscr{C}_n \colon \phi_n(0) = 1\} \tag{1.18}$$

of polynomials with fixed value at the point 0: $\phi_n(0) = 1$. On this set, we define the value

$$E_n[0]_p = \inf\left\{\|\phi_n\|_{L^p(0,\pi/2)} \colon \phi_n \in \mathscr{C}_n[0]\right\} \tag{1.19}$$

of the least deviation from zero of the class of polynomials (1.18) in the space $L^p(0,\pi/2)$. It is clear that

$$E_n[0]_p = 1/M(n,0)_p.$$

Moreover, extremal polynomials in problem (1.19) and inequality (1.16) coincide. More precisely, (every) extremal polynomial in (1.19) is also extremal in (1.16); conversely, if φ is an extremal polynomial of inequality (1.16), then the polynomial $\varphi/\varphi(0)$ is extremal in (1.19). Thus, the problem of sharp inequality (1.16) is equivalent to problem (1.19) on the least deviation from zero of the class (1.18).

1.3.3. Christoffel–Darboux kernel for system (1.13)

Sometimes, we will use the following shorter notation for the functions of system (1.13):

$$\eta_k(x) = \cos(2k+1)x, \quad x \in [0, \pi/2]. \tag{1.20}$$

This system of functions is orthogonal with respect to the inner product

$$\langle f, g \rangle = \frac{2}{\pi} \int_0^{\pi/2} f(t)\overline{g(t)}\, dt.$$

More precisely, as is easy to see, for $k, m \geq 1$, the inner products

$$\delta_{k,m} = \langle \eta_k, \eta_m \rangle = \frac{2}{\pi} \int_0^{\pi/2} \cos(2k+1)t\, \cos(2m+1)t\, dt$$

have the following values: $\delta_{k,m} = 0$, $k \neq m$, and $\delta_{k,k} = 1/2$.

Due to the orthogonality of the system $\{\eta_k\}_{k\geq 1}$, the coefficients of the polynomial

$$\phi(x) = \sum_{k=0}^{n} a_k \cos(2k+1)x \tag{1.21}$$

are expressed in terms of the polynomial itself by the formulas $a_k = 2\langle \phi, \eta_k \rangle$. Substituting these expressions into (1.21), we obtain

$$\phi(x) = \sum_{k=0}^{n} a_k \eta_k(x) = 2\sum_{k=0}^{n} \langle \phi, \eta_k \rangle \eta_k(x) = \Big\langle \phi, 2\sum_{k=0}^{n} \eta_k(x)\eta_k \Big\rangle,$$

which can be written in the form

$$\phi(x) = \frac{2}{\pi} \int_0^{\pi/2} \phi(t)\mathcal{K}_n(x,t) dt,$$

where

$$\mathcal{K}_n(x,t) = 2\sum_{k=0}^{n} \eta_k(x)\eta_k(t) \tag{1.22}$$

is the Christoffel–Darboux kernel for the system $\mathscr{C}_n$. Convolute this kernel. Using the formula

$$2\cos a\, \cos b = \cos(a+b) + \cos(a-b), \tag{1.23}$$

we find

$$\mathcal{K}_n(x,t) = 2\sum_{k=0}^{n} \eta_k(x)\eta_k(t) = 2\sum_{k=0}^{n} \cos((2k+1)x) \cos((2k+1)t) =$$
$$= \sum_{k=0}^{n} \left(\cos((2k+1)(x+t)) + \cos((2k+1)(x-t))\right).$$

Using the formula (see, for example, [27, Part 6, Sect. 3, Problem 16])

$$\sum_{k=0}^{n} \cos(2k+1)\theta = \frac{\sin 2(n+1)\theta}{2\sin\theta}, \quad \theta \neq \nu\pi, \quad \nu \in \mathbb{Z},$$

we obtain the following representation for the Christoffel–Darboux kernel:

$$\mathcal{K}_n(x,t) = \frac{1}{2}\left(\frac{\sin 2(n+1)(x+t)}{\sin(x+t)} + \frac{\sin 2(n+1)(x-t)}{\sin(x-t)}\right).$$

The orthogonality of the system of functions (1.20) implies that, for a pair of polynomials

$$\phi(x) = \sum_{k=0}^{n} a_k \cos(2k+1)x, \quad \psi(x) = \sum_{k=0}^{n} b_k \cos(2k+1)x,$$

a generalized version of Parceval's identity holds:

$$2\langle\phi, g\rangle = \sum_{k=1}^{n} a_k \overline{b}_k.$$

In particular, the norm of the polynomial $\phi \in \mathscr{C}_n$ is expressed in terms of its Fourier coefficients $\{a_k\}$ by Parceval's identity

$$2\|\phi\|^2_{L_2(0,\pi/2)} = \sum_{k=0}^{n} |a_k|^2.$$

Using this equality and Hölder's inequality, we obtain the inequality

$$|\phi(0)| = \left|\sum_{k=0}^{n} a_k\right| \leq \sqrt{n+1}\left(\sum_{k=0}^{n} |a_k|^2\right)^{1/2} = \sqrt{2(n+1)}\,\|\phi\|_{L_2(0,\pi/2)},$$

which at the kernel (1.22) turns into an equality. Thus, in the space $L_2(0,\pi/2)$, we have

$$M(n)_2 = \sqrt{2(n+1)}, \quad n \geq 0. \tag{1.24}$$

It is useful to compare this result with the corresponding result (1.2)–(1.3) for the classical case.

1.4. Main results

The authors consider the following statements to be the main ones in this paper.

1.4.1. Limit behavior of the best constants in inequalities (1.14) and (1.16)

For the best constant in inequality (1.14), we have an analog of the above Theorem A.

Theorem 1. *The following limit relation holds for constants $M(n)_p$ in inequality (1.14) for $0 < p < \infty$:*

$$\lim_{n\to\infty} \frac{M(n)_p}{(2\pi n)^{1/p}} = A_p. \tag{1.25}$$

1.4.2. Characterization of a polynomial extremal in inequality (1.16)

Denote by $\varphi_n^* = \varphi_{n,p}^* \in \mathscr{C}_n$ a polynomial of order $n \geq 1$ with unit leading coefficient that deviates least from zero in the space $L_w^p(0, \pi/2)$ with the weight

$$w(x) = \sin^2 x \tag{1.26}$$

on the interval $(0, \pi/2)$. In other words, φ_n^* is a solution to the problem

$$\min\{\|\phi_n\|_{L_w^p(0,\pi/2)} \colon \phi_n \in \mathscr{C}_n^1\} = \|\varphi_n^*\|_{L_w^p(0,\pi/2)}$$

on the set $\mathscr{C}_n^1$ of polynomials (1.12) of order n with leading coefficient 1: $a_n = 1$.

Theorem 2. *For all $1 \leq p < \infty$ and $n \geq 1$, the polynomial φ_n^* of order n with unit leading coefficient that deviates least from zero in the space $L_w^p(0, \pi/2)$ with weight (1.26) is the unique extremal polynomial in inequality (1.16).*

There are statements similar to Theorem 2 in [5, Theorem 1; 6, Theorem 2; 7, Theorem 2; 9, Theorem 3; 4, Theorem 2; 3].

We will also give some estimates for the best constant $M(n)_p$ in inequality (1.14); see, in particular, Section 2.4.

2. Behavior with respect to n of the best constant in the Nikol'skii inequality for $\mathscr{C}$-polynomials

2.1. Case $p = 2$

According to (1.3) and (1.24), for $n \geq 0$, we have

$$C(n)_2 = \sqrt{2n+1},$$
$$M(n)_2 = \sqrt{2(n+1)}.$$

Thus,

$$\lim_{n\to\infty} \frac{M(n)_2}{n^{1/2}} = \lim_{n\to\infty} \frac{C(n)_2}{n^{1/2}} = \sqrt{2}.$$

Both limits exist and coincide; this fact served as an argument for the authors that this property of the quantities $M(n)_p$ will hold for all $0 < p < \infty$.

2.2. Expression of $\mathscr{C}$-polynomials in terms of the classical trigonometric polynomials

Let $\mathcal{C}_n$ be the set of even (complex) trigonometric polynomials

$$f_n(x) = \sum_{k=0}^{n} a_k \cos kx$$

of order (at most) $n \geq 1$. On this set, we consider the uniform norm

$$\|f_n\|_\infty = \|f_n\|_{C[0,\pi]} = \max\left\{|f_n(x)| \colon x \in [0, \pi]\right\}$$

and the integral p-norm (for $0 < p < \infty$)

$$\|f_n\|_p = \|f_n\|_{L^p(0,\pi)} = \left(\frac{1}{\pi}\int_0^\pi |f_n(x)|^p dx\right)^{1/p}.$$

It is well known (however, it is easy to show) that the best constants $C(n)_p$ in inequality (1.2) and the inequality

$$|f_n(0)| \le C(n)_p \|f_n\|_p, \quad f_n \in \mathcal{C}_n, \tag{2.1}$$

coincide.

Lemma 1. *For $f_n \in \mathcal{C}_n$, the function*

$$\phi_n(x) = f_n(2x)\cos x \tag{2.2}$$

is a $\mathscr{C}_n$-polynomial. Conversely, every $\mathscr{C}_n$-polynomial ϕ_n can be represented in the form (2.2), where $f_n \in \mathcal{C}_n$. Thus, formula (2.2) establishes one-to-one correspondence between $\mathcal{C}_n$ and $\mathscr{C}_n$.

P r o o f. Let $f_n \in \mathcal{C}_n$ be a trigonometric polynomial. Let

$$\phi(x) = f_n(2x)\cos x = \sum_{k=0}^{n} a_k \cos 2kx \cos x.$$

Applying again formula (1.23), we have

$$2\cos 2kx \cos x = \cos(2k+1)x + \cos(2k-1)x.$$

Using this relation, we find

$$2\phi(x) = 2f_n(2x)\cos x = 2\sum_{k=0}^{n} a_k \cos 2kx \cos x =$$

$$= \sum_{k=0}^{n} a_k \cos(2k+1)x + \sum_{k=0}^{n} a_k \cos(2k-1)x = \sum_{k=0}^{n} a_k \cos(2k+1)x + \sum_{k=-1}^{n-1} a_{k+1}\cos(2k+1)x.$$

As a result, we obtain the representation

$$2\phi(x) = \sum_{k=-1}^{n} (a_k + a_{k+1})\cos(2k+1)x = (2a_0 + a_1)\cos x + \sum_{k=1}^{n}(a_k + a_{k+1})\cos(2k+1)x, \tag{2.3}$$

where $a_{-1} = 0$ and $a_{n+1} = 0$. The function (2.3) is a $\mathscr{C}_n$-polynomial.

Let us prove the inverse statement, i.e., let us prove that an arbitrary $\mathscr{C}_n$-polynomial

$$\phi(x) = \sum_{k=0}^{n} \lambda_k \cos(2k+1)x$$

can be represented in the form (2.2). Rewrite this polynomial in the form

$$\phi(x) = \sum_{k=-1}^{n} \lambda'_k \cos(2k+1)x, \tag{2.4}$$

where $\lambda'_{-1} = \lambda'_0 = \lambda_0/2$ and $\lambda'_k = \lambda_k$, $1 \le k \le n$.

It suffices to represent the polynomial 2ϕ in the form (2.3). The latter means that the coefficients $\{\lambda'_k\}_{k=-1}^n$ of the polynomial (2.4) can be represented as

$$\lambda'_k = a_k + a_{k+1}, \quad -1 \le k \le n; \quad a_{-1} = 0, \quad a_{n+1} = 0. \tag{2.5}$$

It is easy to see that the formulas

$$a_k = \sum_{\ell=0}^{n-k} (-1)^\ell \lambda'_{k+\ell} = \lambda'_k - a_{k+1}, \quad k = n, n-1, \dots, 0,$$

give a solution to system (2.5). Lemma 1 is proved. □

Representation (2.2) implies that inequality (1.16) is equivalent to the inequality

$$|f_n(0)| \le C(n,\sigma)_p \|f_n\|_{L^p_\sigma(0,\pi)}, \quad f_n \in \mathcal{C}_n, \tag{2.6}$$

on the set $\mathcal{C}_n$ with weight $\sigma(t) = \cos^p(t/2)$. More exactly, the following assertion holds.

Lemma 2. *For $0 < p < \infty$ and $n \ge 1$, inequality (1.16) on the set $\mathscr{C}_n$ and inequality (2.6) on $\mathcal{C}_n$ are equivalent; specifically:*

(1) *the best constants in inequalities (2.6) and (1.16) are related by the equality*

$$C(n,\sigma)_p = M(n,0)_p; \tag{2.7}$$

(2) *extremal polynomials in these inequalities are related by (2.2).*

P r o o f. Using relation (2.2), we find that, for an arbitrary polynomial $\phi_n \in \mathscr{C}_n$,

$$|f_n(0)| = |\phi_n(0)| \le M(n,0)_p \|\phi_n\|_{L^p(0,\pi/2)},$$

where

$$\|\phi_n\|_{L^p(0,\pi/2)} = \left(\frac{2}{\pi}\int_0^{\pi/2} |f_n(2x)\cos x|^p dx\right)^{1/p} = \left(\frac{1}{\pi}\int_0^{\pi} |f_n(t)\cos(t/2)|^p dt\right)^{1/p} = \|f_n\|_{L^p_\sigma(0,\pi)}.$$

Lemma 2 is proved. □

The following statement contains a quantitative relation between the constants in inequalities (1.14), (1.16), and (1.2).

Lemma 3. *For $0 < p < \infty$ and $n \ge 1$, the best constants in inequalities (1.14), (1.16), and (2.1) (or, equivalently, (1.2)) are related as follows:*

$$C(n)_p \le M(n,0)_p, \tag{2.8}$$

$$M(n)_p \le C(n+1)_p. \tag{2.9}$$

P r o o f. For polynomials $f_n \in \mathcal{C}_n$, we have $\|f_n\|_{L^p_\sigma(0,\pi)} \le \|f_n\|_{L^p(0,\pi)}$. Therefore, the best constants in (2.1) and (2.6) are related by the inequality $C(n)_p \le C(n,\sigma)_p$. This and (2.7) imply (2.8).

Let us prove inequality (2.9). A polynomial $\phi_n \in \mathscr{C}_n$ has the form

$$\phi_n(x) = \sum_{k=0}^n a_k \cos(2k+1)x.$$

Writing it in exponential form, we find

$$2\phi_n(x) = \sum_{k=0}^{n} a_k \left(e^{i(2k+1)x} + e^{-i(2k+1)x}\right) = \sum_{k=0}^{n} a_k e^{i(2k+1)x} + \sum_{k=0}^{n} a_k e^{-i(2k+1)x} =$$

$$= e^{-ix} \sum_{k=0}^{n} a_k e^{i2(k+1)x} + e^{-ix} \sum_{k=0}^{n} a_k e^{-i2kx} = e^{-ix} \left(\sum_{k=1}^{n+1} a_{k-1} e^{i2kx} + \sum_{k=0}^{n-1} a_{k+1} e^{-i2kx} \right).$$

The function

$$g_{n+1}(x) = \sum_{k=1}^{n+1} a_{k-1} e^{ikx} + \sum_{k=0}^{n-1} a_{k+1} e^{-ikx}$$

is a trigonometric polynomial of order $n+1$. The functions $2\phi_n$ and g_{n+1} are related as $2\phi_n(x) = e^{-ix} g_{n+1}(2x)$, $x \in \mathbb{R}$, or, equivalently, as

$$2\phi_n(x/2) = e^{-ix/2} g_{n+1}(x), \quad x \in \mathbb{R}. \tag{2.10}$$

The following inequality holds for the polynomial g_{n+1} (cf. (1.2)):

$$\|g_{n+1}\|_{C_{2\pi}} \le C(n+1)_p \|g_{n+1}\|_{\mathcal{L}^p_{2\pi}}. \tag{2.11}$$

As a consequence of (2.10), we have $\|g_{n+1}\|_{C_{2\pi}} = 2\|\phi_n\|_{C[0,\pi/2]}$ and

$$\|g_{n+1}\|_{\mathcal{L}^p_{2\pi}} = \left(\frac{1}{2\pi} \int_0^{2\pi} |g_{n+1}(x)|^p dx \right)^{1/p} = [x/2 = t] = 2 \left(\frac{1}{\pi} \int_0^{\pi} |\phi_n(t)|^p dt \right)^{1/p} =$$

$$= 2 \left(\frac{2}{\pi} \int_0^{\pi/2} |\phi_n(t)|^p dt \right)^{1/p} = 2\|\phi_n\|_{L^p(0,\pi/2)}.$$

Consequently, inequality (2.11) is equivalent to the inequality

$$\|\phi_n\|_{C[0,\pi/2]} \le C(n+1)_p \|\phi_n\|_{L^p(0,\pi/2)}.$$

Comparing this inequality with (1.14), we conclude that inequality (2.9) holds. Lemma 3 is proved completely. □

2.3. Proof of Theorem 1

By Lemma 3, for all $0 < p < \infty$ and $n \ge 1$, we have the inequalities

$$C(n)_p \le M(n)_p \le C(n+1)_p.$$

As a consequence, we have

$$\frac{C(n)_p}{(2\pi n)^{1/p}} \le \frac{M(n)_p}{(2\pi n)^{1/p}} \le \frac{C(n+1)_p}{(2\pi(n+1))^{1/p}} \left(\frac{n}{n+1} \right)^{1/p}.$$

Passing here to the limit as $n \to \infty$ and using the result (1.10) of Theorem A, we obtain (1.25). Theorem 1 is proved. □

2.4. Estimates

2.4.1. Monotonicity of a constant in (1.14) in p

Let $0 < p_1 < p_2 < \infty$. For an arbitrary $\phi_n \in \mathscr{C}_n$, we have

$$\|\phi\|_{p_2} = \left(\frac{2}{\pi}\int_0^{\pi/2} |\phi(t)|^{p_2} dt\right)^{1/p_2} = \left(\frac{2}{\pi}\int_0^{\pi/2} |\phi(t)|^{p_1} |\phi(t)|^{p_2-p_1} dt\right)^{1/p_2} \le$$
$$\le \left(\frac{2}{\pi}\int_0^{\pi/2} |\phi(t)|^{p_1} dt\right)^{1/p_2} \left(\|\phi\|_{C[0,\pi/2]}\right)^{(p_2-p_1)/p_2} = (\|\phi\|_{p_1})^{p_1/p_2} \left(\|\phi\|_{C[0,\pi/2]}\right)^{(p_2-p_1)/p_2}.$$

Consider inequality (1.14) with the parameter p_2:

$$\|\phi\|_{C[0,\pi/2]} \le M(n)_{p_2} \|\phi\|_{p_2} \le M(n)_{p_2} (\|\phi\|_{p_1})^{p_1/p_2} \left(\|\phi\|_{C[0,\pi/2]}\right)^{(p_2-p_1)/p_2}.$$

Dividing by the latter factor, we obtain the inequality

$$\left(\|\phi\|_{C[0,\pi/2]}\right)^{p_1/p_2} \le M(n)_{p_2} (\|\phi\|_{p_1})^{p_1/p_2}.$$

Raise it to the power p_2/p_1:

$$\|\phi\|_{C[0,\pi/2]} \le (M(n)_{p_2})^{p_2/p_1} \|\phi\|_{p_1}.$$

This inequality holds for any polynomial $\phi \in \mathscr{C}_n$; hence,

$$M(n)_{p_1} \le (M(n)_{p_2})^{p_2/p_1},$$

or

$$(M(n)_{p_1})^{p_1} \le (M(n)_{p_2})^{p_2}, \quad 0 < p_1 < p_2 < \infty; \tag{2.12}$$

this is the required property of monotonicity.

2.4.2. Estimates for a constant in (1.14)

As a special case of (2.12), we have the inequality

$$M(n)_p \ge (M(n)_1)^{1/p}, \quad 1 < p < \infty.$$

Using now inequalities (2.8) and (1.6), we obtain the lower estimate

$$M(n)_p \ge (n+1)^{1/p}, \quad 1 \le p < \infty.$$

Using (2.12) and (1.24), we obtain the estimates

$$M(n)_p \le (2(n+1))^{1/p}, \quad 0 < p \le 2,$$
$$M(n)_p \ge (2(n+1))^{1/p}, \quad p \ge 2.$$

Finally, inequalities (2.9) and (1.5) imply the estimate

$$M(n)_p \le 4(n+1)^{1/p}, \quad 0 < p < \infty.$$

3. Characterization of an extremal $\mathscr{C}$-polynomial in the Nikol'skii inequality

The primary purpose of this section is to prove Theorem 2, which characterizes the extremal $\mathscr{C}_n$-polynomial in the Nikol'skii inequality (1.16). As mentioned above, statements similar to Theorem 2 can be found in the authors' papers, personal and with co-authors: [5–7, 9]. The ideas contained in these statements have been summarized in [3]. The fact that we need [3] holds for inequalities similar to inequality (1.16) for algebraic polynomials in spaces L^p with weight. Because of this, we first rewrite inequality (1.16) in terms of algebraic polynomials, use Theorem 4 from [3], and then make a conclusion related to inequality (1.16).

3.1. Equivalent to (1.16) inequality for algebraic polynomials on an interval

We can associate inequalities (1.14) and (1.16) on the set $\mathscr{C}_n$ with equivalent inequalities on the set of algebraic polynomials. For this, we describe $\mathscr{C}$-polynomials in terms of algebraic polynomials. Denote by $\mathcal{P}_n$ the set of algebraic polynomials

$$\rho_n(t) = \sum_{\ell=0}^{n} c_\ell t^\ell \tag{3.1}$$

of degree (at most) n with complex coefficients. In further consideration, we will need some properties of the Chebyshev polynomials

$$T_\nu(t) = \cos(\nu x), \quad x = \arccos t, \quad t \in [-1, 1], \tag{3.2}$$

of the first kind of degree $\nu \ge 0$; these properties can be found, for example, in [21, Ch. 2, Sect. 2.2]). A polynomial T_ν, $\nu \ge 1$, can be represented as

$$T_\nu(t) = \frac{\nu}{2} \sum_{k=0}^{[\nu/2]} \frac{(-1)^k}{\nu - k} C_{\nu-k}^k (2t)^{\nu-k} = 2^{\nu-1} t^\nu - \nu 2^{\nu-3} t^{\nu-2} + \cdots . \tag{3.3}$$

A polynomial T_ν is even or odd in accordance with the evenness of the number ν, and its leading coefficient (for $\nu \ge 1$) is $2^{\nu-1}$.

Lemma 4. *For $n \ge 1$, the relation*

$$\phi_n(x) = t\rho_n(t^2), \quad t = \cos x, \quad x \in [0, \pi], \quad t \in [-1, 1], \tag{3.4}$$

establishes a bijection between the set of polynomials $\phi_n \in \mathscr{C}_n$ of order n and the set $\mathcal{P}_n$ of algebraic polynomials ρ_n of degree n. Under this correspondence, the leading coefficient $a_n(\phi_n)$ of a $\mathscr{C}$-polynomial ϕ_n and the leading coefficient $c_n(\rho_n)$ of the polynomial ρ_n are related by the formula

$$c_n(\rho_n) = 2^{2n} a_n(\phi_n); \tag{3.5}$$

in addition, the following equalities hold:

$$\phi_n(0) = \rho_n(1), \tag{3.6}$$

$$\|\phi_n\|_{L^p(0,\pi/2)} = \|\rho_n\|_{L^p_\upsilon(0,1)}, \quad \upsilon(u) = \upsilon(u)_p = \frac{u^{(p-1)/2}}{\pi\sqrt{1-u}}. \tag{3.7}$$

P r o o f. First, let us show that an arbitrary $\mathscr{C}$-polynomial

$$\phi_n(x) = \sum_{k=0}^{n} a_k \cos(2k+1)x \tag{3.8}$$

of order $n \geq 0$ can be presented in the form (3.4), where ρ_n is an algebraic polynomial (3.1) of degree n. Replace $\cos(2k+1)x$ with $T_{2k+1}(t)$, $t = \cos x$, $t \in [-1,1]$, in (3.8). A polynomial T_{2k+1} is odd, and consequently, can be represented in the form $T_{2k+1}(t) = tR_k(t^2)$, where R_k is a real algebraic polynomial of degree k. The leading coefficient of the polynomial R_k is 2^{2k} and $R_k(1) = 1$. Therefore, polynomial (3.8) can be represented as (3.4), where ρ_n is some algebraic polynomial (3.1) of degree n; the leading coefficients of the polynomials ϕ_n and ρ_n satisfy the relation (3.5).

Conversely, let $\rho_n \in \mathcal{P}_n$. Consider the function $f(x) = \cos x\, \rho_n(\cos^2 x)$. Based on the representation (3.1), we have

$$f(x) = \cos x\, \rho_n(\cos^2 x) = \sum_{\ell=0}^{n} c_\ell \cos^{2\ell+1} x. \tag{3.9}$$

The function $\cos^{2\ell+1} x$ is an even trigonometric polynomial of order $2\ell+1$:

$$\cos^{2\ell+1} x = \sum_{k=0}^{\ell} \rho_{\ell,k} \cos(2k+1)x. \tag{3.10}$$

This fact is, of course, known. However, it is easy to obtain (by induction on ℓ) starting from the representations (3.2) and (3.3). In particular, it follows from (3.3) that the leading coefficient of the representation (3.10) is $\rho_{\ell,\ell} = 2^{-2\ell}$. Thus, function (3.9) is a $\mathscr{C}$-polynomial of the form (3.8).

Relation (3.4) establishes a bijection between $\mathscr{C}_n$ and $\mathcal{P}_n$. Formula (3.6) is obvious. Let us verify (3.7):

$$\|\phi_n\|_{L^p(0,\pi/2)} = \left(\frac{2}{\pi}\int_0^{\pi/2} |\phi_n(x)|^p dx\right)^{1/p} = \left(\frac{2}{\pi}\int_0^{\pi/2} |\cos x \rho_n(\cos^2 x)|^p dx\right)^{1/p} = [\cos x = t] =$$
$$= \left(\frac{2}{\pi}\int_0^{1} |t\rho_n(t^2)|^p \frac{dt}{\sqrt{1-t^2}}\right)^{1/p} = \left[t^2 = u\right] = \left(\frac{1}{\pi}\int_0^{1} |\rho_n(u)|^p \frac{u^{(p-1)/2}du}{\sqrt{1-u}}\right)^{1/p} =$$
$$= \left(\int_0^1 |\rho_n(u)|^p \upsilon(u)du\right)^{1/p}, \quad \upsilon(u) = \upsilon(u)_p = \frac{u^{(p-1)/2}}{\pi\sqrt{1-u}}.$$

Thus, we obtain equality (3.7). Lemma 4 is proved completely. □

Corollary 1. *The representation* (3.4) *implies that for* $n \geq 1$ *the polynomial* $\phi \in \mathscr{C}_n$, $\phi \not\equiv 0$, *can have on* $[0,\pi/2)$ *at most* n *zeros, taking into account their multiplicities, i.e.,* $\mathscr{C}_n$ *is the Chebyshev system on* $[0,\pi/2)$.

Corollary 2. *Lemma 4 implies that inequalities* (1.14) *and* (1.16) *can be associated with equivalent sharp inequalities on the set of algebraic polynomials*:

$$\|\rho_n\|_\infty \leq \mathcal{M}(n)_p \|\rho_n\|_{L^p_\upsilon(0,1)}, \quad \rho_n \in \mathcal{P}_n, \tag{3.11}$$

with constant $\mathcal{M}(n)_p = M(n)_p$ *and*

$$|\rho_n(1)| \leq \mathcal{M}(n,1)_p \|\rho_n\|_{L^p_\upsilon(0,1)}, \quad \rho_n \in \mathcal{P}_n, \tag{3.12}$$

with constant $\mathcal{M}(n,1)_p = M(n,0)_p$. *Moreover, a polynomial* $\varrho_n \in \mathcal{P}_n$ *is extremal in inequalities* (3.11) *and* (3.12) *if and only if the polynomial* $\varphi_n \in \mathscr{C}_n$ *related to* ϱ_n *by formula* (3.4) *is extremal in inequalities* (1.14) *and* (1.16).

Let us reformulate the problem of studying inequality (3.12) as an approximation problem. Consider the set

$$\mathcal{P}_n[1] = \{\rho_n \in \mathcal{P}_n \colon \rho_n(1) = 1\} \tag{3.13}$$

of polynomials with a fixed unit value at unity: $\rho_n(1) = 1$. On this set, we define the value

$$e_n[1]_p = \inf\{\|\rho_n\|_{L^p_\upsilon(0,1)} \colon \rho_n \in \mathcal{P}_n[1]\} \tag{3.14}$$

of the least deviation from zero in the space $L^p_\upsilon(0,1)$ of the class of polynomials (3.13). It is clear that $e_n[1]_p = 1/\mathcal{M}(n,1)_p$. Moreover, polynomials extremal in problem (3.14) and in inequality (3.12) coincide (up to a multiplicative constant). Thus, the problem on the sharp inequality (3.12) is equivalent to problem (3.14) on the least deviation from zero of the class (3.13).

Lemma 5. *For $1 \le p < \infty$ and $n \ge 1$, the following statements hold for a polynomial $\varrho_n \in \mathcal{P}_n$ extremal in inequality (3.12) and such that $\varrho_n(1) = 1$.*

(1) *The polynomial ϱ_n is characterized by the property*

$$\int_0^1 \rho_{n-1}(u)(1-u)\upsilon(u)|\varrho_n(u)|^{p-1}\operatorname{sign}\varrho_n(u)du = 0, \quad \rho_{n-1} \in \mathcal{P}_{n-1}. \tag{3.15}$$

(2) *The polynomial ϱ_n has degree n, all its n roots are real, simple, and lie on the interval $(0,1)$; in this sense, the polynomial ϱ_n is real.*

(3) *The polynomial ϱ_n is unique.*

P r o o f. Most of the results of this lemma are contained in [3]. We find it difficult in some cases to make precise references to this paper. Therefore, we have to repeat some arguments from [3] with clarifications and explanations.

In Theorem 4 of [3], in particular, the following properties of a polynomial ϱ_n extremal in the inequality (3.12) were proved.

(1′) The polynomial ϱ_n is real; more exactly, it has real coefficients and, hence, takes real values on the real axis.

(2′) The polynomial ϱ_n is characterized by the property (3.15).

It follows that the polynomial ϱ_n has n sign changes on the interval $(0,1)$. Otherwise, the polynomial with simple zeros at the sign change points of the polynomial ϱ_n would have order at most $n-1$ and the property (3.15) would not hold on it.

Finally, let us verify that an extremal polynomial ϱ_n is unique for all $1 \le p < \infty$. In fact, let ϱ_n and ζ_n be two polynomials that solve problem (3.14). By the inequality $\|\varrho_n + \zeta_n\|_{L^p_\upsilon} \le \|\varrho_n\|_{L^p_\upsilon} + \|\zeta_n\|_{L^p_\upsilon}$, their half-sum $(\varrho_n + \zeta_n)/2$ has the same property; hence, we have the equality $\|\varrho_n + \zeta_n\|_{L^p_\upsilon} = \|\varrho_n\|_{L^p_\upsilon} + \|\zeta_n\|_{L^p_\upsilon}$. For $1 < p < \infty$, since the space $L^p_\upsilon(0,1)$ is strictly normalized, it immediately follows that $\zeta_n = \varrho_n$. If $p = 1$, then we can only assert so far that the signs of the polynomials ζ_n and ϱ_n coincide almost everywhere on $[0,1]$. The zeros of these polynomials are simple and lie on the interval $(0,1)$, so the polynomials ζ_n and ϱ_n have the same set of zeros and the same value at the point $u = 1$: $\zeta_n(1) = \varrho_n(1) = 1$, hence, these polynomials coincide. Thus, the extremal polynomial is unique for $p = 1$ too.

Lemma 5 is proved completely. □

Based on the weight υ defined in (3.7), let us define the following weight on the interval $(0,1)$:

$$\varpi(x) = (1-u)\,\upsilon(u) = \frac{u^{(p-1)/2}\sqrt{1-u}}{\pi}. \tag{3.16}$$

Consider the problem on the least deviation from zero

$$u(\mathcal{P}_n^1)_{L^p_\varpi(0,1)} = \min\{\|\rho_n\|_{L^p_\varpi(0,1)} \colon \rho_n \in \mathcal{P}_n^1\} \tag{3.17}$$

in the space $L^p_\varpi(0,1)$ of the set $\mathcal{P}_n^1$ of algebraic polynomials of degree n whose leading coefficient is 1. Denote by $\varrho_n^* = \varrho_{n,\varpi,p}^*$ a polynomial solving this problem:

$$u(\mathcal{P}_n^1)_{L^p_\varpi(0,1)} = \|\varrho_n^*\|_{L^p_\varpi(0,1)};$$

it is called a polynomial of degree n with unit leading coefficient that deviates least from zero in the space $L^p_\varpi(0,1)$.

Theorem 3. *For all $1 \le p < \infty$ and $n \ge 1$, the polynomial ϱ_n^* of degree n with unit leading coefficient that deviates least from zero in the space $L^p_\varpi(0,1)$ with weight (3.16) is the unique extremal polynomial in inequality (3.12).*

P r o o f. The polynomial ϱ_n^* is characterized by the property that the function $|\varrho_n^*|^{p-1}\operatorname{sign}\varrho_n^*$ is orthogonal to the space $\mathcal{P}_{n-1}$ (see, for example, [20, Ch. 3, Sect. 3.3, Theorems 3.3.1 and 3.3.2]):

$$\int_0^1 \varpi(x)\, \rho_{n-1}(x) |\varrho_n^*(x)|^{p-1} \operatorname{sign} \varrho_n^*(x)\, dx = 0, \quad \rho_{n-1} \in \mathcal{P}_{n-1}.$$

This property is the same as property (3.15). Therefore, the polynomials ϱ_n and ϱ_n^* can differ only by a multiplicative constant. Theorem 3 is proved. □

3.2. Characterization of a $\mathscr{C}$-polynomial extremal in inequality (1.16)

Let us apply the results of the previous section to describing the characteristic properties of $\mathscr{C}_n$-polynomials extremal in inequalities (1.14) and (1.16).

3.2.1. An analog of Lemma 5 in the set of $\mathscr{C}$-polynomials

Let us reformulate Lemma 5 for the extremal $\mathscr{C}_n$-polynomial of inequality (1.16).

Lemma 6. *For $1 \le p < \infty$ and $n \ge 1$, the following statements hold for polynomials $\varphi_n \in \mathscr{C}_n$ extremal in inequality (1.16) and such that $\varphi_n(0) = 1$.*

(1) *The polynomial φ_n is characterized by the property*

$$\int_0^{\pi/2} \phi_{n-1}(t)(\sin^2 t)\, |\varphi_n(t)|^{p-1} \operatorname{sign} \varphi_n(t) dt = 0, \quad \phi_{n-1} \in \mathscr{C}_{n-1}. \tag{3.18}$$

(2) *The polynomial φ_n has order n. The polynomial φ_n has n simple roots on the interval $(0, \pi/2)$.*

(3) *The polynomial φ_n is unique.*

P r o o f. Let us employ the statements of Lemmas 4 and 5. Let $\varrho_n \in \mathcal{P}_n$ be an extremal (algebraic) polynomial in inequality (1.16) with the property $\varrho_n(1) = 1$. The polynomial ϱ_n and the polynomial φ_n are related by (3.4).

Let us check that the relation (3.18) coincides with (is equivalent to) (3.15). For this, we transform the functional on the left-hand side of (3.15). For a polynomial $\rho_{n-1} \in \mathcal{P}_{n-1}$, we define a polynomial $\phi_{n-1} \in \mathscr{C}_{n-1}$ that is expressed in terms of ρ_{n-1} by formula (3.4). Based on the left-hand side of (3.15), we find

$$\int_0^1 \rho_{n-1}(u)(1-u)v(u)|\varrho_n(u)|^{p-1}\text{sign}\,\varrho_n(u)du =$$
$$= \frac{1}{\pi}\int_0^1 \rho_{n-1}(u)(1-u)\frac{1}{\sqrt{1-u}}|\sqrt{u}\varrho_n(u)|^{p-1}\text{sign}\,\varrho_n(u)du =$$
$$= \frac{1}{\pi}\int_0^1 \rho_{n-1}(u)\sqrt{1-u}|\sqrt{u}\varrho_n(u)|^{p-1}\text{sign}\,\varrho_n(u)du = \left[u=t^2\right] =$$
$$= \frac{2}{\pi}\int_0^1 t\rho_{n-1}(t^2)\sqrt{1-t^2}|t\varrho_n(t^2)|^{p-1}\text{sign}\,\varrho_n(t^2)dt = [t=\cos x] =$$
$$= \frac{2}{\pi}\int_0^{\pi/2} \phi_{n-1}(x)\sin^2 x\,|\varphi_n(x)|^{p-1}\text{sign}\,\varphi_n(x)dt.$$

Now you can see that the conditions (3.18) and (3.15) hold or do not hold simultaneously. Lemma 6 is proved. □

Consider a problem similar (3.17) on the value

$$U(\mathscr{C}_n^1)_{L_\sigma^p(0,\pi/2)} = \min\{\|\phi_n\|_{L_\sigma^p(0,\pi/2)} \colon \phi_n \in \mathscr{C}_n^1\} \tag{3.19}$$

of the least deviation from zero in the space $L_\sigma^p(0,\pi/2)$ with weight $\sigma(x) = \sin^2 x$ of the set $\mathscr{C}_n^1$ of algebraic polynomials of degree n whose leading coefficient is 1. Denote by $\varphi_n^* = \varphi_{n,\sigma,p}^*$ the polynomial of degree n with unit leading coefficient that deviates least from zero in the space $L_\sigma^p(0,\pi/2)$, i.e., the polynomial that solves problem (3.19):

$$U(\mathscr{C}_n^1)_{L_\sigma^p(0,\pi/2)} = \|\varphi_n^*\|_{L_\sigma^p(0,\pi/2)}.$$

Lemma 7. *For $1 \le p < \infty$ and $n \ge 1$, the following statements hold for problems (3.17) and (3.19).*

(1) *The values of the problems are related by the equality*

$$U(\mathscr{C}_n^1)_{L_\sigma^p(0,\pi/2)} = 2^{2n}\,u(\mathcal{P}_n^1)_{L_\varpi^p(0,1)}.$$

(2) *A polynomial $\varrho_n^* \in \mathcal{P}_n^1$ extremal in problem (3.17) and a polynomial $\varphi_n^* \in \mathscr{C}_n^1$ extremal in problem (3.19) are related by the equality*

$$\varphi_n^*(x) = 2^{2n}t\,\varrho_n^*(t^2), \quad t=\cos x, \quad x\in[0,\pi/2], \quad t\in[0,1].$$

P r o o f. Let $\rho_n \in \mathcal{P}_n$, and let $\phi_n \in \mathscr{C}_n$ be the polynomial expressed in terms of ρ_n by formula (3.4). We have

$$\|\rho_n\|_{L_\varpi^p(0,1)}^p = \frac{1}{\pi}\int_0^1 u^{(p-1)/2}\sqrt{1-u}|\rho_n(u)|^p du = [u=t^2] =$$
$$= \frac{2}{\pi}\int_0^1 \sqrt{1-t^2}|t\rho_n(t^2)|^p dt = [t=\cos x] = \frac{2}{\pi}\int_0^{\pi/2}\sin^2 x|\phi_n(x)|^p dx = \|\phi_n\|_{L_\sigma^p(0,\pi/2)}^p.$$

Consequently, the norms satisfy the equality

$$\|\rho_n\|_{L_\varpi^p(0,1)} = \|\phi_n\|_{L_\sigma^p(0,\pi/2)}.$$

According to (3.5), we have

$$\phi_n \in \mathscr{C}_n^1 \quad \Longleftrightarrow \quad 2^{-2n}\rho_n \in \mathcal{P}_n^1.$$

From here, all the assertions of Lemma 7 follow. Lemma 7 is proven. □

3.2.2. The proof of Theorem 2

For an algebraic polynomial $\rho \in \mathcal{P}_n$ and a $\mathscr{C}_n$-polynomial $\phi \in \mathscr{C}_n$ related by (3.4), we will say that they are (3.4)-related.

(1) According to Corollary 2, the polynomials $\varrho \in \mathcal{P}_n$ and $\phi \in \mathscr{C}_n$ extremal in the Nikol'skii inequalities (1.14) and (1.16) and inequalities (3.11) and (3.12), respectively, are (3.4)-related. According to Lemmas 5 and 6, these polynomials are unique up to numerical factors.

(2) Let $\varrho^* \in \mathcal{P}_n^1$ and $\varphi_n^* \in \mathscr{C}_n^1$ be extremal polynomials, i.e., polynomials that deviate least from zero in problems (3.17) and (3.19), respectively. According to Lemma 7, the polynomials ρ_n^* and $2^{2n}\varphi^*$ are also (3.4)-related.

(3) According to Theorem 3, the polynomial ϱ^* is extremal in inequalities (3.11) and (3.12). Consequently, φ^* is also extremal in inequalities (1.14) and (1.16).

Thus, Theorem 2 is proved. □

4. Conclusions

The system of functions (1.20) is in some sense a "quarter" of the classical trigonometric system. However, as it turned out (see Theorem 1 and Lemma 3), the best constants in inequalities (1.2) and (1.14) are very close. The reason for this is not clear to the authors. What will be the situation with other extremal problems in these systems, the authors also do not know.

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A CHARACTERIZATION OF DERIVATIONS AND AUTOMORPHISMS ON SOME SIMPLE ALGEBRAS

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Abstract: In the present paper, we study simple algebras, which do not belong to the well-known classes of algebras (associative algebras, alternative algebras, Lie algebras, Jordan algebras, etc.). The simple finite-dimensional algebras over a field of characteristic 0 without finite basis of identities, constructed by Kislitsin, are such algebras. In the present paper, we consider two such algebras: the simple seven-dimensional anticommutative algebra $\mathcal{D}$ and the seven-dimensional central simple commutative algebra $\mathcal{C}$. We prove that every local derivation of these algebras $\mathcal{D}$ and $\mathcal{C}$ is a derivation, and every 2-local derivation of these algebras $\mathcal{D}$ and $\mathcal{C}$ is also a derivation. We also prove that every local automorphism of these algebras $\mathcal{D}$ and $\mathcal{C}$ is an automorphism, and every 2-local automorphism of these algebras $\mathcal{D}$ and $\mathcal{C}$ is also an automorphism.

Keywords: Simple algebra, Derivation, Local derivation, 2-Local derivation, Automorphism, Local automorphism, 2-Local automorphism, Basis of identities.

1. Introduction

In the present paper, we study local and 2-local derivations and automorphisms of simple finite-dimensional algebras without finite basis of identities, constructed by Kislitsin in [19] and [20]. Kadison in [12] introduced and investigated a notion of local derivations. He proved that each continuous local derivation from a von Neumann algebra into its dual Banach bimodule is a derivation. Šemrl introduced a similar notion of 2-local derivations. He proved that any 2-local derivation of the algebra $B(H)$ of all bounded linear operators on the infinite-dimensional separable Hilbert space H is a derivation [24]. After, numerous new results related to the description of local and 2-local derivations of associative algebras have appeared. For example, papers [1, 3, 4, 15, 16, 22] are devoted to local and 2-local derivations of associative algebras.

The study of local and 2-local derivations of nonassociative algebras was initiated in papers [5, 6] of Ayupov and Kudaybergenov (for the case of Lie algebras). They proved that each local and 2-local derivation on a semisimple finite-dimensional Lie algebra are derivations. In [8], examples of 2-local derivations on nilpotent Lie algebras that are not derivations are given. After the cited works, the study of local and 2-local derivations was continued for Leibniz algebras [7] and Jordan algebras [2]. Local and 2-local automorphisms were also studied in many cases. For example, local and 2-local automorphisms on Lie algebras have been studied in [5, 10].

The variety of Malcev algebras is a generalization of the variety of Lie algebras [23]. It is closely related to other classes of nonassociative structures: it is a proper subvariety of binary Lie algebras, and, under the multiplication $ab - ba$, an alternative algebra is a Malcev algebra. Moreover, it is connected with various classes of algebraic systems such as Moufang loops, Poisson–Malcev algebras, etc. The study of generalizations of derivations of simple Malcev algebras was initiated by Filippov in [11] and continued in some papers of Kaygorodov and Popov [13, 14].

Now, a linear operator ∇ on $\mathcal{A}$ is called a local automorphism if, for every $x \in \mathcal{A}$, there exists an automorphism ϕ_x of $\mathcal{A}$, depending on x, such that $\nabla(x) = \phi_x(x)$. The concept of local automorphism was introduced by Larson and Sourour [21] in 1990. They proved that invertible local automorphisms of the algebra of all bounded linear operators on an infinite-dimensional Banach space X are automorphisms.

A similar notion, which characterizes non-linear generalizations of automorphisms, was introduced by Šemrl in [24] as 2-local automorphisms. Namely, a map $\Delta : \mathcal{A} \to \mathcal{A}$ (not necessarily linear) is called a 2-local automorphism if, for every $x, y \in \mathcal{A}$, there exists an automorphism $\phi_{x,y} : \mathcal{A} \to \mathcal{A}$ such that $\Delta(x) = \phi_{x,y}(x)$ and $\Delta(y) = \phi_{x,y}(y)$. After the work of Šemrl, it appeared numerous new results related to the description of local and 2-local automorphisms of algebras (see, for example, [5, 7, 9, 10, 16]).

In the present paper, we continue the study of derivations and automorphisms of simple algebras. We study derivations and automorphisms of simple algebras, which do not belong to well-known classes of algebras (commutative, associative, alternative, Lie, Jordan, etc.). The simple finite-dimensional algebras without finite basis of identities, constructed by Kislitsin are such algebras. Namely, we prove that any local derivation (automorphism) of the simple finite-dimensional algebras without finite basis of identities, constructed by Kislitsin in [19] and [20], is a derivation (an automorphism, respectively), and every 2-local derivation (automorphism) of these algebras is also a derivation (an automorphism, respectively). Note that central simple finite-dimensional algebras which has no finite basis of identities were considered in the works [17] and [18] of Isaev and Kislitsin.

2. A simple finite-dimensional algebra without finite basis of identities

Let $\mathcal{D} = \langle e, v_1, v_2, e_{11}, e_{12}, e_{22}, p\rangle_{\mathbb{F}}$ be an algebra over a field $\mathbb{F}$ of characteristic 0 whose nonzero products of basis elements from

$$\{e, v_1, v_2, e_{11}, e_{12}, e_{22}, p\} \tag{2.1}$$

are defined by the rules

$$v_i e_{ij} = -e_{ij} v_i = v_j, \quad v_2 p = -p v_2 = e, \quad v_i e = -e v_i = v_i,$$
$$e_{ij} e = -e e_{ij} = e_{ij}, \quad pe = -ep = p.$$

Then $\mathcal{D}$ is a simple anticommutative algebra without finite basis of identities [20]. Let a be an element in $\mathcal{D}$. Then we can write

$$a = a_1 e + a_2 v_1 + a_3 v_2 + a_4 e_{11} + a_5 e_{12} + a_6 e_{22} + a_7 p$$

for some elements $a_1, a_2, a_3, a_4, a_5, a_6$, and a_7 in $\mathbb{F}$. Throughout the paper, let

$$\overline{a} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7)^T.$$

Conversely, if $v = (a_1, a_2, a_3, a_4, a_5, a_6, a_7)^T$ is a column vector with $a_1, a_2, a_3, a_4, a_5, a_6$, and a_7 in $\mathbb{F}$, then, throughout the paper, we will denote by $\widehat{v}$ the element

$$a_1 e + a_2 v_1 + a_3 v_2 + a_4 e_{11} + a_5 e_{12} + a_6 e_{22} + a_7 p;$$

i.e.,

$$\widehat{v} = a_1 e + a_2 v_1 + a_3 v_2 + a_4 e_{11} + a_5 e_{12} + a_6 e_{22} + a_7 p.$$

Let $\mathcal{A}$ be an algebra. A linear map $D\colon \mathcal{A} \to \mathcal{A}$ is called a derivation if

$$D(xy) = D(x)y + xD(y)$$

for any two elements $x, y \in \mathcal{A}$.

Our principal tool for the description of local and 2-local derivations of $\mathcal{D}$ is the following proposition.

Proposition 1. *A linear map $D\colon \mathcal{D} \to \mathcal{D}$ is a derivation if and only if the matrix of D in the standard basis (2.1) has the following form:*

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{2,2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2,2}+a_{5,5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{5,5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -(a_{2,2}+a_{5,5}) \end{pmatrix}.$$

Here the action of D corresponds to multiplying the matrix by a column on the right.

P r o o f. The proof is carried out by checking the derivation property on the algebra $\mathcal{D}$.

Let $A = (a_{i,j})_{i,j=1}^7$ be the matrix of the derivation D. Then

$$A\overline{v_i e_{ij}} = -A\overline{e_{ij} v_i} = A\overline{v_j}, \quad A\overline{v_2 p} = -A\overline{p v_2} = A\overline{e}, \quad A\overline{v_i e} = -A\overline{e v_i} = A\overline{v_i},$$
$$A\overline{e_{ij} e} = -A\overline{e e_{ij}} = A\overline{e_{ij}}, \quad A\overline{pe} = -A\overline{ep} = A\overline{p}.$$

On the other hand,

$$\widehat{A\overline{v_i e_{ij}}} = \widehat{A\overline{v_i}} e_{ij} + v_i \widehat{A\overline{e_{ij}}}.$$

Hence,

$$\widehat{A\overline{v_j}} = \widehat{A\overline{v_i}} e_{ij} + v_i \widehat{A\overline{e_{ij}}}.$$

So,

$$\widehat{A\overline{v_1}} = \widehat{A\overline{v_1}} e_{11} + v_1 \widehat{A\overline{e_{11}}},$$
$$a_{1,2} e + a_{2,2} v_1 + a_{3,2} v_2 + a_{4,2} e_{11} + a_{5,2} e_{12} + a_{6,2} e_{22} + a_{7,2} p$$
$$= (a_{1,2} e + a_{2,2} v_1 + a_{3,2} v_2 + a_{4,2} e_{11} + a_{5,2} e_{12} + a_{6,2} e_{22} + a_{7,2} p) e_{11}$$
$$+ v_1 (a_{1,4} e + a_{2,4} v_1 + a_{3,4} v_2 + a_{4,4} e_{11} + a_{5,4} e_{12} + a_{6,4} e_{22} + a_{7,4} p)$$

if $i=1$, $j=1$, and

$$\begin{aligned}&a_{1,2}e+a_{2,2}v_1+a_{3,2}v_2+a_{4,2}e_{11}+a_{5,2}e_{12}+a_{6,2}e_{22}+a_{7,2}p\\&=-a_{1,2}e_{11}+a_{2,2}v_1+a_{1,4}v_1+a_{4,4}v_1+a_{5,4}v_2.\end{aligned}$$

This implies that

$$a_{1,2}=0,\quad a_{1,4}+a_{4,4}=0,\quad a_{3,2}=a_{5,4},\quad a_{4,2}=-a_{1,2}=0,\quad a_{5,2}=0,\quad a_{6,2}=0,\quad a_{7,2}=0.$$

In addition, if $i=1$ and $j=2$, then

$$\widehat{A\overline{v_2}}=\widehat{A\overline{v_1}}e_{12}+v_1\widehat{A\overline{e_{12}}}$$

and

$$\begin{aligned}&a_{1,3}e+a_{2,3}v_1+a_{3,3}v_2+a_{4,3}e_{11}+a_{5,3}e_{12}+a_{6,3}e_{22}+a_{7,3}p\\&=(a_{1,2}e+a_{2,2}v_1+a_{3,2}v_2+a_{4,2}e_{11}+a_{5,2}e_{12}+a_{6,2}e_{22}+a_{7,2}p)e_{12}\\&+v_1(a_{1,5}e+a_{2,5}v_1+a_{3,5}v_2+a_{4,5}e_{11}+a_{5,5}e_{12}+a_{6,5}e_{22}+a_{7,5}p)\\&=-a_{1,2}e_{12}+a_{2,2}v_2+a_{1,5}v_1+a_{4,5}v_1+a_{5,5}v_2.\end{aligned}$$

This implies that

$$\begin{gathered}a_{1,3}=0,\quad a_{2,3}=a_{1,5}+a_{4,5},\quad a_{3,3}=a_{2,2}+a_{5,5},\\a_{4,3}=0,\quad a_{5,3}=-a_{1,2},\quad a_{6,3}=0,\quad a_{7,3}=0.\end{gathered}$$

Besides, if $i=2$ and $j=2$, then

$$\widehat{A\overline{v_2}}=\widehat{A\overline{v_2}}e_{22}+v_2\widehat{A\overline{e_{22}}}$$

and

$$\begin{aligned}&a_{1,3}e+a_{2,3}v_1+a_{3,3}v_2+a_{4,3}e_{11}+a_{5,3}e_{12}+a_{6,3}e_{22}+a_{7,3}p\\&=(a_{1,3}e+a_{2,3}v_1+a_{3,3}v_2+a_{4,3}e_{11}+a_{5,3}e_{12}+a_{6,3}e_{22}+a_{7,3}p)e_{22}\\&+v_2(a_{1,6}e+a_{2,6}v_1+a_{3,6}v_2+a_{4,6}e_{11}+a_{5,6}e_{12}+a_{6,6}e_{22}+a_{7,6}p)\\&=-a_{1,3}e_{22}+a_{3,3}v_2+a_{1,6}v_2+a_{6,6}v_2+a_{7,6}e.\end{aligned}$$

This implies that

$$\begin{gathered}a_{1,3}=a_{7,6}=0,\quad a_{2,3}=0,\quad a_{1,6}+a_{6,6}=0,\quad a_{4,3}=0,\\a_{5,3}=0,\quad a_{6,3}=-a_{1,3}=0,\quad a_{7,3}=0.\end{gathered}$$

Similarly, we have

$$\begin{gathered}a_{3,3}=-a_{7,7},\quad a_{2,1}=0,\quad a_{1,7}=0,\quad a_{6,7}=0,\quad a_{4,1}=0,\quad a_{5,1}=0,\quad a_{6,1}=0,\\a_{1,2}=0,\quad a_{4,1}=0,\quad a_{5,1}=0,\quad a_{1,3}=0,\quad a_{7,1}=0,\quad a_{6,1}=0,\\a_{1,4}=0,\quad a_{2,1}=0,\quad a_{1,1}=0,\quad a_{1,5}=0,\quad a_{1,6}=0,\quad a_{3,1}=0,\quad a_{1,7}=0,\\a_{6,2}=0,\quad a_{7,2}=0,\quad a_{4,3}=0,\quad a_{3,6}=0,\quad a_{3,7}=0,\quad a_{3,5}=0,\quad a_{2,7}=0,\\a_{3,4}=0,\quad a_{1,2}=0,\quad a_{3,2}=0,\quad a_{4,7}=0,\quad a_{5,7}=0,\quad a_{2,6}=0,\quad a_{3,4}=0,\quad a_{2,4}=0,\\a_{2,5}=0,\quad a_{1,2}=0,\quad a_{5,6}=0,\quad a_{4,6}=0,\quad a_{1,3}=0,\quad a_{2,3}=0,\quad a_{6,5}=0,\quad a_{7,5}=0,\\a_{6,4}=0,\quad a_{7,4}=0.\end{gathered}$$

As a result, we get the matrix from Proposition 1. The proof is complete. □

Let $\mathcal{A}$ be an algebra. A linear map $\nabla\colon\mathcal{A}\to\mathcal{A}$ is called a local derivation if, for any element $x\in\mathcal{A}$, there exists a derivation $D\colon\mathcal{A}\to\mathcal{A}$ such that $\nabla(x)=D(x)$.

Theorem 1. *Each local derivation on the simple algebra $\mathcal{D}$ is a derivation.*

P r o o f. Let ∇ be a local derivation on $\mathcal{D}$, and let $A = (a_{i,j})_{i,j=1}^7$ be the matrix of ∇. Then

$$\nabla(v_1) = a_{2,2}^{v_1} v_1 = a_{2,2} v_1, \quad \nabla(v_2) = (a_{2,2}^{v_2} + a_{5,5}^{v_2}) v_2 = a_{3,3} v_2,$$
$$\nabla(e_{1,2}) = a_{5,5}^{e_{1,2}} e_{1,2} = a_{5,5} e_{1,2}, \quad \nabla(p) = -(a_{2,2}^{p} + a_{5,5}^{p}) p = a_{7,7} p,$$

and the remaining components of the matrix A are equal to zero. At the same time,

$$\nabla(v_1 + v_2 + e_{1,2} + p) = \nabla(v_1) + \nabla(v_2) + \nabla(e_{1,2}) + \nabla(p) \tag{2.2}$$

and

$$\nabla(v_1 + v_2 + e_{1,2} + p) = a_{2,2}^{v_1+v_2+e_{1,2}+p} v_1 + (a_{2,2}^{v_1+v_2+e_{1,2}+p} + a_{5,5}^{v_1+v_2+e_{1,2}+p}) v_2$$
$$+ a_{5,5}^{v_1+v_2+e_{1,2}+p} e_{1,2} - (a_{2,2}^{v_1+v_2+e_{1,2}+p} + a_{5,5}^{v_1+v_2+e_{1,2}+p}) p.$$

By 2.2, we have

$$a_{2,2}^{v_1+v_2+e_{1,2}+p} v_1 + (a_{2,2}^{v_1+v_2+e_{1,2}+p} + a_{5,5}^{v_1+v_2+e_{1,2}+p}) v_2$$
$$+ a_{5,5}^{v_1+v_2+e_{1,2}+p} e_{1,2} - (a_{2,2}^{v_1+v_2+e_{1,2}+p} + a_{5,5}^{v_1+v_2+e_{1,2}+p}) p$$
$$= a_{2,2}^{v_1} v_1 + (a_{2,2}^{v_2} + a_{5,5}^{v_2}) v_2 + a_{5,5}^{e_{1,2}} e_{1,2} - (a_{2,2}^{p} + a_{5,5}^{p}) p.$$

Hence,

$$a_{2,2}^{v_1+v_2+e_{1,2}+p} = a_{2,2}^{v_1}, \quad a_{2,2}^{v_1+v_2+e_{1,2}+p} + a_{5,5}^{v_1+v_2+e_{1,2}+p} = a_{2,2}^{v_2} + a_{5,5}^{v_2},$$
$$a_{5,5}^{v_1+v_2+e_{1,2}+p} = a_{5,5}^{e_{1,2}}, \quad a_{2,2}^{v_1+v_2+e_{1,2}+p} + a_{5,5}^{v_1+v_2+e_{1,2}+p} = a_{2,2}^{p} + a_{5,5}^{p}.$$

This implies that

$$a_{2,2}^{v_2} + a_{5,5}^{v_2} = a_{2,2}^{v_1} + a_{5,5}^{e_{1,2}}, \quad a_{2,2}^{p} + a_{5,5}^{p} = a_{2,2}^{v_1} + a_{5,5}^{e_{1,2}}$$

and

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{2,2}^{v_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2,2}^{v_1} + a_{5,5}^{e_{1,2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{5,5}^{e_{1,2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -(a_{2,2}^{v_1} + a_{5,5}^{e_{1,2}}) \end{pmatrix}.$$

Hence, by Proposition 1, ∇ is a derivation. This completes the proof. □

We give another characterization of derivations on the algebra $\mathcal{D}$ in the following theorem.

Let $\mathcal{A}$ be an algebra. A (not necessary linear) map $\Delta : \mathcal{A} \to \mathcal{A}$ is called a 2-local derivation if, for all elements $x, y \in \mathcal{A}$, there exists a derivation $D_{x,y} : \mathcal{A} \to \mathcal{A}$ such that $\Delta(x) = D_{x,y}(x)$ and $\Delta(y) = D_{x,y}(y)$.

Theorem 2. *Each 2-local derivation on the simple algebra $\mathcal{D}$ is a derivation.*

P r o o f. Suppose that Δ is a 2-local derivation on $\mathcal{D}$ and, for elements $a, b \in \mathcal{D}$, $D_{a,b}$ is a derivation on $\mathcal{D}$ such that $D_{a,b}(a) = \Delta(a)$ and $D_{a,b}(b) = \Delta(b)$. Let $A_{a,b} = (a_{i,j}^{a,b})_{i,j=1}^{7}$ be the matrix of $D_{a,b}$.

Let

$$a = \lambda_1 e + \lambda_2 v_1 + \lambda_3 v_2 + \lambda_4 e_{1,1} + \lambda_5 e_{1,2} + \lambda_6 e_{2,2} + \lambda_7 p$$

be an arbitrary element from $\mathcal{D}$. For every $v \in \mathcal{D}$, there exists a derivation $D_{v,a}$ such that

$$\Delta(v) = D_{v,a}(v), \quad \Delta(a) = D_{v,a}(a).$$

Then from

$$D_{v_1,v}(v_1) = D_{v_1,a}(v_1), \quad v \in \mathcal{D},$$

it follows that

$$a_{2,2}^{v_1,v} v_1 = a_{2,2}^{v_1,a} v_1.$$

Hence,

$$a_{2,2}^{v_1,v} = a_{2,2}^{v_1,a}.$$

Therefore,

$$\Delta(a) = D_{v_1,a}(a) = a_{2,2}^{v_1,v}\lambda_2 v_1 + (a_{2,2}^{v_1,a} + a_{5,5}^{v_1,a})\lambda_3 v_2 + a_{5,5}^{v_1,a}\lambda_5 e_{1,2} - (a_{2,2}^{v_1,a} + a_{5,5}^{v_1,a})\lambda_7 p.$$

Similarly, from

$$D_{v_2,v}(v_2) = D_{v_2,a}(v_2), \quad v \in \mathcal{D},$$

it follows that

$$\Delta(a) = D_{v_2,a}(a) = a_{2,2}^{v_2,a}\lambda_2 v_1 + (a_{2,2}^{v_2,v} + a_{5,5}^{v_2,v})\lambda_3 v_2 + a_{5,5}^{v_2,a}\lambda_5 e_{1,2} - (a_{2,2}^{v_2,a} + a_{5,5}^{v_2,a})\lambda_7 p.$$

Similarly, we have

$$\Delta(a) = D_{e_{1,2},a}(a) = a_{2,2}^{e_{1,2},a}\lambda_2 v_1 + (a_{2,2}^{e_{1,2},a} + a_{5,5}^{e_{1,2},a})\lambda_3 v_2 + a_{5,5}^{e_{1,2},v}\lambda_5 e_{1,2} - (a_{2,2}^{e_{1,2},a} + a_{5,5}^{e_{1,2},a})\lambda_7 p,$$
$$\Delta(a) = D_{p,a}(a) = a_{2,2}^{p,a}\lambda_2 v_1 + (a_{2,2}^{p,a} + a_{5,5}^{p,a})\lambda_3 v_2 + a_{5,5}^{p,a}\lambda_5 e_{1,2} - (a_{2,2}^{p,v} + a_{5,5}^{p,v})\lambda_7 p.$$

Hence,

$$\Delta(a) = D_{v_1,a}(a) = D_{v_2,a}(a) = D_{e_{1,2},a}(a) = D_{p,a}(a) =$$
$$a_{2,2}^{v_1,v}\lambda_2 v_1 + (a_{2,2}^{v_2,w} + a_{5,5}^{v_2,w})\lambda_3 v_2 + a_{5,5}^{e_{1,2},z}\lambda_5 e_{1,2} - (a_{2,2}^{p,t} + a_{5,5}^{p,t})\lambda_7 p$$

for any $v, w, z, t \in \mathcal{D}$. Note that the components in the last sum do not depend on the element a. Therefore, the map Δ is linear and it is a local derivation. The linear operator Δ has the following matrix:

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{2,2}^{v_1,v} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2,2}^{v_2,w} + a_{5,5}^{v_2,w} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{5,5}^{e_{1,2},z} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -(a_{2,2}^{p,t} + a_{5,5}^{p,t}) \end{pmatrix}.$$

From $\Delta(v_2 + p) = \Delta(v_2) + \Delta(p)$, we get

$$(a_{2,2}^{a,v_2+p} + a_{5,5}^{a,v_2+p})v_2 - (a_{2,2}^{a,v_2+p} + a_{5,5}^{a,v_2+p})p = (a_{2,2}^{v_2,w} + a_{5,5}^{v_2,w})v_2 - (a_{2,2}^{p,t} + a_{5,5}^{p,t})p.$$

Hence,

$$a_{2,2}^{a,v_2+p} + a_{5,5}^{a,v_2+p} = a_{2,2}^{v_2,w} + a_{5,5}^{v_2,w} = a_{2,2}^{p,t} + a_{5,5}^{p,t}. \tag{2.3}$$

From $\Delta(v_1 + v_2 + e_{1,2}) = \Delta(v_1) + \Delta(v_2) + \Delta(e_{1,2})$, we get

$$a_{2,2}^{a,v_1+v_2+e_{1,2}} = a_{2,2}^{v_1,v},$$
$$a_{2,2}^{a,v_1+v_2+e_{1,2}} + a_{5,5}^{a,v_1+v_2+e_{1,2}} = a_{2,2}^{v_2,w} + a_{5,5}^{v_2,w},$$
$$a_{5,5}^{a,v_1+v_2+e_{1,2}} = a_{5,5}^{e_{1,2},z}.$$

Hence,

$$a_{2,2}^{v_2,w} + a_{5,5}^{v_2,w} = a_{2,2}^{v_1,v} + a_{5,5}^{e_{1,2},z}.$$

By (2.3), we also have

$$a_{2,2}^{p,t} + a_{5,5}^{p,t} = a_{2,2}^{v_1,v} + a_{5,5}^{e_{1,2},z}.$$

Thus,

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{2,2}^{v_1,v} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2,2}^{v_1,v} + a_{5,5}^{e_{1,2},z} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{5,5}^{e_{1,2},z} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -(a_{2,2}^{v_1,v} + a_{5,5}^{e_{1,2},z}) \end{pmatrix}.$$

Therefore, by Proposition 1, Δ is a derivation. This completes the proof. □

Let $\mathcal{A}$ be an algebra. A linear bijective map $\Phi\colon \mathcal{A} \to \mathcal{A}$ is called an automorphism if $\Phi(xy) = \Phi(x)\Phi(y)$ for any two elements $x, y \in \mathcal{A}$.

Our principal tool for the description of local and 2-local automorphisms of $\mathcal{D}$ is the following proposition.

Proposition 2. *A linear map $\Phi\colon \mathcal{D} \to \mathcal{D}$ is an automorphism if and only if the matrix of Φ in the standard basis (2.1) has the following form:*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{2,2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2,2}a_{5,5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{5,5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{a_{2,2}a_{5,5}} \end{pmatrix},$$

where $a_{2,2}$ and $a_{5,5}$ are nonzero elements from $\mathbb{F}$. Here the action of Φ corresponds to multiplying the matrix by a column on the right.

P r o o f. Let $B = (b_{i,j})_{i,j=1}^{7}$ be the matrix of the automorphism Φ. Then there exists a derivation D such that

$$B = e^A,$$

where A is the matrix of D. It is known that

$$e^A = E + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots,$$

where E is the unit matrix. Hence,

$$B = E + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots. \tag{2.4}$$

By (2.4) and Proposition 1, B is equal to

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sum_{i=0}^{\infty} \frac{a_{2,2}^i}{i!} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sum_{i=0}^{\infty} \frac{(a_{2,2}+a_{5,5})^i}{i!} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sum_{i=0}^{\infty} \frac{a_{5,5}^i}{i!} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sum_{i=0}^{\infty} \frac{(-1)^i(a_{2,2}+a_{5,5})^i}{i!} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{a_{2,2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{a_{2,2}+a_{5,5}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{a_{5,5}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{-(a_{2,2}+a_{5,5})} \end{pmatrix}.$$

The latter matrix gives the desired form. This completes the proof. □

Let $\mathcal{A}$ be an algebra. A linear map $\nabla : \mathcal{A} \to \mathcal{A}$ is called a local automorphism if, for every element $x \in \mathcal{A}$, there exists an automorphism $\phi_x : \mathcal{A} \to \mathcal{A}$ such that $\nabla(x) = \phi_x(x)$.

Theorem 3. *Each local automorphism on the simple algebra $\mathcal{D}$ is an automorphism.*

P r o o f. Let ∇ be a local automorphism on $\mathcal{D}$, and let $A = (a_{i,j})_{i,j=1}^{7}$ be the matrix of ∇. Then

$$\nabla(v_1) = a_{2,2}^{v_1} v_1 = a_{2,2} v_1, \quad \nabla(v_2) = a_{2,2}^{v_2} a_{5,5}^{v_2} v_2 = a_{3,3} v_2,$$
$$\nabla(e_{1,2}) = a_{5,5}^{e_{1,2}} e_{1,2} = a_{5,5} e_{1,2}, \quad \nabla(p) = \frac{1}{a_{2,2}^p a_{5,5}^p} p = a_{7,7} p$$

and the remaining components of the matrix A are equal to zero. At the same time,

$$\nabla(v_1 + v_2 + e_{1,2} + p) = \nabla(v_1) + \nabla(v_2) + \nabla(e_{1,2}) + \nabla(p) \tag{2.5}$$

and

$$\nabla(v_1+v_2+e_{1,2}+p)=a_{2,2}^{v_1+v_2+e_{1,2}+p}v_1+a_{2,2}^{v_1+v_2+e_{1,2}+p}a_{5,5}^{v_1+v_2+e_{1,2}+p}v_2+$$
$$a_{5,5}^{v_1+v_2+e_{1,2}+p}e_{1,2}+\frac{1}{a_{2,2}^{v_1+v_2+e_{1,2}+p}a_{5,5}^{v_1+v_2+e_{1,2}+p}}p.$$

By (2.5), we have

$$a_{2,2}^{v_1+v_2+e_{1,2}+p}v_1+a_{2,2}^{v_1+v_2+e_{1,2}+p}a_{5,5}^{v_1+v_2+e_{1,2}+p})v_2+a_{5,5}^{v_1+v_2+e_{1,2}+p}e_{1,2}+\frac{1}{a_{2,2}^{v_1+v_2+e_{1,2}+p}a_{5,5}^{v_1+v_2+e_{1,2}+p}}p$$
$$=a_{2,2}^{v_1}v_1+a_{2,2}^{v_2}a_{5,5}^{v_2}v_2+a_{5,5}^{e_{1,2}}e_{1,2}+\frac{1}{a_{2,2}^{p}a_{5,5}^{p}}p.$$

Hence,

$$a_{2,2}^{v_1+v_2+e_{1,2}+p}=a_{2,2}^{v_1},\quad a_{2,2}^{v_1+v_2+e_{1,2}+p}a_{5,5}^{v_1+v_2+e_{1,2}+p}=a_{2,2}^{v_2}a_{5,5}^{v_2},$$
$$a_{5,5}^{v_1+v_2+e_{1,2}+p}=a_{5,5}^{e_{1,2}},\quad a_{2,2}^{v_1+v_2+e_{1,2}+p}a_{5,5}^{v_1+v_2+e_{1,2}+p}=a_{2,2}^{p}a_{5,5}^{p}.$$

This implies that

$$a_{2,2}^{v_2}a_{5,5}^{v_2}=a_{2,2}^{v_1}a_{5,5}^{e_{1,2}},\quad a_{2,2}^{p}a_{5,5}^{p}=a_{2,2}^{v_1}a_{5,5}^{e_{1,2}}$$

and

$$A=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & a_{2,2}^{v_1} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & a_{2,2}^{v_1}a_{5,5}^{e_{1,2}} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & a_{5,5}^{e_{1,2}} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{a_{2,2}^{v_1}a_{5,5}^{e_{1,2}}} \end{pmatrix}.$$

Hence, by Proposition 2, ∇ is an automorphism. This completes the proof. □

A (not necessary linear) map $\Delta:\mathcal{A}\to\mathcal{A}$ is called a 2-local automorphism if, for all elements $x,y\in\mathcal{A}$, there exists an automorphism $\phi_{x,y}:\mathcal{A}\to\mathcal{A}$ such that $\Delta(x)=\phi_{x,y}(x)$ and $\Delta(y)=\phi_{x,y}(y)$.

Theorem 4. *Each 2-local automorphism on the simple algebra $\mathcal{D}$ is an automorphism.*

P r o o f. Suppose that Δ is a 2-local automorphism on $\mathcal{D}$ and, for elements $a,b\in\mathcal{D}$, $\Phi_{a,b}$ is an automorphism on $\mathcal{D}$ such that $\Phi_{a,b}(a)=\Delta(a)$ and $\Phi_{a,b}(b)=\Delta(b)$. Let $A_{a,b}=(a_{i,j}^{a,b})_{i,j=1}^{7}$ be the matrix of $\Phi_{a,b}$. Then, for all $v,z\in\mathcal{D}$, there exists an automorphism $\Phi_{v,z}$ such that

$$\Delta(v)=\Phi_{v,z}(v),\quad \Delta(z)=\Phi_{v,z}(z).$$

Let $A_{v,z}=(a_{i,j}^{v,z})_{i,j=1}^{n}$ be the matrix of the automorphism $\Phi_{v,z}$.

Let

$$a=\lambda_1 e+\lambda_2 v_1+\lambda_3 v_2+\lambda_4 e_{1,1}+\lambda_5 e_{1,2}+\lambda_6 e_{2,2}+\lambda_7 p$$

be an arbitrary element from $\mathcal{D}$. For every $v\in\mathcal{D}$, there exists an automorphism $\Phi_{v,a}$ such that

$$\Delta(v)=\Phi_{v,a}(v),\quad \Delta(a)=\Phi_{v,a}(a).$$

Then from

$$\Phi_{v_1,v}(v_1) = \Phi_{v_1,a}(v_1), \quad v \in \mathcal{D},$$

it follows that

$$a_{2,2}^{v_1,v} v_1 = a_{2,2}^{v_1,a} v_1.$$

Hence,

$$a_{2,2}^{v_1,v} = a_{2,2}^{v_1,a}.$$

Therefore,

$$\begin{aligned}\Delta(a) = \Phi_{v_1,a}(a) = \lambda_1 e + a_{2,2}^{v_1,v}\lambda_2 v_1 + a_{2,2}^{v_1,a} a_{5,5}^{v_1,a}\lambda_3 v_2 + \lambda_4 e_{1,1}\\ + a_{5,5}^{v_1,a}\lambda_5 e_{1,2} + \lambda_6 e_{2,2} + \frac{1}{a_{2,2}^{v_1,a} a_{5,5}^{v_1,a}}\lambda_7 p.\end{aligned}$$

Similarly, from

$$\Phi_{v_2,v}(v_2) = \Phi_{v_2,a}(v_2), \quad v \in \mathcal{D},$$

it follows that

$$\begin{aligned}\Delta(a) = \Phi_{v_2,a}(a) = \lambda_1 e + a_{2,2}^{v_2,a}\lambda_2 v_1 + a_{2,2}^{v_2,v} a_{5,5}^{v_2,v}\lambda_3 v_2 + \lambda_4 e_{1,1}\\ + a_{5,5}^{v_2,a}\lambda_5 e_{1,2} + \lambda_6 e_{2,2} + \frac{1}{a_{2,2}^{v_2,a} a_{5,5}^{v_2,a}}\lambda_7 p.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\Delta(a) = \Phi_{e_{1,2},a}(a) = +\lambda_1 e + a_{2,2}^{e_{1,2},a}\lambda_2 v_1 + a_{2,2}^{e_{1,2},a} a_{5,5}^{e_{1,2},a}\lambda_3 v_2\\ + \lambda_4 e_{1,1} + a_{5,5}^{e_{1,2},v}\lambda_5 e_{1,2} + \lambda_6 e_{2,2} + \frac{1}{a_{2,2}^{e_{1,2},a} a_{5,5}^{e_{1,2},a}}\lambda_7 p,\\ \Delta(a) = \Phi_{p,a}(a) = \lambda_1 e + a_{2,2}^{p,a}\lambda_2 v_1 + a_{2,2}^{p,a} a_{5,5}^{p,a}\lambda_3 v_2\\ + \lambda_4 e_{1,1} + a_{5,5}^{p,a}\lambda_5 e_{1,2} + \lambda_6 e_{2,2} + \frac{1}{a_{2,2}^{p,v} a_{5,5}^{p,v}}\lambda_7 p.\end{aligned}$$

Hence,

$$\begin{aligned}\Delta(a) = \Phi_{v_1,a}(a) = \Phi_{v_2,a}(a) = \Phi_{e_{1,2},a}(a) = \Phi_{p,a}(a) =\\ \lambda_1 e + a_{2,2}^{v_1,v}\lambda_2 v_1 + a_{2,2}^{v_2,w} a_{5,5}^{v_2,w}\lambda_3 v_2 + \lambda_4 e_{1,1} + a_{5,5}^{e_{1,2},z}\lambda_5 e_{1,2} + \lambda_6 e_{2,2} + \frac{1}{a_{2,2}^{p,t} a_{5,5}^{p,t}}\lambda_7 p\end{aligned}$$

for any $v, w, z, t \in \mathcal{D}$. Note that the components in the last sum do not depend on the element a. Therefore, the map Δ is linear and it is a local automorphism. The linear operator Δ has the following matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & a_{2,2}^{v_1,v} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & a_{2,2}^{v_2,w} a_{5,5}^{v_2,w} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & a_{5,5}^{e_{1,2},z} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{a_{2,2}^{p,t} a_{5,5}^{p,t}} \end{pmatrix}.$$

From $\Delta(v_2+p)=\Delta(v_2)+\Delta(p)$, we get

$$a_{2,2}^{a,v_2+p}a_{5,5}^{a,v_2+p}v_2+\frac{1}{a_{2,2}^{a,v_2+p}a_{5,5}^{a,v_2+p}}p=a_{2,2}^{v_2,w}a_{5,5}^{v_2,w}v_2+\frac{1}{a_{2,2}^{p,t}a_{5,5}^{p,t}}p.$$

Hence,

$$a_{2,2}^{a,v_2+p}a_{5,5}^{a,v_2+p}=a_{2,2}^{v_2,w}a_{5,5}^{v_2,w}=a_{2,2}^{p,t}a_{5,5}^{p,t}. \tag{2.6}$$

From $\Delta(v_1+v_2+e_{1,2})=\Delta(v_1)+\Delta(v_2)+\Delta(e_{1,2})$, we get

$$a_{2,2}^{a,v_1+v_2+e_{1,2}}=a_{2,2}^{v_1,v},\quad a_{2,2}^{a,v_1+v_2+e_{1,2}}a_{5,5}^{a,v_1+v_2+e_{1,2}}=a_{2,2}^{v_2,w}a_{5,5}^{v_2,w},$$

$$a_{5,5}^{a,v_1+v_2+e_{1,2}}=a_{5,5}^{e_{1,2},z}.$$

Hence,

$$a_{2,2}^{v_2,w}a_{5,5}^{v_2,w}=a_{2,2}^{v_1,v}a_{5,5}^{e_{1,2},z}.$$

By (2.6), we also have

$$a_{2,2}^{p,t}a_{5,5}^{p,t}=a_{2,2}^{v_1,v}a_{5,5}^{e_{1,2},z}.$$

Thus,

$$A=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & a_{2,2}^{v_1,v} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & a_{2,2}^{v_1,v}a_{5,5}^{e_{1,2},z} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & a_{5,5}^{e_{1,2},z} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{a_{2,2}^{v_1,v}a_{5,5}^{e_{1,2},z}} \end{pmatrix}.$$

Therefore, by Proposition 2, Δ is an automorphism. This completes the proof. □

3. A simple central commutative algebra with no finite basis of identities

Let $\mathcal{C}=\langle \mathbf{1},v_1,v_2,e_{11},e_{12},e_{22},p\rangle_{\mathbb{F}}$ be an algebra over a field $\mathbb{F}$ of characteristic 0, where $\mathbf{1}$ is unity and nonzero products of basis elements

$$\{\mathbf{1},v_1,v_2,e_{11},e_{12},e_{22},p\} \tag{3.1}$$

other than $\mathbf{1}$ are defined as follows:

$$v_ie_{ij}=e_{ij}v_i=v_j,\quad v_2p=pv_2=\mathbf{1}.$$

Then the algebra $\mathcal{C}$ is a simple central commutative algebra with no finite basis of identities [19]. Let a be an element in $\mathcal{C}$. Then we can write

$$a=a_1e+a_2v_1+a_3v_2+a_4e_{11}+a_5e_{12}+a_6e_{22}+a_7p,$$

for some elements $a_1, a_2, a_3, a_4, a_5, a_6$, and a_7 in $\mathbb{F}$. Throughout the paper, let

$$\overline{a} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7)^T.$$

Conversely, if $v = (a_1, a_2, a_3, a_4, a_5, a_6, a_7)^T$ is a column vector with $a_1, a_2, a_3, a_4, a_5, a_6$, and a_7 in $\mathbb{F}$, then, throughout the paper, we will denote by $\widehat{v}$ the element

$$a_1 e + a_2 v_1 + a_3 v_2 + a_4 e_{11} + a_5 e_{12} + a_6 e_{22} + a_7 p,$$

i.e.,

$$\widehat{v} = a_1 e + a_2 v_1 + a_3 v_2 + a_4 e_{11} + a_5 e_{12} + a_6 e_{22} + a_7 p.$$

Our principal tool for the description of local and 2-local derivations of $\mathcal{C}$ is the following proposition.

Proposition 3. *A linear map $D\colon \mathcal{C} \to \mathcal{C}$ is a derivation if and only if the matrix of D in the basis (3.1) has the following form:*

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{2,2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2,2}+a_{5,5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{5,5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -(a_{2,2}+a_{5,5}) \end{pmatrix}.$$

Here the action of D corresponds to multiplying the matrix by a column on the right.

P r o o f. The proof of this proposition is similar to the proof of Proposition 1. □

Theorem 5. *Each local (2-local) derivation on the simple algebra $\mathcal{C}$ is a derivation.*

P r o o f. The proof of this theorem is similar to the proofs of Theorems 1 and 2. □

Proposition 4. *A linear map $\Phi\colon \mathcal{C} \to \mathcal{C}$ is an automorphism if and only if the matrix of Φ in the standard basis (3.1) has the following form:*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{2,2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2,2}a_{5,5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{5,5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{a_{2,2}a_{5,5}} \end{pmatrix},$$

where $a_{2,2}$ and $a_{5,5}$ are nonzero elements from $\mathbb{F}$. Here the action of Φ corresponds to multiplying the matrix by a column on the right.

Theorem 6. *Each local (2-local) automorphism on the simple algebra $\mathcal{C}$ is an automorphism.*

P r o o f. The proof of this theorem is similar to the proofs of Theorems 3 and 4. □

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APPROXIMATE CONTROLLABILITY OF IMPULSIVE STOCHASTIC SYSTEMS DRIVEN BY ROSENBLATT PROCESS AND BROWNIAN MOTION

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Abstract: In this paper we consider a class of impulsive stochastic functional differential equations driven simultaneously by a Rosenblatt process and standard Brownian motion in a Hilbert space. We prove an existence and uniqueness result and we establish some conditions ensuring the approximate controllability for the mild solution by means of the Banach fixed point principle. At the end we provide a practical example in order to illustrate the viability of our result.

Keywords: Approximate controllability, Fixed point theorem, Rosenblatt process, Mild solution stochastic impulsive systems.

1. Introduction

It is well known that approximate controllability is one of the fundamental concepts in mathematical control theory for infinite differential systems and plays a significant role in both deterministic and in stochastic dynamical systems. Approximate controllability means that the system can be moved to an arbitrary small neighborhood of the final state. Some recent researches on the existence results of approximate controllability are [8, 9, 14, 25].

Recently, there has been increasing interest in the analysis of control synthesis problems for impulsive systems due to their significance both in theory and applications, for example, in problems of sudden environmental changes, radiation of electromagnetic waves and changes in the interconnections of subsystems. For some recent researches on the existence results for impulsive stochastic differential equations, we refer the reader to monographs [3–5, 10, 23, 24, 29]. In these models, the processes are characterized by the fact that they undergo abrupt changes of state at certain moments of time between intervals of continuous evolution. For basic concepts about the impulsive systems see [12, 17].

In recent years, there has been a growing interest in stochastic functional differential equations driven by the Rosenblatt process [2, 19, 20, 22]. The theory of Rosenblatt process has been developed accordingly due to its nice properties see [13, 16, 27]. Tudor [28] investigated the Rosenblatt process which is Gaussian and the calculus for it is much easier than other processes. However, in concrete situations where the Gaussianity is not plausible for the model, one can employ the Rosenblatt process. There is corresponding literature devoted to various theoretical aspects of impulse systems controlled by Rosenblatt processes [7, 15, 18, 20].

Some dynamical systems of a special kind require a mixed process to model their dynamics [1, 26].

Inspired by the above studies, this article is devoted to demonstrating the approximate controllability of a soft solution for a class of neutral functional-stochastic differential equations controlled by a Wiener process and a Rosenblatt process independent of the form

$$\begin{cases} dx(t) = Ax(t)dt + Bu(t)dt + f(t, x(t))\,dt + g(t, x(t))\,dW(t) + \sigma(t)dZ_H(t), \\ t \in [0,T], \quad t \neq t_k, \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad k = 1,2,...,m, \\ x(0) = x_0 \in X, \end{cases} \tag{1.1}$$

where $x(\cdot)$ takes values in the separable Hilbert space X, $A : D(A) \subset X \to X$ is a closed, linear, and densely defined operator on X. Let B be a bounded linear operator from the Hilbert space U into X.

Let the control $u \in \mathcal{L}_2^{\mathcal{F}}([0,T],U)$ which is the Hilbert space of all square integrable and $\mathcal{F}_t$-adapted processes with values in U. Let Q_K be a positive, self adjoint and trace class operator on K and let $\mathcal{L}_2(K,X)$ be the space of all Q_K-Hilbert–Schmidt operators acting between K and X equipped with the Hilbert–Schmidt norm $\|.\|_{\mathcal{L}_2}$. The W is a Q_K-Wiener process on Hilbert space K.

Let Q be a positive, self adjoint and trace class operator on Y and let $\mathcal{L}_2^0(Y,X)$ be the space of all Q -Hilbert–Schmidt operators acting between Y and X equipped with the Hilbert–Schmidt norm $\|.\|_{\mathcal{L}_2^0}$. Let Z_H be a Q-Rosenblatt process on a Hilbert space Y. The process W and Z_H are independent. The functions f, g and σ will be specified later. Moreover, the fixed moments of times t_k satisfy $0 = t_0 < t_1 < ... < t_m < t_{m+1} = T$, $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$. Here $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ represents the jump in the state x at time t_k, where I_k determines the size of the jump.

Let $(\Omega, \mathcal{F}_T, P)$ be the complete probability space with the natural filtration $\{\mathcal{F}_t \mid t \in [0,T]\}$ generated by random variables $\{Z_H(s),\ W(s),\ s \in [0,T]\}$. Let x_0 be an $\mathcal{F}_0$-measurable random variable independent of W and Z_H satisfying $\mathbf{E}\,\|x_0\|^2 < \infty$. We define the following classes of functions: let $\mathcal{L}_2(\Omega, \mathcal{F}_T, X)$ be the Hilbert space of all $\mathcal{F}_T-$measurable, square integrable variables with values in X, $\mathcal{L}_2^{\mathcal{F}}([0,T],X)$ is the Hilbert space of all square integrable and $\mathcal{F}_t-$adapted processes with values in X.

The space $C([0,T], \mathcal{L}_2(\Omega, \mathcal{F}_T, X))$ is the Banach space of continuous maps except for a finite number of points t_k at which $x(t_k^-)$ and $x(t^+)$ exists and $x(t_k^-) = x(t_k)$ satisfying the condition

$$\sup_{t\in[0,T]} \mathbf{E}\,\|x(t)\|^2 < \infty$$

and $\mathbf{\Lambda}_2^T$ is the closed subspace of $C([0,T], \mathcal{L}_2(\Omega, \mathcal{F}_T, X))$ consisting of measurable and $\mathcal{F}_t$-adapted processes $x(t)$, then $\mathbf{\Lambda}_2^T$ is a Banach space with the norm defined by

$$\|x\|_{\mathbf{\Lambda}_2^T} = \left(\sup_{t\in[0,T]} \mathbf{E}\,\|x(t)\|^2\right)^{1/2}.$$

Let $\{Z_H(t),\ t \in [0,T]\}$ be the one-dimensional Rosenblatt process with parameter $H \in (1/2,1)$, Z_H has the following representation (see Tudor [28])

$$Z_H(t) = d(H)\int_0^t \int_0^t \left[\int_{y_1\vee y_2}^t \frac{\partial K^{H'}}{\partial u}(u,y_1)\frac{\partial K^{H'}}{\partial u}(u,y_2)du\right] dB(y_1)dB(y_2),$$

where

$$\begin{cases} B(t)_{t\in[0,T]} & \text{is the Wiener process,} \\ \mathcal{B}(\cdot,\cdot) & \text{is the Beta function,} \\ H' = \dfrac{H+1}{2}, \quad d(H) = \dfrac{1}{H+1}\sqrt{\dfrac{H}{2(2H-1)}}, & c_H = \sqrt{\dfrac{H(2H-1)}{\mathcal{B}(2-2H, H-1/2)}}, \\ K^H(t,s) = 1_{\{t>s\}} c_H s^{1/2-H} \displaystyle\int_s^t (u-s)^{H-3/2} u^{H-1/2} du. \end{cases}$$

Let X and Y be two real separable Hilbert spaces, $\mathcal{L}(Y;X)$ be the space of bounded linear operator from Y to X, $Q \in \mathcal{L}(Y;X)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace

$$\operatorname{tr} Q = \sum_{n=1}^{\infty} \lambda_n < \infty, \quad \lambda_n \geq 0$$

and $\{e_n\}$ is a complete orthonormal basis in Y.

We define the infinite dimensional Q-Rosenblatt process on Y as

$$Z_H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n z_n(t),$$

where $(z_n)_{n\geq 0}$ is a family of real independent Rosenblatt processes. Consider the following fundamental inequality.

Lemma 1 [21]. *If $\phi : [0,T] \to \mathcal{L}_2^0(Y;X)$ satisfies*

$$\int_0^T \|\phi(s)\|^2_{\mathcal{L}_2^0} \, ds < \infty,$$

then we have

$$E\left\| \int_0^t \phi(s) dZ_H(s) \right\|^2 \leq 2Ht^{2H-1} \int_0^t \|\phi(s)\|^2_{\mathcal{L}_2^0} \, ds.$$

Definition 1. *For each $u \in \mathcal{L}_2^{\mathcal{F}}([0,T],U)$, a stochastic process $x \in \mathbf{\Lambda}_2^T$ is a mild solution of (1.1) if we have*

$$x(t) = S(t)x_0 + \int_0^t S(t-s)\left(Bu(s) + f(s,x(s))\right) ds$$
$$+ \int_0^t S(t-s)g(s,x(s))dW(s) + \int_0^t S(t-s)\sigma(s)dZ_H(s) + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-)).$$

Let $x(T;u)$ be the state value of system (1.1) at terminal time T corresponding to control u. The set

$$R(T) = \left\{x(T;u) : u \in \mathcal{L}_2^{\mathcal{F}}([0,T],U)\right\}$$

is called the reachable set of (1.1) at the terminal time T.

Definition 2. *The stochastic control system (1.1) is called approximately controllable on the interval $[0,T]$ if*

$$\overline{R(T)} = \mathcal{L}_2(\Omega, \mathcal{F}_T, X).$$

For the proof of the main result, we impose the following conditions on data of the problem.

(Hyp 1) A is the infinitesimal generator of a compact semigroup $\{S(t),\ t \geq 0\}$ on X such that $\|S(t)\| \leq M$, for some constant $M > 0$.

(Hyp 2) **1.** The function $f : [0,T] \times X \to X$ is continuous and there exists a constant C_f such that for $x, y \in X$ and $t \in [0,T]$

$$\|f(t,x)\|^2 \leq C_f(1 + \|x\|^2),$$
$$\|f(t,x) - f(t,y)\|^2 \leq C_f \|x-y\|^2.$$

2. The function $g : [0, T] \times X \to \mathcal{L}_2(K, X)$ is continuous and there exists a constant C_g such that for $x, y \in X$ and $t \in [0, T]$

$$\|g(t,x)\|^2_{\mathcal{L}_2} \le C_g(1 + \|x\|^2),$$
$$\|g(t,x) - g(t,y)\|^2_{\mathcal{L}_2} \le C_g \|x - y\|^2 .$$

(Hyp 3) The function $\sigma : [0, T] \to \mathcal{L}^0_2$ is bounded by a positive constant L for all $t \in [0, T]$.

(Hyp 4) $I_k : X \to X$ is continuous and there exist constants $d_k,\ q_k > 0$ such that, for $x, y \in X$

$$(i)\ \|I_k(x) - I_k(y)\|^2 \le d_k \|x - y\|^2 , \quad k \in \{1, ..., m\} ,$$
$$(ii)\ \|I_k(x)\|^2 \le q_k \left(1 + \|x\|^2\right), \quad k \in \{1, ..., m\} ,$$
$$(iii)\ M^2 m \Big(\sum_{k=1}^{m} d_k\Big) < \frac{1}{4}.$$

(Hyp 5) For each $0 \le t < T$, the operator $\alpha(\alpha I + \Gamma^T_t)^{-1} \to 0$ in the strong operator topology as $\alpha \to 0^+$, with $\Gamma^T_s \in \mathcal{L}(X, X)$ and

$$\Gamma^T_s = \int_s^T S(T - t)BB^*S^*(T - t)dt.$$

(Hyp 6) **1.** The function $f : [0, T] \times X \to X$ is continuous and there exists a constant C_f such that for $x, y \in X$ and $t \in [0, T]$

$$\|f(t,x) - f(t,y)\|^2 \le C_f \|x - y\|^2 .$$

2. The function $g : [0, T] \times X \to \mathcal{L}_2(K, X)$ is continuous and there exists a constant C_g such that for $x, y \in X$ and $t \in [0, T]$

$$\|g(t,x) - g(t,y)\|^2_{\mathcal{L}_2} \le C_g \|x - y\|^2 .$$

3. The functions f and g are uniformly bounded, then there exists $C > 0$ such that

$$\|f(s, x(s))\|^2 + \|g(s, x(s))\|^2_{\mathcal{L}_2} \le C.$$

Lemma 2 [6]. *For any $x_T \in \mathcal{L}_2(\Omega, \mathcal{F}_T, X)$ there exists a unique $\Psi \in \mathcal{L}^{\mathcal{F}}_2([0, T]; \mathcal{L}_2(K, X))$ such that*

$$x_T = \mathbf{E}(x_T) + \int_0^T \Psi(s)dW(s).$$

For any $\alpha > 0$ and an arbitrary function $x(.)$, we define the control function for system (1.1) in the following form

$$\begin{aligned} u^\alpha(t, x) &= B^*S^*(T - t)(\alpha I + \Gamma^T_0)^{-1} (\mathbf{E}(x_T) - S(T)x_0) \\ &+ B^*S^*(T - t)\int_0^t (\alpha I + \Gamma^T_s)^{-1}\Psi(s)dW(s) - B^*S^*(T - t)\int_0^t (\alpha I + \Gamma^T_s)^{-1}S(T - s)\sigma(s))dZ_H(s) \\ &- B^*S^*(T - t)\int_0^t (\alpha I + \Gamma^T_s)^{-1}S(T - s)f(s, x(s))ds \\ &- B^*S^*(T - t)\int_0^t (\alpha I + \Gamma^T_s)^{-1}S(T - s)g(s, x(s))dW(s) \\ &- B^*S^*(T - t)(\alpha I + \Gamma^T_0)^{-1} \sum_{0<t_k<t} S(t - t_k)I_k(x(t_k^-)), \end{aligned}$$

the function $u^\alpha(t, x)$ is defined so that the system driven by this command has a unique solution (see Theorem 1) and moreover the system is approximately controllable (see Theorem 2).

Lemma 3. *There exists positive real constant M_u such that, for all $x,\ y \in \mathbf{\Lambda}_2^T$ we have*

$$\mathbf{E}\left\|u^{\alpha}(t,x)-u^{\alpha}(t,y)\right\|^2 \leq \frac{M_u}{\alpha^2}\left\|x-y\right\|^2_{\mathbf{\Lambda}_2^T}, \tag{1.2}$$

$$\mathbf{E}\left\|u^{\alpha}(t,x)\right\|^2 \leq \frac{M_u}{\alpha^2}\left(1+\left\|x\right\|^2_{\mathbf{\Lambda}_2^T}\right). \tag{1.3}$$

P r o o f. Let $x,\ y \in \mathbf{\Lambda}_2^T$, we have

$$\mathbf{E}\Big\|u^{\alpha}(t,x)-u^{\alpha}(t,y)\Big\|^2 \leq 3\mathbf{E}\Big\|B^*S^*(T-t)\int_0^t(\alpha I+\Gamma_s^T)^{-1}S(T-s)\left[f(s,x(s))-f(s,y(s))\right]ds\Big\|^2$$
$$+3\mathbf{E}\Big\|B^*S^*(T-t)\int_0^t(\alpha I+\Gamma_s^T)^{-1}S(T-s)\left[g(s,x(s))-g(s,y(s))\right]dW(s)\Big\|^2$$
$$+3\mathbf{E}\Big\|B^*S^*(T-t)(\alpha I+\Gamma_0^T)^{-1}\sum_{k=1}^{m}S(T-t_k)\left[I_k(x(t_k^-))-I_k(y(t_k^-))\right]\Big\|^2.$$

Using the Holder inequality, Ito isometric theorem and the assumptions on the data, we obtain

$$\mathbf{E}\left\|u^{\alpha}(t,x)-u^{\alpha}(t,y)\right\|^2 \leq \frac{3}{\alpha^2}\left\|B\right\|^2 M^4TC_f\int_0^t\mathbf{E}\left\|x(s)-y(s)\right\|^2ds$$
$$+\frac{3}{\alpha^2}\left\|B\right\|^2M^4C_g\int_0^t\mathbf{E}\left\|x(s)-y(s)\right\|^2ds+\frac{3}{\alpha^2}\left\|B\right\|^2M^4m\Big(\sum_{k=1}^{m}d_k\Big)\mathbf{E}\left\|\left[I_k(x(t_k^-))-I_k(y(t_k^-))\right]\right\|^2$$
$$\leq \frac{3}{\alpha^2}\left\|B\right\|^2M^4TC_fT\sup_{s\in[0,T]}\mathbf{E}\left\|x(s)-y(s)\right\|^2$$
$$+\frac{3}{\alpha^2}\left\|B\right\|^2M^4C_gT\sup_{s\in[0,T]}\mathbf{E}\left\|x(s)-y(s)\right\|^2+m\Big(\sum_{k=1}^{m}d_k\Big)\sup_{s\in[0,T]}\mathbf{E}\left\|x(s)-y(s)\right\|^2$$
$$\leq \frac{3}{\alpha^2}\left\|B\right\|^2M^4\Big[T^2C_f+TC_g+m\Big(\sum_{k=1}^{m}d_k\Big)\Big]\left\|x-y\right\|^2_{\mathbf{\Lambda}_2^T}$$
$$=\frac{M_u}{\alpha^2}\left\|x-y\right\|^2_{\mathbf{\Lambda}_2^T},$$

where

$$M_{\mu}=3\left\|B\right\|^2M^4\Big[T^2C_f+TC_g+m\Big(\sum_{k=1}^{m}d_k\Big)\Big].$$

The proof of the second (1.3) is similar. □

2. Approximate controllability

For any $\alpha>0$, define the operator $F_{\alpha}:\mathbf{\Lambda}_2^T\to\mathbf{\Lambda}_2^T$ by

$$(F_{\alpha}x)(t)=S(t)x_0+\int_0^tS(t-s)\left(Bu^{\alpha}(s,x)+f(s,x(s))\right)ds$$
$$+\int_0^tS(t-s)g(s,x(s))dW(s)+\int_0^tS(t-s)\sigma(s)dZ_H(s)+\sum_{0<t_k<t}S(t-t_k)I_k(x(t_k^-)).$$

The first main result is the following theorem.

Theorem 1. *Under assumptions* (Hyp 1)–(Hyp 5)*, the system* (1.1) *has a mild solution on* $[0, T]$.

P r o o f. **Step 1.** Let $0 \leq t_1 \leq t_2 \leq T$. Then for any fixed $x \in \mathbf{\Lambda}_2^T$

$$\mathbf{E}\left\|(F_\alpha x)(t_2) - (F_\alpha x)(t_1)\right\|^2 \leq 6\mathbf{E}\left\|\left(S(t_2) - S(t_1)\right)x_0\right\|^2$$
$$+6\mathbf{E}\Big\|\int_0^{t_2} S(t_2 - s)f(s, x(s))ds - \int_0^{t_1} S(t_1 - s)f(s, x(s))ds\Big\|^2$$
$$+6\mathbf{E}\Big\|\int_0^{t_2} S(t_2 - s)g(s, x(s))dW(s) - \int_0^{t_1} S(t_1 - s)g(s, x(s))dW(s)\Big\|^2$$
$$+6\mathbf{E}\Big\|\int_0^{t_2} S(t_2 - s)\sigma(s)dZ_H(s) - \int_0^{t_1} S(t_1 - s)\sigma(s)dZ_H(s)\Big\|^2$$
$$+6\mathbf{E}\Big\|\sum_{0<t_k<t_2} S(t_2 - t_k)I_k(x(t_k^-)) - \sum_{0<t_k<t_1} S(t_1 - t_k)I_k(x(t_k^-))\Big\|^2$$
$$+6\mathbf{E}\Big\|\int_0^{t_2} S(t_2 - s)Bu^\alpha(s, x(s))ds - \int_0^{t_1} S(t_1 - s)Bu^\alpha(s, x(s))ds\Big\|^2$$
$$= 6\left(J_1 + J_2 + J_3 + J_4 + J_5 + J_6\right).$$

Thus we obtain by Holder inequality, Ito isometric theorem and the assumptions (Hyp 1)–(Hyp 5)

$$J_1 \leq \left\|S(t_2) - S(t_1)\right\|^2 \mathbf{E}\left\|x_0\right\|^2,$$
$$J_2 \leq 2\mathbf{E}\Big\|\int_0^{t_1}\left(S(t_2 - s) - S(t_1 - s)\right)f(s, x(s))ds\Big\|^2 + 2\mathbf{E}\Big\|\int_{t_1}^{t_2} S(t_2 - s)f(s, x(s))ds\Big\|^2$$
$$\leq 2t_1\int_0^{t_1}\mathbf{E}\left\|\left(S(t_2 - s) - S(t_1 - s)\right)f(s, x(s))\right\|^2 ds + 2M^2(t_2 - t_1)\int_{t_1}^{t_2}\mathbf{E}\left\|f(s, x(s))\right\|^2 ds,$$
$$J_3 \leq 2\mathbf{E}\Big\|\int_0^{t_1}\left(S(t_2 - s) - S(t_1 - s)\right)g(s, x(s))dW(s)\Big\|^2 2 + \mathbf{E}\Big\|\int_{t_1}^{t_2} S(t_2 - s)g(s, x(s))dW(s)\Big\|^2$$
$$\leq 2\int_0^{t_1}\mathbf{E}\Big\|\left(S(t_2 - s) - S(t_1 - s)\right)g(s, x(s))\Big\|_{\mathcal{L}_2}^2 ds + 2M^2\int_{t_1}^{t_2}\mathbf{E}\left\|g(s, x(s))\right\|_{\mathcal{L}_2}^2 ds,$$
$$J_4 \leq 2\mathbf{E}\Big\|\int_0^{t_1}\left(S(t_2 - s) - S(t_1 - s)\right)\sigma(s)dZ_H(s)\Big\|^2 + 2\mathbf{E}\Big\|\int_{t_1}^{t_2} S(t_2 - s)\sigma(s)dZ_H(s)\Big\|^2$$
$$\leq 4Ht_1^{2H-1}\int_0^{t_1}\mathbf{E}\left\|\left(S(t_2 - s) - S(t_1 - s)\right)\sigma(s)\right\|_{\mathcal{L}_2^0}^2 ds + 4M^2H\left(t_2^{2H-1} - t_1^{2H-1}\right)\int_{t_1}^{t_2}\left\|\sigma(s)\right\|_{\mathcal{L}_2^0}^2 ds,$$
$$J_5 \leq 2m\sum_{t_1<t_k<t_2}\mathbf{E}\left\|S(t_2 - s)I_k(x(t_k^-))\right\|^2 + 2m\sum_{0<t_k<t_1}\mathbf{E}\left\|\left(S(t_2 - s) - S(t_1 - s)\right)I_k(x(t_k^-))\right\|^2$$
$$\leq 2mM^2\sum_{t_1<t_k<t_2}\mathbf{E}\left\|I_k(x(t_k^-))\right\|^2 + 2m\sum_{0<t_k<t_1}\mathbf{E}\left\|\left(S(t_2 - s) - S(t_1 - s)\right)I_k(x(t_k^-))\right\|^2,$$
$$J_6 \leq 2\mathbf{E}\Big\|\int_0^{t_1}\left(S(t_2 - s) - S(t_1 - s)\right)Bu^\alpha(s, x)ds\Big\|^2 + 2\mathbf{E}\Big\|\int_{t_1}^{t_2} S(t_2 - s)Bu^\alpha(s, x)ds\Big\|^2$$
$$\leq 2t_1\int_0^{t_1}\mathbf{E}\left\|\left(S(t_2 - s) - S(t_1 - s)\right)Bu^\alpha(s, x)\right\|^2 ds + 2M^2\left\|B\right\|^2(t_2 - t_1)\int_{t_1}^{t_2}\mathbf{E}\left\|u^\alpha(s, x)\right\|^2 ds.$$

Consequently, using the strong continuity of $S(t)$, as well as the Lebesgue's dominated convergence theorem, we conclude that the right side of the above inequality tends to zero when $t_2 - t_1 \to 0$. Thus we conclude that $(F_\alpha x)(t)$ is continuous in $[0, T]$.

Step 2. Let $x \in \mathbf{\Lambda}_2^T$, then we have

$$\mathbf{E}\left\|(F_\alpha x)(t)\right\|^2 \leq 6\mathbf{E}\left\|S(t)x_0\right\|^2 + 6\mathbf{E}\Big\|\int_0^t S(t-s)Bu^\alpha(s,x)ds\Big\|^2$$
$$+6\mathbf{E}\Big\|\int_0^t S(t-s)f(s,x(s))ds\Big\|^2 + 6\mathbf{E}\Big\|\int_0^t S(t_2-s)g(s,x(s))dW(s)\Big\|^2$$
$$+6\mathbf{E}\Big\|\int_0^t S(t-s)\sigma(s)dZ_H(s)\Big\|^2 + 6\mathbf{E}\Big\|\sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-))\Big\|^2.$$

By Holder inequality, Lemma 3, Ito isometric theorem and the assumptions (Hyp 1)–(Hyp 5), we have

$$\mathbf{E}\left\|(F_\alpha x)(t)\right\|^2 \leq 6\mathbf{E}\left\|S(t)x_0\right\|^2 + 6M^2\left\|B\right\|^2 T\mathbf{E}\int_0^t \left\|u^\alpha(s,x)\right\|^2 ds$$
$$+6M^2T\mathbf{E}\int_0^t \left\|f(s,x(s))\right\|^2 ds + 6M^2\mathbf{E}\int_0^t \left\|g(s,x(s))\right\|^2_{\mathcal{L}_2} ds$$
$$+12M^2HT^{2H-1}\mathbf{E}\int_0^t \left\|\sigma(s)\right\|^2_{\mathcal{L}_2^0} ds + 6mM^2\sum_{k=1}^m \mathbf{E}\left\|I_k(x(t_k^-))\right\|^2.$$

Hence

$$\mathbf{E}\left\|(F_\alpha x)(t)\right\|^2 \leq 6M^2\mathbf{E}\left\|x_0\right\|^2 + 6M^2\left\|B\right\|^2 T^2\frac{M_u}{\alpha^2}\left(1+\left\|x\right\|^2_{\mathbf{\Lambda}_2^T}\right)$$
$$+6M^2T^2C_f\left(1+\left\|x\right\|^2_{\mathbf{\Lambda}_2^T}\right) + 6M^2TC_g\left(1+\left\|x\right\|^2_{\mathbf{\Lambda}_2^T}\right)$$
$$+12M^2HT^{2H-1}TL + 6mM^2\Big(\sum_{k=1}^m q_k\Big)\left(1+\left\|x\right\|^2_{\mathbf{\Lambda}_2^T}\right)$$
$$\leq 6M^2\left(\mathbf{E}\left\|x_0\right\|^2 + 2HT^{2H-1}TL\right)$$
$$+6M^2\Big(\left\|B\right\|^2 T^2\Big[\frac{M_u}{\alpha^2}+C_f\Big] + TC_g + m\Big(\sum_{k=1}^m q_k\Big)\Big)\left(1+\left\|x\right\|^2_{\mathbf{\Lambda}_2^T}\right),$$

we thus obtain that $\left\|(F_\alpha x)\right\|^2_{\mathbf{\Lambda}_2^T} < \infty$. Since $(F_\alpha x)(t)$ is continuous on $[0,T]$, therefore F_α maps $\mathbf{\Lambda}_2^T$, in itself.

Step 3. Let $x, y \in \mathbf{\Lambda}_2^T$, then for any fixed $t \in [0,T]$ we have

$$\left\|(F_\alpha x)(t) - (F_\alpha y)(t)\right\|^2 \leq 4\mathbf{E}\Big\|\int_0^t S(t-s)B\left(u^\alpha(s,x) - u^\alpha(s,y)\right)ds\Big\|^2$$
$$+4\mathbf{E}\Big\|\int_0^t S(t-s)\left(f(s,x(s)) - f(s,y(s))\right)ds\Big\|^2$$
$$+4\mathbf{E}\Big\|\int_0^t S(t-s)\left(g(s,x(s)) - g(s,y(s))\right)dW(s)\Big\|^2$$
$$+4\mathbf{E}\Big\|\sum_{0<t_k<t} S(t-t_k)\left(I_k(x(t_k^-)) - I_k(y(t_k^-))\right)\Big\|^2.$$

By assumptions (Hyp 1)–(Hyp 5) combined with Hölder's inequality, Lemma 3 and Ito isometric theorem, we get that

$$\begin{aligned}&\|(F_\alpha x)(t)-(F_\alpha y)(t)\|^2\\ &\le 4M^2\|B\|^2 t\int_0^t \|u^\alpha(s,x)-u^\alpha(s,y)\|^2 ds+4M^2t\int_0^t\|f(s,x(s))-f(s,y(s))\|^2 ds\\ &+4M^2\int_0^t\|g(s,x(s))-g(s,y(s))\|^2_{\mathcal{L}_2} ds+4M^2m\Big(\sum_{k=1}^m d_k\Big)\left\|I_k(x(t_k^-))-I_k(y(t_k^-))\right\|^2.\end{aligned}$$

Therefore,

$$\begin{aligned}&\|(F_\alpha x)(t)-(F_\alpha y)(t)\|^2\\ &\le 4M^2\|B\|^2 t\frac{M_\mu}{\alpha^2}\int_0^t\|x(s)-y(s)\|^2 ds+4M^2tC_f\int_0^t\|x(s)-y(s)\|^2 ds\\ &+4M^2C_g\int_0^t\|x(s)-y(s)\|^2 ds+4M^2m\Big(\sum_{k=1}^m d_k\Big)\left\|x(t_k^-)-y(t_k^-)\right\|^2.\end{aligned}$$

Then we have

$$\begin{aligned}&\sup_{s\in[0,t]}\mathbf{E}\|(F_\alpha x)(t)-(F_\alpha y)(t)\|^2\\ &\le 4M^2\bigg(\|B\|^2t^2\frac{M_\mu}{\alpha^2}+t(tC_f+C_g)+m\Big(\sum_{k=1}^m d_k\Big)\bigg)\sup_{s\in[0,t]}\mathbf{E}\|x(s)-y(s)\|^2\\ &=\varphi(t)\sup_{s\in[0,t]}\mathbf{E}\|x(s)-y(s)\|^2,\end{aligned}$$

where

$$\varphi(t)=4M^2\|B\|^2t^2\frac{M_\mu}{\alpha^2}+4M^2t(tC_f+C_g)+4M^2m\Big(\sum_{k=1}^m d_k\Big).$$

We have (see (Hyp 4)–(iii))

$$\varphi(0)=4M^2m\left(\sum_{k=1}^m d_k\right)<1.$$

So there is T_1 with $0<T_1\le T$ such that $0<\varphi(T_1)<1$ and F_α is a contraction mapping on $\mathbf{\Lambda}_2^{T_1}$ and consequently has a unique fixed point. So by repeating the procedure, we extend the solution to the interval $[0,T]$ in several finite steps. □

The second main result is the following theorem.

Theorem 2. *Under assumptions* (Hyp 1), (Hyp 3), (Hyp 4), (Hyp 5) *and* (Hyp 6), *the system* (1.1) *is approximately controllable on* $[0,T]$.

P r o o f. Let x_α the solution of system (1.1) corresponding to $\mu(t,x)=\mu^\alpha(t,x)$. We obtain by the stochastic Fubini theorem

$$\begin{aligned}&x_\alpha(T)=x_T-\alpha(\alpha I+\Gamma_0^T)^{-1}(\mathbf{E}\bar{x}_T-S(T)x_0)\\ &+\alpha\int_0^T(\alpha I+\Gamma_s^T)^{-1}S(T-s)f(s,x(s)ds+\alpha\int_0^T(\alpha I+\Gamma_s^T)^{-1}[S(T-s)g(s,x(s)-\Psi(s)]\,dW(s)\\ &+\alpha\int_0^T(\alpha I+\Gamma_s^T)^{-1}S(T-s)\sigma(s)dZ_H(s)+\alpha(\alpha I+\Gamma_0^T)^{-1}\sum_{k=1}^m S(T-t_k)I_k(x^\alpha(t_k^-)).\end{aligned}$$

By the hypotheses (Hyp 6–2), there is a subsequence still designated by $\{f(s, x_\alpha(s), g(s, x_\alpha(s)\}$ which converges weakly to some $\{f(s), g(s)\}$ in $X \times \mathcal{L}_2$ and $\left\{I_k(x^\alpha(t_k^-))\right\}$ weakly converging to $\{I_k(w)\}$ in X. By the compactness of $\{S(t) : t \geq 0\}$, we have

$$S(T-s)f(s, x_\alpha(s) \to S(T-s)f(s),$$
$$S(T-s)g(s, x_\alpha(s) \to S(T-s)g(s),$$
$$S(T-t_k)I_k(x^\alpha(t_k^-)) \to S(T-t_k)I_k(w).$$

By hypothesis (Hyp 5), we have

$$\begin{cases} \alpha(\alpha I + \Gamma_s^T)^{-1} \to 0 \quad \text{strongly} \quad \text{as} \quad \alpha \to 0^+, \quad \text{for all} \quad 0 \leq s \leq T, \\ \left\|\alpha(\alpha I + \Gamma_s^T)^{-1}\right\| \leq 1. \end{cases}$$

So, by the Lebesgue dominated convergence theorem we obtain

$$\begin{aligned}
\mathbf{E}\left\|x_\alpha(T) - x_T\right\|^2 &\leq 9\mathbf{E}\left\|\alpha(\alpha I + \Gamma_0^T)^{-1}\left(\mathbf{E}x_T - S(T)x_0\right)\right\|^2 + 9\mathbf{E}\int_0^T \left\|\alpha(\alpha I + \Gamma_s^T)^{-1}\Psi(s)\right\|_{\mathcal{L}_2}^2 ds \\
&+18HT^{2H-1}\int_0^T \left\|\alpha(\alpha I + \Gamma_s^T)^{-1}S(T-s)\sigma(s)\right\|_{\mathcal{L}_2^0}^2 ds + 9\mathbf{E}\Big(\int_0^T \left\|\alpha(\alpha I + \Gamma_s^T)^{-1}S(T-s)f(s)\right\| ds\Big)^2 \\
&+9\mathbf{E}\Big(\int_0^T \left\|\alpha(\alpha I + \Gamma_s^T)^{-1}\right\| \left\|S(T-s)\left(f(s, x_\alpha(s)) - f(s)\right)\right\| ds\Big)^2 \\
&+9\mathbf{E}\int_0^T \left\|\alpha(\alpha I + \Gamma_s^T)^{-1}S(T-s)g(s)\right\|_{\mathcal{L}_2}^2 ds \\
&+9\mathbf{E}\int_0^T \left\|\alpha(\alpha I + \Gamma_s^T)^{-1}\right\|^2 \left\|S(T-s)\left(g(s, x_\alpha(s)) - g(s)\right)\right\|_{\mathcal{L}_2}^2 ds \\
&+9\mathbf{E}\Big\|\sum_{k=1}^m \alpha(\alpha I + \Gamma_s^T)^{-1}S(T-t_k)I_k(w)\Big\|^2 \\
&+9\mathbf{E}\left\|\alpha(\alpha I + \Gamma_s^T)^{-1}\right\|^2 \Big\|\sum_{k=1}^m S(T-t_k)I_k(x^\alpha(t_k^-)) - \sum_{k=1}^m S(T-t_k)I_k(w)\Big\|^2 \to 0 \quad as \quad \alpha \to 0^+.
\end{aligned}$$

Then the system (1.1) is approximately controllable. □

3. Example

In this section we present an example. Let $X = L_2[0, \pi]$, $U = L_2[0, \pi]$ and $x_0 \in L_2[0, \pi]$. Let $A \subset D(A) : X \to X$ be the linear operator given by $Ay = y''$, where

$$D(A) = \{y \in X \ / \ y,\ y' \text{ are absolutely continuous } y'' \in X, \quad y(0) = y(\pi) = 0\}.$$

Let $B \in L(\mathbb{R}, X)$ be defined as

$$(Bu)(z) = b(x)u, \quad 0 \leq z \leq \pi, \quad u \in \mathbb{R}, \quad b(x) \in L_2[0, \pi].$$

Here $W(t)$ denotes a one dimensional standard Brownian motion and Z_H is a Rosenblatt process, the processes W and Z_H are independent.

Consider the control system driven by the process W and Z_H to illustrate the obtained theory

$$\begin{cases} dx(t,z) = \Big(\dfrac{\partial^2}{\partial z^2}x(t,z) + b(z)u(t) + f_1\left(t, x(t,z)\right)\Big)dt \\ \qquad\qquad +g_1\left(t, x(t,z)\right)dw(t) + \sigma(t)dZ_H, \quad t \in [0,T], \quad z \in [0,\pi], \\ \Delta x(t_k, z) = x(t_k^+, z) - x(t_k^-, z) = \dfrac{1}{2^k}x(t_k, z), \quad t = t_k, \quad k = 1, ..., m, \\ x(t,0) = x(t,\pi) = 0, \quad t \in [0,T], \\ x(0,z) = x_0(z), \quad z \in [0,\pi]. \end{cases} \tag{3.1}$$

Suppose f_1, g_1: $\mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ are continuous, satisfy the Lipschitz condition and the linear growth condition and are uniformly bounded.

First of all, note that there exists a complete orthonormal set $\{e_n\}_{n\geq 1}$ of eigenvectors of A with

$$e_n(z) = \sqrt{(2/\pi)}\sin nz, \quad 0 \leq z \leq \pi, \quad n = 1, 2, ...$$

and the compact semigroup $S(t),\ t \geq 0$, that is generated by A such that

$$Ay = -\sum_{n=1}^{\infty} n^2 \langle y, e_n\rangle\, e_n(y), \quad y \in D(A),$$
$$S(t)y = \sum_{n=1}^{\infty} e^{-n^2 t} \langle x, e_n\rangle\, e_n(y), \quad y \in X.$$

Now define the functions: $f : [0,T] \times X \to X,\ g : [0,T] \times X \to L(K;X)$ as follows

$$f(t,x)(z) = f_1(t, x(z)),$$
$$g(t,x)(z) = g_1(t, x(z))$$

for $t \in [0,T],\ x \in X$ and $0 < z < \pi$. Consequently, by [11, Theorem 4.1.7], we have that the deterministic linear system (3.1) is approximately controllable on every $[0,t],\ t > 0$, provided that

$$\int_0^{\pi} b(z)e_n(z)dz \neq 0, \quad \text{for} \quad n = 1, 2, 3, \ldots.$$

Hence, all conditions of Theorem 2 are satisfied, and consequently system (3.1) is approximately controllable on $[0,T]$.

4. Conclusion

Approximate controllability of a class of impulsive stochastic functional differential equations driven simultaneously by a Rosenblatt process and standard Brownian motion in a Hilbert space are obtained. The controllability problem is transformed into a fixed point problem for an appropriate nonlinear operator in a function space. By using some famous fixed point theorems and the approximating technique some new existence and controllability results are obtained.

We also remark that the same idea can be used to study the controllability and the exponential stability of impulsive stochastic functional differential equations driven simultaneously by a Rosenblatt process and standard Brownian motion under non-Lipschitz condition and with non local conditions.

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PERIODIC SOLUTIONS OF A CLASS OF SECOND ORDER NEUTRAL DIFFERENTIAL EQUATIONS WITH MULTIPLE DIFFERENT DELAYS

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Abstract: The present work mainly probes into the existence and uniqueness of periodic solutions for a class of second-order neutral differential equations with multiple delays. Our approach is based on using Banach and Krasnoselskii's fixed point theorems as well as the Green's function method. Besides, two examples are exhibited to validate the effectiveness of our findings which complement and extend some relevant ones in the literature.

Keywords: Fixed point theorem, Green's function, Neutral differential equation, Periodic solutions.

1. Introduction

We frequently encounter neutral delay differential equations in the modeling of many phenomena in various domains such as physics, biology, population dynamics, medicine, epidemiology, economics, etc.

The investigation on such equations has been one of the most attracting topics in the literature. Recently, these equations have received a considerable attention and many researchers have sought to study them. For some related works, we refer the interested reader to some of them [1, 2, 4, 6, 8–10, 12, 13] and the references cited therein.

Stimulated by the aforementioned publications, we propose the following class of second order neutral differential equations

$$\frac{d^2}{dt^2}x\left(t\right)+p\left(t\right)\frac{d}{dt}x\left(t\right)+q\left(t\right)x\left(t\right)+\frac{d^2}{dt^2}\Big[k\left(t\right)x\left(t\right)-\sum_{\ell=1}^{n}c_{\ell}\left(t\right)x\left(t-\tau_{\ell}\left(t\right)\right)\Big]=e(t), \tag{1.1}$$

where $p, q \in \mathcal{C}(\mathbb{R}, (0, \infty))$, $k, c_\ell, \tau_\ell \in \mathcal{C}^2(\mathbb{R}, (0, \infty))$, $\ell = \overline{1, n}$ and $e \in \mathcal{C}(\mathbb{R}, [0, \infty))$ are T-periodic functions.

In the current work, the authors aim is to establish sufficient conditions under which Banach and Krasnoselskii's fixed point theorems are guaranteed to work and hence the existence and uniqueness of periodic solutions of the equation (1.1) are proved. The general idea of our technique is to convert the equation (1.1) into an equivalent integral one in order to pave the way for the application of Banach and Krasnoselskii's fixed point theorems. Indeed, this last one with the help of Arzelà-Ascoli theorem and some properties of the obtained Green's kernel, is a proper means for achieving our desired goals.

The key contributions of this work can be summarized as follows.

(i) New sufficient conditions that ensure the existence of periodic solutions of the equation (1.1) are established.

(ii) The studied problems in [1, 3–5, 7, 9, 12] are with globally Lipschitz source terms while this condition is not required here.

The basic frame of this paper is as follows. Section 2, provides some preliminary results and prerequisites that will be used in the sequel. Section 3 is dedicated to the statements and the proofs of our main results. In Section 4, we present two examples to which our main findings can be applied. The conclusion is included in the last section.

2. Preliminaries

Let

$$P_T = \left\{x \in \mathcal{C}(\mathbb{R}, \mathbb{R}t),\ x(t+T) = x(t)\right\}, \quad T > 0,$$

endowed with the supremum norm

$$\|x\| = \sup_{t\in\mathbb{R}} |x(t)| = \sup_{t\in[0,T]} |x(t)|,$$

be a Banach space.

Throughout this paper we will assume that the following hypothesis are fulfilled.
Here p, q, k, e, c_ℓ and τ_ℓ are T-periodic real-valued functions such that

$$\begin{gathered} p(t+T) = p(t), \quad q(t+T) = q(t), \quad k(t+T) = k(t), \\ e(t+T) = e(t), \quad c_\ell(t+T) = c_\ell(t), \quad \tau_\ell(t+T) = \tau_\ell(t), \quad \ell = \overline{1, n}, \end{gathered} \tag{2.1}$$

and

$$\int_0^T p(s)ds > 0, \quad \int_0^T q(s)ds > 0, \quad \tau_\ell(t) \geq \tau_\ell^* > 0, \quad \ell = \overline{1, n}. \tag{2.2}$$

Lemma 1 [10]. *If* (2.1) *and* (2.2) *hold and*

$$\frac{R_1\left[\exp\left(\int_0^T p(u)\,du\right) - 1\right]}{Q_1 T} \geq 1, \tag{2.3}$$

where

$$R_1 = \max_{t\in[0,T]}\left|\int_t^{t+T} \frac{\exp\left(\int_t^s p(u)\,du\right)}{\exp\left(\int_0^T p(u)\,du\right) - 1} q(s)ds\right|,$$

and

$$Q_1 = \left(1 + \exp\left(\int_0^T p(u)\,du\right)\right)^2 R_1^2,$$

then there are continuous and T-periodic functions a and b such that

$$b(t) > 0, \quad \int_0^T a(u)du > 0, \quad a(t) + b(t) = p(t),$$

and

$$\frac{d}{dt}b(t) + a(t)b(t) = q(t),$$

for all $t \in \mathbb{R}$. Furthermore, if $\phi \in P_T$ then the equation

$$x''(t) + p(t)x'(t) + q(t)x(t) = \phi(t)$$

has a T-periodic solution. Moreover, the periodic solution can be expressed as

$$x(t) = \int_t^{t+T} G(t,s)\phi(s)ds,$$

where

$$\begin{aligned} G(t,s) = {} & \frac{\displaystyle\int_t^s \exp\left[\int_t^u b(v)\,dv + \int_u^s a(v)\,dv\right] du}{\left[\exp\left(\displaystyle\int_0^T a(u)\,du\right) - 1\right]\left[\exp\left(\displaystyle\int_0^T b(u)\,du\right) - 1\right]} \\ & + \frac{\displaystyle\int_s^{t+T} \exp\left[\int_t^u b(v)\,dv + \int_u^{s+T} a(v)\,dv\right] du}{\left[\exp\left(\displaystyle\int_0^T a(u)\,du\right) - 1\right]\left[\exp\left(\displaystyle\int_0^T b(u)\,du\right) - 1\right]}. \end{aligned} \tag{2.4}$$

Corollary 1 [12]. *If G is the Green's function given by (2.4), then G satisfies*

$$G(t,t+T) = G(t,t), \ G(t+T,s+T) = G(t,s),$$

$$\frac{\partial}{\partial s}G(t,s) = a(s)G(t,s) - \frac{\exp\left(\displaystyle\int_t^s b(v)\,dv\right)}{\exp\left(\displaystyle\int_0^T b(v)\,dv\right) - 1},$$

$$\frac{\partial}{\partial t}G(t,s) = -b(t)G(t,s) + \frac{\exp\left(\displaystyle\int_t^s a(v)\,dv\right)}{\exp\left(\displaystyle\int_0^T b(v)\,dv\right) - 1},$$

$$\frac{\partial^2}{\partial s^2}G(t,s) = \left(a(s) + a'(s)\right)G(t,s) - \left(a(s) + b(s)\right)\frac{\exp\left(\displaystyle\int_t^s b(v)\,dv\right)}{\exp\left(\displaystyle\int_0^T b(v)\,dv\right) - 1}.$$

Furthermore, by putting

$$A=\int_0^T p\left(u\right)du,\quad B=T^2\exp\left(\frac{1}{T}\int_0^T \ln\left(q\left(u\right)\right)du\right),$$
$$M_1=\frac{1}{2}\left(A-\sqrt{A^2-4b}\right),\quad M_2=\frac{1}{2}\left(A+\sqrt{A^2+4b}\right),$$
$$\alpha_1=\frac{T}{\left(e^{M_2}-1\right)^2},\quad \alpha_2=\frac{T\exp\left(\int_0^T p\left(u\right)du\right)}{\left(e^{M_1}-1\right)^2},$$
$$H\left(t,s\right)=\frac{\exp\left(\int_t^s b\left(v\right)dv\right)}{\exp\left(\int_0^T b\left(v\right)dv\right)-1},\quad \beta=\frac{\exp\left(\int_0^T b\left(v\right)dv\right)}{\exp\left(\int_0^T b\left(v\right)dv\right)-1},$$
$$H^*\left(t,s\right)=\frac{\exp\left(\int_t^s a\left(v\right)dv\right)}{\exp\left(\int_0^T b\left(v\right)dv\right)-1},\quad \beta^*=\frac{\exp\left(\int_0^T a\left(v\right)dv\right)}{\exp\left(\int_0^T b\left(v\right)dv\right)-1},$$

and if $A^2\geq 4B$, then we have

$$0<\alpha_1\leq G(t,s)\leq\alpha_2,\quad |H(t,s)|\leq\beta,\quad |H^*(t,s)|\leq\beta^*.$$

3. Existence and uniqueness of periodic solutions

Lemma 2. *Suppose that (2.1)–(2.3) hold. If $x\in P_T\cap\mathcal{C}^2\left(\mathbb{R},\mathbb{R}\right)$, then x is a solution of (1.1) if and only if x is a solution of the following equation*

$$\begin{aligned}x\left(t\right)&=\frac{1}{1+k\left(t\right)}\sum_{\ell=1}^{n}c_\ell\left(t\right)x\left(t-\tau_\ell\left(t\right)\right)+\frac{1}{1+k\left(t\right)}\int_t^{t+T}e\left(s\right)G\left(t,s\right)ds\\&+\int_t^{t+T}\frac{a(s)+b(s)}{1+k\left(t\right)}\left[k\left(s\right)x\left(s\right)-\sum_{\ell=1}^{n}c_\ell\left(s\right)x\left(s-\tau_\ell\left(s\right)\right)\right]H\left(t,s\right)ds\\&-\int_t^{t+T}\frac{a(s)+a'(s)}{1+k\left(t\right)}\left[k\left(s\right)x\left(s\right)-\sum_{\ell=1}^{n}c_\ell\left(s\right)x\left(s-\tau_\ell\left(s\right)\right)\right]G(t,s)ds.\end{aligned}\tag{3.1}$$

P r o o f. Let $x\in P_T\cap\mathcal{C}^2\left(\mathbb{R},\mathbb{R}\right)$. From Lemma 1, we get

$$\begin{aligned}x\left(t\right)&=\int_t^{t+T}\left\{\frac{\partial}{\partial s}\left[k\left(s\right)x\left(s\right)-\sum_{\ell=1}^{n}c_\ell\left(s\right)x\left(s-\tau_\ell\left(s\right)\right)\right]\right\}\frac{\partial}{\partial s}G\left(t,s\right)ds+\int_t^{t+T}e\left(s\right)G\left(t,s\right)ds\\&=\left[k\left(s\right)x\left(s\right)-\sum_{\ell=1}^{n}c_\ell\left(s\right)x\left(s-\tau_\ell\left(s\right)\right)\right]\frac{\partial}{\partial s}G\left(t,s\right)\Big|_t^{t+T}\\&-\int_t^{t+T}\left[k\left(s\right)x\left(s\right)-\sum_{\ell=1}^{n}c_\ell\left(s\right)x\left(s-\tau_\ell\left(s\right)\right)\right]\frac{\partial^2}{\partial s^2}G\left(t,s\right)ds+\int_t^{t+T}e\left(s\right)G\left(t,s\right)ds.\end{aligned}$$

Since

$$\left[k(s)x(s)-\sum_{\ell=1}^{n}c_\ell(s)x(s-\tau_\ell(s))\right]\frac{\partial}{\partial s}G(t,s)\Big|_t^{t+T}$$
$$=-k(t)x(t)+\sum_{\ell=1}^{n}c_\ell(t)x(t-\tau_\ell(t)),$$

and

$$\frac{\partial^2}{\partial s^2}G(t,s)=\big(a(s)+a'(s)\big)G(t,s)-(a(s)+b(s))H(t,s),$$

then

$$(1+k(t))x(t)=\sum_{\ell=1}^{n}c_\ell(t)x(t-\tau_\ell(t))+\int_t^{t+T}e(s)G(t,s)\,ds$$
$$+\int_t^{t+T}(a(s)+b(s))\left[k(s)x(s)-\sum_{\ell=1}^{n}c_\ell(s)x(s-\tau_\ell(s))\right]H(t,s)\,ds$$
$$-\int_t^{t+T}\big(a(s)+a'(s)\big)\left[k(s)x(s)-\sum_{\ell=1}^{n}c_\ell(s)x(s-\tau_\ell(s))\right]G(t,s)ds.$$

Dividing both sides of the above equation by $1+k(t)$, we obtain (3.1). The converse implication can be obtained by the derivation of (3.1). □

Fore ease of exposition, we will use the following notations

$$\lambda_1=\max_{t\in[0,T]}|a(t)|,\quad \lambda_1^*=\max_{t\in[0,T]}\left|a'(t)\right|,\quad \sigma=\max_{t\in[0,T]}|e(t)|,$$
$$\mu_1=\max_{t\in[0,T]}|b(t)|,\quad \delta_\ell=\max_{t\in[0,T]}|c_\ell(t)|,\quad \ell=\overline{1,n},$$
$$\rho_0=\min_{t\in[0,T]}|k(t)|,\quad \rho_1=\max_{t\in[0,T]}|k(t)|,\quad \rho_1^*=\max_{t\in[0,T]}\left|k'(t)\right|.$$

Furthermore, we suppose that

$$\Gamma_1=\frac{1}{1+\rho_0}\sum_{\ell=1}^{n}\delta_\ell<1, \tag{3.2}$$

and there exists $L>0$ which satisfies the following estimate

$$\Gamma_2=\frac{T\alpha_2\sigma}{1+\rho_0}+\Gamma_3L\le L, \tag{3.3}$$

where

$$\Gamma_3=\frac{1}{1+\rho_0}\left(T\Big(\rho_1+\sum_{\ell=1}^{n}\delta_\ell\Big)\Big(\beta(\lambda_1+\mu_1)+\alpha_2(\lambda_1+\lambda_1^*)\Big)+\sum_{\ell=1}^{n}\delta_\ell\right).$$

For employing Krasnoselskii's fixed point theorem, we need to define an operator that can be expressed as a sum of two operators, one of which is continuous and compact and the other is a contraction.

Indeed, from Lemma 2, we can define an operator $\mathcal{S}:P_T\longrightarrow P_T$ as follows

$$(\mathcal{S}\varphi)(t)=(\mathcal{S}_1\varphi)(t)+(\mathcal{S}_2\varphi)(t),$$

where

$$(\mathcal{S}_1\varphi)(t) = \frac{1}{1+k(t)}\sum_{\ell=1}^{n} c_\ell(t)\,\varphi(t-\tau_\ell(t)) + \frac{1}{1+k(t)}\int_t^{t+T} e(s)\,G(t,s)ds,$$

and

$$\begin{aligned}(\mathcal{S}_2\varphi)(t) = &\int_t^{t+T} \frac{a(s)+b(s)}{1+k(t)}\left[k(s)\,\varphi(s) - \sum_{\ell=1}^{n} c_\ell(s)\,\varphi(s-\tau_\ell(s))\right] H(t,s)\,ds \\ &- \int_t^{t+T} \frac{a(s)+a'(s)}{1+k(t)}\left[k(s)\,\varphi(s) - \sum_{\ell=1}^{n} c_\ell(s)\,\varphi(s-\tau_\ell(s))\right] G(t,s)ds.\end{aligned}$$

Clearly, $(\mathcal{S}_i\varphi)(t+T) = (\mathcal{S}_i\varphi)(t)$, $i = 1, 2$ which shows that operators $\mathcal{S}_i$ are well defined.

To reach our target, it suffices to prove the existence of at least one fixed point of the operator $\mathcal{S}_1 + \mathcal{S}_2$. This is due to the fact that the sought solution of equation (1.1) is just a fixed point of $\mathcal{S}_1 + \mathcal{S}_2$ and vice versa.

Theorem 1. *Suppose that conditions (2.1)–(2.3), (3.2) and (3.3) hold. Then equation (1.1) admits at least one periodic solution $x \in P_T$ which satisfies $\|x\| \le L$.*

P r o o f. For establishing the existence of periodic solutions, we use Krasnoselskii's fixed point theorem ([11]). The proof will be made in three steps.

Step 1. We show that $\mathcal{S}_1$ is a contraction mapping.

Let $\varphi_1, \varphi_2 \in P_T$, we have

$$|(\mathcal{S}_1\varphi_1)(t) - (\mathcal{S}_1\varphi_2)(t)| \le \sum_{\ell=1}^{n} \frac{c_\ell(t)}{1+k(t)}\,|\varphi_1(t-\tau_\ell(t)) - \varphi_2(t-\tau_\ell(t))| \le \Gamma_1\,\|\varphi_1-\varphi_2\|.$$

From (3.2), we deduce that $\mathcal{S}_1$ is a contraction mapping.

Step 2. We show that $\mathcal{S}_2$ is continuous and compact mapping.

Let $\varphi_1, \varphi_2 \in P_T$. For $\varepsilon > 0$ and $\eta = \Lambda\varepsilon$, where

$$\Lambda = \frac{1+\rho_0}{T\left(\rho_1 + \sum\limits_{\ell=1}^{n}\delta_\ell\right)(\beta(\lambda_1+\mu_1) + \alpha_2(\lambda_1+\lambda_1^*))},$$

we obtain

$$\|\varphi_1-\varphi_2\| \le \eta \Longrightarrow \|\mathcal{S}_2\varphi_1 - \mathcal{S}_2\varphi_2\| < \varepsilon,$$

which shows the continuity of $\mathcal{S}_2$.

On the other hand, let $\hbar > 0$, $\mathbb{K} = \{\varphi \in P_T,\ \|\varphi\| \le \hbar\}$ and $\{\varphi_n\}_{n\in\mathbb{N}}$ be a sequence from $\mathbb{K}$. We have

$$\|\mathcal{S}_2\varphi_n\| \le \frac{T\hbar\left(\rho_1 + \sum\limits_{\ell=1}^{n}\delta_\ell\right)}{1+\rho_0}(\beta(\lambda_1+\mu_1) + \alpha_2(\lambda_1+\lambda_1^*)), \tag{3.4}$$

and

$$\frac{d}{dt}\left(\mathcal{S}_2\varphi_n\right)(t) = \frac{a(t)+b(t)}{1+k(t)}\left[k(t)\,\varphi_n(t) - \sum_{\ell=1}^{n} c_\ell(t)\,\varphi_n(t-\tau_\ell(t))\right]$$
$$-\frac{b(t)(1+k(t))+k'(t)}{(1+k(t))^2}\int_t^{t+T}(a(s)+b(s))\left[k(s)\,\varphi_n(s) - \sum_{\ell=1}^{n} c_\ell(s)\,\varphi_n(s-\tau_\ell(s))\right]H(t,s)\,ds$$
$$+\frac{b(t)(1+k(t))+k'(t)}{(1+k(t))^2}\int_t^{t+T}(a(s)+a'(s))\left[k(s)\,\varphi_n(s) - \sum_{\ell=1}^{n} c_\ell(s)\,\varphi_n(s-\tau_\ell(s))\right]G(t,s)ds$$
$$-\frac{1}{1+k(t)}\int_t^{t+T}(a(s)+a'(s))\left[k(s)\,\varphi_n(s) - \sum_{\ell=1}^{n} c_\ell(s)\,\varphi_n(s-\tau_\ell(s))\right]H^*(t,s)\,ds.$$

Hence

$$\left|\frac{d}{dt}\left(\mathcal{S}_2\varphi_n\right)(t)\right| \le \Gamma_4, \tag{3.5}$$

where

$$\Gamma_4 = \hbar\Big(\rho_1 + \sum_{\ell=1}^{n}\delta_\ell\Big)\Bigg(\frac{(\lambda_1+\mu_1)+T\beta^*(\lambda_1+\lambda_1^*)}{1+\rho_0}$$
$$+T\frac{(\mu_1(1+\rho_1)+\rho_1^*)(\beta(\lambda_1+\mu_1)+\alpha_2(\lambda_1+\lambda_1^*))}{(1+\rho_0)^2}\Bigg).$$

It follows from (3.4), (3.5) and the Arzelà-Ascoli theorem [14] that $\mathcal{S}_2$ is a compact operator.

Step 3. If L is defined as in (3.3), let

$$\mathbb{M} = \{\varphi \in P_T,\ \|\varphi\| \le L\}.$$

In view of (3.3), if $\varphi_1, \varphi_2 \in \mathbb{M}$, then

$$\|\mathcal{S}_1\varphi_1 + \mathcal{S}_2\varphi_2\| \le \Gamma_2 \le L,$$

which proves that

$$\mathcal{S}_1\varphi_1 + \mathcal{S}_2\varphi_2 \in \mathbb{M}, \quad \forall \varphi_1, \varphi_2 \in \mathbb{M}.$$

From these three steps, we conclude that the operator $\mathcal{S}_2 + \mathcal{S}_2$ has at least one fixed point $x \in P_T$ with $\|x\| \le L$. Consequently, the equation (1.1) has at least one periodic solution in $\mathbb{M}$. □

Theorem 2. *Suppose that conditions (2.1)–(2.3) and (3.2) hold. If $\Gamma_3 < 1$, then the equation (1.1) has a unique periodic solution $x \in P_T$.*

P r o o f. Let $\varphi_1, \varphi_2 \in P_T$, we have

$$\left|\left(\mathcal{S}\varphi_1\right)(t) - \left(\mathcal{S}\varphi_2\right)(t)\right| \le \Gamma_3 \|\varphi_1 - \varphi_2\|.$$

Since $\Gamma_3 < 1$, the Banach fixed point theorem [11] guarantees that the operator $\mathcal{S}$ has a unique fixed point which is the unique periodic solution of the equation (1.1). □

4. Examples

Example 1. Let $L = 3\pi$. We consider the following equation

$$\begin{aligned} x''(t) + \frac{5}{12}x'(t) + \frac{1}{24}x(t) + \Bigg(\frac{1}{100}x(t) - \left(\frac{1}{120}\sin^2 2\pi t\right) x\left(t - \pi \sin^2 2\pi t\right) \\ - \left(\frac{1}{150}\cos^2 2\pi t\right) x\left(t - 2\pi\cos^4 2\pi t\right)\Bigg)'' = \frac{1}{10}\sin^4 2\pi t. \end{aligned} \tag{4.1}$$

Here

$$p(t) = \frac{5}{12}, \quad p(t) = \frac{1}{24}, \quad k(t) = \frac{1}{100}, \quad c_1(t) = \frac{1}{120}\sin^2 2\pi t,$$
$$c_2(t) = \frac{1}{150}\cos^2 2\pi t, \quad \tau_1(t) = \pi\sin^2 2\pi t, \quad \tau_2(t) = 2\pi\cos^4 2\pi t,$$
$$e(t) = \frac{100}{1010}\sin^4 2\pi t, \quad T = 1,$$

which implies

$$A = \frac{5}{12}, \quad B = \frac{1}{24}, \quad A^2 = \frac{25}{144} > 4B^2 = \frac{1}{6}, \quad R_1 = \frac{1}{10},$$
$$Q_1 = \frac{1}{100}\left(e^{5/12}+1\right)^2, \quad \frac{R_1\left[\exp\left(\int_0^T p(u)\,du\right) - 1\right]}{Q_1 T} \simeq 22.367 > 1,$$
$$M_1 = \frac{1}{6}, \quad M_2 = \frac{1}{4}, \quad \alpha_2 \simeq 46.118, \quad \beta \simeq 6.5139, \quad \Gamma_1 = \frac{3}{202} < 1,$$
$$\Gamma_2 \simeq 8.0746 < L = 3\pi, \quad \Gamma_3 \simeq 0.36742 < 1.$$

It follows from Theorem 2 that the equation (4.1) has a unique solution $x \in P_T$ which satisfies $\|x\| \leq 3\pi$.

The following example shows the usefulness of Theorem 1 when the Banach fixed point theorem cannot be applied.

Example 2. We consider the following equation

$$\begin{aligned} x''(t) + \frac{5}{12}x'(t) + \frac{1}{24}x(t) + \Bigg(\left(6\frac{\left(e^{1/6}-1\right)^2}{5e^{1/3} - 5e^{1/6} + 3e^{5/12}}x(t)\right) \\ - \left(2\frac{\left(e^{1/6}-1\right)^2}{5e^{1/3} - 5e^{1/6} + 3e^{5/12}}\sin^2 2\pi t\right) x\left(t - \pi\sin^2 2\pi t\right) \\ - \left(4\frac{\left(e^{1/6}-1\right)^2}{5e^{1/3} - 5e^{1/6} + 3e^{5/12}}\sin^2 2\pi t\right) x\left(t - 2\pi\cos^4 2\pi t\right)\Bigg)'' = 0. \end{aligned} \tag{4.2}$$

Here

$$p(t) = \frac{5}{12}, \quad p(t) = \frac{1}{24}, \quad k(t) = 6\frac{\left(e^{1/6}-1\right)^2}{5e^{1/3} - 5e^{1/6} + 3e^{5/12}},$$
$$c_1(t) = 2\frac{\left(e^{1/6}-1\right)^2}{5e^{1/3} - 5e^{1/6} + 3e^{5/12}}\sin^2 2\pi t, \quad c_2(t) = 4\frac{\left(e^{1/6}-1\right)^2}{5e^{1/3} - 5e^{1/6} + 3e^{5/12}}\cos^2 2\pi t,$$
$$\tau_1(t) = \pi\sin^2 2\pi t, \quad \tau_2(t) = 2\pi\cos^4 2\pi t, \quad e(t) = 0, \quad T = 1,$$

which implies

$$A = \frac{5}{12}, \quad B = \frac{1}{24}, \quad A^2 = \frac{25}{144} > 4B^2 = \frac{1}{6}, \quad R_1 = \frac{1}{10},$$

$$Q_1 = \frac{1}{100}\left(e^{5/12}+1\right)^2, \quad \frac{R_1\left[\exp\left(\int_0^T p(u)\,du\right)-1\right]}{Q_1 T} \simeq 22.367 > 1,$$

$$M_1 = \frac{1}{6}, \quad M_2 = \frac{1}{4}, \quad \alpha_2 \simeq 46.118, \quad \beta \simeq 6.5139, \quad \Gamma_1 = 0.03391 < 1,$$

$$\Gamma_2 = L \leq L, \quad \forall L > 0, \quad \Gamma_3 = 1.$$

Since $\Gamma_3 = 1$, we can not use Theorem 2, but $\Gamma_2 = L \leq L$, so we can apply Theorem 1 to prove that the equation (4.2) has at least one periodic solution $x \in P_T$ which satisfies $\|x\| \leq L$.

5. Conclusion

In this paper, by utilizing both the Banach and Krasnoselskii's fixed point theorems and the Green's functions method, a class of second-order neutral differential equations with multiple delays has been investigated. To be more precise, we have discussed the existence and uniqueness of periodic solutions by transforming the equation (1.1) into an equivalent integral one and then by using the Banach and Krasnoselskii's fixed point theorems.

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RESTRAINED ROMAN REINFORCEMENT NUMBER IN GRAPHS

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Abstract: A restrained Roman dominating function (RRD-function) on a graph $G = (V, E)$ is a function f from V into $\{0, 1, 2\}$ satisfying: (i) every vertex u with $f(u) = 0$ is adjacent to a vertex v with $f(v) = 2$; (ii) the subgraph induced by the vertices assigned 0 under f has no isolated vertices. The weight of an RRD-function is the sum of its function value over the whole set of vertices, and the restrained Roman domination number is the minimum weight of an RRD-function on G. In this paper, we begin the study of the restrained Roman reinforcement number $r_{rR}(G)$ of a graph G defined as the cardinality of a smallest set of edges that we must add to the graph to decrease its restrained Roman domination number. We first show that the decision problem associated with the restrained Roman reinforcement problem is NP-hard. Then several properties as well as some sharp bounds of the restrained Roman reinforcement number are presented. In particular it is established that $r_{rR}(T) = 1$ for every tree T of order at least three.

Keywords: Restrained Roman domination, Restrained Roman reinforcement.

1. Introduction

For definitions and notations not given here we refer the reader to [8]. We consider simple graphs G with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* of G is $n = n(G) = |V|$. The *open neighborhood* of a vertex v, denoted by $N(v)$ (or $N_G(v)$ to refer to G) is the set $\{u \in V(G) \mid uv \in E\}$ and its *closed neighborhood* is the set $N[v] = N_G[v] = N(v) \cup \{v\}$. The *degree* of vertex $v \in V$ is $d(v) = d_G(v) = |N(v)|$. The *maximum* and *minimum degree* in G are denoted by $\Delta = \Delta(G)$ and $\delta = \delta(G)$, respectively. A vertex of degree one is called a *leaf* and its neighbor is called a *support vertex*. As usual, the *path* (*cycle*, *complete p-partite graph*, respectively) of order n is denoted by P_n (C_n, $K_{n_1,n_2,\ldots,n_p}$, respectively). A *star* of order $n \geq 2$ is the graph $K_{1,n-1}$. For a subset $S \subseteq V$, the subgraph induced by S in G is denoted as $G[S]$.

A subset $S \subseteq V$ is a *dominating set* of G if every vertex in $V \setminus S$ has a neighbor in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G.

As an application, in the design of networks for example, it is essential to study the effect of some modifications of the graph parameters on its structure. These modifications can be deletion or addition of vertices, deletion or addition of edges. We refer the reader to chapter 7 of [8] when the graph parameter is the domination number. The *reinforcement number* $r(G)$ of a graph G is the minimum number of edges that have to be added to the graph G in order to decrease the domination number. Of course for graphs G with domination number one it was assumed that $r(G) = 0$. The concept of the reinforcement number was introduced in 1990 by Kok and Mynhardt [10], and since then it has been defined and studied for several other domination parameters, such as Roman domination [9], total Roman domination [1], quasi-total Roman domination [5], Italian domination [7], double Roman domination [4] and rainbow domination [3, 13].

In 2015, Leely Pushpam and Padmapriea [11] introduced the concept of restrained Roman domination as a new variation of Roman domination. A *restrained Roman dominating function* (RRD-function, for short) on a graph G is a function $f : V \longrightarrow \{0, 1, 2\}$ having the properties that (i) every vertex u with $f(u) = 0$ is adjacent to a vertex v with $f(v) = 2$; and (ii) the subgraph induced by the vertices assigned 0 under f has no isolated vertices. The weight of an RRD-function f is the sum

$$w(f) = \sum_{v \in V(G)} f(v)$$

and the *restrained Roman domination number* of G denoted by $\gamma_{rR}(G)$, is the minimum weight of an RRD-function on G. Any RRD-function f on G can simply be referred as $f = (V_0, V_1, V_2)$, where $V_i = \{v \in V(G) : f(v) = i\}$ for $i \in \{0, 1, 2\}$. For further studies on restrained Roman domination and its variants, see [2, 12, 14–16].

In this paper, we are interested in starting the study of the *restrained Roman reinforcement number* $r_{rR}(G)$ of a graph G defined as the cardinality of a smallest set of edges $F \subseteq E(\overline{G})$ such that $\gamma_{rR}(G + F) < \gamma_{rR}(G)$, where $\overline{G}$ denotes the complement graph of G. If there is no subset of edges F satisfying $\gamma_{rR}(G + F) < \gamma_{rI}(G)$, then we define $r_{rR}(G) = 0$. Since for any nontrivial connected graph G, $\gamma_{rR}(G) \geq 2$, we deduce that $r_{rR}(G) = 0$ for all nontrivial connected graphs with $\gamma_{rR}(G) = 2$. Moreover, a subset $E' \subseteq E(\overline{G})$ is called an $r_{rR}(G)$-set if $|E'| = r_{rR}(G)$ and $\gamma_{rR}(G + E') < \gamma_{rR}(G)$.

Further, we will prove that the decision problem associated with the Restrained Roman reinforcement is NP-hard. Then various properties of the restrained Roman reinforcement number are investigated and some sharp bounds on it are presented.

We finish this section by observing that any $r_{rR}(G)$-set of a connected graph G with $\gamma_{rR}(G) \geq 3$ can decrease the restrained Roman domination number of G by at most two.

Proposition 1. *Let G be a connected graph with $\gamma_{rR}(G) \geq 3$. If F is an $r_{rR}(G)$-set, then*

$$\gamma_{rR}(G) - 2 \leq \gamma_{rR}(G + F) \leq \gamma_{rR}(G) - 1.$$

Both bounds are sharp.

P r o o f. By assumption, $\gamma_{rR}(G + F) < \gamma_{rR}(G)$, whence the upper bound follows. To show the lower bound, let us assume that

$$\gamma_{rR}(G + F) \leq \gamma_{rR}(G) - 3.$$

Let f be a $\gamma_{rR}(G+F)$-function and let $uv \in F$ such that $0 \in \{f(u), f(v)\}$. If such an edge does not exist, then f is an RRD-function of G leading to the contradiction

$$\gamma_{rR}(G) \leq \gamma_{rR}(G+F).$$

Hence we suppose that uv exists, and let $F' = F - \{uv\}$. Without loss of generality, suppose that $f(u) = 0$. If $f(v) = 1$, then f is an RRD-function of G leading to the contradiction

$$\gamma_{rR}(G) \leq \gamma_{rR}(G+F)$$

too. Hence assume that $f(v) \neq 1$.

First let $f(v) = 2$. If u has a neighbor w in $G+F'$ with $f(w) \geq 1$, then the function g defined by $g(w) = 2$ and $g(x) = f(x)$ otherwise, is an RRD-function of $G+F'$ yielding as above to the contradiction $\gamma_{rR}(G+F') < \gamma_{rR}(G)$. Hence we assume that each neighbor u in $G+F'$ is assigned 0 under f. Let $x_1, \dots, x_k$ be the neighbors of u in $G+F'$. If $k = 1$ and x_1 has a neighbor assigned 0 other than u, then the function $g(u) = 1$ and $g(x) = f(x)$ otherwise, is an RRD-function of $G+F'$ yielding

$$\gamma_{rR}(G+F') \leq \gamma_{rR}(G+F) + 1 < \gamma_{rR}(G),$$

this is a contradiction. If $k = 1$ and x_1 has no neighbor assigned 0 other than u, then the function $g(u) = g(x_1) = 1$ and $g(x) = f(x)$ otherwise, is an RRD-function of $G+F'$ and thus

$$\gamma_{rR}(G+F') \leq \gamma_{rR}(G+F) + 2 < \gamma_{rR}(G),$$

it is a contradiction too. Hence assume that $k \geq 2$. If some x_i has no neighbor assigned 0 other than u, then the function $g(x_i) = 2$ and $g(x) = f(x)$ otherwise, is an RRD-function of $G+F'$ yielding again $\gamma_{rR}(G+F') < \gamma_{rR}(G)$. Hence we assume that for each i, x_i has at least two neighbors assigned 0 under f. In this case, we have $g(u) = 1$ and $g(x) = f(x)$ otherwise, it is an RRD-function of $G+F'$ and thus

$$\gamma_{rR}(G+F') < \gamma_{rR}(G).$$

Finally, assume that $f(v) = 0$. Since F is an $r_{rR}(G)$-set, we can suppose, without loss of generality, that all neighbors of u in $G+F'$ have positive labels under f. Now, if v has a neighbor with weight 0 in $G+F'$, then the function $g(u) = 1$ and $g(x) = f(x)$ otherwise, it is an RRD-function of $G+F'$ while if v has no neighbor with weight 0 in $G+F'$, then the function $g(u) = g(v) = 1$ and $g(x) = f(x)$ otherwise, is an RRD-function of $G+F'$. Both situations yield the contradiction $\gamma_{rR}(G+F') < \gamma_{rR}(G)$. Consequently,

$$\gamma_{rR}(G+F) \geq \gamma_{rR}(G) - 2.$$

The upper bound of Proposition 1 is attained for the cycle C_4, while the lower bound is attained for the cycle C_6. □

2. NP-hardness result

The aim of this section, is to show that the decision problem associated with the Restrained Roman reinforcement is NP-hard. Consider the following decision problem.

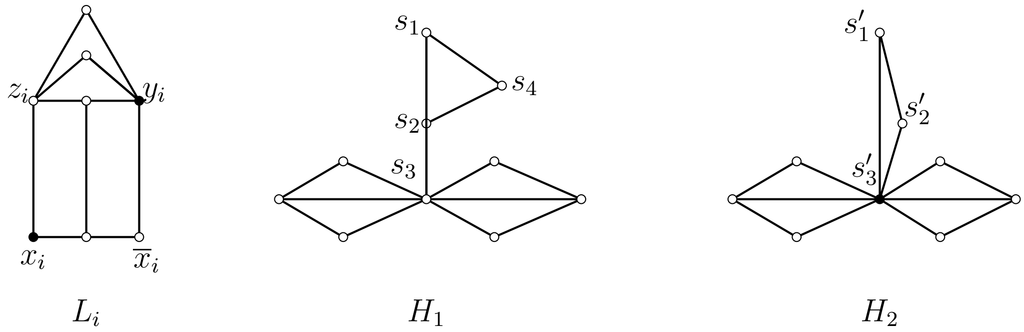

Figure 1. The graphs L_i and $H = H_1 \cup H_2$.

Restrained Roman reinforcement problem (RR-reinforcement)

Instance: *A nonempty graph G and a positive integer k.*
Question: *Is $r_{rR}(G) \leq k$?*

We show that the NP-hardness of the RR-reinforcement problem by transforming the well-known 3-SAT problem to it in polynomial time. Recall that the 3-SAT problem specified below was proven to be NP-complete in [6].

3-SAT problem

Instance: A collection $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ of clauses over a finite set X of variables such that $|C_j| = 3$ for every $j \in \{1, 2, \ldots, m\}$.
Question: Is there a truth assignment for X that satisfies all the clauses in $\mathcal{C}$?

Theorem 1. *Problem RR-reinforcement is NP-hard for an arbitrary graph.*

P r o o f. Let $X = \{x_1, x_2, \ldots, x_n\}$ and $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ be an arbitrary instance of 3-SAT problem. We will build a graph G and a positive integer k such that $r_{rR}(G) \leq k$ if and only if $\mathcal{C}$ is satisfiable.

For each $i \in \{1, 2, \ldots, n\}$, we associate to the variable $x_i \in X$ a copy of the graph L_i as depicted in Figure 1, and for each $j \in \{1, 2, \ldots, m\}$, we associate to the clause $C_j = \{u_j, v_j, w_j\} \in \mathcal{C}$ a vertex c_j by adding the edge-set $E_j = \{c_j u_j, c_j v_j, c_j w_j\}$. Finally, we enclose the graph H illustrated in Figure 1 by connecting vertices s_1, s'_1 to every vertex c_j. Clearly, the resulting graph G is of order $8n+m+19$ and size $11n+5m+27$ and hence G can be built in polynomial time. Set $k = 1$. Figure 2 provides an example of the resulting graph when $X = \{x_1, x_2, x_3, x_4\}$ and $\mathcal{C} = \{C_1, C_2, C_3\}$, where $C_1 = \{x_1, x_2, \bar{x}_3\}$, $C_2 = \{\bar{x}_1, x_2, x_4\}$ and $C_3 = \{\bar{x}_2, x_3, x_4\}$.

It is easy to verify that for any $\gamma_{rR}(G)$-function g we must have

$$\sum_{v \in V(L_j)} g(v) \geq 4$$

for each $j \in \{1, 2, \ldots, n\}$. Moreover, to restrained Roman dominate all vertices of $V(H)$, we need that

$$\sum_{i=1}^{m} g(c_i) + g(V(H)) \geq 6.$$

Therefore

$$\gamma_{rR}(G) = w(g) \geq 4n + 6.$$

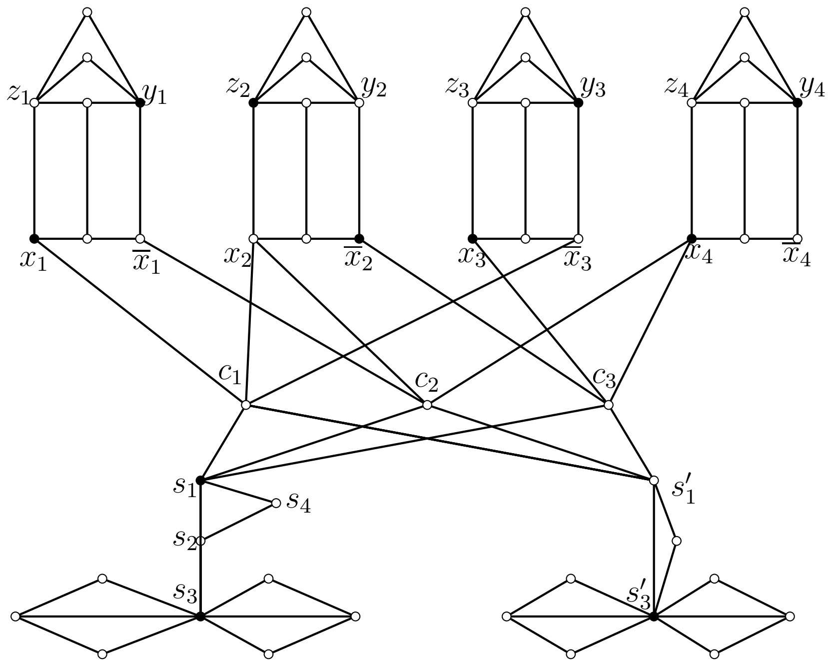

Figure 2. An instance of the restrained Roman reinforcement number problem resulting from an instance of 3-SAT. Here $k = 1$ and $\gamma_{rR}(G) = 22$, where the black vertex p means there is a RRDF f with $f(p) = 2$.

Basing on the assignment given to the graph in Figure 2, one can easily define an RRD-function of G with weight $4n + 6$, which consequently leads to $\gamma_{rR}(G) = 4n + 6$.

In the following, we show that $\mathcal{C}$ is satisfiable if and only if $r_{rR}(G) = 1$. Let $\mathcal{C}$ be satisfiable and $t : X \to \{T, F\}$ a satisfying function for $\mathcal{C}$. We build a subset S of vertices of G as follows. If $t(x_i) = T$, then put the vertices x_i and y_i in S; while if $t(x_i) = F$, then put the vertices $\overline{x_i}$ and z_i in S. So $|S| = 2n$. Define the function h on $V(G)$ by $h(x) = 2$ for every $x \in S$, $h(s_1) = 1$, $h(s_3) = h(s'_3) = 2$ and $h(y) = 0$ for the remaining vertices. It is easy to verify that h is an RRD-function of $G + s_4s_3$ of weight

$$4n + 5 < \gamma_{rR}(G) = 4n + 6,$$

and hence $r_{rR}(G) = 1$.

Conversely, let $r_{rR}(G) = 1$. Then, there is an edge $e = uv \in E(\overline{G})$ for which

$$\gamma_{rR}(G + e) < 4n + 6.$$

Let $g = (V_0, V_1, V_2)$ be a $\gamma_{rR}(G + e)$-function. Since whatever the added edge e, we have $g(V(L_i)) \geq 4$, and thus vertices u and v cannot both belong to $V(L_i)$ (for otherwise $\gamma_{rR}(G + e) \geq 4n + 6$). On the other hand, since $r_{rR}(G) = 1$ and $g(V(L_i)) \geq 4$, we must have

$$\sum_{j=1}^{m} g(c_j) + g(V(H)) < 6.$$

Since also whatever the added edge e, we have $g(V(H)) \geq 5$, we conclude that $g(V(H)) = 5$. In particular, this is only possible if $g(s'_1) = 0$, $g(s_1) \leq 1$ and

$$\sum_{j=1}^{m} g(c_j) = 0.$$

In addition, we note that if $\{x_i, \bar{x_i}\} \subseteq V_2$ or $\{x_i, \bar{x_i}\} \cap V_1 \neq \emptyset$ for some i, then $g(V(L_i)) \geq 5$ which results in the contradiction

$$\gamma_{rR}(G+e) \geq 4n+6.$$

Thus, $|\{x_i, \bar{x_i}\} \cap V_2| \leq 1$ and $\{x_i, \bar{x_i}\} \cap V_1 = \emptyset$ for every $i \in \{1, \dots, n\}$. Therefore each vertex c_j must have a neighbor in $\{x_i, \bar{x_i}\}$ for some i which is assigned a 2. In this case, define the mapping $t : X \to \{T, F\}$ by

$$t(x_i) = \begin{cases} T & \text{if} \quad f(x_i) = 2, \\ F & \text{otherwise} \end{cases} \tag{2.1}$$

for $i \in \{1, \dots, n\}$.

We show that t satisfies the truth assignment for $\mathcal{C}$. It is enough to show that every clause in $\mathcal{C}$ is satisfied by t. Consider an arbitrary clause $C_j \in \mathcal{C}$ for some $j \in \{1, \dots, m\}$. If c_j is dominated by x_i, then $g(x_i) = 2$ and so $t(x_i) = T$. If c_j is dominated by $\bar{x_i}$, then $g(\bar{x_i}) = 2$ and hence $t(x_i) = F$ and $t(\bar{x_i}) = T$. Therefore, in either case the clause C_j is satisfied. The arbitrariness of j shows that all clauses in $\mathcal{C}$ are satisfied by t, that is, $\mathcal{C}$ is satisfiable. This completes the proof of the theorem. □

3. Exact values

In this section, we determine the restrained Roman reinforcement number of some classes of graphs including paths, cycles and complete p-partite graphs for any integer $p \geq 2$. As observed in [11], for every connected graph G of order $n \geq 2$, we have $2 \leq \gamma_{rR}(G) \leq n$. A characterization of all connected graphs of order n with $\gamma_{rR}(G) \in \{2, 3, n\}$ was provided in [11, 14] as follows.

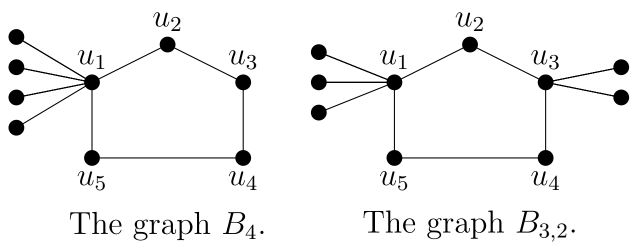

Figure 3. Graphs B_4 and $B_{3,2}$.

Let $C := (u_1u_2u_3u_4u_5)$ be a cycle of length 5 and let B_p be the graph obtained from C by adding $p \geq 1$ new vertices attached by edges at u_1 and let $B_{p,q}$ be the graph obtained from C by adding $p \geq 1$ new vertices attached by edges at u_1 and $q \geq 1$ other new vertices attached by edges at u_3 (see Fig. 3). Recall that the *diameter,* $\operatorname{diam}(G)$, of a graph G is the maximum distance between the pair of vertices.

Proposition 2 [11]. *Let G be a connected graph of order $n \geq 2$. Then*

(a) $\gamma_{rR}(G) = 2$ *if and only if* $n = 2$ *or* $\Delta(G) = n - 1$ *and* $\delta(G) \geq 2$;

(b) $\gamma_{rR}(G) = n$ *if and only if* $G \simeq C_4, C_5, B_p, B_{p,q}$ *or G is a tree with* $\operatorname{diam}(G) \leq 5$.

Proposition 3 [14]. *Let G be a connected graph of order $n \geq 4$. Then $\gamma_{rR}(G) = 3$ if and only if G satisfies one of the following conditions*:

(i) $\Delta(G) = n-1$ *and* G *has exactly one leaf*;

(ii) $\Delta(G) = n-2$ *and* G *has a vertex* u *of degree* $n-2$ *such that the induced subgraph* $G[N(u)]$ *has no isolated vertex.*

On the other hand, the exact values of the restrained Roman domination number have been established in [11] for paths, cycles and complete p-partite graphs.

Proposition 4 [11]. *The following conditions holds*:

(a) $\gamma_{rR}(P_n) = n$ *for* $1 \le n \le 6$ *and* $\gamma_{rR}(P_n) = \lceil (2n+1)/3 \rceil + 1$ *for* $n \ge 7$;

(b) $\gamma_{rR}(C_n) = 2\lceil n/3 \rceil$ *when* $n \not\equiv 2 \pmod 3$ *and* $\gamma_{rR}(C_n) = 2\lceil n/3 \rceil + 1$ *otherwise*;

(c) $\gamma_{rR}(K_{m,n}) = 4$ *for* $m, n \ge 2$;

(d) *if* $K_{n_1,n_2,\ldots,n_p}$ *is the complete* p*-partite graph such that* $p \ge 3$ *and* $n_1 \le n_2 \le \ldots \le n_p$, *then* $\gamma_{rR}(K_{1,n_2,\ldots,n_p}) = 2$, $\gamma_{rR}(K_{2,n_2,\ldots,n_p}) = 3$ *and* $\gamma_{rR}(K_{n_1,n_2,\ldots,n_p}) = 4$ *for* $n_1 \ge 3$.

Now we are ready to find the restrained Roman reinforcement number for paths, cycles and complete p-partite graphs, $p \ge 2$.

Proposition 5. *For* $n \ge 3$, $r_{rR}(P_n) = 1$.

P r o o f. Let $P_n := w_1 w_2 \ldots w_n$. If $n \equiv 0 \pmod 3$, then the function g defined by

$$g(w_{3i+1}) = 2$$

for $0 \le i \le (n-3)/3$ and $g(w) = 0$ otherwise, is an RRD-function of $P_n + w_1 w_n$ of weight $2n/3$. If $n \equiv 2 \pmod 3$, then the function g defined by $g(w_n) = 2$, $g(w_{3i+1}) = 2$ for $0 \le i \le (n-5)/3$ and $g(x) = 0$ otherwise, is an RRD-function of $P_n + w_1 w_{n-2}$ of weight $(2n+2)/3$. Finally, if $n \equiv 1 \pmod 3$, then the function g defined by $g(w_n) = 1$, $g(w_{3i+1}) = 2$ for $0 \le i \le (n-4)/3$ and $g(w) = 0$ otherwise, is an RRD-function of $P_n + w_1 w_{n-1}$ of weight $(2n+1)/3$. All considered cases show that $r_{rR}(P_n) = 1$. □

Proposition 6. For $n \ge 4$,

$$r_{rR}(C_n) = \begin{cases} 2 & \text{if } n \equiv 0 \pmod 3, \\ 1 & \text{otherwise.} \end{cases}$$

P r o o f. Assume that $C_n := (w_1 w_2 \ldots w_n)$ be a cycle on n vertices. If $n \equiv r \pmod 3$ with $r \in \{1, 2\}$, then by a similar argument to that used in the proof of Proposition 5, we can see that $r_{rR}(C_n) = 1$. Hence we assume that $n \equiv 0 \pmod 3$. First, since the function g defined by $g(w_{n-2}) = 1$, $g(w_{3i+1}) = 2$ for $0 \le i \le (n-6)/3$ and $g(x) = 0$ otherwise, is an RRD-function of $C_n + \{w_1 w_{n-1}, w_1 w_{n-3}\}$ of weight $(2n-3)/3 = \gamma_{rR}(C_n) - 1$ (Proposition 4-(b)), we deduce that $r_{rR}(C_n) \le 2$.

Now we prove the inverse inequality. For this purpose, we need only to show that adding an arbitrary edge e cannot decrease $\gamma_{rR}(C_n)$. Observe that for any edge $e \in \overline{C_n}$,

$$\gamma_{rR}(C_n + e) \le \gamma_{rR}(C_n).$$

Let e be an arbitrary edge in $\overline{C_n}$ and let f be a $\gamma_{rR}(C_n + e)$-function. Suppose first that there are three consecutive vertices w_i, w_{i+1}, w_{i+2} such that $f(w_i) = f(w_{i+1}) = f(w_{i+2}) = 0$, say for $i = 1$.

Then the edge e must join w_2 to some vertex assigned 2, say w_k, with $k \notin \{1,3\}$. Also, to restrained Roman dominate w_1 and w_3, we must also have $f(w_4) = f(w_n) = 2$.

Consider the cycles $C' := (w_2 w_3 \dots w_k)$ of order $k-1$ and $C'' := (w_2 w_k \dots w_n w_1)$ of order $n-k+3$. Let $k-1 \equiv s_1 \pmod 3$ and $n-k+3 \equiv s_2 \pmod 3$. Notice that $s_1 = 0$ and $s_2 = 2$; $s_1 = 2$ and $s_2 = 0$ or $s_1 = s_2 = 1$. Assume that $k-1 \equiv 0 \pmod 3$ (the case $n-k+3 \equiv 0 \pmod 3$) is similar). Then $n-k+3 \equiv 2 \pmod 3$, and since the restrictions of f on $V(C')$ and $V(C'')$ are RRD-functions, we deduce from Proposition 4-(b), that

$$\gamma_{rR}(C_n + e) = f(V(C')) + f(V(C'')) - 2$$
$$\geq \gamma_{rR}(C') + \gamma_{rR}(C'') - 2 = \frac{2(k-1)}{3} + \frac{2(n-k+3)+3+2}{3} - 2 = \frac{2n+3}{3} > \gamma_{rR}(C_n).$$

Assume now that $s_1 = s_2 = 1$. Then, as above, it follows from Proposition 4-(b) that

$$\gamma_{rR}(C_n + e) = f(V(C')) + f(V(C'')) - 2$$
$$\geq \gamma_{rR}(C') + \gamma_{rR}(C'') - 2 = \frac{2(k-1)+3+1}{3} + \frac{2(n-k+3)+3+1}{3} - 2 = \frac{2n+6}{3} > \gamma_{rR}(C_n).$$

Thus in either case we obtain a contradiction. Next suppose there are three consecutive vertices w_i, w_{i+1}, w_{i+2} such that $f(w_i) + f(w_{i+1}) + f(w_{i+2}) = 1$, say for $i = 1$.

If $f(w_2) = 1$, then $f(w_1) = f(w_3) = 0$ and each of w_1 and w_3 must be adjacent a vertex assigned 2 as well as to a vertex assigned 0. This possible only if $e = w_1 w_3$ and so

$$H = (C_n + e) - w_2$$

is a cycle on $n-1$ vertices, where the restriction of f to H is an RRD-function. It follows that

$$\gamma_{rR}(C_n + e) = f(V(H)) + 1 \geq \gamma_{rR}(H) + 1,$$

and by Proposition 4-(b), we obtain

$$\gamma_{rR}(C_n + e) \geq \frac{2(n-1)+3+2}{3} + 1 > \gamma_{rR}(C_n)$$

which is a contradiction. Hence we can assume that $f(w_2) = 0$. Without loss of generality, let $f(w_1) = 1$ and $f(w_3) = 0$. To restrained Roman dominate w_2, the edge e must join w_2 to a vertex with label 2, say w_k. Likewise for w_3 we must have $f(w_4) = 2$. Now, consider the cycles $C' := (w_2 w_3 \dots w_k)$ of order $k-1$ and the path $P' := w_k \dots w_n w_1$ of order $n-k+2$.

Let $k-1 \equiv s_1 \pmod 3$ and $n-k+2 \equiv s_2 \pmod 3$. Notice that $s_1 = 0$ and $s_2 = 1$; $s_1 = 1$ and $s_2 = 0$ or $s_1 = s_2 = 2$. Notice also that the restrictions of f on $V(C')$ and $V(P')$ are RRD-functions, and thus

$$\gamma_{rR}(C_n + e) = f(V(C')) + f(V(P')) - 2 \geq \gamma_{rR}(C') + \gamma_{rR}(P') - 2.$$

Now using Propositions 4-(a,b), we get a contradiction as before.

Finally, let

$$f(w_i) + f(w_{i+1}) + f(w_{i+2}) \geq 2$$

for each $1 \leq i \leq n$, where the sum in indices is taken modulo n. Then we have

$$\gamma_{rR}(C_n + e) = \frac{1}{3} \sum_{i=1}^{n} (f(w_i) + f(w_{i+1}) + f(w_{i+2})) \geq \frac{2n}{3} = \gamma_{rR}(C_n),$$

and therefore, $\gamma_{rR}(C_n + e) = \gamma_{rR}(G)$. Consequently, $r_{rR}(C_n) = 2$ as desired. □

Proposition 7. For integers $1 \le r \le s$ with $r + s \ge 3$,

$$r_{rR}(K_{r,s}) = \begin{cases} 1 & \text{if} \quad r = 1, 2, 3, \\ r - 2 & \text{if} \quad r \ge 4. \end{cases}$$

P r o o f. Let $X = \{x_1, x_2, \dots, x_r\}$ and $Y = \{y_1, y_2, \dots, y_s\}$ be the partite sets of $K_{r,s}$.

If $r = 1$, then the function g defined by $g(x_1) = 2$, $g(y_1) = g(y_2) = 0$ and $g(x) = 1$ otherwise, is an RRD-function of $K_{1,s} + y_1y_2$ of weight $n - 1$ and it follows from Proposition 2-(b) that $r_{rR}(K_{1,s}) = 1$.

If $r = 2$, then the function g defined by $g(x_1) = 2$ and $g(x) = 0$ otherwise, is an RRD-function of $K_{2,s} + x_1x_2$ of weight 2 and we get from Proposition 2-(a) that $r_{rR}(K_{2,s}) = 1$.

If $r = 3$, then the function g defined by $g(x_1) = 2, g(x_2) = 1$ and $g(x) = 0$ otherwise, is an RRD-function of $K_{3,s} + x_1x_3$ of weight 3 and by Proposition 4-(c), we have $r_{rR}(K_{3,s}) = 1$.

Let $r \ge 4$. First we observe that the function g defined by $g(x_1) = 2$, $g(x_2) = 1$ and $g(x) = 0$ otherwise, is an RRD-function of $K_{r,s} + \{x_1x_i \mid 3 \le i \le r\}$ of weight 3 and thus by Proposition 4-(c), $r_{rR}(K_{r,s}) \le r - 2$.

To show that $r_{rR}(K_{r,s}) \ge r - 2$, let F be an $r_{rR}(K_{r,s})$-set. Then

$$2 \le \gamma_{rR}(K_{r,s} + F) \le 3.$$

By Propositions 2-(a) and 3 we must have $\Delta(K_{r,s} + F) \ge r + s - 2$ and this implies that $|F| \ge r - 2$. Therefore $r_{rR}(K_{r,s}) = r - 2$ and the proof is complete. □

Proposition 8. *Let $K_{n_1,n_2,\dots,n_p}$ be the complete p-partite graph such that $p \ge 3$ and $3 \le n_1 \le n_2 \le \dots \le n_p$. Then $r_{rR}(K_{n_1,n_2,\dots,n_p}) = n_1 - 2$.*

P r o o f. Let $G = K_{n_1,n_2,\dots,n_p}$ and $X_1 = \{x_1, \dots, x_{n_1}\}$, $X_2 = \{y_1, \dots, y_{n_2}\}, \dots, X_p$ be the partite sets of G. Let F be an $r_{rR}(G)$-set. By Proposition 4-(d) we deduce that $\gamma_{rR}(G+F) \in \{2, 3\}$, and by Propositions 2-(a) and 3 we must have

$$\Delta(G + F) \ge n_1 + \dots + n_p - 2$$

implying that $|F| \ge n_1 - 2$. On the other hand, the function g defined by $f(x_1) = 2$, $f(x_2) = 1$ and $f(x) = 0$ otherwise, is an RRD-function of $G + \{x_ix_1 \mid 3 \le i \le n_1\}$ yielding $r_{rR}(G) \le |F| = n_1 - 2$. Consequently, $r_{rR}(G) = n_1 - 2$. □

4. Graphs with small restrained Roman reinforcement number

In this section, we study graphs with small restrained Roman reinforcement number. We begin with the following lemma.

Lemma 1. *If G is a connected graph of order $n \ge 3$ with $\gamma_{rR}(G) = n$, then $r_{rR}(G) = 1$.*

P r o o f. By Proposition 2, $G \simeq C_4, C_5, B_p, B_{p,q}$ or G is a tree with $\text{diam}(G) \le 5$. If $G \in \{C_4, C_5\}$, then the desired result follows from Proposition 6. If $G \in \{B_p, B_{p,q}\}$, then the function g defined by $g(u_1) = 2$, $g(u_2) = g(u_5) = 0$ and $g(x) = 1$ otherwise, is an RRD-function of $G + u_2u_5$ and hence $r_{rR}(G) = 1$. Hence, we assume that G is a tree with diameter at most 5.

Let $v_1v_2 \dots v_k$ ($k \ge 3$) be a diametral path in G. Define the function f by $f(v_1) = f(v_3) = 0$, $f(v_2) = 2$ and $f(x) = 1$ for the remaining vertices. Clearly, f is an RRD-function of $G + v_1v_3$ and hence $r_{rR}(G) = 1$. □

Proposition 9. *Let G be a connected graph of order $n \geq 4$ with $\gamma_{rI}(G) \geq 3$. If $f = (V_0, V_1, V_2)$ is a $\gamma_{rR}(G)$-function with $V_1 \neq \emptyset$, then*

$$r_{rR}(G) = 1.$$

P r o o f. Let $f = (V_0, V_1, V_2)$ be a $\gamma_{rR}(G)$-function such that $V_1 \neq \emptyset$. If $\gamma_{rR}(G) = n$, then the desired result comes from Lemma 1.

Hence assume that $\gamma_{rR}(G) < n$. Then $V_0 \neq \emptyset$ and so $V_2 \neq \emptyset$. Since G is connected and $V_1 \neq \emptyset$, there exists a vertex $w \in V_1$ such that w is dominated by $V_0 \cup V_2$. Note that if w has a neighbor in V_0 and another one in V_2, then reassigning w provides an RRD-function with weight $\gamma_{rR}(G) - 1$, a contradiction.

Now, if w is adjacent to a vertex in V_2, then the function g defined by $g(w) = 0$ and $g(x) = f(x)$ otherwise, is an RRD-function of $G + wz$ where $z \in V_0$, of weight less than $\gamma_{rR}(G)$, and thus $r_{rR}(G) = 1$. If w is adjacent to a vertex in V_0, then the function g defined by $g(w) = 0$ and $g(x) = f(x)$ otherwise, is an RRD-function of $G + wu$ where $u \in V_2$, of weight less than $\gamma_{rR}(G)$ and so $r_{rR}(G) = 1$. This completes the proof. □

Proposition 10. *Let G be a connected graph of order n with $\gamma_{rR}(G) \geq 3$. Then $r_{rR}(G) = 1$ if and only if $\gamma_{rR}(G) = n$ or G has a function $f = (V_0, V_1, V_2)$ of weight less than $\gamma_{rR}(G)$ such that one of the following conditions holds*:

(i) *$G[V_0]$ has at most two isolated vertices and V_2 dominates V_0;*

(ii) *$G[V_0]$ has no isolated vertices and there is exactly one vertex $v \in V_0$ which is not dominated by V_2.*

P r o o f. If $\gamma_{rR}(G) = n$, then by Lemma 1 we have $r_{rR}(G) = 1$. Hence suppose that $\gamma_{rR}(G) < n$, and let $f = (V_0, V_1, V_2)$ be a function on G with weight less than $\gamma_{rR}(G)$ satisfying (i) or (ii). Since $\omega(f) < \gamma_{rR}(G) \leq n - 2$, we have $|V_0| \geq 2$. In the case V_2 is non-empty, let $u \in V_2$. Now, if (ii) holds, then $V_2 \neq \emptyset$ and f is an RRD-function of $G + uv$.

Assume now that (i) holds. If $G[V_0]$ has two isolated vertices w, v, then f is an RRD-function of $G + \{wv\}$ and if $G[V_0]$ has exactly one isolated vertex, say w, then f is an RRD-function of $G + \{wz\}$, where z is any vertex in $V_0 - \{w\}$. Hence in either case $r_{rR}(G) = 1$.

Conversely, let $r_{rR}(G) = 1$ and suppose that $\{uv\}$ is an $r_{rR}(G)$-set. If $\gamma_{rR}(G) = n$, then we are done. Hence suppose that $\gamma_{rR}(G) \leq n - 1$ and let f be a $\gamma_{rR}(G + \{uv\})$-function. Notice that vertices u and v cannot be assigned both positive values under f (otherwise f is an RRD-function of G). Without loss of generality, assume that $f(u) = 0$. If $f(v) = 0$, then f is a function satisfying (i). Hence assume that $f(v) \geq 1$. If u is adjacent to a vertex with label 2 other than v, then f is an RRD-function of G. Hence u is not dominated by V_2 in G and so f is a function satisfying (ii). This completes the proof. □

Proposition 11. *Let G be a connected graph of order n with $\gamma_{rR}(G) \geq 3$. If $\delta(G) = 1$, then $r_{rR}(G) = 1$.*

P r o o f. First note that $n \geq 3$, since $\gamma_{rR}(G) \geq 3$. If $\gamma_{rR}(G) = n$, then the result comes from Lemma 1. Hence we assume that $\gamma_{rR}(G) < n$, and let $f = (V_0, V_1, V_2)$ be a $\gamma_{rR}(G)$-function. We have $V_0 \neq \emptyset$ (because $\gamma_{rR}(G) < n$) and thus $V_2 \neq \emptyset$.

Let u be a support vertex of G and u_1 a leaf neighbor of u. By definition we have $f(u_1) \geq 1$. If $f(u_1) = 1$ or $f(u) = 1$, then the desired result comes from Proposition 9.

Hence we assume that $f(u_1) = 2$ and $f(u) \neq 1$. The minimality of f implies that $f(u) = 0$. Note that u_1 is the only neighbor of u which assigned a 2, for otherwise u_1 can be reassigned the value 1 instead of 2. Let w be a neighbor of u with label 0. To Roman dominate w, there is a vertex v such that $f(v) = 2$. Then the function g defined on $G + uv$ by $g(u_1) = 1$ and $g(x) = f(x)$ otherwise, is an RRD-function of $G + \{uv\}$ with weight $\omega(f) - 1$. Consequently, $r_{rR}(G) = 1$. □

Corollary 1. *For any tree T of order $n \geq 3$, $r_{rR}(T) = 1$.*

5. Bounds on $r_{rR}(G)$

In this section, we present some sharp upper bounds on the restrained Roman reinforcement number of a graph. Given a set $S \subseteq V$ of vertices in a graph G and a vertex $v \in S$, the *external private neighborhood* of v with respect to S in the set

$$\mathrm{epn}(v, S) = \big\{u \in V - S \mid N(u) \cap S = \{v\}\big\}.$$

Proposition 12. *Let G be a connected graph with $\gamma_{rI}(G) \geq 3$. If $f = (V_0, V_1, V_2)$ is a $\gamma_{rR}(G)$-function with $V_2 \neq \emptyset$, then*

$$r_{rR}(G) \leq \min\big\{|\mathrm{epn}(v, V_2) \cap V_0| \;:\; v \in V_2\big\}.$$

P r o o f. Let $f = (V_0, V_1, V_2)$ be a $\gamma_{rR}(G)$-function with $V_2 \neq \emptyset$. If $|\mathrm{epn}(v, V_2) \cap V_0| = 0$ for some vertex $v \in V_2$, then reassigning v the value 1 instead of 2 provides an RRD-function of weight less than $\gamma_{rR}(G)$ leading to a contradiction. Hence $|\mathrm{epn}(v, V_2) \cap V_0| \geq 1$ for every $v \in V_2$. Let u be a vertex in V_2 such that

$$|\mathrm{epn}(u, V_2) \cap V_0| = \min\{|\mathrm{epn}(v, V_2) \cap V_0| : \; v \in V_2\}$$

and let $\mathrm{epn}(u, V_2) \cap V_0 = \{u_1, \ldots, u_\epsilon\}$. If $|V_2| \geq 2$ and $w \in V_2 - \{u\}$, then the function g defined by $g(u) = 1$ and $g(x) = f(x)$ otherwise, is an RRD-function of $G + \{wx \mid x \in \mathrm{epn}(v, V_2) \cap V_0\}$ of weight less than $\gamma_{rR}(G)$ and so

$$r_{rR}(G) \leq \min\{|\mathrm{epn}(v, V_2) \cap V_0| : \; v \in V_2\}.$$

Hence assume that $V_2 = \{u\}$. Then u dominates all vertices in V_0. Since $\gamma_{rR}(G) \geq 3$, we have $V_1 \neq \emptyset$ and the desired result follows from Proposition 9. □

We observe that for any $\gamma_{rR}(G)$-function $f = (V_0, V_1, V_2)$, every vertex u of V_2 can have at most $d_G(u)$ neighbors in V_0. Whence we have the following corollary.

Corollary 2. *Let G be a connected graph with $\gamma_{rR}(G) \geq 3$ and $f = (V_0, V_1, V_2)$ a $\gamma_{rR}(G)$-function with $|V_2| \geq 1$. Then $r_{rR}(G) \leq \Delta$.*

Corollary 3. *Let G be a connected graph with $\gamma_{rR}(G) \geq 3$ containing a path $v_1v_2v_3v_4v_5$ in which $d_G(v_i) = 2$ for $i \in \{2, 3, 4\}$. Then $r_{rR}(G) \leq 2$.*

P r o o f. If $\gamma_{rR}(G) = n$, then the result is immediate from Lemma 1. Hence we assume that $\gamma_{rR}(G) < n$, and let $f = (V_0, V_1, V_2)$ be a $\gamma_{rR}(G)$-function. By Proposition 9, we may assume that $V_1 = \emptyset$. Then we must have $2 \in \{f(v_2), f(v_3), f(v_4)\}$ and the result follows from Proposition 12. □

Using Propositions 9 and 12 we obtain the next result.

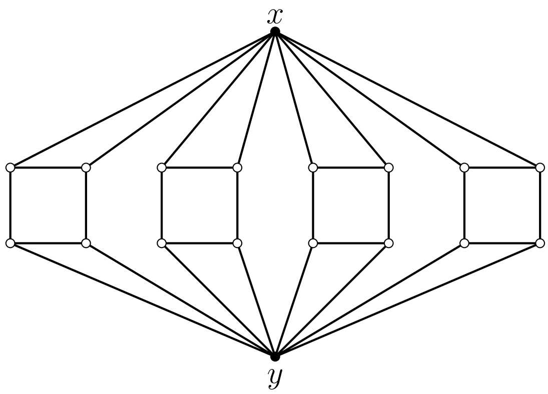

Figure 4. A graph G of order 18 and $r_{rR}(G) = 4$.

Theorem 2. *For any graph G of order $n \geq 3$, we have*

$$r_{rR}(G) \leq \max\{1, (2n - \gamma_{rR}(G))/\gamma_{rR}(G)\}.$$

Moreover, the bound is sharp.

P r o o f. If $\gamma_{rR}(G) = 2$, then $r_{rR}(G) = 0$ and the result is true.

If $\gamma_{rR}(G) = n$, then by Lemma 1, $r_{rR}(G) = 1$ and the desired result follows.

Hence we assume that $3 \leq \gamma_{rR}(G) < n$, and let $f = (V_0, V_1, V_2)$ be a $\gamma_{rR}(G)$-function. If $V_1 \neq \emptyset$, then the result follows from Proposition 9. Thus suppose that $V_1 = \emptyset$. Then $\gamma_{rR}(G)/2 = |V_2| \geq 2$ and clearly

$$|\mathrm{epn}(u, V_2) \cap V_0| \leq (2n - \gamma_{rR}(G))/\gamma_{rR}(G)$$

for some $u \in V_2$. Now, the result is immediate by Proposition 9.

To show the sharpness, consider the graph G illustrated in Figure 4. It is easy to see that $\gamma_{rR}(G) = 4$ and the function f on G defined by $f(x) = f(y) = 2$ and $f(z) = 0$ otherwise, is the unique $\gamma_{rR}(G)$-function. Then

$$r_{rR}(G) \leq (2n - \gamma_{rR}(G))/\gamma_{rR}(G) = 8.$$

Now let F be an $r_{rR}(G)$-set. Then $\gamma_{rR}(G+F) \leq 3$ and so $\Delta(G+F) \geq n-2$ (see Propositions 2-(a) and 3). This implies that $|F| \geq 8$, and consequently,

$$r_{rR}(G) = 8 = (2n - \gamma_{rR}(G))/\gamma_{rR}(G).$$

□

6. Conclusion

The main objective of this paper was to start the study of the restrained Roman reinforcement number $r_{rR}(G)$ of a graph G. We first showed that the decision problem associated with the restrained Roman reinforcement problem is NP-hard, and then various properties as well as some sharp bounds of the restrained Roman reinforcement number have been established. In particular we showed that $r_{rR}(T) = 1$ for every tree T of order at least three and that $r_{rR}(G) \leq \Delta(G)$ for any connected graph G with $\gamma_{rR}(G) \geq 3$. As a future work, one can focus on the problem of characterizing all connected graphs G such that $r_{rR}(G) = \Delta(G)$.

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ON LOCAL IRREGULARITY OF THE VERTEX COLORING OF THE CORONA PRODUCT OF A TREE GRAPH

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Abstract: Let $G = (V, E)$ be a graph with a vertex set V and an edge set E. The graph G is said to be with a local irregular vertex coloring if there is a function f called a local irregularity vertex coloring with the properties: (i) $l : (V(G)) \to \{1, 2, ..., k\}$ as a vertex irregular k-labeling and $w : V(G) \to N$, for every $uv \in E(G)$, $w(u) \neq w(v)$ where $w(u) = \sum_{v \in N(u)} l(i)$ and (ii) $\mathrm{opt}(l) = \min\{\max\{l_i : l_i \text{ is a vertex irregular labeling}\}\}$. The chromatic number of the local irregularity vertex coloring of G denoted by $\chi_{lis}(G)$, is the minimum cardinality of the largest label over all such local irregularity vertex colorings. In this paper, we study a local irregular vertex coloring of $P_m \bigodot G$ when G is a family of tree graphs, centipede C_n, double star graph $(S_{2,n})$, Weed graph $(S_{3,n})$, and E graph $(E_{3,n})$.

Keywords: Local irregularity, Corona product, Tree graph family.

1. Introduction

Let $G(V, E)$ be a connected and simple graph with a vertex set V and an edge set E. In this paper, we combine two concepts, namely the local antimagic vertex coloring and the distance irregular labelling, with a local irregularity of vertex coloring. This concept firstly was introduced by Kristiana [2, 3], et. al. The latest research was conducted by Azzahra [4], who examined the local irregularity vertex coloring of a grid graph family. In this paper we study the local irregularity of vertex coloring of corona product graph of a tree graph family.

Definition 1. *Suppose $l : V(G) \to \{1, 2, ..., k\}$ and $w : V(G) \to N$, where*

$$w(u) = \sum_{v \in N(u)} l(v),$$

then $l(v)$ is called the vertex irregular k-labeling and $w(u)$ is called the local irregularity of vertex coloring if

(i) $\mathrm{opt}(l) = \min\{\max\{l_i\} :\ l_i \text{ vertex irregular labeling}\}$;

(ii) *for every $uv \in E(G), w(u) \neq w(v)$.*

Definition 2. *The chromatic number of local irregular graph G denoted by $\chi_{lis}(G)$, is the minimum of cardinality of the local irregularity of vertex coloring.*

In this paper, we will use the following lemma which gives a lower bound on the chromatic number of local irregular vertex coloring:

Lemma 1 [2]. *Let G be a simple and connected graph, then $\chi_{lis}(G) \geq \chi(G)$.*

Proposition 1 [2]. *Let G be a graph each two adjacent vertices of which have a different vertex degree then* $\text{opt}(l) = 1$.

Proposition 2 [2]. *Let G be a graph each two adjacent vertices have the same vertex degree then* $\text{opt}(l) \geq 2$.

Definition 3 [1]. *Let G and H be two connected graphs. Let o be a vertex of H. The corona product of the combination of two graphs G and H is defined as the graph obtained by taking a duplicate of graph G and $|V(G)|$ a duplicate of graph H, namely H_i; $i = 1, 2, 3, ...|V(G)|$ then connects each vertex i in G to each vertex in H_i. The corona product of the graphs G and H is denoted by $G \bigodot H$.*

2. Result and discussion

In this paper, we analyze the new result of the chromatic number of local irregular vertex coloring of corona product by family of tree graph ($P_m \bigodot G$) where G is centipede graph (C_n), double star graph ($S_{2,n}$), and Weed graph ($S_{3,n}$).

Theorem 1. *Let $G = P_m \bigodot Cp_n$, be a corona product of a path graph of order m and a centipede graph of order n for $n, m \geq 2$, then*

$$\chi_{lis}(P_m \bigodot Cp_n) = \begin{cases} 5, & \text{for} \quad m = 3 \quad \text{and} \quad n = 2, 3, \\ 6, & \text{for} \quad m = 2 \quad \text{and} \quad n = 2, 3 \quad \text{or} \quad \text{for} \quad m = 3 \quad \text{and} \quad n \geq 4, \\ 7, & \text{for} \quad m = 2 \quad \text{and} \quad n \geq 4 \quad \text{or} \quad \text{for} \quad m \geq 4 \quad \text{and} \quad n = 2, 3, \\ 8, & \text{for} \quad m \geq 4 \quad \text{and} \quad n \geq 4, \end{cases}$$

with $\text{opt}(l)$ *defined as*

$$\text{opt}(l)(P_m \bigodot Cp_n) = \begin{cases} 1, & \text{for} \quad m = 3 \quad \text{and} \quad n = 3, \\ 1, 2, & \text{for} \quad m = 2 \quad \text{and} \quad n = 2 \quad \text{or} \\ & \text{for} \quad m = 3 \quad \text{and} \quad n = 2 \quad \text{or} \\ & \text{for} \quad m \geq 3 \quad \text{and} \quad n \geq 4. \end{cases}$$

P r o o f. Vertex set is

$$V(P_m \bigodot CP_n) = \{x_i; 1 \leq i \leq m\} \cup \{x_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{y_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}$$

and the edge set is

$$E(P_m \bigodot CP_n) = \{x_i x_{i+1}; 1 \leq i \leq m-1\} \cup \{x_{ij} x_{ij+1}; 1 \leq i \leq m, 1 \leq j \leq n-1\}$$
$$\cup \{x_{ij} y_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{x_i x_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{x_i y_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\},$$

the *order* and *size* respectively are $2mn + m$ and $4mn - 1$.

Case 1: $m \neq p, \ m \geq 2, \ p \geq 2, \ n \geq 3$.

First step to prove this theorem is to find the lower bound of $V(P_m \bigodot CP_n)$. Based on Lemma 1, we have $\chi_{lis}(P_m \bigodot Cp_n) \geq \chi(P_m \bigodot Cp_n) = 3$.

Assume $\chi_{lis}(P_m \bigodot Cp_n) = 4$, let $\chi_{lis}(P_m \bigodot Cp_n) = 4$, if $l(x_1) = l(x_3) = 1$, $l(x_2) = 2$, $l(x_{ij}) = l(y_{ij}) = 1$ then $w(x_1) = w(x_2)$, then there are 2 adjacent vertices that have the same color, it contradicts the definition of vertex coloring. If

$$l(x_{ij}) = 1, \quad l(y_{ij}) = 1, \quad 1 \leq i \leq 3, \quad j = 1, \quad l(y_{ij}) = 2, \quad 1 \leq i \leq 3, \quad j = 2,$$
$$l(x_i) = 1 \rightarrow w(x_i) \neq w(x_{i+1}), \quad w(x_{i1}) \neq (x_{i2}),$$

then $\chi_{lis}(P_m \bigodot Cp_n) \geq 5$. Based on this, we have the lower bound $\chi_{lis}(P_m \bigodot Cp_n) \geq 5$.

After that, we will find the upper bound of $\chi_{lis}(P_m \bigodot Cp_n)$. Furthermore, the upper bound for the chromatic number of local irregular $(P_m \bigodot Cp_n)$, we define $l : V(P_m \bigodot Cp_n) \rightarrow \{1, 2\}$ with the vertex irregular 2-labelling as follows:

$$l(x_i) = 1, \quad l(x_{ij}) = 1,$$
$$l(y_{ij}) = \begin{cases} 1, & \text{for} \quad 1 \leq i \leq 3 \text{ and } j = 1, \\ 2, & \text{for} \quad 1 \leq i \leq 3 \text{ and } j = 2. \end{cases}$$

Hence, opt$(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 6, & \text{for } i = 1, 3, \\ 8, & \text{for } i = 2, \end{cases}$$
$$w(x_{ij}) = \begin{cases} 3, & \text{for } 1 \leq i \leq 3 \text{ and } j = 1, \\ 4, & \text{for } 1 \leq i \leq 3 \text{ and } j = 2, \end{cases}$$
$$w(y_{ij}) = 2, \quad \text{for } 1 \leq i \leq 3 \text{ and } j = 1, 2.$$

The upper bound is true: $\chi_{lis}(P_m \bigodot Cp_n) \leq 5$, and we have $5 \leq \chi_{lis}(P_m \bigodot Cp_n) \leq 5$, so $\chi_{lis}(P_m \bigodot Cp_n) = 5$ for $m = 3$ and $n = 2$.

Case 2: $m = n = 3$.

Based on Proposition 1, opt$(l) = 1$. So the lower bound of $(P_m \bigodot Cp_n)$ is

$$\chi_{lis}(P_m \bigodot Cp_n) \geq 5.$$

Hence opt$(l) = 1$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 7, & \text{for } i = 1, 3, \\ 8, & \text{for } i = 2, \end{cases}$$
$$w(y_{ij}) = \begin{cases} 3, & \text{for } 1 \leq i \leq 3 \text{ and } j \equiv 1, 3 \pmod 4, \\ 4, & \text{for } 1 \leq i \leq 3 \text{ and } j = 2, \end{cases}$$
$$w(x_{ij}) = 2, \quad \text{for } 1 \leq i \leq 3 \text{ and } 1 \leq j \leq 3.$$

The upper bound is true: $\chi_{lis}(P_m \bigodot Cp_n) \leq 5$. We have $5 \leq \chi_{lis}(P_m \bigodot Cp_n) \leq 5$, so $\chi_{lis}(P_m \bigodot Cp_n) = 5$ for $m = 3$ and $n = 3$.

Case 3: $m = n = 2$.

First step here is to find the lower bound of $V(P_m \bigodot CP_n)$. Based on Lemma 1, we have $\chi_{lis}(P_m \bigodot Cp_n) \geq \chi(P_m \bigodot Cp_n) = 3$.

Assume $\chi_{lis}(P_m \bigodot Cp_n) = 5$, if $l(x_1) = 1$, $l(x_2) = 2$, $l(x_{ij}) = l(y_{ij}) = 1$, then $w(x_{11}) = w(x_{12})$ and there are 2 adjacent vertices, that have the same color, it contradicts the definition of vertex coloring. If

$$l(x_1) = 1, \quad l(x_2) = 2, \quad l(x_{ij}) = 1, \quad l(y_{i1}) = 1, \quad i = 1, 2, \quad l(y_{i2}) = 2, \quad i = 1, 2,$$

then $w(x_1) \neq w(x_2)$, $w(x_{i1}) \neq w(x_{i2})$. Based on that we have the lower bound $\chi_{lis}(P_m \bigodot Cp_n) \geq 6$.

After that, we will find the upper bound of $\chi_{lis}(P_m \bigodot Cp_n)$.

Furthermore, we define $l : V(P_m \bigodot Cp_n) \to \{1, 2\}$ with the vertex irregular 2-labelling as follows:

$$l(x_i) = \begin{cases} 1, & \text{for } i = 1, \\ 2, & \text{for } i = 2, \end{cases} \quad l(x_{ij}) = 1, \quad l(y_{ij}) = \begin{cases} 1, & \text{for } i = 1, 2 \text{ and } j = 1, \\ 2, & \text{for } i = 1, 2 \text{ and } j = 2. \end{cases}$$

Hence, $\text{opt}(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 6, & \text{for } i = 2, \\ 7, & \text{for } i = 1, \end{cases}$$

$$w(x_{ij}) = \begin{cases} 3, & \text{for } i = 1 \text{ and } j = 1 \\ 4, & \text{for } i = 1 \text{ and } j = 1, \text{ or for } i = 2 \text{ and } j = 1, \\ 5, & \text{for } i = 2 \text{ and } j = 2, \end{cases}$$

$$w(y_{ij}) = \begin{cases} 2, & \text{for } i = 1 \text{ and } j = 1, 2, \\ 3, & \text{for } i = 2 \text{ and } j = 1, 2. \end{cases}$$

We have the following upper bound $\chi_{lis}(P_m \bigodot Cp_n) \leq 6$. We have $6 \leq \chi_{lis}(P_m \bigodot Cp_n) \leq 6$, so $\chi_{lis}(P_m \bigodot Cp_n) = 6$ for $m = 2$ and $n = 2$.

Case 4: $m = 2$ and $n = 3$.

First step here is to find the lower bound of $V(P_m \bigodot CP_n)$. Based on Lemma 1, we have $\chi_{lis}(P_m \bigodot Cp_n) \geq \chi(P_m \bigodot Cp_n) = 3$.

Assume $\chi_{lis}(P_m \bigodot Cp_n) = 5$, if

$$l(x_i) = l(x_{ij}) = 1, \quad l(y_{1j}) = 1, \quad l(y_{2j}) = 1, \quad j = 1, 2, \quad l(y_{i3}) = 2,$$

then $w(x_{22}) = w(x_{23})$, so there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_1) = 1, \quad l(x_2) = 2, \quad l(x_{ij}) = 1, \quad l(y_{ij}) = 1,$$

then $w(x_1) \neq w(x_2)$, $w(x_{i,1}) \neq w(x_{i,2})$. Based on that we have the lower bound $\chi_{lis}(P_m \bigodot Cp_n) \geq 6$.

After that, we will find the upper bound of $\chi_{lis}(P_m \bigodot Cp_n)$.

Furthermore, we define $l : V(P_m \bigodot Cp_n) \to \{1, 2\}$ with the vertex irregular 2-labelling as follows:

$$l(x_i) = \begin{cases} 1, & \text{for } i = 1, \\ 2, & \text{for } i = 2, \end{cases} \quad l(x_{ij}) = 1, \quad w(y_{ij}) = 1.$$

Hence, opt$(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 7, & \text{for } i = 2, \\ 8, & \text{for } i = 1, \end{cases}$$

$$w(x_{ij}) = \begin{cases} 3, & \text{for } i = 1 \text{ and } j = 1,3, \\ 4, & \text{for } i = 1 \text{ and } j = 2, \text{ or for } i = 2 \text{ and } j = 1,3, \\ 5, & \text{for } i = 2 \text{ and } j = 2, \end{cases}$$

$$w(y_{ij}) = \begin{cases} 2, & \text{for } i = 1 \text{ and } 1 \le j \le 3, \\ 3, & \text{for } i = 2 \text{ and } 1 \le j \le 3. \end{cases}$$

The upper bound is true: $\chi_{lis}(P_m \bigodot Cp_n) \le 6$. So we have $\chi_{lis}(P_m \bigodot Cp_n) = 6$ for $m = 2$ and $n = 3$.

Case 5: $m = 3$ and $n \ge 4$.

First step to prove this theorem in this case is to find the lower bound of $V(P_m \bigodot CP_n)$. Based on Lemma 1, we have $\chi_{lis}(P_m \bigodot Cp_n) \ge \chi(P_m \bigodot Cp_n) = 3$.

Assume $\chi_{lis}(P_m \bigodot Cp_n) = 5$, if $l(x_1) = l(x_3) = 1$, $l(x_2) = 2$, $l(x_{ij}) = l(y_{ij}) = 1$, then $w(x_1) = w(x_2)$ so there are 2 adjacent vertices with the have same color, it contradicts the definition of vertex coloring. If

$$l(x_i) = l(x_{ij}) = 1, \quad l(y_{ij}) = 1, \quad 1 \le i \le 3, \quad j = 1, n, \quad j \equiv 0 \pmod 2,$$
$$l(y_{ij}) = 2, \quad 1 \le i \le 3, \quad j \equiv 1,3 \pmod 4, \quad j \ne 1, n,$$

with the $w(x_i) \ne w(x_{i+1})$, $w(x_{ij}) = w(x_{ij+1})$. Therefore we have the lower bound $\chi_{lis}(P_m \bigodot Cp_n) \ge 6$.

After that, we will find the upper bound for $\chi_{lis}(P_m \bigodot Cp_n)$.

Furthermore, we define $l : V(P_m \bigodot Cp_n) \to \{1, 2\}$ with the vertex irregular 2-labelling as follows:

$$l(y_{ij}) = \begin{cases} 1, & \text{for } 1 \le i \le 3 \text{ and } j = 1, n \quad \text{or} \quad \text{for } 1 \le i \le 3 \text{ and } j \equiv 0 \pmod 2, \\ 2, & \text{for } 1 \le i \le 3 \text{ and } j \equiv 1,3 \pmod 4, \ j \ne 1, n. \end{cases}$$

Hence, opt$(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 2n + n/2, & \text{for } i = 1,3 \text{ and } n \equiv 0 \pmod 2, \\ 2n + \lfloor n/2 \rfloor, & \text{for } i = 1,3 \text{ and } n \equiv 1,3 \pmod 4, \\ 3n + 1 - n/2, & \text{for } i = 2 \text{ and } n \equiv 0 \pmod 2, \\ 3n + 1 - \lceil n/2 \rceil, & \text{for } i = 2 \text{ and } n \equiv 1,3 \pmod 4, \end{cases}$$

$$w(x_{ij}) = \begin{cases} 3, & \text{for } 1 \le i \le 3 \text{ and } j = 1, n, \\ 4, & \text{for } 1 \le i \le 3 \text{ and } j \equiv 0 \pmod 2, \\ 5, & \text{for } 1 \le i \le 3 \text{ and } j \equiv 1,3 \pmod 4, \quad j \ne 1, n, \end{cases}$$

$$w(y_{ij}) = 2.$$

The upper bound is true: $\chi_{lis}(P_m \bigodot Cp_n) \le 6$. So $\chi_{lis}(P_m \bigodot Cp_n) = 6$ for $m = 3$ and $n \ge 4$.

Case 6: $m \equiv 0 \pmod 2$, $m \ge 4$ and $n = 2$.

First step here is to find the lower bound of $V(P_m \bigodot CP_n)$. Based on Lemma 1, we have $\chi_{lis}(P_m \bigodot Cp_n) \geq \chi(P_m \bigodot Cp_n) = 3$.

Assume $\chi_{lis}(P_m \bigodot Cp_n) = 5$, let $\chi_{lis}(P_m \bigodot Cp_n) = 5$, if

$$l(x_i) = 1, \quad i \equiv 1,3 \pmod 4, \quad i \equiv 2 \pmod 4, \quad l(x_i) = 2, \quad i \equiv 0 \pmod 4, \quad l(x_{ij}) = l(y_{ij}) = 1$$

then $w(x_{ij}) = w(x_{ij+1})$, then there are 2 adjacent vertices that have same color, this contradicts the definition of vertex coloring. If

$$l(x_i) = 1, \quad i \equiv 1,3 \pmod 4, \quad i \equiv 2 \pmod 4, \quad l(x_i) = 2, \quad i \equiv 0 \pmod 4,$$
$$l(x_{ij}) = 1 l(y_{ij}) = 1, \quad j = 2, \quad l(y_{ij}) = 2, \quad j = 1,$$

then $w(x_{ij}) \neq w(x_{ij+1}); w(x_{i+1}) \neq w(x_{i+2})$. So we have the lower bound $\chi_{lis}(P_m \bigodot Cp_n) \geq 7$.

After that, we will find the upper bound of $\chi_{lis}(P_m \bigodot Cp_n)$.

Furthermore, we define $l : V(P_m \bigodot Cp_n) \to \{1,2\}$ with the vertex irregular 2-labelling as follows:

$$l(x_i) = \begin{cases} 1, & \text{for } i \equiv 1,3 \pmod 4 \text{ or for } i \equiv 2 \pmod 4, \\ 2, & \text{for } i \equiv 0 \pmod 4, \end{cases}$$

$$l(x_{ij}) = 1, \quad l(y_{ij}) = \begin{cases} 1, & \text{for } 1 \leq i \leq m \text{ and } j = 1, \\ 2, & \text{for } 1 \leq i \leq m \text{ and } j = 2. \end{cases}$$

Hence, $\text{opt}(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 6, & \text{for } i = 1, m, \\ 7, & \text{for } i \equiv 0 \pmod 2, \quad i \neq m, \\ 8, & \text{for } i \equiv 1,3 \pmod 4, \quad i \neq 1, \end{cases}$$

$$w(x_{ij}) = \begin{cases} 3, & \text{for } i \equiv 1,3 \pmod 4 \text{ and } j = 1 \quad \text{or} \quad \text{for } i \equiv 2 \pmod 4 \text{ and } j = 1, \\ 4, & \text{for } i \equiv 1,3 \pmod 4 \text{ and } j = 2 \quad \text{or} \quad \text{for } i \equiv 2 \pmod 4 \text{ and } j = 2 \text{ or} \\ & \quad \text{for } i \equiv 0 \pmod 4 \text{ and } j = 1, \\ 5, & \text{for } i \equiv 0 \pmod 4 \text{ and } j = 2, \end{cases}$$

$$w(y_{ij}) = \begin{cases} 2, & \text{for } i \equiv 1,3 \pmod 4 \text{ and } j = 1,2 \quad \text{or} \quad \text{for } i \equiv 2 \pmod 4 \text{ and } j = 1,2, \\ 3, & \text{for } i = 0 \pmod 4 \text{ and } j = 1,2. \end{cases}$$

We have the upper bound $\chi_{lis}(P_m \bigodot Cp_n) \leq 7$. So $\chi_{lis}(P_m \bigodot Cp_n) = 7$ for $m \geq 4$ and $n = 2$.

Case 7: $m \equiv 0 \pmod 2$, $m \geq 4$ and $n = 3$.

First step to prove this theorem is to find the lower bound of $V(P_m \bigodot CP_n)$. Based on Lemma 1, we have $\chi_{lis}(P_m \bigodot Cp_n) \geq \chi(P_m \bigodot Cp_n) = 3$.

Assume $\chi_{lis}(P_m \bigodot Cp_n) = 5$, in this case if

$$l(x_i) = l(x_{ij}) = 1, \quad l(y_{ij}) = 1, \quad 1 \leq i \leq m, \quad j = 3, \quad l(y_{ij}) = 2, \quad 1 \leq i \leq m, \quad j = 1,2,$$

then $w(x_i) = w(x_{i+1})$, then there are 2 adjacent vertices that have the same color, this contradicts the definition of vertex coloring. If

$$l(x_i) = 1 \quad i \equiv 1,3 \pmod 4, \quad i \equiv 2 \pmod 4, \quad l(x_i) = 2, \quad i \equiv 0 \pmod 2, \quad l(y_{ij}) = l(x_{ij}) = 1,$$

then $w(x_{i+1}) \neq w(x_{i+2})$, $w(x_{i1}) \neq w(x_{i2})$, $w(x_{i1}) \neq w(y_{i2})$. Therefore we have the lower bound $\chi_{lis}(P_m \bigodot Cp_n) \geq 7$.

After that, we will find the upper bound of $\chi_{lis}(P_m \bigodot Cp_n)$.

Furthermore, we define $l : V(P_m \bigodot Cp_n) \to \{1, 2\}$ with the vertex irregular 2-labelling as follows:

$$l(x_i) = \begin{cases} 1, & \text{for } i \equiv 1,3 \pmod 4 \text{ or for } i \equiv 2 \pmod 4, \\ 2, & \text{for } i \equiv 0 \pmod 4, \end{cases}$$

$$l(x_{ij}) = 1, \quad l(y_{ij}) = 1.$$

Hence, $\text{opt}(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 7, & \text{for } i = 1, m, \\ 8, & \text{for } i \equiv 0 \pmod 2,\ i \neq m, \\ 9, & \text{for } i \equiv 1,3 \pmod 4,\ i \neq 1, \end{cases}$$

$$w(x_{ij}) = \begin{cases} 3, & \text{for } i \equiv 1,3 \pmod 4 \text{ and } j = 1,3 \text{ or for } i \equiv 2 \pmod 4 \text{ and } j = 1,3, \\ 4, & \text{for } i \equiv 1,3 \pmod 4 \text{ and } j = 2 \text{ or for } i \equiv 2 \pmod 4 \text{ and } j = 2 \text{ or} \\ & \quad \text{for } i \equiv 0 \pmod 4 \text{ and } j = 1,3, \\ 5, & \text{for } i \equiv 0 \pmod 4 \text{ and } j = 2, \end{cases}$$

$$w(y_{ij}) = \begin{cases} 2, & \text{for } i \equiv 1,3 \pmod 4 \text{ and } 1 \leq j \leq 3 \text{ or for } i \equiv 2 \pmod 4 \text{ and } 1 \leq j \leq 3, \\ 3, & \text{for } i = 0 \pmod 4 \text{ and } 1 \leq j \leq 3. \end{cases}$$

We have the upper bound $\chi_{lis}(P_m \bigodot Cp_n) \leq 7$. So $\chi_{lis}(P_m \bigodot Cp_n) = 7$ for $m \geq 4$ and $n = 3$.

Case 8: $m = 2$ and $n \geq 4$.

First step here is to find the lower bound of $V(P_m \bigodot CP_n)$. Based on Lemma 1, we have $\chi_{lis}(P_m \bigodot Cp_n) \geq \chi(P_m \bigodot Cp_n) = 3$.

Assume $\chi_{lis}(P_m \bigodot Cp_n) < 7$, let $\chi_{lis}(P_m \bigodot Cp_n) = 6$, if

$$l(x_1) = 1, \quad l(x_2) = 2, \quad l(x_{ij}) = l(y_{ij}) = 1,$$

then $w(x_{ij+1}) = w(x_{ij+2})$, then there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_1) = 1, \quad l(x_2) = 2, \quad l(x_{ij}) = 1, \quad l(y_{ij}) = 1, \quad j \equiv 0 \pmod 2, \quad j = 1, n,$$
$$l(y_{ij}) = 2, \quad j \equiv 1,3 \pmod 4, \quad j \neq 1, \quad n \to w(x_1) \neq w(x_2), \quad w(x_{ij+1}) \neq w(x_{ij+2}),$$

then we have the lower bound $\chi_{lis}(P_m \bigodot Cp_n) \geq 7$.

After that, we will find the upper bound of $\chi_{lis}(P_m \bigodot Cp_n)$.

Furthermore, we define $l : V(P_m \bigodot Cp_n) \to \{1, 2\}$ with the vertex irregular 2-labelling as follows:

$$l(x_i) = \begin{cases} 1, & \text{for } i = 1, \\ 2, & \text{for } i = 2, \end{cases}$$

$$l(x_{ij}) = 1,$$

$$l(y_{ij}) = \begin{cases} 1, & \text{for } i = 1,2 \text{ and } j = 1, n \text{ or for } i = 1,2 \text{ and } j \equiv 0 \pmod 2, \\ 2, & \text{for } i = 1,2 \text{ and } j \equiv 1,3 \pmod 4,\ j \neq 1, n. \end{cases}$$

Hence, opt$(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 3n+1-n/2, & \text{for } i=1 \text{ and } n \equiv 0 \pmod 2, \\ 3n+1-\lceil n/2 \rceil, & \text{for } i=1 \text{ and } n \equiv 1,3 \pmod 4, \\ 2n+n/2, & \text{for } i=2 \text{ and } n \equiv 0 \pmod 2, \\ 2n+\lfloor n/2 \rfloor, & \text{for } i=2 \text{ and } n \equiv 1,3 \pmod 4, \end{cases}$$

$$w(x_{ij}) = \begin{cases} 3, & \text{for } i=1 \text{ and } j=1,n, \\ 4, & \text{for } i=1 \text{ and } j \equiv 0 \pmod 2,\ j \neq n \text{ or for } i=2 \text{ and } j=1,n, \\ 5, & \text{for } i=1 \text{ and } j \equiv 1,3 \pmod 4,\ j \neq 1,n \text{ or for } i=2 \text{ and } j \equiv 0 \pmod 2,\ j \neq n, \\ 6, & \text{for } i=2 \text{ and } j \equiv 1,3 \pmod 4,\ j \neq 1,n, \end{cases}$$

$$w(y_{ij}) = \begin{cases} 2, & \text{for } i=1 \text{ and } 1 \leq j \leq n, \\ 3, & \text{for } i=2 \text{ and } 1 \leq j \leq n. \end{cases}$$

The upper bound $\chi_{lis}(P_m \odot Cp_n) \leq 7$ is true. So $\chi_{lis}(P_m \odot Cp_n) = 7$ for $m = 2$ and $n \geq 4$.

Case 9: $m \equiv 1,3 \pmod 4$, $m \geq 5$ and $n = 2$.

First step to prove this theorem in this case is to find the lower bound of $V(P_m \odot CP_n)$. Based on Lemma 1, we have $\chi_{lis}(P_m \odot Cp_n) \geq \chi(P_m \odot Cp_n) = 3$.

Assume $\chi_{lis}(P_m \odot Cp_n) < 7$, and let $\chi_{lis}(P_m \odot Cp_n) = 6$, if

$$l(x_i) = 1, \quad i \equiv 1 \pmod 4, \quad i \equiv 0 \pmod 2, \quad l(x_i) = 2, \quad i \equiv 3 \pmod 4, \quad l(x_{ij}) = l(y_{ij}) = 1,$$

then $w(x_{i1}) = w(x_{i2})$, $w(x_{i+1}) = w(x_{i+2})$, then there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_i) = 1, \quad i \equiv 1 \pmod 4, \quad i \equiv 0 \pmod 2, \quad l(x_i) = 2, \quad i \equiv 3 \pmod 4,$$
$$l(x_{ij}) = 1, \quad l(y_{ij}) = 1, \quad 1 \leq i \leq m, \quad j = 1, \quad l(y_{ij}) = 2, \quad 1 \leq i \leq m, \quad j = 2,$$

then $w(x_{i+1}) \neq w(x_{i+2})$, $w(x_{ij+1}) \neq w(x_{ij+2})$. Therefore we have the lower bound $\chi_{lis}(P_m \odot Cp_n) \geq 7$.

After that, we will find the upper bound of $\chi_{lis}(P_m \odot Cp_n)$.

Furthermore, we define $l : V(P_m \odot Cp_n) \to \{1,2\}$ with the vertex irregular 2-labelling as follows:

$$l(x_i) = \begin{cases} 1, & \text{for } i \equiv 1 \pmod 4 \text{ or for } i \equiv 0 \pmod 2, \\ 2, & \text{for } i \equiv 3 \pmod 4, \end{cases}$$

$$w(x_{ij}) = 1; \quad l(y_{ij}) = \begin{cases} 1, & \text{for } 1 \leq i \leq m \text{ and } j = 1, \\ 2, & \text{for } 1 \leq i \leq m \text{ and } j = 2. \end{cases}$$

Hence, opt$(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 6, & \text{for } i = 1, m, \\ 7, & \text{for } i \equiv 1,3 \pmod 4,\ i \neq 1, \\ 8, & \text{for } i \equiv 0 \pmod 2,\ i \neq m, \end{cases}$$

$$w(x_{ij}) = \begin{cases} 3, & \text{for } i \equiv 1 \pmod 4 \text{ and } j = 1 \text{ or for } i \equiv 0 \pmod 2 \text{ and } j = 1, \\ 4, & \text{for } i \equiv 1 \pmod 4 \text{ and } j = 2 \text{ or for } i \equiv 0 \pmod 2 \text{ and } j = 2 \text{ or} \\ & \quad \text{for } i \equiv 3 \pmod 4 \text{ and } j = 1, \\ 5, & \text{for } i \equiv 3 \pmod 4 \text{ and } j = 2, \end{cases}$$

$$w(y_{ij}) = \begin{cases} 2, & \text{for } i \equiv 1 \pmod 4 \text{ and } j = 1,2 \text{ or for } i \equiv 0 \pmod 2 \text{ and } j = 1,2, \\ 3, & \text{for } i = 3 \pmod 4 \text{ and } j = 1,2. \end{cases}$$

The upper bound $\chi_{lis}(P_m \bigodot Cp_n) \leq 7$ is true. So $\chi_{lis}(P_m \bigodot Cp_n) = 7$ for $m \equiv 1,3 \pmod 4$, $m \geq 5$ and $n = 2$.

Case 10: $m \equiv 1,3 \pmod 4$, $m \geq 5$ and $n = 3$.

First step here is to find the lower bound of $V(P_m \bigodot CP_n)$. Based on Lemma 1, we have $\chi_{lis}(P_m \bigodot Cp_n) \geq \chi(P_m \bigodot Cp_n) = 3$.

Assume $\chi_{lis}(P_m \bigodot Cp_n) < 7$, let $\chi_{lis}(P_m \bigodot Cp_n) = 6$, if

$$l(x_i) = l(x_{ij}) = 1, \quad l(y_{i1}) = 1, \quad l(y_{ij}) = 2, \quad j = 2,3,$$

then $w(x_{i+1}) = w(x_{i+2})$, then we have that there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_i) = 1, \quad i \equiv 1 \pmod 4, \quad i \equiv 0 \pmod 2, \quad l(x_i) = 2, \quad i \equiv 3 \pmod 4, \quad l(x_{ij}) = 1; l(y_{ij}) = 1,$$

then $w(x_{i+1}) \neq w(x_{i+2})$, $w(x_{ij+1}) \neq w(x_{ij+2})$. Based on that we have the lower bound $\chi_{lis}(P_m \bigodot Cp_n) \geq 7$.

After that, we will find the upper bound of $\chi_{lis}(P_m \bigodot Cp_n)$.

Furthermore, we define $l : V(P_m \bigodot Cp_n) \to \{1,2\}$ with the vertex irregular 2-labelling as follows:

$$l(x_i) = \begin{cases} 1, & \text{for } i \equiv 1 \pmod 4 \text{ or for } i \equiv 0 \pmod 2, \\ 2, & \text{for } i \equiv 3 \pmod 4, \end{cases}$$
$$l(x_{ij}) = 1, \quad l(y_{ij}) = 1.$$

Hence, $\text{opt}(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 7, & \text{for } i = 1, m, \\ 8, & \text{for } i \equiv 1,3 \pmod 4, \ i \neq 1, m, \\ 9, & \text{for } i \equiv 0 \pmod 2, \end{cases}$$

$$w(x_{ij}) = \begin{cases} 3, & \text{for } i \equiv 1 \pmod 4 \text{ and } j = 1,3 \text{ or for } i \equiv 0 \pmod 2 \text{ and } j = 1,3, \\ 4, & \text{for } i \equiv 1 \pmod 4 \text{ and } j = 2 \text{ or for } i \equiv 0 \pmod 2 \text{ and } j = 2 \text{ or} \\ & \quad \text{for } i \equiv 3 \pmod 4 \text{ and } j = 1,3, \\ 5, & \text{for } i \equiv 3 \pmod 4 \text{ and } j = 2, \end{cases}$$

$$w(y_{ij}) = \begin{cases} 2, & \text{for } i \equiv 1 \pmod 4 \text{ and } 1 \leq j \leq 3 \text{ or for } i \equiv 0 \pmod 2 \text{ and } 1 \leq j \leq 3, \\ 3, & \text{for } i = 3 \pmod 4 \text{ and } 1 \leq j \leq 3. \end{cases}$$

The upper bound is true: $\chi_{lis}(P_m \bigodot Cp_n) \leq 7$. So $\chi_{lis}(P_m \bigodot Cp_n) = 7$ for $m \equiv 1,3 \pmod 4$, $m \geq 5$ and $n = 3$.

Case 11: $m \equiv 0 \pmod 2$ $m \geq 4$ and $n \geq 4$.

First step to prove this theorem is to find the lower bound of $V(P_m \bigodot CP_n)$. Based on Lemma 1, we have $\chi_{lis}(P_m \bigodot Cp_n) \geq \chi(P_m \bigodot Cp_n) = 3$.

Assume $\chi_{lis}(P_m \bigodot Cp_n) < 8$, let $\chi_{lis}(P_m \bigodot Cp_n) = 7$, if

$$l(x_i) = 1, \quad i \equiv 1,3 \pmod 4, \quad i \equiv 2 \pmod 4, \quad l(x_i) = 2, \quad i \equiv 0 \pmod 4, \quad l(x_{ij}) = l(y_{ij}) = 1,$$

then $w(x_{ij+1}) = w(x_{ij+2})$, so there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_i) = 1, \quad i \equiv 1,3 \pmod 4, \quad i \equiv 2 \pmod 4, \quad l(x_i) = 2, \quad i \equiv 0 \pmod 4, \quad l(x_{ij}) = 1,$$
$$l(y_{ij}) = 1, \quad 1 \leq i \leq m, \quad j = 1, n, \quad j \equiv 0 \pmod 2, \quad l(y_{ij}) = 2,$$
$$1 \leq i \leq m, \quad j \equiv 1,3 \pmod 4, \quad j \neq 1, n,$$

then $w(x_{i+1}) \neq w(x_{i+2})$, $w(x_{ij+1}) \neq w(x_{ij+2})$, $w(x_{ij}) \neq w(y_{ij})$. Based on that we have the lower bound $\chi_{lis}(P_m \bigodot Cp_n) \geq 8$.

After that, we will find the upper bound of $\chi_{lis}(P_m \bigodot Cp_n)$.

Furthermore, we define $l : V(P_m \bigodot Cp_n) \to \{1, 2\}$ with the vertex irregular 2-labelling as follows:

$$l(x_i) = \begin{cases} 1, & \text{for } i \equiv 1,3 \pmod 4 \text{ and } 1 \leq j \leq n \text{ or for } i \equiv 2 \pmod 4 \text{ and } 1 \leq j \leq n, \\ 2, & \text{for } i \equiv 0 \pmod 4 \text{ and } 1 \leq j \leq n, \end{cases}$$

$$l(x_{ij}) = 1,$$

$$l(y_{ij}) = \begin{cases} 1, & \text{for } 1 \leq i \leq m \text{ and } j = 1, n \text{ or for } 1 \leq i \leq m \text{ and } j \equiv 0 \pmod 2, \\ 2, & \text{for } 1 \leq i \leq m \text{ and } j \equiv 1,3 \pmod 4,\ j \neq 1, m. \end{cases}$$

Hence, $\text{opt}(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 2n + n/2, & \text{for } i = 1, m \text{ and } n \equiv 0 \pmod 2, \\ 2n + \lfloor n/2 \rfloor, & \text{for } i = 1, m \text{ and } n \equiv 1,3 \pmod 4, \\ 3n + 1 - n/2, & \text{for } i \equiv 1,3 \pmod 4,\ i \neq 1 \text{ and } n \equiv 0 \pmod 2, \\ 3n + 1 - \lceil n/2 \rceil, & \text{for } i \equiv 0 \pmod 2,\ i \neq m \text{ and } n \equiv 1,3 \pmod 4, \\ 3n + 2 - n/2, & \text{for } i \equiv 1,3 \pmod 4,\ i \neq 1 \text{ and } n \equiv 0 \pmod 2, \\ 3n + 1 - \lfloor n/2 \rfloor, & \text{for } i \equiv 0 \pmod 2,\ i \neq m \text{ and } n \equiv 1,3 \pmod 4, \end{cases}$$

$$w(x_{ij}) = \begin{cases} 3, & \text{for } i \equiv 1,3 \pmod 4 \text{ and } j = 1, n \text{ or for } i \equiv 2 \pmod 4 \text{ and } j = 1, n, \\ 4, & \text{for } i \equiv 1,3 \pmod 4 \text{ and } j \equiv 0 \pmod 2,\ j \neq n \text{ or} \\ & \quad \text{for } i \equiv 2 \pmod 4 \text{ and } j \equiv 0 \pmod 2,\ j \neq n \text{ or for } i \equiv 0 \pmod 4 \text{ and } j = 1, n, \\ 5, & \text{for } i \equiv 1,3 \pmod 4 \text{ and } j \equiv 1,3 \pmod 4,\ j \neq 1, n \text{ or} \\ & \quad \text{for } i \equiv 2 \pmod 4 \text{ and } j \equiv 1,3 \pmod 4,\ j \neq 1, n \text{ or} \\ & \quad \text{for } i \equiv 0 \pmod 4 \text{ and } j \equiv 0 \pmod 2,\ j \neq n, \\ 6, & \text{for } i \equiv 0 \pmod 4 \text{ and } j \equiv 1,3 \pmod 4,\ j \neq 1, n, \end{cases}$$

$$w(y_{ij}) = \begin{cases} 2, & \text{for } i \equiv 1,3 \pmod 4 \text{ and } 1 \leq j \leq n \text{ or for } i \equiv 2 \pmod 4 \text{ and } 1 \leq j \leq n, \\ 3, & \text{for } i \equiv 0 \pmod 4 \text{ and } 1 \leq j \leq n. \end{cases}$$

The upper bound is true: $\chi_{lis}(P_m \bigodot Cp_n) \leq 8$. So $\chi_{lis}(P_m \bigodot Cp_n) = 8$ for $m \equiv 0 \pmod 4$, $m \geq 4$ and $n \geq 4$.

Case 12: $m \equiv 1,3 \pmod 4$, $m \geq 5$ and $n \geq 4$.

First step to prove this theorem in this case is to find the lower bound of $V(P_m \bigodot CP_n)$. Based on Lemma 1, we have $\chi_{lis}(P_m \bigodot Cp_n) \geq \chi(P_m \bigodot Cp_n) = 3$.

Assume $\chi_{lis}(P_m \bigodot Cp_n) < 8$, let $\chi_{lis}(P_m \bigodot Cp_n) = 7$, if

$$l(x_i) = 1, \quad i \equiv 1 \pmod 4, \quad i \equiv 0 \pmod 2, \quad l(x_i) = 2, \quad i \equiv 3 \pmod 4, \quad l(x_{ij}) = l(y_{ij}) = 1,$$

then $w(x_{ij+1}) = w(x_{ij+2})$, so there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_i) = 1, \quad i \equiv 1 \pmod 4, \quad i \equiv 0 \pmod 2, \quad l(x_i) = 2, \quad i \equiv 3 \pmod 4, \quad l(x_{ij}) = 1, \quad l(y_{ij}) = 1,$$
$$1 \leq i \leq m, \quad j = 1, n, \quad j \equiv 0 \pmod 2, \quad l(y_{ij}) = 2, \quad 1 \leq i \leq m, \quad j \equiv 1,3 \pmod 4,$$
$$j \neq 1, n \to w(x_{i+1}) \neq w(x_{i+2}), \quad w(x_{ij+1}) \neq w(x_{ij+2}), \quad w(x_{ij}) \neq w(y_{ij}),$$

therefore we have the lower bound $\chi_{lis}(P_m \bigodot Cp_n) \geq 8$.

After that, we will find the upper bound of $\chi_{lis}(P_m \bigodot Cp_n)$.

Furthermore, we define $l : V(P_m \bigodot Cp_n) \to \{1, 2\}$ with the vertex irregular 2-labelling as follows:

$$l(x_i) = \begin{cases} 1, & \text{for } i \equiv 1 \pmod 4 \text{ or for } i \equiv 0 \pmod 2, \\ 2, & \text{for } i \equiv 3 \pmod 4, \end{cases}$$
$$l(x_{ij}) = 1,$$
$$l(y_{ij}) = \begin{cases} 1, & \text{for } 1 \leq i \leq m \text{ and } j = 1, n \text{ or for } 1 \leq i \leq m \text{ and } j \equiv 0 \pmod 2, \\ 2, & \text{for } 1 \leq i \leq m \text{ and } j \equiv 1,3 \pmod 4,\ j \neq 1, n. \end{cases}$$

Hence, $\mathrm{opt}(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 2n + n/2, & \text{for } i = 1, m \text{ and } n \equiv 0 \pmod 2, \\ 3n + 2 - n/2, & \text{for } i = 0 \pmod 2 \text{ and } n \equiv 0 \pmod 2, \\ 3n + 1 - n/2, & \text{for } i \equiv 1,3 \pmod 4 \text{ and } n \equiv 0 \pmod 2, \\ 2n + \lfloor n/2 \rfloor, & \text{for } i = 1, m \text{ and } n \equiv 1,3 \pmod 4, \\ 3n - \lfloor n/2 \rfloor, & \text{for } i \equiv 1,3 \pmod 4,\ i \neq 1 \text{ and } n \equiv 1,3 \pmod 4, \\ 3n + 1 - \lfloor n/2 \rfloor, & \text{for } i = 0 \pmod 4 \text{ and } n \equiv 1,3 \pmod 4, \end{cases}$$

$$w(x_{ij}) = \begin{cases} 3, & \text{for } i \equiv 1 \pmod 4 \text{ and } j = 1, n \text{ or for } i \equiv 0 \pmod 2 \text{ and } j = 1, n, \\ 4, & \text{for } i \equiv 1 \pmod 4 \text{ and } j \equiv 0 \pmod 2,\ j \neq n \text{ or} \\ & \quad \text{for } i \equiv 0 \pmod 2 \text{ and } j \equiv 0 \pmod 2,\ j \neq n \text{ or} \\ & \quad \text{for } i \equiv 3 \pmod 4 \text{ and } j = 1, n, \\ 5, & \text{for } i \equiv 1 \pmod 4 \text{ and } j \equiv 1,3 \pmod 4,\ j \neq 1, n \text{ or} \\ & \quad \text{for } i \equiv 0 \pmod 2 \text{ and } j \equiv 1,3 \pmod 4,\ j \neq 1, n \text{ or} \\ & \quad \text{for } i \equiv 3 \pmod 4 \text{ and } j \equiv 0 \pmod 2,\ j \neq n, \\ 6, & \text{for } i \equiv 3 \pmod 4 \text{ and } j \equiv 1,3 \pmod 4,\ j \neq 1, n, \end{cases}$$

$$w(y_{ij}) = \begin{cases} 2, & \text{for } i \equiv 1 \pmod 4 \text{ and } 1 \leq j \leq n \text{ or for } i \equiv 0 \pmod 2 \text{ and } 1 \leq j \leq n, \\ 3, & \text{for } i \equiv 3 \pmod 4 \text{ and } 1 \leq j \leq n. \end{cases}$$

The upper bound is true: $\chi_{lis}(P_m \bigodot Cp_n) \leq 8$. So $\chi_{lis}(P_m \bigodot Cp_n) = 8$ for $m \geq 5$ and $n \geq 4$. □

Theorem 2. *Let $G = P_m \bigodot S_{2,n}$ for $n, m \geq 2$, then the chromatic number of local irregular G is*

$$\chi_{lis}(P_m \bigodot S_{2,n}) = \begin{cases} 5, & \text{for } m = 3 \text{ and } n \geq 2, \\ 6, & \text{for } m = 2 \text{ and } n \geq 2, \\ 7, & \text{for } m \geq 4 \text{ and } n \geq 2, \end{cases}$$

with $\text{opt}(l)(P_m \bigodot S_{2,n}) = 1, 2,$ *for* $m \geq 2$ *and* $n \geq 2$.

P r o o f. Vertex set is

$$V(P_n \bigodot S_{2,n}) = \{x_i; 1 \leq i \leq m\} \cup \{a_i; 1 \leq i \leq m\} \cup \{b_i; 1 \leq i \leq m\}$$
$$\cup \{a_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{b_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}$$

and the edge set is

$$E(P_n \bigodot S_{2,n}) = \{x_i x_{i+1},\ 1 \leq i \leq m-1\} \cup \{a_i b_i; 1 \leq i \leq m\} \cup \{x_i a_i; 1 \leq i \leq m\}$$
$$\cup \{x_i b_i; 1 \leq i \leq m\} \cup \{x_i a_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{x_i b_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}$$
$$\cup \{a_i a_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{b_i b_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}.$$

The order and the size respectively are $2mn + 3m$ and $4mn + 4m - 1$. This proof is divided into 4 cases as follows.

Case 1: $m = 3$ and $n \geq 2$.

First step to prove this theorem is to find the lower bound of $V(P_m \bigodot S_{2,n})$. Based on Lemma 1, we have $\chi_{lis}(P_m \bigodot S_{2,n}) \geq \chi(P_m \bigodot S_{2,n}) = 3$.

Assume $\chi_{lis}(P_m \bigodot S_{2,n}) = 4$, if $l(a_i) = l(b_i) = 1$, $l(x_i) = l(a_{ij}) = l(b_{ij}) = 1$ then $w(a_i) = w(b_i)$, then there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_i) = l(a_i) = l(b_i) = l(a_{ij}) = l(b_{ij}) = 1, \quad 1 \leq j \leq n-1, \quad l(b_{in}) = 2,$$

then

$$w(a_i) \neq w(b_i), \quad w(x_1) = w(x_3) \neq w(x_2),$$

therefore we have the lower bound $\chi_{lis}(P_m \bigodot S_{2,n}) \geq 5$.

After that, we will find the upper bound of $\chi_{lis}(P_m \bigodot S_{2,n})$.

Furthermore, we define $l : V(P_m \bigodot S_{2,n}) \to \{1, 2\}$ with the vertex irregular 2-labelling as follows:

$$l(x_i) = 1, \quad l(a_i) = 1, \quad l(b_i) = 1, \quad l(a_{ij}) = 1,$$
$$l(b_{ij}) = \begin{cases} 1, & \text{for } 1 \leq i \leq 3 \text{ and } 1 \leq j \leq n-1, \\ 2, & \text{for } 1 \leq i \leq 3 \text{ and } j = n. \end{cases}$$

Hence, $\text{opt}(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 2n+4, & \text{for } i = 1, 3, \\ 2n+5, & \text{for } i = 2, \end{cases}$$
$$w(a_i) = n + 2, \quad \text{for } 1 \leq i \leq 3,$$
$$w(b_i) = n + 3, \quad \text{for } 1 \leq i \leq 3,$$
$$w(a_{ij}) = 2, \quad w(b_{ij}) = 2.$$

The upper bound $\chi_{lis}(P_m \bigodot S_{2,n}) \leq 5$ is true. So $\chi_{lis}(P_m \bigodot S_{2,n}) = 5$ for $m = 3$ and $n \geq 2$.

Case 2: $m = 2$ and $n \geq 2$.

First step here is to find the lower bound of $V(P_m \bigodot S_{2,n})$. Based on Lemma 1, we have $\chi_{lis}(P_m \bigodot S_{2,n}) \geq \chi(P_m \bigodot S_{2,n}) = 3$.

Assume $\chi_{lis}(P_m \bigodot S_{2,n}) = 5$, if

$$l(x_i) = l(a_i) = l(b_i) = l(a_{ij}) = l(b_{2j}) = 1, \quad l(b_{1j}) = 1, \quad 1 \leq j \leq n-1, \quad l(b_{1n}) = 2,$$

and then $w(a_2) = w(b_2)$, and there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_i) = l(a_i) = l(b_i) = l(a_{ij}) = 1,$$
$$l(b_{1,j}) = 1, \quad l(b_{1,n}) = 2, \quad l(b_{2,j}) = 2, \quad j = 1, n, \quad l(b_{2j}) = 1, \quad 2 \leq j \leq n-1,$$

then $w(a_i) \neq w(b_i)$, $w(x_1) \neq w(x_2)$. Based on that we have the lower bound $\chi_{lis}(P_m \bigodot S_{2,n}) \geq 6$.

After that, we will find the upper bound of $\chi_{lis}(P_m \bigodot S_{2,n})$.

Furthermore, we define $l : V(P_m \bigodot S_{2,n}) \to \{1, 2\}$ with the vertex irregular 2-labelling as follows:

$$l(x_i) = 1, \quad l(a_i) = 1, \quad l(b_i) = 1, \quad l(a_{ij}) = 1,$$
$$l(b_{ij}) = \begin{cases} 1, & \text{for } i = 1 \text{ and } 1 \leq j \leq n-1 \text{ or for } i = 2 \text{ and } 2 \leq j \leq n-1, \\ 2, & \text{for } i = 1 \text{ and } j = n \text{ or for } i = 2 \text{ and } j = 1, n. \end{cases}$$

Hence, $\mathrm{opt}(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 2n+4, & \text{for } i = 1, \\ 2n+5, & \text{for } i = 2, \end{cases}$$
$$w(a_i) = n+2, \quad \text{for } i = 1, 2,$$
$$w(b_i) = \begin{cases} n+3, & \text{for } i = 1, \\ n+4, & \text{for } i = 2, \end{cases}$$
$$w(a_{ij}) = 2, \quad w(b_{ij}) = 2.$$

The upper bound is true: $\chi_{lis}(P_m \bigodot S_{2,n}) \leq 6$. So $\chi_{lis}(P_m \bigodot S_{2,n}) = 6$ for $m = 2$ and $n \geq 2$.

Case 3: $m \equiv 0 \pmod 4$, $m \geq 4$ and $n \geq 2$.

First step to prove this theorem in this case is to find the lower bound of $V(P_m \bigodot S_{2,n})$. Based on Lemma 1, we have $\chi_{lis}(P_m \bigodot S_{2,n}) \geq \chi(P_m \bigodot S_{2,n}) = 3$.

Assume $\chi_{lis}(P_m \bigodot S_{2,n}) = 6$, if

$$l(x_i) = l(a_i) = l(b_i) = l(a_{ij}) = 1, \quad l(b_{ij}) = 1, \quad i \equiv 1, 3 \pmod 4, \quad i \equiv 2 \pmod 4,$$
$$l(b_{ij}) = 1, \quad i \equiv 0 \pmod 4, \quad j \neq 1, n, \quad l(b_{ij}) = 2, \quad i \equiv 0 \pmod 4, \quad j = 1, n,$$

then $w(a_i) = w(b_i)$, so there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_i) = l(a_i) = l(b_i) = l(a_{ij}) = 1, \quad l(b_{ij}) = 1, \quad i \equiv 1, 3 \pmod 4, \quad i = m, \quad 1 \leq j \leq n-1,$$
$$i \equiv 0 \pmod 2, \quad i \neq m, \quad 2 \leq j \leq n-1, \quad l(b_{ij}) = 2, \quad i \equiv 1, 3 \pmod 4,$$
$$i = m, \quad j = n, \quad i \equiv 0 \pmod 2, \quad i \neq m, \quad j = 1, n,$$

then $w(a_i) \neq w(b_i)$, $w(x_i) \neq w(x_{i+1})$. Based on that we have the lower bound $\chi_{lis}(P_m \bigodot S_{2,n}) \geq 7$.

After that, we will find the upper bound of $\chi_{lis}(P_m \bigodot S_{2,n})$

Furthermore, we define $l : V(P_m \bigodot S_{2,n}) \to \{1, 2\}$ with the vertex irregular 2-labelling as follows:

$$l(x_i) = 1, \quad l(a_i) = 1, \quad l(b_i) = 1, \quad l(a_{ij}) = 1,$$

$$l(b_{ij}) = \begin{cases} 1, & \text{for } i \equiv 1, 3 \pmod 4 \text{ and } 1 \leq j \leq n-1 \text{ or} \\ & \quad \text{for } i \equiv 0 \pmod 2,\ i \neq m \text{ and } 2 \leq j \leq n-1 \text{ or} \\ & \quad \text{for } i = m, \text{ and } 1 \leq j \leq n-1, \\ 2, & \text{for } i \equiv 1, 3 \pmod 4 \text{ and } j = n \text{ or} \\ & \quad \text{for } i \equiv 0 \pmod 2,\ i \neq m \text{ and } j = 1, n \text{ or} \\ & \quad \text{for } i = m, \text{ and } j = n. \end{cases}$$

Hence, opt$(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 2n+4, & \text{for } i = 1, m, \\ 2n+5, & \text{for } i \equiv 0 \pmod 2,\ i \neq m, \\ 2n+6, & \text{for } i \equiv 1, 3 \pmod 4,\ i \neq 1, \end{cases}$$

$$w(a_i) = n+2, \quad \text{for } 1 \leq i \leq m,$$

$$w(b_i) = \begin{cases} n+3, & \text{for } i \equiv 1, 3 \pmod 4,\ i = m, \\ n+4, & \text{for } i \equiv 0 \pmod 2,\ i \neq m, \end{cases}$$

$$w(a_{ij}) = 2, \quad w(b_{ij}) = 2.$$

The upper bound is true: $\chi_{lis}(P_m \bigodot S_{2,n}) \leq 7$. So $\chi_{lis}(P_m \bigodot S_{2,n}) = 7$ for $m \equiv 0 \pmod 2$, $m \geq 4$ and $n \geq 2$.

Case 4: $m \equiv 1, 3 \pmod 4$, $m \geq 5$ and $n \geq 2$.

First step here is to find the lower bound of $V(P_m \bigodot S_{2,n})$. Based on Lemma 1, we have

$$\chi_{lis}(P_m \bigodot S_{2,n}) \geq \chi(P_m \bigodot S_{2,n}) = 3.$$

Assume $\chi_{lis}(P_m \bigodot S_{2,n}) = 6$, if

$$l(x_i) = l(a_i) = l(b_i) = l(a_{ij}) = 1, \quad l(b_{ij}) = 1, \quad i \equiv 1 \pmod 4, \quad i \equiv 0 \pmod 2,$$
$$l(b_{ij}) = 1, \quad i \equiv 3 \pmod 4, \quad j \neq n, \quad l(b_{ij}) = 2, \quad i \equiv 3 \pmod 4, \quad j = n,$$

then $w(a_i) = w(b_i)$, and there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_i) = l(a_i) = l(b_i) = 1, \quad l(a_{ij}) = 1, \quad l(b_{ij}) = 1, \quad i \equiv 1, 3 \pmod 4, \quad 1 \leq j \leq n-1,$$
$$i \equiv 0 \pmod 2, \quad 2 \leq j \leq n-1, \quad l(b_{ij}) = 2, \quad i \equiv 1, 3 \pmod 4, \quad j = n, \quad i \equiv 0 \pmod 2, \quad j = 1, n,$$

then $w(a_i) \neq w(b_i), w(x_i) \neq w(x_{i+1})$. Based on that we have the lower bound $\chi_{lis}(P_m \bigodot S_{2,n}) \geq 7$.

After that, we will find the upper bound of $\chi_{lis}(P_m \bigodot S_{2,n})$.

Furthermore, we define $l : V(P_m \bigodot S_{2,n}) \to \{1, 2\}$ with the vertex irregular 2-labelling as follows:

$$l(x_i) = 1, \quad l(a_i) = 1, \quad l(b_i) = 1, \quad l(a_{ij}) = 1,$$

$$l(b_{ij}) = \begin{cases} 1, & \text{for } i \equiv 1, 3 \pmod 4 \text{ and } 1 \leq j \leq n-1 \text{ or for } i \equiv 0 \pmod 2, \text{ and } 2 \leq j \leq n-1, \\ 2, & \text{for } i \equiv 1, 3 \pmod 4 \text{ and } j = n \text{ or for } i \equiv 0 \pmod 2 \text{ and } j = 1, n. \end{cases}$$

Hence, $\text{opt}(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 2n+4, & \text{for } i = 1, m, \\ 2n+5, & \text{for } i \equiv 1, 3 \pmod 4, \\ 2n+6, & \text{for } i \equiv 0 \pmod 2, \end{cases}$$

$$w(a_i) = n+2, \quad \text{for } 1 \le i \le m,$$

$$w(b_i) = \begin{cases} n+3, & \text{for } i \equiv 1, 3 \pmod 4, \\ n+4, & \text{for } i \equiv 0 \pmod 2, \end{cases}$$

$$w(a_{ij}) = 2, \quad w(b_{ij}) = 2.$$

The upper bound is true: $\chi_{lis}(P_m \bigodot S_{2,n}) \le 7$. So $\chi_{lis}(P_m \bigodot S_{2,n}) = 7$ for $m \equiv 1, 3 \pmod 4$, $m \ge 5$ and $n \ge 2$. □

Theorem 3. *Let $G = P_m \bigodot S_{3,n}$ for $n, m \ge 2$, then the chromatic number of local irregular G is*

$$\chi_{lis}(P_m \bigodot S_{3,n}) = \begin{cases} 5, & \text{for } m = 3 \text{ and } n \ge 2, \\ 6, & \text{for } m = 2 \text{ and } n \ge 2, \\ 7, & \text{for } m \ge 4 \text{ and } n \ge 3, \end{cases}$$

with

$$\text{opt}(l)(P_m \bigodot S_{3,n}) = \begin{cases} 1, & \text{for } m = 3 \text{ and } n = 3, \\ 1, 2, & \text{for } m = 2 \text{ and } n = 2 \text{ or for } m = 3 \text{ and } n = 2 \text{ or} \\ & \quad \text{for } m \ge 4 \text{ and } n \ge 2. \end{cases}$$

P r o o f. The vertex set is

$$V(P_m \bigodot S_{3,n}) = \{x_i; 1 \le i \le m\} \cup \{a_i; 1 \le i \le m\} \cup \{b_i; 1 \le i \le m\} \cup \{c_i; 1 \le i \le m\}$$
$$\cup \{a_{ij}; 1 \le i \le m, 1 \le j \le n\} \cup \{b_{ij}; 1 \le i \le m, 1 \le j \le n\} \cup \{c_{ij}; 1 \le i \le m, 1 \le j \le n\}$$

and the edge set is

$$V(P_m \bigodot S_{3,n}) = \{x_i x_{i+1}; 1 \le i \le m-1\} \cup \{x_i y_i; 1 \le i \le m\} \cup \{x_i a_i; 1 \le i \le m\}$$
$$\cup \{x_i b_i; 1 \le i \le m\} \cup \{x_i c_i; 1 \le i \le m\} \cup \{y_i a_i; 1 \le i \le m\} \cup \{y_i b_i; 1 \le i \le m\}$$
$$\cup \{y_i c_i; 1 \le i \le m\} \cup \{x_i a_{ij}; 1 \le i \le m; 1 \le j \le n\} \cup \{x_i b_{ij}; 1 \le i \le m; 1 \le j \le n\}$$
$$\cup \{x_i c_{ij}; 1 \le i \le m; 1 \le j \le n\} \cup \{a_i a_{ij}; 1 \le i \le m; 1 \le j \le n\}$$
$$\cup \{b_i b_{ij}; 1 \le i \le m; 1 \le j \le n\} \cup \{c_i c_{ij}; 1 \le i \le m; 1 \le j \le n\}.$$

The order and size respectively are $3mn + 5m$ and $6mn + 8n - 1$. This proof can be divided into 8 following cases.

Case 1: $m = 3$ and $n = 2$.

First step to prove this theorem is to find the lower bound of $V(P_m \bigodot S_{3,n})$. Based on Lemma 1, we have $\chi_{lis}(P_m \bigodot S_{3,n}) \ge \chi(P_m \bigodot S_{3,n}) = 3$.

Assume $\chi_{lis}(P_m \bigodot S_{3,n}) = 4$, if

$$l(a_i) = l(b_i) = l(c_i) = l(y_i) = l(a_{ij}) = l(b_{ij}) = l(c_{ij}) = 1,$$

then $w(a_i) = w(b_i) = w(c_i) = w(y_i)$, and there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(a_i) = l(b_i) = l(c_i) = 1, \quad l(a_{ij}) = l(b_{ij}) = l(c_{ij}) = 1, \quad l(y_i) = 2,$$

then $(w(a_i) = w(b_i) = w(c_i)) \neq w(y_i)$, $w(x_1) \neq w(x_2)$. Therefore we have the lower bound $\chi_{lis}(P_m \bigodot S_{3,n}) \geq 5$.

After that, we will find the upper bound of $\chi_{lis}(P_m \bigodot S_{3,n})$.

Furthermore, we define $l : V(P_m \bigodot S_{3,n}) \to \{1, 2\}$ with vertex irregular 2-labelling as follows:

$$l(x_i) = 1, \quad l(y_i) = 2, \quad l(a_i) = 1, \quad l(b_i) = 1, \quad l(c_i) = 1, \quad l(a_{ij}) = 1, \quad l(b_{ij}) = 1, \quad l(c_{ij}) = 1.$$

Hence, $\text{opt}(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 12, & \text{for } i = 1, 3, \\ 13, & \text{for } i = 2, \end{cases}$$

$$w(y_i) = 4, \quad w(a_i) = 5, \quad w(b_i) = 5, \quad w(c_i) = 5, \quad w(a_{ij}) = 2, \quad w(b_{ij}) = 2, \quad w(c_{ij}) = 2.$$

The upper bound is true: $\chi_{lis}(P_m \bigodot S_{3,n}) \leq 5$. So $\chi_{lis}(P_m \bigodot S_{3,n}) = 5$ for $m = 3$ and $n = 2$.

Case 2: $m = 3$ and $n = 3$.

Based on Proposition 1, we have $\text{opt}(l) = 1$. So the lower bound $(P_m \bigodot S_{3,n})$ is $\chi_{lis}(P_m \bigodot S_{3,n}) \geq 5$

Since $\text{opt}(l) = 1$, the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 3n + 5, & \text{for } i = 1, 3, \\ 3n + 6, & \text{for } i = 2, \end{cases}$$

$$\begin{aligned} w(y_i) &= 4, \\ w(a_i) &= n + 1 \quad \text{for } 1 \leq i \leq 3, \\ w(b_i) &= n + 1 \quad \text{for } 1 \leq i \leq 3, \\ w(c_i) &= n + 1 \quad \text{for } 1 \leq i \leq 3, \\ w(a_{ij}) &= 2, \quad w(b_{ij}) = 2, \quad w(c_{ij}) = 2. \end{aligned}$$

The upper bound is true: $\chi_{lis}(P_m \bigodot S_{3,n}) \leq 5$. So $\chi_{lis}(P_m \bigodot S_{3,n}) = 5$ for $m = 3$ and $n \geq 2$.

Case 3: $m = 2$ and $n = 2$.

First step to prove this theorem is to find the lower bound of $V(P_m \bigodot S_{3,n})$. Based on Lemma 1, we have $\chi_{lis}(P_m \bigodot S_{3,n}) \geq \chi(P_m \bigodot S_{3,n}) = 3$.

Assume $\chi_{lis}(P_m \bigodot S_{3,n}) = 5$, if

$$l(a_i) = l(b_i) = l(c_i) = l(a_{ij}) = l(b_{ij}) = l(c_{ij}) = 1, \quad l(y_1) = 1, \quad l(y_2) = 2,$$

then $w(a_2) = w(b_2) = w(c_2) = w(y_2)$ and there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_i) = l(a_i) = l(b_i) = l(c_i) = l(b_{ij}) = 1, \quad l(y_i) = 2, \quad l(c_{1j}) = 1, \quad l(c_{2,1}) = 1, \quad l(c_{2,2}) = 2,$$

then $w(x_1) \neq w(x_2)$, $w(y_i) \neq ((w(a_i) = w(b_i) = w(c_i))$. Based on that we have the lower bound $\chi_{lis}(P_m \bigodot S_{3,n}) \geq 6$.

After that, we will find the upper bound of $\chi_{lis}(P_m \bigodot S_{3,n})$.

Furthermore, we define $l : V(P_m \bigodot S_{3,n}) \to \{1, 2\}$ with vertex irregular 2-labelling as follows:

$$l(x_i) = 1, \quad l(y_i) = 2, \quad l(a_i) = 1, \quad l(b_i) = 1, \quad l(c_i) = 1, \quad l(a_{ij}) = 1, \quad l(b_{ij}) = 1,$$

$$l(c_{ij}) = \begin{cases} 1, & \text{for } i = 1 \text{ and } j = 1, 2 \text{ or for } i = 2 \text{ and } j = 1, \\ 2, & \text{for } i = 2 \text{ and } j = 2. \end{cases}$$

Hence, opt$(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 12, & \text{for } i = 1, \\ 13, & \text{for } i = 2, \end{cases}$$

$$w(y_i) = 4, \quad w(a_i) = 5, \quad w(b_i) = 5,$$

$$w(c_i) = \begin{cases} 5, & \text{for } i = 1, \\ 6, & \text{for } i = 2, \end{cases}$$

$$w(a_{ij}) = 2, \quad w(b_{ij}) = 2, \quad w(c_{ij}) = 2.$$

The upper bound is true: $\chi_{lis}(P_m \bigodot S_{2,n}) \leq 6$. So $\chi_{lis}(P_m \bigodot S_{2,n}) = 6$ for $m = 2$ and $n = 2$.

Case 4: $m = 2$ and $n \geq 3$.

First step here is to find the lower bound of $V(P_m \bigodot S_{3,n})$. Based on Lemma 1, we have $\chi_{lis}(P_m \bigodot S_{3,n}) \geq \chi(P_m \bigodot S_{3,n}) = 3$.

Assume $\chi_{lis}(P_m \bigodot S_{3,n}) = 5$, if

$$l(a_i) = l(b_i) = l(c_i) = l(y_i) = 1, \quad l(a_{ij}) = l(b_{ij}) = 1, \quad l(c_{ij}) = 1, \quad i = 1, 2, \quad 1 \leq j \leq n-1$$
$$l(c_{ij}) = 2, \quad i = 1, 2, \quad j = n,$$

then $w(x_1) = w(x_2)$, then there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_i) = l(a_{ij}) = l(b_{ij}) = 1, \quad l(c_{ij}) = 1, \quad i = 1, \quad 1 \leq j \leq n, \quad i = 2, \quad 1 \leq j \leq n-1,$$
$$l(c_{ij}) = 2, \quad i = 2, \quad j = n,$$

then $w(x_1) \neq w(x_2), w(y_i) \neq ((w(a_i) = w(b_i) = w(c_i))$. Therefore we have the lower bound $\chi_{lis}(P_m \bigodot S_{3,n}) \geq 6$.

After that, we will find the upper bound of $\chi_{lis}(P_m \bigodot S_{3,n})$.

Furthermore, we define $l : V(P_m \bigodot S_{3,n}) \to \{1, 2\}$ with vertex irregular 2-labelling as follows:

$$l(x_i) = 1, \quad l(y_i) = 1, \quad l(a_i) = 1, \quad l(b_i) = 1, \quad l(c_i) = 1, \quad l(a_{ij}) = 1, \quad l(b_{ij}) = 1,$$

$$l(c_{ij}) = \begin{cases} 1, & \text{for } i = 1 \text{ and } 1 \leq j \leq n \text{ or for } i = 2 \text{ and } 1 \leq j \leq n-1, \\ 2, & \text{for } i = 2 \text{ and } j = n. \end{cases}$$

Hence, opt$(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 3n+5, & \text{for } i = 1, \\ 3n+6, & \text{for } i = 2, \end{cases}$$

$$w(y_i) = 4,$$

$$w(a_i) = n+1, \quad \text{for } i = 1, 2,$$

$$w(b_i) = n+1, \quad \text{for } i = 1, 2,$$

$$w(c_i) = \begin{cases} n+1, & \text{for } i = 1, \\ n+2, & \text{for } i = 2, \end{cases}$$

$$w(a_{ij}) = 2, \quad w(b_{ij}) = 2, \quad w(c_{ij}) = 2.$$

The upper bound is true: $\chi_{lis}(P_m \odot S_{3,n}) \leq 6$. So $\chi_{lis}(P_m \odot S_{3,n}) = 6$ for $m = 2$ and $n \geq 3$.

Case 5: $m \equiv 0 \pmod 2$ $m \geq 4$ and $n = 2$.

First step to prove this theorem in this case is to find the lower bound of $V(P_m \odot S_{3,n})$. Based on Lemma 1, we have $\chi_{lis}(P_m \odot S_{3,n}) \geq \chi(P_m \odot S_{3,n}) = 3$.

Assume $\chi_{lis}(P_m \odot S_{3,n}) = 6$, if

$$l(a_i) = l(b_i) = l(c_i) = l(y_i) = 1, \quad l(a_{ij}) = l(b_{ij}) = 1, \quad l(c_{ij}) = 1, \quad i \equiv 1,3 \pmod 4,$$
$$j = 1,2, \quad i \equiv 0 \pmod 4, \quad j = 1,2, \quad l(c_{ij}) = 1, \quad i \equiv 2 \pmod 4,$$
$$j = 1, \quad l(c_{ij}) = 2, \quad i \equiv 2 \pmod 4, \quad j = 2,$$

then $w(y_i) = w(a_i)$. Then there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_i) = 1, \quad l(a_i) = l(b_i) = l(c_i) = 1, \quad l(y_i) = 2, \quad l(a_{ij}) = l(b_{ij}) = 1, \quad l(c_{ij}) = 1,$$
$$i \equiv 0 \pmod 2, \quad j = 1, \quad i \neq m, \quad i \equiv 1,3 \pmod 4, \quad j = 1,2, \quad l(c_{ij}) = 2,$$
$$i \equiv 0 \pmod 2, \quad i \neq m, \quad j = 2,$$

then $w(x_{i+1}) \neq w(x_{i+2}; w(y_i) \neq w(a_i)$. Therefore we have the lower bound $\chi_{lis}(P_m \odot S_{3,n}) \geq 7$. After that, we will find the upper bound of $\chi_{lis}(P_m \odot S_{3,n})$.

Furthermore, we define $l : V(P_m \odot S_{3,n}) \to \{1,2\}$ with vertex irregular 2-labelling as follows:

$$l(x_i) = 1, \quad l(y_i) = 2, \quad l(a_i) = 1, \quad l(b_i) = 1, \quad l(c_i) = 1, \quad l(a_{ij}) = 1, \quad l(b_{ij}) = 1,$$
$$l(c_{ij}) = \begin{cases} 1, & \text{for } i \equiv 1,3 \pmod 4 \text{ and } j = 1,2 \text{ or for } i = m \text{ and } j = 1,2 \text{ or} \\ & \quad \text{for } i \equiv 0 \pmod 2,\ i \neq m \text{ and } j = 1, \\ 2, & \text{for } i \equiv 0 \pmod 2,\ i \neq m \text{ and } j = 2. \end{cases}$$

Hence, opt$(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 12, & \text{for } i = 1, m, \\ 13, & \text{for } i \equiv 1,3 \pmod 4,\ i \neq 1, \\ 14, & \text{for } i \equiv 0 \pmod 2,\ i \neq m, \end{cases}$$
$$w(y_i) = 4, \quad w(a_i) = 5, \quad w(b_i) = 5,$$
$$w(c_i) = \begin{cases} 5, & \text{for } i \equiv 1,3 \pmod 4, \\ 6, & \text{for } i \equiv 0 \pmod 2,\ i \neq m, \end{cases}$$
$$w(a_{ij}) = 2, \quad w(b_{ij}) = 2, \quad w(c_{ij}) = 2,$$

The upper bound is true: $\chi_{lis}(P_m \odot S_{3,n}) \leq 7$. So $\chi_{lis}(P_m \odot S_{3,n}) = 7$ for $m \equiv 0 \pmod 2$; $m \geq 4$ and $n = 2$.

Case 6: $m \equiv 1,3 \pmod 4$, $m \geq 5$ and $n = 2$.

First step here is to find the lower bound of $V(P_m \odot S_{3,n})$. Based on Lemma 1, we have $\chi_{lis}(P_m \odot S_{3,n}) \geq \chi(P_m \odot S_{3,n}) = 3$.

Assume $\chi_{lis}(P_m \odot S_{3,n}) = 6$, if

$$l(a_i) = l(b_i) = l(c_i) = l(y_i) = 1, \quad l(a_{ij}) = l(b_{ij}) = 1, \quad l(c_{ij}) = 1,$$
$$i \equiv 1 \pmod 4, \quad j = 1,2, \quad i \equiv 0 \pmod 2, \quad j = 1,2, \quad l(c_{ij}) = 1,$$
$$i \equiv 3 \pmod 4, \quad j = 1, \quad l(c_{ij}) = 2, \quad i \equiv 3 \pmod 4, \quad j = 2,$$

then $w(y_i) = w(a_i)$, then there are 2 adjacent vertices that have same color, it contradicts to definition of vertex coloring. If

$$l(x_i) = 1, \quad l(a_{ij}) = l(b_{ij}) = 1, \quad l(c_{ij}) = 1, \quad i \equiv 1, 3 \pmod 4,$$
$$j = 1, 2, \quad i \equiv 0 \pmod 2, \quad j = 1, \quad l(c_{ij}) = 2, \quad i \equiv 0 \pmod 2, \quad j = 2, \quad l(y_i) = 2,$$

then

$$w(x_{i+1}) \neq w(x_{i+2}, \quad w(y_i) \neq w(a_i), \quad w(y_i) \neq w(b_i), \quad w(y_i) \neq w(c_i).$$

We have the lower bound $\chi_{lis}(P_m \bigodot S_{3,n}) \geq 7$.

After that, we will find the upper bound of $\chi_{lis}(P_m \bigodot S_{3,n})$.

Furthermore, we define $l : V(P_m \bigodot S_{3,n}) \to \{1, 2\}$ with vertex irregular 2-labelling as follows:

$$l(x_i) = 1, \quad l(y_i) = 2, \quad l(a_i) = 1, \quad l(b_i) = 1, \quad l(c_i) = 1, \quad l(a_{ij}) = 1, \quad l(b_{ij}) = 1,$$
$$l(c_{ij}) = \begin{cases} 1, & \text{for } i \equiv 1, 3 \pmod 4 \text{ and } j = 1, 2 \text{ or} \\ & \quad \text{for } i \equiv 0 \pmod 2,\ i \neq m \text{ and } j = 1, \\ 2, & \text{for } i \equiv 0 \pmod 2,\ i \neq m \text{ and } j = 2. \end{cases}$$

Hence, $\text{opt}(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 12, & \text{for } i = 1, m \\ 13, & \text{for } i \equiv 1, 3 \pmod 4,\ i \neq 1, m, \\ 14, & \text{for } i \equiv 0 \pmod 2, \end{cases}$$
$$w(y_i) = 4, \quad w(a_i) = 5, \quad w(b_i) = 5,$$
$$w(c_i) = \begin{cases} 5, & \text{for } i \equiv 1, 3 \pmod 4,\ i = 1, m, \\ 6, & \text{for } i \equiv 0 \pmod 2, \end{cases}$$
$$w(a_{ij}) = 2, \quad w(b_{ij}) = 2, \quad w(c_{ij}) = 2.$$

The upper bound is true: $\chi_{lis}(P_m \bigodot S_{3,n}) \leq 7$. So $\chi_{lis}(P_m \bigodot S_{3,n}) = 7$ for $m \equiv 1, 3 \pmod 4$; $m \geq 5$ and $n = 2$.

Case 7: $m \equiv 0 \pmod 2$ $m \geq 4$ and $n \geq 3$.

First step to prove this theorem is to find the lower bound of $V(P_m \bigodot S_{3,n})$. Based on Lemma 1, we have $\chi_{lis}(P_m \bigodot S_{3,n}) \geq \chi(P_m \bigodot S_{3,n}) = 3$.

Assume $\chi_{lis}(P_m \bigodot S_{3,n}) = 6$, it is true if

$$l(a_i) = l(b_i) = l(c_i) = l(y_i) = l(a_{ij}) = l(b_{ij}) = 1, \quad l(c_{ij}) = 1, \quad 1 \leq i \leq m,$$
$$1 \leq j \leq n - 1, \quad l(c_{ij}) = 2, \quad 1 \leq i \leq m, \quad j = n,$$

then $w(x_{i+1}) = w(x_{i+2})$, then there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_i) = 1, \quad l(a_{ij}) = l(b_{ij}) = l(y_i) = 1, \quad l(c_{ij}) = 1, \quad i \equiv 1, 3 \pmod 4, \quad 1 \leq j \leq n,$$
$$i \equiv 0 \pmod 2, \quad i \neq m, \quad 1 \leq j \leq n - 1, \quad i = m, \quad 1 \leq j \leq n, \quad l(c_{ij}) = 2, \quad i \equiv 0 \pmod 2,$$
$$i \neq m, \quad i \neq m, \quad j = n, \quad w(x_{i+1}) \neq w(x_{i+2}),$$

we have the lower bound of $\chi_{lis}(P_m \bigodot S_{3,n}) \geq 7$. After that, we will find the upper bound $\chi_{lis}(P_m \bigodot S_{3,n})$.

Furthermore, we define $l : V(P_m \bigodot S_{3,n}) \to \{1,2\}$ with vertex irregular 2-labelling as follows:

$$l(x_i) = 1, \quad l(y_i) = 1, \quad l(a_i) = 1, \quad l(b_i) = 1, \quad l(c_i) = 1, \quad l(a_{ij}) = 1, \quad l(b_{ij}) = 1,$$

$$l(c_{ij}) = \begin{cases} 1, & \text{for } i \equiv 1,3 \pmod 4 \text{ and } 1 \leq j \leq n \text{ or} \\ & \quad \text{for } i \equiv 0 \pmod 2,\ i \neq m, \text{ and } 1 \leq j \leq n-1 \text{ or} \\ & \quad \text{for } i = m, \text{ and } 1 \leq j \leq n, \\ 2, & \text{for } i \equiv 0 \pmod 2,\ i \neq m \text{ and } j = n. \end{cases}$$

Hence, $\mathrm{opt}(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 3n+5, & \text{for } i = 1, m, \\ 3n+6, & \text{for } i \equiv 1,3 \pmod 4,\ i \neq 1, \\ 3n+7, & \text{for } i \equiv 0 \pmod 2,\ i \neq m, \end{cases}$$

$$w(y_i) = 4,$$

$$w(a_i) = n+2, \quad \text{for } 1 \leq i \leq m,$$

$$w(b_i) = n+2, \quad \text{for } 1 \leq i \leq m,$$

$$w(c_i) = \begin{cases} n+2, & \text{for } i = m, \text{ or for } i \equiv 1,3 \pmod 4, \\ n+3, & \text{for } i \equiv 0 \pmod 2,\ i \neq m, \end{cases}$$

$$w(a_{ij}) = 2, \quad w(b_{ij}) = 2, \quad w(c_{ij}) = 2,$$

The upper bound $\chi_{lis}(P_m \bigodot S_{3,n}) \leq 7$. So $\chi_{lis}(P_m \bigodot S_{3,n}) = 7$ for $m \equiv 0 \pmod 2$; $m \geq 4$ and $n \geq 3$.

Case 8: $m \equiv 1,3 \pmod 4$, $m \geq 5$ and $n \geq 3$.

First step to prove the theorem in this case is to find the lower bound of $V(P_m \bigodot S_{3,n})$. Based on Lemma 1, we have $\chi_{lis}(P_m \bigodot S_{3,n}) \geq \chi(P_m \bigodot S_{3,n}) = 3$.

Assume $\chi_{lis}(P_m \bigodot S_{3,n}) = 6$, if

$$l(x_i) = l(y_i) = l(a_{ij}) = l(b_{ij}) = l(c_{ij}) = 1, \quad l(a_i) = l(b_i) = 1, \quad l(c_i) = 2,$$

then $w(x_{i+1}) = w(x_{i+2})$. Then there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_i) = l(a_i) = l(b_i) = l(c_i) = 1, \quad l(a_{ij}) = l(b_{ij}) = l(y_i) = 1, \quad l(c_{ij}) = 1, \quad i \equiv 1,3 \pmod 4,$$
$$1 \leq j \leq n, \quad i \equiv 0 \pmod 2, \quad 1 \leq j \leq n-1, \quad l(c_{ij}) = 2, \quad i \equiv 0 \pmod 2, \quad j = n,$$

then $w(x_{i+1}) \neq w(x_{i+2})$. Based on that we have the lower bound $\chi_{lis}(P_m \bigodot S_{3,n}) \geq 7$.

After that, we will find the upper bound of $\chi_{lis}(P_m \bigodot S_{3,n})$.

Furthermore, we define $l : V(P_m \bigodot S_{3,n}) \to \{1,2\}$ with vertex irregular 2-labelling as follows:

$$l(x_i) = 1, \quad l(y_i) = 1, \quad l(a_i) = 1, \quad l(b_i) = 1, \quad l(c_i) = 1, \quad l(a_{ij}) = 1, \quad l(b_{ij}) = 1,$$

$$l(c_{ij}) = \begin{cases} 1, & \text{for } i \equiv 1,3 \pmod 4, \text{ and } 1 \leq j \leq n \text{ or} \\ & \quad \text{for } i \equiv 0 \pmod 2, \text{ and } 1 \leq j \leq n-1, \\ 2, & \text{for } i \equiv 0 \pmod 2, \text{ and } j = n. \end{cases}$$

Hence, $\mathrm{opt}(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 3n+5, & \text{for } i = 1, m, \\ 3n+6, & \text{for } i \equiv 1,3 \pmod 4,\ i \neq 1, m, \\ 3n+7, & \text{for } i \equiv 0 \pmod 2, \end{cases}$$

$$w(y_i) = 4,$$
$$w(a_i) = n + 2, \text{ for } 1 \le i \le m,$$
$$w(b_i) = n + 2, \text{ for } 1 \le i \le m,$$
$$w(c_i) = \begin{cases} n+2, & \text{for } i \equiv 1,3 \pmod 4, \\ n+3, & \text{for } i \equiv 0 \pmod 2, \end{cases}$$
$$w(a_{ij}) = 2, \quad w(b_{ij}) = 2, \quad w(c_{ij}) = 2.$$

The upper bound is true: $\chi_{lis}(P_m \bigodot S_{3,n}) \le 7$. So $\chi_{lis}(P_m \bigodot S_{3,n}) = 7$ for $m \equiv 1,3 \pmod 4$, $m \ge 5$ and $n \ge 3$. □

3. Conclusion

In this paper, we have studied the coloring of the vertices of the local irregular corona product by the graph of the family tree. We determined the exact value of the local irregular chromatic number of the corona product from the graph of the family tree, namely $\chi_{lis}(P_m \bigodot Cp_n)$, $\chi_{lis}(P_m \bigodot S_{2,n})$ and $\chi_{lis}(P_m \bigodot S_{3,n})$.

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COMBINED ALGORITHMS FOR CONSTRUCTING A SOLUTION TO THE TIME-OPTIMAL PROBLEM IN THREE-DIMENSIONAL SPACE BASED ON THE SELECTION OF EXTREME POINTS OF THE SCATTERING SURFACE[1]

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Abstract: A class of time-optimal control problems in three-dimensional space with a spherical velocity vector is considered. A smooth regular curve Γ is chosen as the target set. We distinguish pseudo-vertices that are characteristic points on Γ and responsible for the appearance of a singularity in the function of the optimal result. We reveal analytical relationships between pseudo-vertices and extreme points of a singular set belonging to the family of bisectors. The found analytical representation for the extreme points of the bisector is taken as the basis for numerical algorithms for constructing a singular set. The effectiveness of the developed approach for solving non-smooth dynamic problems is illustrated by an example of numerical-analytical construction of resolving structures for the time-optimal control problem.

Keywords: Time-optimal problem, Dispersing surface, Bisector, Pseudo-vertex, Extreme point, Curvature, Singular set, Frenet–Serret frame (TNB frame).

1. Introduction

This study continues the series of works by the authors on the development of methods and algorithms for constructing solutions to time-optimal control problems with a constant velocity vector and various geometry of target sets [6, 19]. Previously accumulated experience in solving plane problems [13] was transferred to three-dimensional space [3, 14], expanded, and supplemented with new methods and constructions. In this paper, the authors consider a time-optimal control problem in which a sufficiently smooth regular spatial curve is chosen as the target set. The optimal result function is not differentiable over the entire domain of consideration [2]. A combined approach is applied in its construction, which combines analytical methods for identifying the features of the solution of the problem and numerical algorithms for constructing the solution as a whole. To find

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singularities of the optimal result function, elements of differential geometry are used, in particular, the moving Frenet frame and the main invariants of space curves [9, 10]. Also, a significant role in the constructions is played by the angular characteristic of the point's nonconvexity with respect to the target set and its measure of nonconvexity [16]. The measure of the non-convexity of a set determines the nature of the breaks in the wave fronts generated by this set [1]. The wave fronts in the problem under consideration coincide with the level surfaces of the optimal result function [4]. The key element in constructing a solution is the selection of a singular set — the bisector of the target set [7]. In the problem under consideration, the bisector generally consists of the union of two-dimensional, one-dimensional, and zero-dimensional manifolds [11, 12]. The simulation of the non-smooth solution of the problem was carried out with the help of modernized computational procedures, previously created for solving flat problems of time-optimal control [18]. The developed procedures can be used in constructing generalized solutions of first-order partial differential equations [15], as well as in theoretical mechanics, geometric optics, seismology, and economics [5].

2. Problem statement

The paper is devoted to the study of a time-optimal problem for a 3D system consisting of a single point with the speed limited as follows:

$$\dot{\mathbf{x}} \in U(\mathbf{0},1) \subset \mathbb{R}^3, \tag{2.1}$$

where $U(\mathbf{c},r)$ is a ball in $\mathbb{R}^3$ centered at a point $\mathbf{c}$ of radius $r>0$,

$$\mathbf{x}=\mathbf{x}(\tau) \triangleq (x(\tau),y(\tau),z(\tau)), \quad \dot{\mathbf{x}}=\frac{d\mathbf{x}}{d\tau},$$

and τ is a scalar interpreted as time. For an arbitrary point $\mathbf{x}$, the optimal trajectory is a line segment connecting it to the nearest point in the Euclidean metric of the target closed set $A \subset \mathbb{R}^3$. The optimal result function [17] is

$$u(\mathbf{x})=\rho(\mathbf{x},A) \triangleq \min_{\mathbf{a}\in A} \|\mathbf{x}-\mathbf{a}\|.$$

The time-optional problem under consideration is tightly connected with the Hamilton–Jacobi differential equations

$$\min_{(v_1,v_2,v_3)\in U(\mathbf{0},1)} \left(v_1\frac{\partial u}{\partial x}+v_2\frac{\partial u}{\partial y}+v_3\frac{\partial u}{\partial z}\right)+1=0 \tag{2.2}$$

and Eikonal equations

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=1 \tag{2.3}$$

with a boundary condition

$$u|_{\partial A}=0, \tag{2.4}$$

where ∂A is the boundary of A.

The restriction of the optimal result function $u=u(x,y,z)$ to the closure $\mathrm{cl}(\mathbb{R}^3\setminus A)$ of the set $\mathbb{R}^3\setminus A$ coincides with the generalized (minimax) solution of the Dirichlet problem [15] for equation (2.2) with a boundary condition (2.4). A more detailed proof for an arbitrary finite-dimensional Euclidean space is given in [6]. The fundamental (generalized) solution $u_k(\mathbf{x})$ of the Dirichlet problem for equation (2.3) with boundary condition (2.4) (introduced by S.N. Kruzhkov [5]) is equal to the function $u(x,y,z)$ on $\mathbb{R}^3\setminus A$ in absolute value but has the opposite sign:

$$u_k(x,y)=-\rho\big((x,y,z),A\big).$$

It should be noted that equation (2.3) is used to describe light propagation in a homogeneous medium, provided that the speed is normalized and reduced to 1. The wave front at the time point $\tau > 0$ coincides with the level surface

$$\Phi(\tau) \triangleq \{\mathbf{x} \in \mathbb{R}^3 \colon u(\mathbf{x}) = \tau\}$$

of the optimal result function $u(\mathbf{x})$. In the whole space $\mathbb{R}^3$, the function $u(\mathbf{x})$ satisfies the Lipschitz condition with the constant $L = 1$.

3. Basic notation and definitions

Let $A \subset \mathbb{R}^3$ be a closed set in $\mathbb{R}^3$. We denote by $\Omega_A(\mathbf{x})$ the union of all points closest to $\mathbf{x}$ in the set A.

Definition 1 [19]. *A set*

$$L(A) \triangleq \left\{\mathbf{x} \in \mathbb{R}^3 \colon \operatorname{card} \Omega_A(\mathbf{x}) > 1\right\}$$

is called a bisector of a closed non-empty set A.

Here, $\operatorname{card} \Omega_A(\mathbf{x})$ is the cardinality of the set $\Omega_A(\mathbf{x})$.

The bisector is a specific case of a symmetric set on which the wave front loses its smoothness [1]. In English academic sources, similar sets are termed as “conflict set” [13], “symmetry set”, and “medial axe” [3]. Their geometric properties in 3D space were studied, for example, in [14]. Some topological properties of non-smooth wave front sets in Euclidean spaces of small dimensions (2 to 6) were investigated by V. D. Sedykh in [11, 12].

According to control theory, $L(A)$ is classified as a dispersing surface [4, ex. 6.10.1] in the time-optimal problem for dynamic systems (2.1). More than one optimal trajectory directed differently to the surface, e.g., line segments $[\mathbf{x}, \mathbf{y}_i]$, $i = \overline{1, k}$, where $\mathbf{y}_i \in \Omega_A(\mathbf{x})$, $k = \operatorname{card} \Omega_A(\mathbf{x})$, originates from each of its points. This determines that the optimal result function $u(\mathbf{x})$ is non-differentiable on the set $L(A)$. It should be mentioned that, for $u(\mathbf{x})$ as a function of the Euclidean distance, the superdifferential $D^+u(\mathbf{x})$ is defined at points $\mathbf{x} \in L(A)$, for more details see [2, Ch. II, Sect. S 8]. The value $D^+u(\mathbf{x})$ is used in [6] to prove that the function restriction to the set $\mathbb{R}^3 \setminus A$ is a generalized solution of the Hamilton–Jacobi equation (2.2).

Definition 2 [7]. *Non-coinciding points $\mathbf{y}_i^- \in A$ and $\mathbf{y}_i^+ \in A$ are called quasi-symmetric if*

$$\exists \mathbf{x} \in L(A) \colon \left\{\mathbf{y}_i^-, \mathbf{y}_i^+\right\} \subseteq \Omega_A(\mathbf{x}).$$

In this case, the point $\mathbf{x}$ is called generated by the pair of points $\mathbf{y}_i^-$ and $\mathbf{y}_i^+$.

Definition 3 [19]. *The point $\mathbf{y}_0$ is called a pseudo-vertex of the set A if there exists a sequence of pairs of quasi-symmetric points $\left\{\mathbf{y}_i^-, \mathbf{y}_i^+\right\}_{i=1}^{\infty} \subset A$ and a sequence of points $\mathbf{x}_i \subset L(A)$, for which the following conditions hold:*

$$\forall i \in \mathbb{N} \ \left\{\mathbf{y}_i^-,\ \mathbf{y}_i^+\right\} \subseteq \Omega_A(\mathbf{x}_i)$$

and

$$\lim_{i \to \infty} \left\{\mathbf{y}_i^-, \mathbf{y}_i^+\right\} = \left\{\mathbf{y}_0, \mathbf{y}_0\right\}.$$

If there is an additional limit

$$\lim_{i \to \infty} \mathbf{x}_i = \mathbf{x}_0,$$

then, $\mathbf{x}_0$ is an extreme point of the bisector corresponding to the pseudo-vertex $\mathbf{y}_0$.

Remark 1. The union of the bisector’s extreme points forms the edge of the surface coinciding with the closure of $L(A)$. In general, the dispersing surface is not a closed set, and the extreme points do not belong to it, but they determine its geometry.

4. Singular set characteristics

Hereinafter, we consider the case of a set A whose boundary Γ is a curve defined by the parametric equation:

$$\Gamma = \left\{ \mathbf{r}(t) \in \mathbb{R}^3 \colon t \in T \right\}, \tag{4.1}$$

where $T \subseteq \mathbb{R}$ is a closed connected interval.

Condition 1. We assume that the vector-valued function $\mathbf{r}(t)$ is three times differentiable on T, and the following biregularity condition is satisfied:

$$\forall t \in T \quad \left[\mathbf{r}'(t), \mathbf{r}''(t) \right] \neq \mathbf{0}, \tag{4.2}$$

where $[\cdot,\cdot]$ is the vector product, whereas the function $\mathbf{r}(t)$ satisfies a Lipschitz condition.

Condition (4.2) ensures that, for any $t \in T$, a TNB frame [9] consisting of three unit vectors is defined:

$$\mathbf{e}_1(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}, \tag{4.3}$$

$$\mathbf{e}_2(t) = \frac{\left[\left[\mathbf{r}'(t), \mathbf{r}''(t) \right], \mathbf{r}'(t) \right]}{\left\| \left[\left[\mathbf{r}'(t), \mathbf{r}''(t) \right], \mathbf{r}'(t) \right] \right\|}, \tag{4.4}$$

$$\mathbf{e}_3(t) = \frac{\left[\mathbf{r}'(t), \mathbf{r}''(t) \right]}{\left\| \left[\mathbf{r}'(t), \mathbf{r}''(t) \right] \right\|}. \tag{4.5}$$

According to the classification used in differential geometry, $\mathbf{e}_1(t)$ is a tangent unit vector, $\mathbf{e}_2(t)$ is a normal unit vector, and $\mathbf{e}_3(t)$ is a binormal unit vector. The curve Γ is characterized by two parameters at a point. They are its curvature

$$k(t) = \frac{\left\| \left[\mathbf{r}'(t), \mathbf{r}''(t) \right] \right\|}{\|\mathbf{r}'(t)\|^3} \tag{4.6}$$

and torsion

$$\varkappa(t) = \frac{\left(\mathbf{r}'(t), \mathbf{r}''(t), \mathbf{r}'''(t) \right)}{\left\| \left[\mathbf{r}'(t), \mathbf{r}''(t) \right] \right\|^2}. \tag{4.7}$$

Here $(\cdot,\cdot,\cdot)$ is a triple scalar product.

Definition 4. *The line*

$$V(t) = \left\{ \mathbf{r}(t) + k^{-1}(t)\mathbf{e}_2(t) + \xi \mathbf{e}_3(t) \in \mathbb{R}^3 \colon \xi \in \mathbb{R} \right\} \tag{4.8}$$

is called conjugate to the curve (4.1) *at the point* $\mathbf{r}(t)$.

The biregularity condition (4.2) ensures that, for any t, the curvature is defined and has a non-zero solution. Hence, the conjugate line (4.8) is also defined. It should be noted that a normal at the point $\mathbf{r}(t)$ is to be constructed as

$$P(t) = \left\{ \mathbf{z} \in \mathbb{R}^3 \colon \left\langle \mathbf{z} - \mathbf{r}(t), \mathbf{r}'(t) \right\rangle = 0 \right\}, \tag{4.9}$$

whereas the wave front $\Phi(\tau)$ generated by the point $\mathbf{r}(t)$ falls partly inside the circle:

$$\Theta(t,\tau) = \partial U(\mathbf{r}(t), \tau) \cap P(t), \tag{4.10}$$

where $\tau > 0$.

Lemma 1. *If a sequence of pairs of quasi-symmetric parameters* $\{t_i^-, t_i^+\}_{i=1}^{\infty} \subset T$, *a sequence of points* $\{\mathbf{x}_i\}_{i=1}^{\infty} \subset L(\Gamma)$, *a parameter* $t_0 \in T$, *and a point* $\mathbf{x}_0 \in L(\Gamma)$ *satisfy the conditions*

$$\forall i \in \mathbb{N}\ \{\mathbf{r}(t_i^-), \mathbf{r}(t_i^+)\} \subseteq \Omega_\Gamma(\mathbf{x}_i), \tag{4.11}$$

$$\lim_{i\to\infty} \{t_i^-, t_i^+\} = \{t_0, t_0\}, \tag{4.12}$$

$$\lim_{i\to\infty} \mathbf{x}_i = \mathbf{x}_0, \tag{4.13}$$

then the following relation is true:

$$\lim_{i\to\infty} \big(\langle \mathbf{x}_i - \mathbf{r}(t_i^-), \mathbf{e}_2(t_i^-)\rangle - k^{-1}(t_i^-)\big) = 0. \tag{4.14}$$

P r o o f. Consider a Frenet–Serret frame (trihedron). We should note that if $\overline{\mathbf{r}}(s) = (\overline{x}(s), \overline{y}(s), \overline{z}(s))$ is a vector-valued function that is three times differentiable on the interval $S \subset \mathbb{R}$ and defined by a natural parameter (arc length) $s \geqslant 0$, then the following Taylor expansion is true for any $s \in S$ and sufficiently small increments of Δs:

$$\overline{\mathbf{r}}(s+\Delta s) = \overline{\mathbf{r}}(s) + \overline{\mathbf{r}}'(s)\Delta s + \frac{1}{2}\overline{\mathbf{r}}''(s)\Delta s^2 + \frac{1}{6}\overline{\mathbf{r}}'''(s)\Delta s^3 + \mathbf{o}(\Delta s^3). \tag{4.15}$$

Here, $\mathbf{o}(\delta)$ is a vector-valued function with $\|\mathbf{o}(\delta)\| = o(\delta)$; $o(\delta)$ being an infinitesimal with a higher order of smallness with respect to $\delta \in \mathbb{R}$.

Consider a classical orthonormal Frenet–Serret frame $\{\overline{\mathbf{e}}_1(s), \overline{\mathbf{e}}_2(s), \overline{\mathbf{e}}_3(s)\}$ and specify the coordinates of the vector $\overline{\mathbf{r}}(s+\Delta s) = \big(\overline{x}(s+\Delta s),\ \overline{y}(s+\Delta s), \overline{z}(s+\Delta s)\big)$ based on (4.15) (for more details, see [10, Ch. 5]):

$$\begin{aligned}
\overline{x}(s+\Delta s) &= \overline{x}(s) + \Delta s - \frac{1}{6}\overline{k}^2(s)\Delta s^3 + o(\Delta s^3),\\
\overline{y}(s+\Delta s) &= \overline{y}(s) + \frac{1}{2}\overline{k}(s)\Delta s^2 + \frac{1}{6}k'(s)\Delta s^3 + o(\Delta s^3),\\
\overline{z}(s+\Delta s) &= \overline{z}(s) + \frac{1}{6}\overline{k}(s)\overline{\varkappa}(s)\Delta s^3 + o(\Delta s^3),
\end{aligned}$$

where $\overline{k}(s)$ and $\overline{\varkappa}(s)$ are the curvature and torsion of the curve at the point $\overline{\mathbf{r}}(s)$.

In what follows, to achieve the result stated, it is sufficient to use only the lower terms of the above expansions:

$$\begin{aligned}
\overline{x}(s+\Delta s) &= \overline{x}(s) + \Delta s + o(\Delta s),\\
\overline{y}(s+\Delta s) &= \overline{y}(s) + \frac{1}{2}\overline{k}(s)\Delta s^2 + o(\Delta s^2),\\
\overline{z}(s+\Delta s) &= \overline{z}(s) + \frac{1}{6}\overline{k}(s)\overline{\varkappa}(s)\Delta s^3 + o(\Delta s^3).
\end{aligned}$$

Let us turn to the original curve described by means of the parameter $t \in \mathbb{R}$. We have

$$\mathbf{r}(t) = \overline{\mathbf{r}}\big(s(t)\big),$$

where

$$s'(t) = \|\mathbf{r}'(t)\|.$$

The coordinates of the vector

$$\mathbf{r}(t+\Delta t) = \big(x(t+\Delta t), y(t+\Delta t), z(t+\Delta t)\big),$$

where

$$\mathbf{r}(t+\Delta t) \triangleq \overline{\mathbf{r}}\big(s(t+\Delta t)\big) = \big(\overline{x}\big(s(t+\Delta t)\big), \overline{y}\big(s(t+\Delta t)\big), \overline{z}\big(s(t+\Delta t)\big)\big)\,,$$

are calculated in the orthonormal basis

$$\big\{\overline{\mathbf{e}}_1\big(s(t)\big), \overline{\mathbf{e}}_2\big(s(t)\big), \overline{\mathbf{e}}_3\big(s(t)\big)\big\} = \{\mathbf{e}_1(t), \mathbf{e}_2(t), \mathbf{e}_3(t)\}$$

as follows:

$$x(t+\Delta t) = x(t) + \big\|\mathbf{r}'(t)\big\|\,\Delta t + o(\Delta t), \tag{4.16}$$

$$y(t+\Delta t) = y(t) + \frac{\|\mathbf{r}'(t)\|^2\,\Delta t^2 k(t)}{2} + o(\Delta t^2), \tag{4.17}$$

$$z(t+\Delta t) = z(t) + \frac{k(t)\varkappa(t)\,\|\mathbf{r}'(t)\|^3\,\Delta t^3}{6} + o(\Delta t^3). \tag{4.18}$$

Here, $k(t) = \overline{k}\big(s(t)\big)$ and $\varkappa(t) = \overline{\varkappa}\big(s(t)\big)$. When deriving formulas (4.16)–(4.18), it is taken into account that $\Delta s = s'(t)\Delta t + o(\Delta t)$ with $\Delta t \to 0$ as $\Delta s \to 0$.

Let us relate the moving coordinate system to the point $t = t_i^-$. Provided that $\Delta t = t_i^+ - t_i^-$, we obtain the following equalities by (4.16)–(4.18):

$$x(t_i^+) = x(\widehat{t}_i) + \|\mathbf{r}'(\widehat{t}_i)\|\Delta t + o(\Delta t), \tag{4.19}$$

$$y(t_i^+) = y(t_i^-) + \frac{1}{2}k(t_i^-)\|\mathbf{r}'(t_i^-)\|^2\Delta t^2 + o(\Delta t^2), \tag{4.20}$$

$$z(t_i^+) = z(t_i^-) + \frac{1}{6}k(t_i^-)\varkappa(t_i^-)\|\mathbf{r}'(t_i^-)\|^3\Delta t^3 + o(\Delta t^3). \tag{4.21}$$

Let us calculate the derivatives of the coordinates at the point $t = t_i^+$ up to infinitesimals:

$$x'(t_i^+) = \|\mathbf{r}'(t_i^-)\| + \varepsilon(\Delta t), \tag{4.22}$$

$$y'(t_i^+) = k(t_i^-)\|\mathbf{r}'(t_i^-)\|^2\Delta t + o(\Delta t), \tag{4.23}$$

$$z'(t_i^+) = o(\Delta t). \tag{4.24}$$

Here, $\varepsilon(t)$ is an infinitesimal.

Denote the coordinates of the point $\mathbf{x}_i$ in the proposed coordinate system by (x_i^*, y_i^*, z_i^*). By the conditions, the sequences $\{t_i^-, t_i^+\}_{i=1}^{\infty}$ and $\{\mathbf{x}_i\}_{i=1}^{\infty}$ are bounded, and the function $\mathbf{r}(t)$ is Lipschitz, hence, the sequence $\{(x_i^*, y_i^*, z_i^*)\}_{i=1}^{\infty}$ is bounded. Therefore,

$$\exists \mu > 0\colon\ \forall i \in \mathbb{N} \quad |x_i^*| + |y_i^*| + |z_i^*| \leqslant \mu. \tag{4.25}$$

Since, by construction, $\mathbf{x}_i \in P(t_i^-)$, we have

$$x_i^* = 0. \tag{4.26}$$

On the other hand, if $\mathbf{x}_i \in P(t_i^+)$, then

$$\langle \mathbf{x}_i - \mathbf{r}(t_i^+), \mathbf{r}'(t_i^+)\rangle = 0. \tag{4.27}$$

Based on the equality $t_i^+ = t_i^- + \Delta t$ and representations (4.19)–(4.21) as well as (4.22)–(4.24), we can write equality (4.27) in the form

$$\begin{gathered}\big(x_i^* - \big(\big\|\mathbf{r}'(t_i^-)\big\|\,\Delta t + o(\Delta t)\big)\,\big)\,\big(\big\|\mathbf{r}'(t_i^-)\big\| + \varepsilon(\Delta t)\big) + \\ + (y_i^* - o(\Delta t))\,\Big(\big\|\mathbf{r}'(t_i^-)\big\|^2\,k(t_i^-)\Delta t + o(\Delta t)\Big) + (z_i^* - o(\Delta t))o(\Delta t) = 0.\end{gathered} \tag{4.28}$$

From (4.25), it follows that $|z_i^*| \leqslant \mu$; hence $z_i^* o(\Delta t) = o(\Delta t)$. Therefore, grouping all infinitesimals of a higher order than Δt and substituting the value x_i^* from (4.26) into equality (4.28), we can transform (4.28) to the following form:

$$-\|\mathbf{r}'(t_i^-)\|^2 \Delta t + y_i^* \left\|\mathbf{r}'(t_i^-)\right\|^2 k(t_i^-)\Delta t + o(\Delta t) = 0. \tag{4.29}$$

Let us express $y_i^* k(t_i^-)$ from (4.29). Thus, we get the limit relation

$$\lim_{i\to\infty} y_i^* k(t_i^-) = \lim_{i\to\infty} \frac{\left\|\mathbf{r}'(t_i^-)\right\|^2 \Delta t - o(\Delta t)}{\left\|\mathbf{r}'(t_i^-)\right\|^2 \Delta t} = \lim_{\Delta t\to 0} \frac{\left\|\mathbf{r}'(t_i^-)\right\|^2 \Delta t - o(\Delta t)}{\left\|\mathbf{r}'(t_i^-)\right\|^2 \Delta t} = 1. \tag{4.30}$$

Since, in the adopted coordinate system, the positive direction of the ordinate axis coincides with the unit vector $\mathbf{e}_2(t_i^-)$, we have

$$y_i^* = \left\langle \mathbf{e}_2(t_i^-), \mathbf{x}_i - \mathbf{r}(t_i^-)\right\rangle. \tag{4.31}$$

From (4.30) and (4.31), it follows that

$$\lim_{i\to\infty} \left\langle \mathbf{e}_2(t_i^-), \mathbf{x}_i - \mathbf{r}(t_i^-)\right\rangle k(t_i^-) = 1. \tag{4.32}$$

We move the number 1 to the left side of (4.32) under the limit sign and divide the expression obtained under the limit sign by $k(t_i^-) \neq 0$. As a result, we get (4.14). □

Lemma 1 enables formulating a statement about the coordinates of the extreme points generated by the pseudo-vertex of the spatial curve.

Theorem 1. *Let there be a pseudo-vertex* $\mathbf{r}(t_0)$ *on the curve* (4.1). *If the extreme point of the bisector* $\mathbf{x}_0$ *corresponds to the pseudo-vertex* $\mathbf{r}(t_0)$, *then the following inclusion holds*:

$$\mathbf{x}_0 \in V(t_0). \tag{4.33}$$

P r o o f. If $\mathbf{x}_0$ is the extreme point of the bisector corresponding to the pseudo-vertex $\mathbf{r}(t_0)$ of the set Γ, then Definition 3 implies the existence of a sequence of pairs of non-coinciding numbers $\{t_i^-, t_i^+\}_{i=1}^{\infty} \subset T$ and a sequence of points $\{\mathbf{x}_i\}_{i=1}^{\infty} \subset L(\Gamma)$ satisfying the conditions (4.11)–(4.13). Since it follows from (4.11) that the point $\mathbf{x}_i$ for any i lies in the normal plane (4.9) (constructed at the point $\mathbf{r}(t_i^-)$), we have

$$\forall i \in \mathbb{N} \ \left\langle \mathbf{x}_i - \mathbf{r}(t_i^-), \mathbf{r}'(t_i^-)\right\rangle = 0. \tag{4.34}$$

The vector product is a continuous function of two vector variables, and $\mathbf{r}(t)$ is a three times differentiable function. Therefore, it is possible to calculate the value of the limit as follows:

$$\lim_{i\to\infty} \left\langle \mathbf{x}_i - \mathbf{r}(t_i^-), \mathbf{r}'(t_i^-)\right\rangle = \left\langle \lim_{i\to\infty} \mathbf{x}_i - \lim_{i\to\infty} \mathbf{r}(t_i^-), \lim_{i\to\infty} \mathbf{r}'(t_i^-)\right\rangle = \left\langle \mathbf{x}_0 - \mathbf{r}(t_0), \mathbf{r}'(t_0)\right\rangle. \tag{4.35}$$

According to (4.34) and (4.35), the following equality is true:

$$\left\langle \mathbf{x}_0 - \mathbf{r}(t_0), \mathbf{r}'(t_0)\right\rangle = 0. \tag{4.36}$$

The biregularity condition ensures that the curvature (4.6) at any point of the curve is continuous and strictly positive; hence, the inverse function $k^{-1}(t)$ is continuous in some neighborhood of t_0. Consider the function (4.4). Its numerator represents a composition of vector products of continuous vector-valued functions, and the denominator is equal to the norm of the numerator. In this case, the numerator is different from 0 according to the condition (4.2). Therefore,

$$\begin{gathered}\lim_{i\to\infty}\left(\left\langle \mathbf{x}_i - \mathbf{r}(t_i^-), \mathbf{e}_2(t_i^-)\right\rangle - k^{-1}(t_i^-)\right) = \\ = \left\langle \lim_{i\to\infty}\mathbf{x}_i - \lim_{i\to\infty}\mathbf{r}(t_i^-), \lim_{i\to\infty}\mathbf{e}_2(t_i^-)\right\rangle - \lim_{i\to\infty}k^{-1}(t_i^-) = \langle \mathbf{x}_0 - \mathbf{r}(t_0), \mathbf{e}_2(t_0)\rangle - k^{-1}(t_0).\end{gathered} \tag{4.37}$$

From (4.14) and (4.37), it follows that

$$\langle \mathbf{x}_0 - \mathbf{r}(t_0), \mathbf{e}_2(t_0)\rangle - k^{-1}(t_0) = 0. \tag{4.38}$$

It should be noted that (4.8) for $t = t_0$ can be represented as a set of points, for which the following conditions hold:

$$\mathbf{z} \in P(t_0) \tag{4.39}$$

and

$$\langle \mathbf{z} - \mathbf{r}(t_0), \mathbf{e}_2(t_0)\rangle = k^{-1}(t_0). \tag{4.40}$$

Equality (4.36) is equivalent to the condition (4.39), and equality (4.38) is equivalent to the condition (4.40). Hence, (4.33) holds. □

Remark 2. Equations (4.8) and (4.33) for the extreme points of a singular set are generalizations to three-dimensional equations for the extreme points of a singular set for solving the corresponding planar time-optimal control problem (see (4.1) and (4.2) from [18]).

Remark 3. Strictly speaking, a Frenet–Serret frame is not unique. Depending on the parameters, the vectors (4.3) and (4.5) can be directed differently. However, the vector (4.4) is always coincides with the direction, in which the curve (4.1) is locally convex in the neighborhood of the point $\mathbf{r}(t)$. Therefore, the equation of the conjugate line (4.8) is an invariant and is determined solely by certain characteristics of the curve Γ.

5. Example of solving the time-optimal problem (2.1)

To construct singular sets in 3D space, the authors have upgraded a software package [8], previously used to solve flat tome-optimal problems. It is based on algorithms for calculating the parameters t^- and t^+, which define pairs of quasi-symmetric points $\mathbf{r}(t^-)$ and $\mathbf{r}(t^+)$ and the points $\mathbf{x} \in L(\Gamma)$ generated by them. A key element is searching for pseudo-vertices of the target set. Finding a pseudo-vertex makes it possible, using the results of Section 4, to construct sets of extreme points of the bisector. These sets help to numerically construct the singular set itself. The level surface $\Phi(\tau)$ of the optimal result function $u(\mathbf{x})$ corresponding to the time point $\tau > 0$ is constructed as a union of circles (4.10), from which the parts cut off by the bisector $L(\Gamma)$ are removed. For each circle $\Theta(t,\tau)$, $t \in T$, it is required to find out, which arcs on it get into $\Phi(\tau)$.

Example 1. Consider an example of a time-optimal problem with a target set represented by the curve (4.1), where the function

$$\mathbf{r}(t) = \left(\cos t, \sin t, \frac{\cos 3t}{3}\right) \tag{5.1}$$

is defined on $T = [0, 2\pi]$. The function (5.1) satisfies Condition 1 and the Lipschitz condition with constant $L = 3$. An analysis of its first-order derivatives

$$\mathbf{r}'(t) = (-\sin t, \cos t, -\sin 3t) \tag{5.2}$$

and its second-order derivatives

$$\mathbf{r}''(t) = (-\cos t, -\sin t, -3\cos 3t) \tag{5.3}$$

allows us to prove that the biregularity condition (4.2) holds. We should note that the torsion (4.7) is not identically zero; hence, the curve Γ is not flat. Although, $\varkappa(t) = 0$ is possible at some points $t \in T$.

Modeling the wave front propagation makes it possible to define that the set (4.1) has six pseudo-vertices corresponding to the values of the parameter

$$t_1 = 0, \quad t_2 = \pi/3, \quad t_3 = 2\pi/3, \quad t_4 = \pi, \quad t_5 = 4\pi/3, \quad t_6 = 5\pi/3.$$

According to Theorem 1, the extreme points of the bisector lie on the lines conjugate to Γ at the pseudo-vertices. Fig. 1 shows the curve Γ as a purple line, its pseudo-vertices $\mathbf{r}(t_i)$, $i = \overline{1,6}$, as bubbles, and the dispersing surface $L(\Gamma)$ as the translucent blue surface. The sets of extreme points W_i corresponding to the pseudo-vertices $\mathbf{r}(t_i)$, $i = \overline{1,6}$, are found by means of the derivatives of the vector-valued function of the first-order (5.2) and second-order (5.3):

$$W_1 = \Big\{ \Big(\xi, 0, \frac{1-\xi}{3} \Big) \in \mathbb{R}^3 \colon \xi \in [0, \infty) \Big\},$$
$$W_2 = \Big\{ \Big(\frac{\sqrt{3}}{2}\xi, \frac{\xi}{2}, \frac{\xi-1}{3} \Big) \in \mathbb{R}^3 \colon \xi \in [0, \infty) \Big\},$$
$$W_3 = \Big\{ \Big(-\frac{\xi}{2}, \frac{\sqrt{3}}{2}\xi, \frac{1-\xi}{3} \Big) \in \mathbb{R}^3 \colon \xi \in [0, \infty) \Big\},$$
$$W_4 = \Big\{ \Big(-\xi, 0, \frac{\xi-1}{3} \Big) \in \mathbb{R}^3 \colon \xi \in [0, \infty \Big\},$$
$$W_5 = \Big\{ \Big(-\frac{\sqrt{3}}{2}\xi, -\frac{\xi}{2}, \frac{1-\xi}{3} \Big) \in \mathbb{R}^3 \colon \xi \in [0, \infty) \Big\},$$
$$W_6 = \Big\{ \Big(\frac{\xi}{2}, -\frac{\sqrt{3}}{2}\xi, \frac{\xi-1}{3} \Big) \in \mathbb{R}^3 \colon \xi \in [0, \infty) \Big\}.$$

The sets W_i, $i = \overline{1,6}$, are shown by red lines in Fig. 1. The embedding $W_i \subset V(t_i)$ is valid for all $i = \overline{1,6}$.

The wave front $\Phi(\tau)$ corresponding to the time point $\tau = 0.5$ (that is, the set of points for which the optimal result function is equal to τ) is shown in Fig. 2 as a surface with colors changing from blue to red as they grow along the Z axis. The wave front $\Phi(\tau)$ corresponding to the time point $\tau = 1$ is shown in Fig. 3.

The dispersing surface is characterized by 6 sheets:

$$L_1 = \Big\{ (x,y,z) \in \mathbb{R}^3 \colon x = \xi,\ y = 0,\ z < \frac{1-\xi}{3},\ \xi \in [0, \infty) \Big\},$$
$$L_2 = \Big\{ (x,y,z) \in \mathbb{R}^3 \colon x = \frac{\sqrt{3}}{2}\xi,\ y = \frac{\xi}{2},\ z > \frac{\xi-1}{3},\ \xi \in [0, \infty) \Big\},$$
$$L_3 = \Big\{ (x,y,z) \in \mathbb{R}^3 \colon x = -\frac{\xi}{2},\ y = \frac{\sqrt{3}}{2}\xi,\ z < \frac{1-\xi}{3},\ \xi \in [0, \infty) \Big\},$$
$$L_4 = \Big\{ (x,y,z) \in \mathbb{R}^3 \colon x = 0,\ y = \xi,\ z > \frac{\xi-1}{3},\ \xi \in [0, \infty) \Big\},$$
$$L_5 = \Big\{ (x,y,z) \in \mathbb{R}^3 \colon x = \frac{\xi}{2},\ y = -\frac{\sqrt{3}}{2}\xi,\ z < \frac{1-\xi}{3},\ \xi \in [0, \infty) \Big\},$$
$$L_6 = \Big\{ (x,y,z) \in \mathbb{R}^3 \colon x = \frac{\sqrt{3}}{2}\xi,\ y = -\frac{\xi}{2}, z > \frac{\xi-1}{3},\ \xi \in [0, \infty) \Big\},$$

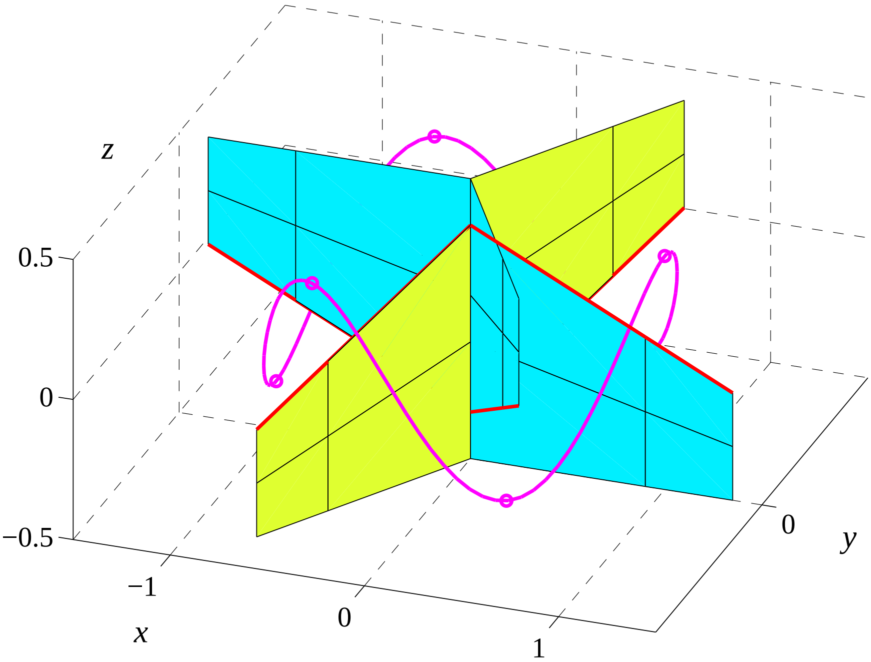

Figure 1. The curve Γ, the pseudo-vertices, and the dispersing surface $L(\Gamma)$.

All sheets have a non-empty intersection

$$L^* = \bigcap_{i=\overline{1,6}} L_i = \Big\{(x,y,z) \in \mathbb{R}^3 \colon x=0,\ y=0,\ z \in \big(-\frac{1}{3},\ \frac{1}{3}\big)\Big\}.$$

We have $\mathrm{card}(\Omega_\Gamma(\mathbf{x})) = 6$ for all points $\mathbf{x} \in L^*$, and $\mathrm{card}(\Omega_\Gamma(\mathbf{x})) = 3$ for all other points

$$\big\{(x,y,z) \in \mathbb{R}^3 \colon\ x=0,\ y=0,\ |z| \geqslant 1/3\big\}$$

on the applicate axis.

Remark 4. The resolving constructions in Example 1 can be considered as a problem solution for an Eikonal equation with the boundary condition given on the graph of the vector-valued function (5.1). In this case, wave fronts represent light propagation surfaces in a homogeneous medium with the source distributed uniformly along the curve Γ. The bisector $L(\Gamma)$ is the union of non-smoothness points of the wave fronts due to the fact that the radiation comes from different points on the curve Γ.

6. Conclusion

One class of time-optimal problems in 3D space with a spherical velocity vectogram is investigated in the case of the target set coinciding with a curve Γ defined by the parametric equation. Characteristic points, such as pseudo-vertices responsible for the origin of the singular set $L(\Gamma)$, are identified. The optimal result function $u(\mathbf{x})$ loses its smoothness on the surface $L(\Gamma)$. Analytical expressions are obtained for the coordinates of the extreme points of the bisector corresponding to a pseudo-vertex. The equations are written in terms of the curvature, principal normal, and

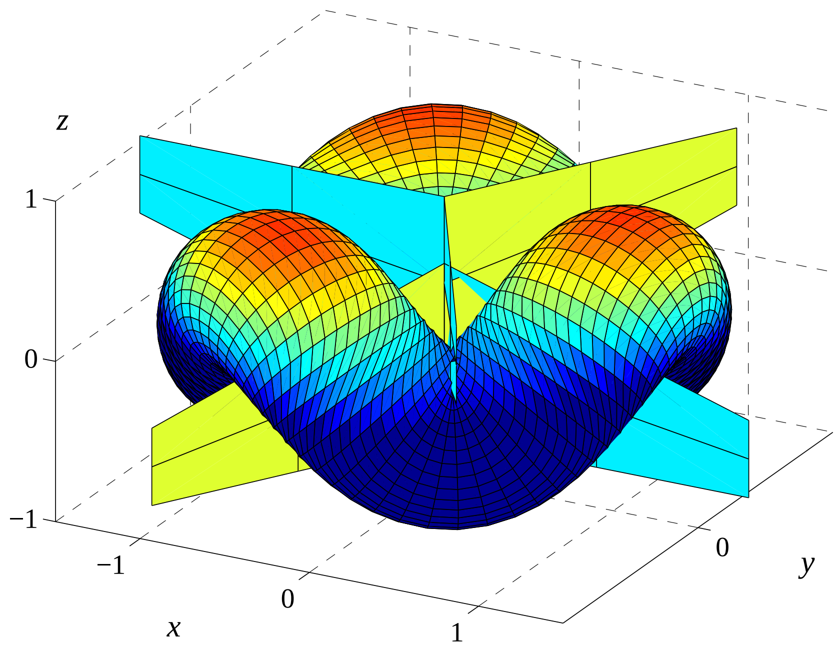

Figure 2. The level surface $\Phi(0.5)$ of the optimal result function and the dispersing surface $L(\Gamma)$.

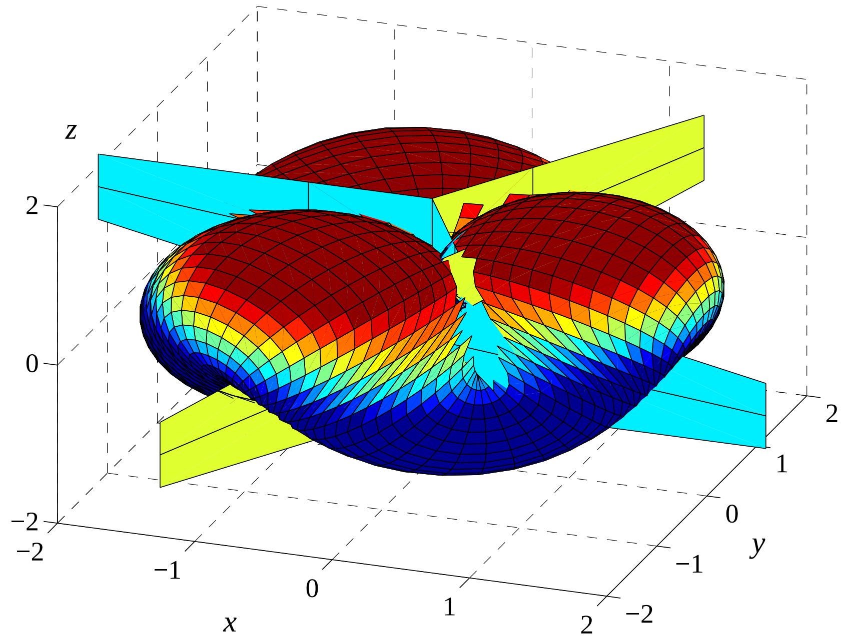

Figure 3. The level surface $\Phi(1)$ of the optimal result function and the dispersing surface $L(\Gamma)$.

binormal of the curve Γ. An example of modeling the construction of a solution to a time-optimal problem with a closed curve taken as the target set is given. Four pseudo-vertices and the sets W_i, $i = \overline{1,6}$, of extreme points corresponding to them, which are rays on lines conjugate to $L(\Gamma)$, are found. Based on the sets W_i, $i = \overline{1,6}$, a bisector is constructed, which is the union of two plane sets lying in orthogonal planes and having a common line segment. The level surfaces $\Phi(\tau)$ are constructed at various time points τ. We should note that, in the previously studied problems on the plane, only one bisector point can correspond to each pseudo-vertex (or two in a very special case, e.g., in [7]). In 3D space, an infinite set of extreme points corresponding to one pseudo-vertex can exist. In the future, it is planned to extend the developed algorithms to solve problems with more complex geometry.

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ON DISTANCE-REGULAR GRAPHS OF DIAMETER 3 WITH EIGENVALUE $\theta = 1$

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Abstract: For a distance-regular graph Γ of diameter 3, the graph Γ_i can be strongly regular for $i = 2$ or 3. J. Kulen and co-authors found the parameters of a strongly regular graph Γ_2 given the intersection array of the graph Γ (independently, the parameters were found by A.A. Makhnev and D.V. Paduchikh). In this case, Γ has an eigenvalue $a_2 - c_3$. In this paper, we study graphs Γ with strongly regular graph Γ_2 and eigenvalue $\theta = 1$. In particular, we prove that, for a Q-polynomial graph from a series of graphs with intersection arrays $\{2c_3 + a_1 + 1, 2c_3, c_3 + a_1 - c_2; 1, c_2, c_3\}$, the equality $c_3 = 4(t^2 + t)/(4t + 4 - c_2^2)$ holds. Moreover, for $t \leq 100000$, there is a unique feasible intersection array $\{9, 6, 3; 1, 2, 3\}$ corresponding to the Hamming (or Doob) graph $H(3, 4)$. In addition, we found parametrizations of intersection arrays of graphs with $\theta_2 = 1$ and $\theta_3 = a_2 - c_3$.

Keywords: Strongly regular graph, Distance-regular graph, Intersection array.

1. Introduction

We consider undirected graphs without loops and multiple edges.

Let Γ be a connected graph. The *distance* $d(a, b)$ between two vertices a, b of Γ is the length of a shortest path between a and b in Γ. For a vertex a of Γ, denote by $\Gamma_i(a)$ the induced subgraph on the set of all vertices at distance i from a in Γ. Let Γ be a graph with diameter d and let a and b be vertices of Γ at distance i $(0 \leq i \leq d)$. Then the number of vertices that are at distance j from a and h from b is denoted by $p^i_{jh}(a, b)$ $(0 \leq i, j, h \leq d)$ and is called an intersection number of Γ. Note that $p^i_{jh}(a, b) = |\Gamma_j(a) \cap \Gamma_h(b)|$. Consider the numbers $c_i(a, b) = p^i_{i'1,1}(a, b)$, $a_i(a, b) = p^i_{i1}(a, b)$, and $b_i(a, b) = p^i_{i+1,1}(a, b)$. If the intersection numbers do not depend on the choice of a and b but only on i, then these numbers are denoted simply by p^i_{jh} $(0 \leq i, j, h \leq d)$. In this case, Γ of diameter d is called a *distance-regular* graph with intersection array $(b_0, b_1, \ldots, b_{d-1}; c_1, \ldots, c_d)$.

If a and b are vertices of the graph Γ, then we denote by $d(a, b)$ the distance between a and b. Given a vertex a in a graph Γ, we denote by $\Gamma_i(a)$ the subgraph induced by Γ on the set of all vertices at the distance i from a. The subgraph $\Gamma_1(a)$ is called the *neighbourhood of the vertex a* and is denoted by $[a]$, if the graph Γ is fixed.

Let Γ be a graph of diameter d and $i \in \{1, 2, 3, \ldots, d\}$. The graph Γ_i have the same set of vertices, and vertices u and w are adjacent in Γ_i if $d_\Gamma(u, w) = i$. For a subset of vertices Y from Γ, we denote by $\Gamma_i(Y)$ the subgraph with the set of vertices Y in which PI vertices u and w are adjacent if $d_\Gamma(u, w) = i$.

An incidence system with a set of points P and a set of lines $\mathcal{L}$ is called an *α-partial geometry of order* (s, t) if each line contains exactly $s + 1$ points, each point lies exactly on $t + 1$ lines, any two points lie on at most one line, and, for any antiflag $(a, l) \in (P, \mathcal{L})$, there is exactly α lines passing through a and intersecting l (the notation is $pG_\alpha(s, t)$).

A point graph of a geometry of points and lines is a graph whose vertices are points of the geometry, and two different vertices are adjacent if they lie on a common line. It is easy to see that a point graph of a partial geometry $pG_\alpha(s, t)$ is strongly regular with parameters $v = (s+1)(1+st/\alpha)$, $k = s(t+1)$, $\lambda = (s-1) + (\alpha-1)t$, and $\mu = \alpha(t+1)$. A strongly regular graph having the above parameters for some positive integers α, s, and t is called a *pseudogeometric graph* for $pG_\alpha(s, t)$.

The direct problem in the theory of distance-regular graphs is, given an intersection array, to find the parameters of a symmetric structure corresponding to a graph with this intersection array. The inverse problem is finding the intersection array of a distance-regular graph given the parameters of the corresponding symmetric structure.

If, for a distance-regular graph Γ of diameter 3, the graph Γ_3 is strongly regular, then, by [1, Lemma 3], the graph $\bar{\Gamma}_3$ is pseudogeometric for $pG_{c_3}(k, b_1/c_2)$. Conversely, for the graph $\bar{\Gamma}_3$, which is pseudogeometric for $pG_\alpha(l, t)$, the graph Γ has an intersection array $\{l, tc_2, l-\alpha+1; 1, c_2, \alpha\}$, where $l > tc_2 \geq l - \alpha + 1$ and $c_2 \leq \alpha$.

Let Γ be a non-bipartite distance-regular graph of diameter 3. By [2, Lemma 3.1], the graph Γ_2 is strongly regular if and only if Γ has the eigenvalue $\theta = a_2 - c_3$.

The inverse problem was solved by A.A. Makhnev and D.V. Paduchickh. Let Γ be a distance-regular graph of diameter 3, for which Γ_2 is a strongly regular graph with parameters $(v, \kappa, \lambda, \mu)$ and eigenvalues κ, r, and $-s$. Then for $x = b_2 + c_2 \leq rs$ and $\mu x \neq rs(r+1)(s-1)$ the parameters of the intersection array of the graph Γ are expressed in terms of $\kappa, \mu, r, -s$, and x ([3, Theorem 2]).

We continue the study of distance-regular graphs Γ of diameter 3 with strongly regular graph Γ_2 and eigenvalue $\theta_2 = 1$.

The following result is obtained in [2, Lemma 4.5].

Proposition 1. *Let Γ be a non-bipartite distance-regular graph of diameter 3 with eigenvalue $\theta_2 = a_2 - c_3 = 1$. The following statements hold:*

(1) *the eigenvalues θ_1 and θ_3 are integer, $\theta_1 + \theta_3 = a_1$;*

(2) $c_3(c_2 + 2) = -(\theta_1 + 1)(\theta_3 + 1)$;

(3) *Γ has the intersection array $\{2c_3 + a_1 + 1, 2c_3, c_3 + a_1 - c_2; 1, c_2, c_3\}$.*

By Proposition 1, the graph Γ with $\theta_2 = a_2 - c_3 = 1$ and $n = a_1^2 + 4(c_2 + 2)c_3 + 4a_1 + 4$ has non-principal eigenvalues 1 and $a_1/2 \pm \sqrt{n}$, where the multiplicity of 1 is equal to

$$(2a_1 - c_2 + 4c_3 + 2)(a_1 + 2c_3 + 1)c_3/(c_2c_3 + 2a_1 + 2c_3).$$

This implies that n is a square and the multiplicity of $a_1/2 \pm \sqrt{n}$ is equal to

$$4(2a_1 - c_2 + 4c_3 + 2)(a_1 - c_2 + c_3)(a_1 + 2c_3 + 1)(a_1 + 2c_3)/\big((2a_1^3 - a_1^2c_2 + 2a_1^2c_3 + 8a_1c_2c_3 - 4c_2^2c_3 + 8c_2c_3^2 + \sqrt{n}(2a_1^2 - a_1c_2 + 2a_1c_3 + 2c_2c_3 + 2c_2) + 8a_1^2 - 4a_1c_2 + 24a_1c_3 - 8c_2c_3 + 16c_3^2 + 8a_1 - 4c_2 + 8c_3)c_2\big).$$

Theorem 1. *Let Γ be a Q-polynomial distance-regular graph of diameter 3 with strongly regular graph Γ_2. If Γ has an eigenvalue $\theta = a_2 - c_3 = 1$, then $c_3 = 4(t^2 + t)/(4t + 4 - c_2^2)$ and Γ has the intersection array $\big\{(c_2^2 + 4c_2 + 4t + 4)(t+1)/(4t + 4 - c_2^2), 8(t+1)t/(4t + 4 - c_2^2), (c_2 + t + 2)c_2^2/(4t + 4 - c_2^2); 1, c_2, 4(t^2 + t)/(4t + 4 - c_2^2)\big\}$.*

For $t \leq 100000$, there is only one feasible intersection array $\{9, 6, 3; 1, 2, 3\}$ ($t = c_2 = 2$) corresponding to the Hamming graph $H(3, 4)$ or the Doob graph with the same parameters.

We found parametrizations of distance-regular graphs of diameter 3 with eigenvalues $\theta_2 = 1 \neq \theta_3 = a_2 - c_3$.

Theorem 2. *Let Γ be a distance-regular graph of diameter 3 with strongly regular graph Γ_2. If Γ has the eigenvalue $\theta_2 = 1 \neq a_2 - c_3$, then Γ has the intersection array $\{(2n+r)t+1, 2(n-1)t, r(t-1); 1, n+r+1, 2nt\}$ or $\{(2n+r)t+n+r+1, (n-1)(2t+1), r(2t-1); 1, n+2r+1, n(2t+1)\}$.*

The following examples of graphs with eigenvalues $\theta_2 = 1 \neq \theta_3 = a_2 - c_3$ are known:

(1) $\{21, 10, 3; 1, 6, 15\}$, half 7-cube with spectrum $21^1, 9^7, 1^{21}, -3^{35}$, $v = 1+21+35+7 = 64$, and Γ_2 is a graph with parameters $(64, 35, 18, 20)$;

(2) $\{111, 88, 9; 1, 12, 99\}$ with spectrum $111^1, 21^{148}, 1^{444}, -9^{407}$, $v = 1 + 111 + 814 + 74 = 1000$, and Γ_2 is a strongly regular graph with parameters $(1000, 814, 663, 660)$.

For graphs from Theorem 2 for $n < 350, t < 1000$, we have only feasible intersection arrays $\{21, 10, 3; 1, 6, 15\}$, $\{111, 88, 9; 1, 12, 99\}$, $\{561, 448, 54; 1, 12, 504\}$, and $\{561, 448, 75; 1, 21, 480\}$.

2. Proof of Theorem 1

Let Γ be a Q-polynomial distance-regular graph of diameter 3 with eigenvalue $\theta_2 = a_2 - c_3 = 1$. By Proposition 1, the graph Γ has integer eigenvalues.

Lemma 1. *$a_1 = (c_2 + 2)c_3/t - t - 2$ for some positive integer t.*

P r o o f. We have

$$(a_1^2 + 4(c_2 + 2)c_3 + 4a_1 + 4) = u^2,$$

where u is a positive integer. Solving the Diophantine equation

$$u^2 - (a_1 + 2)^2 = 4(c_2 + 2)c_3,$$

we get

$$u = (c_2 + 2)c_3/t + t, \quad a_1 = (c_2 + 2)c_3/t - t - 2$$

for some positive integer t. □

Lemma 2. *The inequality $c_3 > t$ holds.*

P r o o f. We have

$$k = (c_2c_3 + 2c_3t - t^2 + 2c_3 - t)/t,$$

hence

$$(c_2c_3 + 2c_3t - t^2 + 2c_3 - t) > 0.$$

Further,

$$k_3 = 2(c_2c_3 + 2c_3t - t^2 + 2c_3 - t)(c_2 + t + 2)(c_3 - t)/(c_2t^2),$$

hence $c_3 > t$. □

Lemma 3. *The graph Γ is not Q-polynomial with respect to E_2.*

P r o o f. Suppose that Γ is a Q-polynomial graph with respect to E_2. Then, by [4], the equality

$$-2(c_2c_3+2c_3t-t^2+2c_3-2t)(c_2+2t+2)(2c_3-t)(c_3+1)/((c_2c_3+2c_3-2t)(t+2)t)$$
$$=-(c_2c_3+2c_3t-t^2+2c_3-2t)(c_2+2t+2)(2c_3-t)(c_3+1)/((c_2c_3+2c_3-2t)(t+2)t)$$

holds and either $c_3=(t^2+2t)/(c_2+2t+2)$, or $c_3=t/2$, or $c_3=-1$.
In any case, we have a contradiction. □

Lemma 4. *If Γ is not Q-polynomial with respect to E_1, then $c_3=4(t^2+t)/(4t+4-c_2^2)$.*

P r o o f. Let Γ be a Q-polynomial graph with respect to E_1. Then, by [4], the following equality holds:

$$-(c_2^2c_3^2-c_2^2c_3t-c_2c_3t^2+4c_2c_3^2-4c_2c_3t+2c_3t^2-2t^3+4c_3^2-4c_3t)(c_2c_3+2c_3t$$
$$-t^2+2c_3-2t)(c_2+2t+2)(2c_3-t)/((c_2c_3+t^2+2c_3)(c_2c_3+2c_3-2t)c_2t^2)$$
$$=-(c_2^4c_3^3+4c_2^3c_3^3t-5c_2^3c_3^2t^2+4c_2^2c_3^3t^2+c_2^3c_3t^3-10c_2^2c_3^2t^3+4c_2^2c_3t^4-4c_2c_3^2t^4$$
$$+4c_2c_3t^5+8c_2^3c_3^3-6c_2^3c_3^2t+24c_2^2c_3^3t-42c_2^2c_3^2t^2+16c_2c_3^3t^2+16c_2^2c_3t^3-40c_2c_3^2t^3$$
$$+24c_2c_3t^4+24c_2^2c_3^3-36c_2^2c_3^2t+48c_2c_3^3t+12c_2^2c_3t^2-108c_2c_3^2t^2+16c_3^3t^2$$
$$+68c_2c_3t^3-40c_3^2t^3-8c_2t^4+32c_3t^4-8t^5+32c_2c_3^3-72c_2c_3^2t+32c_3^3t+48c_2c_3t^2$$
$$-88c_3^2t^2-8c_2t^3+80c_3t^3-24t^4+16c_3^3-48c_3^2t+48c_3t^2-16t^3)(c_2c_3+2c_3t-t^2+2c_3$$
$$-2t)(2c_3-t)/\big((c_2c_3+2c_3t-2t^2+2c_3-2t)(c_2c_3+t^2+2c_3)(c_2c_3+2c_3-2t)c_2t^2\big).$$

Hence,

$$c_3\in\big\{4(t^2+t)/(4t+4-c_2^2),(2t^3+(t^2+2t)c_2+4t^2+4t)/(c_2^2+2c_2(t+2)+2t^2+4t+4),$$
$$(t^2+2t)/(c_2+2t+2),1/2t\big\}.$$

The latter three cases contradict Lemma 2. □

Theorem 1 is proved. □

3. Proof of Theorem 2

Let Γ be a non-bipartite distance-regular graph of diameter 3 with eigenvalues

$$\theta_1=a_1-1,\quad \theta_2=1,\quad \theta_3=a_2-c_3.$$

By [2, Lemma 3.1(v)], we have $b_1=(a_2-c_3+1)c_3/(a_2-c_3)$. This implies the following statement.

Lemma 5. *One of the following equalities holds*:

(1) $c_3=(c_3-a_2)m$, *where m is a positive integer not exceeding* 1;

(2) $k=b_2+c_2+c_3+1$;

(3) $k=b_2+c_2+c_3-1$.

In the second case, we have $a_2 - c_3 = 1$. In the third case, we have $a_2 - c_3 = -1$, a contradiction with [2, Lemma 3.1(b)].

Hence,

$$c_3 = (c_3 - a_2)m, \quad a_2 m = c_3(m-1), \quad a_2 = (m-1)n,$$

$b_1 = mn - m$ for some positive integer n greater than 1.

The non-principal eigenvalues $a_1 - 1$ and 1 are roots of the quadratic equation

$$x^2 - (b_2 + c_2 + m - n - 1)x + c_2 m - (m-1)n - b_2 - c_2 = 0.$$

Hence,

$$a_1 = k - a_2 + m - n - 1$$

and

$$a_1 - 1 = c_2 m - (m-1)n - k + a_2.$$

Hence

$$k = a_1 + 1 + mn - m, \quad k + a_1 - 1 = c_2 m, \quad 2a_1 = m(c_2 - n + 1).$$

If $m = 2t$, then $c_2 = n + r + 1$, $a_1 = t(r+2)$, $b_1 = 2t(n-1)$, and Γ has the intersection array

$$\{t(2n+r)+1, 2t(n-1), rt-r; 1, n+r+1, 2nt\}$$

and the non-principal eigenvalues $rt + 2t - 1, 1$, and $-n$ of multiplicities

$$(2nt+rt+n+1)(2nt+rt+1)(2n+r)(t-1)(n-1)/((rt+n+2t-1)(rt+2t-2)(n+r+1)n),$$
$$(2nt+rt+n+1)(2nt+rt+1)(nt-t+1)(n-1)r/((rt+2t-2)(n+r+1)(n+1)n),$$
$$2(2nt+rt+1)(nt-t+1)(2n+r)t/((rt+n+2t-1)(n+1)n),$$

respectively.

If $m = 2t+1$, then

$$c_2 = n + 2r + 1, \quad a_1 = (2t+1)r, \quad b_1 = (2t+1)(n-1),$$

and Γ has the intersection array

$$\big\{(2t+1)(n+r-1)+1, (2t+1)(n-1), 2rt-r; 1, n+2r+1, 2nt+n\big\}$$

and the non-principal eigenvalues $r(2t+1)+2t, 1$, and $-n$ of multiplicities

$$(2nt+2rt+2n+r+1)(2nt+2rt+n+r+1)(n+r)(n-1)(2t-1)/((2rt+n+r+2t)$$
$$\times(2rt+r+2t-1)(n+2r+1)n),$$
$$(2nt+2rt+2n+r+1)(2nt+2rt+n+r+1)(2nt+n-2t+1)(n-1)r/((2rt+r+2t-1)$$
$$\times(n+2r+1)(n+1)n),$$
$$(2nt+2rt+n+r+1)(2nt+n-2t+1)(n+r)(2t+1)/((2rt+n+r+2t)(n+1)n),$$

respectively.

Theorem 2 is proved. □

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ANALYSIS OF THE GROWTH RATE OF FEMININE MOSQUITO THROUGH DIFFERENCE EQUATIONS

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Abstract: The mosquito life cycle is developed mathematically with the concept of difference equation. The qualitative properties of the life-cycle are analyzed. The Lyapunov function is defined for difference equation to stabilize the system of mosquito life cycle. A novel technique is applied for deriving stability criterion, especially the back-stepping control technique is applied for discrete time system. The bifurcation analysis is also furnished for the model of mosquito life cycle. The new technique is applied in the mosquito life cycle model and its results are examined through MATLAB.

Keywords: Difference Equation, Mosquito, Bifurcation, Equilibrium, Strict Feedback.

1. Introduction

Research on mosquito epidemiology is imperative for the society. All over the world, all governments can pay more attention to mosquito epidemiological research. [1, 2, 4].

Many researchers developed a mathematical model of Plasmodium Life Cycle in Hepatocyte, mosquito midgut malaria transmission, HIV transmission, nitrogen cycle etc., in which the authors explore the complexity, bifurcation and analyze the stability of their model by the presence of an equilibrium point of the system [5, 6]. By constructing suitable conditions through the Lyapunov function, local and global stability analysis are discussed [7–9]. The difference equations have a long journey on the discrete time models of population dynamics [3]. These equations describe typically autonomous, discrete time dynamics and assume that there is only a temporary change in vital rates due to dependence on population density. An individual's important behaviour and activities can similarly change and fluctuate. Such kind of explicit dependencies on time can be modelled by using the difference equation. In the recent years, the difference equations have received more attention in the mathematical areas.

This paper is devotes a mathematical study of mosquito life cycle. The difference equation concept is utilized to construct the model. A novelty is involved in the derivation of stability conditions. Earlier researcher have not considered such type of Lyapunov function for difference equation. Section 2 describes the mathematical model for the mosquito life cycle under difference equation. Section 3 contains the discussion on equilibrium point position. Sections 4 includes the bifurcation analysis of the system of difference equation for the mosquito life cycle. In section 5 we investigate the stability analysis for the system with the conditions of Lypanouv stability, also related results are presented and finally, Section 6 describes the conclusion.

2. The mathematical model

The mathematical model for the Anopheles mosquito life cycle is described by the system of equations with the following assumptions.

- The total population of Anopheles mosquito life cycle consists of four forms, namely, adult, egg, larva and pupa.
- In every stage, the natural death rate μ is considered to be uniform.
- Let N denote the existing population, where ϕ is natural birth rate at adult stage.
- x_1 is the number of population existing at initial stage.
- x_2 is the number of eggs.
- x_3 is the population of larva.
- x_4 is the number of pupa.

The following Figure 1 shows the flow diagram of Anopheles mosquito life cycle.

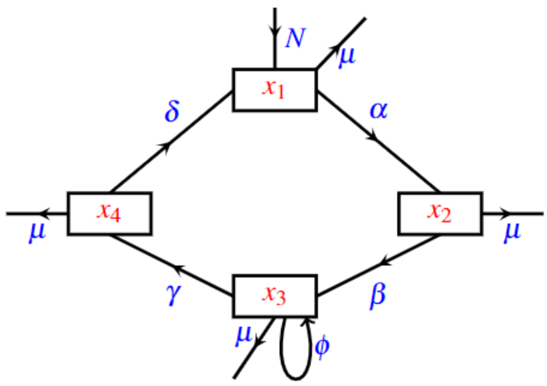

Figure 1. The flow diagram of Anopheles mosquito life cycle

The Anopheles mosquito life cycle is given by the following system of difference equation:

$$\begin{aligned} x_1(n+1) &= (N-\mu-\alpha)\ x_1(n) + \delta\ x_4(n), \\ x_2(n+1) &= \alpha\ x_1(n) - (\mu+\beta)\ x_2(n), \\ x_3(n+1) &= \beta\ x_2(n) - (\mu+\phi+\gamma)\ x_3(n), \\ x_4(n+1) &= \gamma\ x_3(n) - (\mu+\delta)\ x_4(n), \end{aligned} \tag{2.1}$$

where

- $x_1(n+1)$, $x_2(n+1)$, $x_3(n+1)$, $x_4(n+1)$ respectively are the difference equation at each stage,
- α, β, γ, δ are the respective rates of growth from one stage to another stage.

3. Analysis of equilibrium position

The equilibrium points are essential for analysing epidemiological dynamics which revolves around the equilibrium points. In epidemiology, the equilibrium point is a condition in which some identified or non-identified epidemiological form is balanced.

The epidemiological equilibrium points are unchanged from the epidemiological structure [10, 11]. They arise as a combination of corresponding epidemiological variables.

In mosquito epidemiology, adult, egg, lava and pupa are identified as key variables. The equilibrium points are obtained by means of relations

$$\begin{aligned} x_1^*(n) &= -\Big(\frac{\gamma}{N-\mu-\alpha}\Big)(x_4(n)), \\ x_2^*(n) &= -\Big(\frac{\alpha\gamma}{(N-\mu-\alpha)(\mu+\beta)}\Big)(x_4(n)), \\ x_3^*(n) &= -\Big(\frac{\gamma\alpha\delta}{(\mu+\beta)(N-\mu-\alpha)(\mu-\phi+\gamma)}\Big)(x_4(n)). \end{aligned} \tag{3.1}$$

If the pupa $x_4(n)$ state growth is equal to same arbitrary constant then the equilibrium points differ for following cases:

Case 1: If the arbitrary constant $\chi = 0$, then the four states of anopheles mosquito life cycle such as adult $x_1(n)$, eggs $x_2(n)$, larva $x_3(n)$ and pupa $x_4(n)$ are zero, which implies that a zero-equilibrium point.

Case 2: If the pupa growth rate is non-zero, also if

$$\chi > 0, \quad N-\mu-\alpha > 0, \quad \mu-\phi+\gamma > 0, \quad \mu+\beta > 0,$$

then $x_3 = -c_1$, $x_2 = -c_2$, $x_1 = -c_3$, and so $E = (-c_3, -c_2, -c_1, c_4)$ is an equilibrium solution.

Case 3: If

$$\chi < 0, \quad N-\mu-\alpha > 0, \quad \mu-\phi+\gamma > 0,$$

then $x_3 = c_1$, $x_2 = c_2$, $x_1 = c_3$, and so $E = (c_3, c_2, c_1, -c_4)$ is an equilibrium solution.

4. Bifurcation analysis

The purpose of bifurcation analysis is to study a dynamical system with respect to the trajectory represented by system, the occurrence of an equilibrium point and the stability properties of the equilibrium point, when changes occur in a certain parameter of the system of equations. The bifurcation analysis is carried out by linearizing the system of equations.

The Jacobian matrix is obtained as

$$\begin{bmatrix} (N-\mu-\alpha) & 0 & 0 & \delta \\ \alpha & -(\mu+\beta) & 0 & 0 \\ 0 & \beta & -(\mu-\phi+\gamma) & 0 \\ 0 & 0 & \gamma & -(\mu+\delta) \end{bmatrix}. \tag{4.1}$$

The characteristic equation of the above Jacobian matrix given by the equation (4.1) is obtained as

$$\Delta_1\lambda^4 + \Delta_2\lambda^3 + \Delta_3\lambda^2 + \Delta_4\lambda + \Delta_5 = 0,$$

where

$$
\begin{aligned}
\Delta_1 &= 1,\\
\Delta_2 &= \alpha - N + b + \gamma + \delta + 4\mu - \phi,\\
\Delta_3 &= N\phi - N\gamma - Nd - 3N\mu - N\beta + \alpha\beta + \alpha\gamma + \alpha d + \beta\gamma + 3\alpha\mu + \beta d - \alpha\phi + 3\beta\mu\\
&\quad + \gamma\delta - \beta\phi + 3\gamma\mu + 3\delta\mu - \delta\phi - 3\mu\phi + 6\mu^2,\\
\Delta_4 &= 3\alpha\mu^2 - 3N\mu^2 + 3\beta\mu^2 + 3\gamma\mu^2 + 3\delta\mu^2 - 3\mu^2\phi + 4\mu^3 - N\beta\gamma - N\beta d - 2N\beta\mu\\
&\quad - N\gamma\delta + N\beta\phi - 2N\gamma\mu - 2N\delta\mu + N\delta\phi + 2N\mu\phi + \alpha\beta\gamma + \alpha\beta d + 2\alpha\beta\mu\\
&\quad + \alpha\gamma\delta - \alpha\beta\phi + 2\alpha\gamma\mu + \beta\gamma\delta + 2\alpha\delta\mu + 2\beta\gamma\mu - \alpha\delta\phi + 2\beta\delta\mu - 2\alpha\mu\phi\\
&\quad - \beta\delta\phi + 2\gamma\delta\mu - 2\beta\mu\phi - 2\delta\mu\phi,\\
\Delta_5 &= \alpha\mu^3 - N\mu^3 + \beta\mu^3 + \gamma\mu^3 + \delta\mu^3 - \mu^3\phi + \mu^4 - N\beta\mu^2 - N\gamma\mu^2 - N\delta\mu^2 + N\mu^2\phi\\
&\quad + \alpha\beta\mu^2 + \alpha\gamma\mu^2 + \alpha\delta\mu^2 + \beta\gamma\mu^2 + \beta\delta\mu^2 - \alpha\mu^2\phi + \gamma\delta\mu^2 - \beta\mu^2\phi - \delta\mu^2\phi - N\beta\gamma\delta\\
&\quad - N\beta\gamma\mu - N\beta\delta\mu + N\beta\delta\phi - N\gamma\delta\mu + N\beta\mu\phi + N\delta\mu\phi + \alpha\beta\gamma\mu + \alpha\beta\delta\mu - \alpha\beta\delta\phi\\
&\quad + \alpha\gamma\delta\mu - \alpha\beta\mu\phi + \beta\gamma\delta\mu - \alpha\delta\mu\phi - \beta\delta\mu\phi,
\end{aligned}
$$

from the analysis with the different cases.

If any one of the parameter values is equal to zero or $N-\mu-\gamma < 0$ or $\mu+\beta < 0$ or $N-\mu-\phi-\gamma < 0$ or $\mu+\delta < 0$ then all the eigen values of the Jacobian matrix given in equation (4.1) are real. Hence for the linearised form of the system of equations there exists the hyperbolic equilibrium. Therefore the proposed mathematical model for the mosquito life cycle is satisfies the Lyapunov's conditions with respect to the robustness.

By introducing Holling type II parameter [15, 16] in larva stage ($x_3(n)$), the new dimension of the equation becomes,

$$
x_3(n+1) = r\ x_3(n) - \left[0.2x_3^2(n) + \frac{0.375x_3(n)}{1+x_3^2(n)}\right],
$$

where $r = -(\mu + \phi + \gamma)$ and the transmission rate from the state is

$$
\beta x_2(n) = \left[0.2x_3^2(n) + \frac{0.375x_3(n)}{1+x_3^2(n)}\right].
$$

The bifurcation exists at the larva state x_3 when the value of the parameter r varies between 2.5 and 4. Figure 2 shows the existence of bifurcation on the Anopheles mosquito life cycle at the larva state x_3.

5. Stability analysis of anopheles mosquito life cycle

In epidemiology the stability analysis of the system is possible to create a new example and explore new options. The stability analysis of anopheles mosquito life cycle is developing a balance of its cycle [12–14]. The following theorem gives the stability of the described model and the following relation establishes the condition for the anopheles mosquito life cycle.

Theorem 1. *The system of equation (2.1) for the anopheles mosquito life cycle is stabilized, if the following conditions exist for the system namely*

$$
\begin{aligned}
(N-\mu-\alpha)x_1(n) &= x_1(n) - \delta x_4(n) - x_1^2(n+1),\\
(\mu+\beta)x_2(n) &= \alpha x_1(n) - x_2(n) + x_2^2(n+1),\\
(\mu-\phi+\gamma)x_3(n) &= \beta x_2(n) - x_3(n) + x_3^2(n+1),\\
(\mu+\gamma)x_4(n) &= \gamma x_3(n) - x_4(n) + x_4^2(n+1).
\end{aligned}
\tag{5.1}
$$

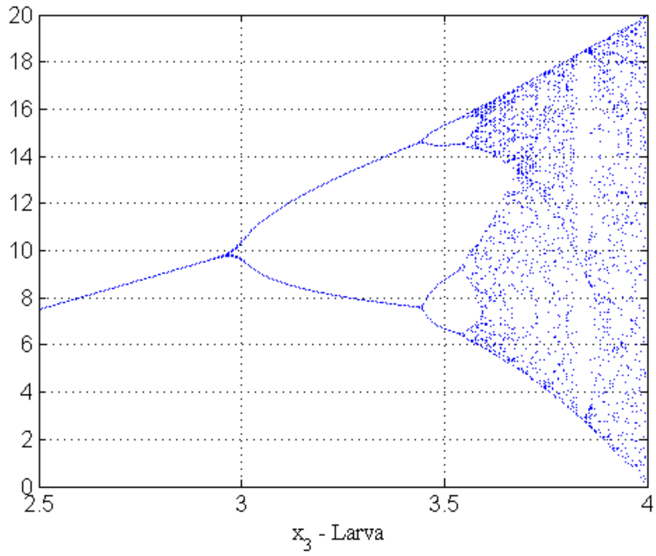

Figure 2. Existence of bifurcation in the Anopheles mosquito life cycle at the state x_3

P r o o f. Consider the Lyapunov function

$$V(x_n) = \sum_{i=1}^{4}(x_i(n)).$$

Take the difference equation (2.1), we obtain

$$\Delta V(x_n) = \sum_{i=1}^{4} \Delta\big(x_i(n)\big) \sum_{i=1}^{4} \big(x_i(n+1) - x_i(n)\big).$$

Substitutions of (5.1) in (2.1) leads to the relation

$$\Delta V = -x_i^2(n+1) \quad \text{for} \quad i = 1, 2, 3, 4.$$

Hence

$$\Delta V < 0,$$

which shows that V is a negative definite function. By Laselle's invariance principle, the model (2.1) is asymptotically stable. □

5.1. Stability analysis for Anopheles life cycle by using backward strict-feedback

The stability analysis helps to know how long the life can be accumulated and accelerated about the condition without any degradation. This study helps to determine the mean life of the mosquito. The strict-feedback control gives more accuracy to the system.

Theorem 2. *The system of equations* (2.1) *for the anopheles mosquito life cycle with the backward strict feedback mechanism under the concept of difference equation is globally asymptotically stable if*

$$\begin{aligned} u_1 &= (\mu + \delta + 1)x_4(n) - x_4^2(n), \\ u_2 &= -w_2^2(n), \\ u_3 &= -w_3^2(n), \\ u_4 &= -\delta x_4(n) - w_4^2. \end{aligned} \tag{5.2}$$

P r o o f. The backward strict feedback is applied to the system equation (2.1) to get the accuracy and so, consider the following difference equations

$$\begin{aligned} x_4(n+1) &= \gamma x_3(n) - (\mu+\delta)x_4(n) + u_1, \\ x_3(n+1) &= \beta x_2(n) - (\mu-\phi+\gamma)x_3(n) + u_2, \\ x_2(n+1) &= \alpha x_1(n) - (\mu+\beta)x_2(n) + u_3, \\ x_1(n+1) &= (N-\mu-\alpha)x_1(n) + \delta x_4(n) + u_4. \end{aligned}$$

Consider the stability of the pupa state

$$x_4(n+1) = \gamma x_3(n) - (\mu+\delta)x_4(n),$$

where $x_3(n)$ is regraded as a virtual controller.

Define the Lyapunov function

$$V_1(n) = x_4(n) \tag{5.3}$$

and the difference of the above equation (5.3) as follows

$$\Delta V_1(n) = \Delta x_4(n) = x_4(n+1) - x_4(n) = \gamma x_3(n) - (\mu+\delta)x_4(n) - x_4(n) + u_1. \tag{5.4}$$

Assume the virtual controller $x_3(n) = \kappa_1$ then we have

$$\Delta V_1(n) = \gamma\kappa_1 - (\mu+\delta)x_4(n) - x_4(n) + u_1.$$

By applying the controller,

$$u_1 = (\mu+\delta+1)x_4(n) - x_4^2(n)$$

and the virtual control $\kappa_1 = 0$ then the difference equation (5.4) becomes

$$\Delta V_1(n) = -x_4^2(n) < 0,$$

which is the negative definite function. Hence the pupa state x_4 is globally asymptotically stable.

Thus, the controller $\kappa_1(x_4(n))$ is an estimative when $x_4(n)$ is regarded as virtual controller.

The relation between x_3 and $k_1(x_4(n))$ is

$$w_2(n) = x_3(n) - \kappa_1.$$

Consider the $(x_4(n), w_2(n))$ subsystem (pupa and larva states)

$$\begin{aligned} x_4(n) &= -x_4(n) - x_4^2(n), \\ w_2(n+1) &= \beta\kappa_2 + w_2(n) + u_2. \end{aligned} \tag{5.5}$$

Let $x_2(n)$ be a virtual controller for the subsystem (5.5) and assume that the subsystem (5.5) is globally asymptotically stable when the state $x_2(n) = \kappa_2$.

Define the Lyapunov function

$$V_2(n) = x_4(n) + w_2(n).$$

The difference equation of $V_2(n)$ is

$$\Delta V_2(n) = \Delta x_4(n) + \Delta w_2(n) = x_4(n+1) - x_4(n) + w_2(n+1) - w_2(n). \tag{5.6}$$

Substituting the equation (5.5) in the difference equation (5.6), also taking $\kappa_2 = 0$ and $u_2 = -w_2^2(n)$, then the equation (5.6) leads to

$$\Delta V_2(n) = -x_4^2(n) - w_2^2(n).$$

Consequently V_2 is the negative definite function. Hence the system of equation (5.5) is globally asymptotically stable.

Thus, the function $w_2(n)$ is estimative, when the state $x_2(n)$ is consider as a virtual controller. Then the relation between $w_3(n)$ and $w_2(x_4(n), w_2(n))$ is

$$w_3(n) = x_2(n) - \kappa_2.$$

Consider the $(w_3(n), w_2(n), w_4(n))$ subsystem

$$\begin{aligned} w_3(n+1) &= \alpha x_1(n) + w_3(n) + u_3, \\ w_2(n+1) &= w_2(n) - w_2^2(n), \\ x_4(n+1) &= x_4(n) - x_4^2(n). \end{aligned} \tag{5.7}$$

Let $x_1(n)$ be a virtual controller in (5.7) and assume that the subsystem (5.7) is globally asymptotically stable, when $x_1(n) = \kappa_3$.

Let us define the Lyapunov function

$$V_3(n) = V_2(n) + w_3(n). \tag{5.8}$$

The differences from of the above equation (5.8) gives

$$\Delta V_3(n) = \Delta x_4(n) + \Delta w_2(n) + \Delta w_3(n). \tag{5.9}$$

Assume the controller $x_1(n) = \kappa_3$.

If $\kappa_3 = 0$, and $u_3 = -w_3^2(n)$, then the difference equation (5.9) leads to

$$\Delta V_3(n) = -x_4^2(n) w_2^2(n) - w_3^2(n) < 0,$$

which is the negative definite function. Hence the subsystem of equation (5.7) is globally asymptotically stable.

Thus, the function $w_4(n)$ is estimative when $x_1(n)$ is taking as virtual controller, then the relation between $x_1(n)$ and κ_3 is

$$w_4(n) = x_1(n) - \kappa_3.$$

Consider the $(w_4(n), w_2(n), w_3(n), w_4(n))$ subsystem

$$\begin{aligned} w_4(n+1) &= \gamma x_4(n) + w_4(n) + u_4, \\ w_3(n+1) &= w_3(n) - w_3^2(n), \\ w_2(n+1) &= w_2(n) - w_2^2(n), \\ x_1(n+1) &= x_1(n) - x_1^2(n). \end{aligned}$$

Let us assume the Lyapunov function is as follows

$$V_4(n) = V_3(n) + w_4(n). \tag{5.10}$$

The difference equation of $V_4(n)$ is

$$\Delta V_4(n) = \Delta x_4(n) + \Delta w_2(n) + \Delta w_3(n) + \Delta w_4(n). \tag{5.11}$$

Choose the controller as follows

$$u_4 = -\delta x_4(n) - w_4^2$$

substituting the controller u_4 in the equation (5.10), then the difference equation (5.11) becomes

$$\Delta V_4(n) = -x_4^2(n) - w_2^2(n) - w_3^2(n) - w_4^2(n) < 0,$$

which is negative definite function on $\mathbb{R}^4$. Thus by the concept of Lyapunov stability theory, the Anopheles mosquito life cycle (2.1) is globally asymptotically stable.

5.2. Numerical simulation

A numerical result is required in this section to validate the model's analytical result. MATLAB tool is utilised to confirm the theoretical results obtained in our model via backsteeping control technique analysis. Here the stability of the model is composed respect to two different initial conditions with the backstepping controllers is as follows in the system of equations (5.2).

The sensitive depend on initial condition is used to identify the stability and internal equilibrium that have a large influence on the each life cycle states.

To perform the sensitivity depend on initial conditions, the parameter values are considered as

$$\alpha = 0.341, \quad \beta = 0.567, \quad \gamma = 0.197, \quad \delta = 0.907.$$

The natural death rate $\mu = 0.4$ is considered to be uniform in all states and the total population N is considered as 10000000.

First, the initial conditions of the model is taken as

$$x_1(0) = 1.28, \quad x_2(0) = 8.76, \quad x_3(0) = 9.87, \quad x_4(0) = 8.23.$$

Figure 3 shows the stability on the internal equilibrium points. From Figure 3, the adult state x_1 is stable at 1.3869, the egg state x_2 is stable at 0.4063, the larva state x_3 is stable at 0.2019 and the pupa state x_4 is stable at 0.0305.

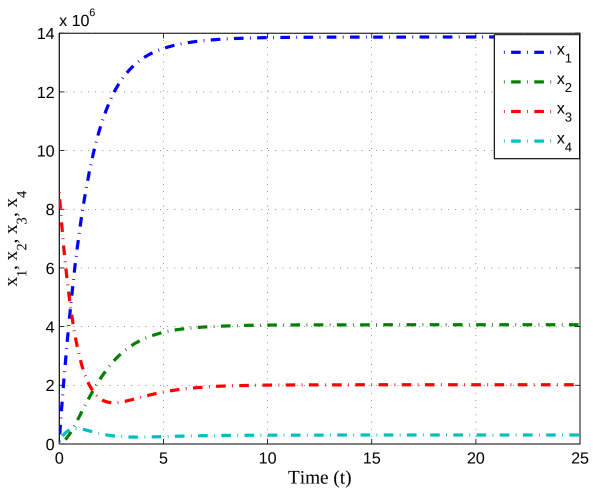

Figure 3. Stability at the internal equilibrium points

Second, the initial conditions of the model are taken as

$$x_1(0) = 86198, \quad x_2(0) = 27564, \quad x_3(0) = 8584367, \quad x_4(0) = 48975.$$

Figure 4 shows the stability on the internal equilibrium points. From the Figure 4, the adult state x_1 is stable at 1.3869, the egg state x_2 is stable at 0.4063, the larva state x_3 is stable at 0.2019 and the pupa state x_4 is stable at 0.0305.

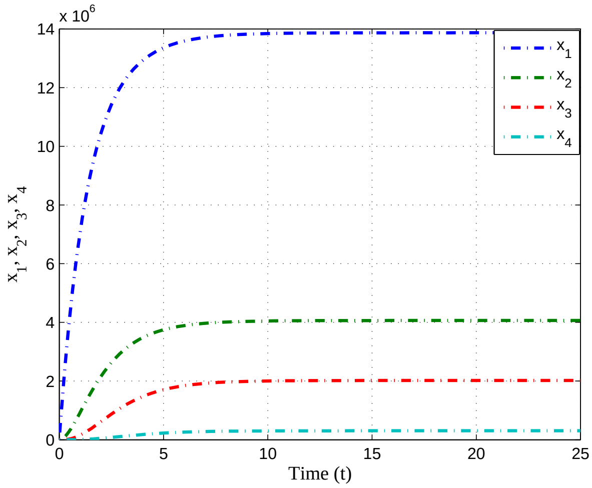

Figure 4. Stability on the internal equilibrium points

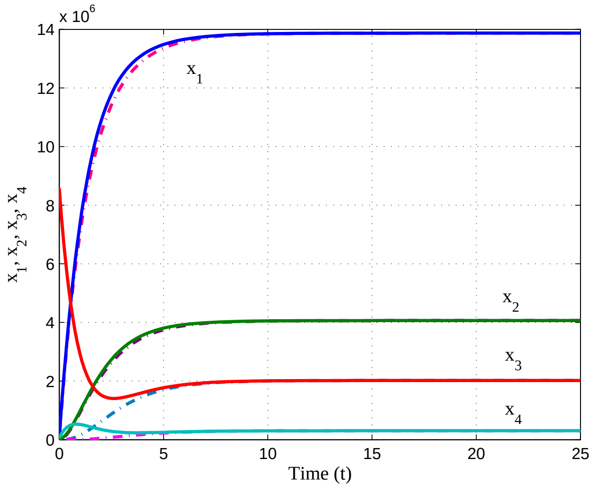

Figure 5. Sensitive dependance on initial conditions and internal equilibrium points

From the Figure 5, the Anopheles mosquito life cycle is stable at the internal equilibrium points, for this two different initial conditions were considered and the model is stable at the internal equilibrium points $x_1^*(n) = 1.3869,\ x_2^*(n) = 0.4063,\ x_3^*(n) = 0.2019,\ x_4^*(n) = 0.030.$

6. Conclusion

The Anopheles mosquito life cycle is modeled under the concept of difference equation. The stability of the model is estimated based on the Lyapunov conditions. The designing of the Lyapunov function is a new development in the difference equation concept. The strict feedback technique is also applied for a proposed mathematical model. Numerical results are furnished to supports the theory.

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INEQUALITIES PERTAINING TO RATIONAL FUNCTIONS WITH PRESCRIBED POLES

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Abstract: Let $\Re_n$ be the set of all rational functions of the type $r(z) = p(z)/w(z)$, where $p(z)$ is a polynomial of degree at most n and $w(z) = \prod_{j=1}^{n}(z - a_j)$, $|a_j| > 1$ for $1 \le j \le n$. In this paper, we set up some results for rational functions with fixed poles and restricted zeros. The obtained results bring forth generalizations and refinements of some known inequalities for rational functions and in turn produce generalizations and refinements of some polynomial inequalities as well.

Keywords: Rational functions, Polynomials, Inequalities.

1. Introduction

Let P_n denote the class of all complex polynomials of degree at most n. For $a_j \in \mathbb{C}$, $j = 1, 2, \ldots, n$, we write

$$w(z) := \prod_{j=1}^{n}(z - a_j), \quad B(z) := \prod_{j=1}^{n}\left(\frac{1 - \overline{a_j}z}{z - a_j}\right)$$

and

$$\Re_n := \Re_n(a_1, a_2, \ldots, a_n) = \left\{\frac{p(z)}{w(z)};\ p \in P_n\right\}.$$

Then $\Re_n$ is the set of all rational functions with poles $a_j, j = 1, 2, \ldots, n$ at most and with finite limit at infinity. It is clear that $B(z) \in \Re_n$ and $|B(z)| = 1$ for $|z| = 1$. Throughout this paper, we shall assume that all the poles a_j, $j = 1, 2, \ldots, n$ lie in $|z| > 1$.

If $p \in P_n$, then concerning the estimate of $|p'(z)|$ on the unit disk $|z| \le 1$, we have the following famous result known as Bernstein's inequality [3].

Theorem 1 [3]. *If $p \in P_n$, then*

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|$$

with equality only for $p(z) = \lambda z^n$, $\lambda \neq 0$ being a complex number.

For polynomials having all their zeros in $|z| \leq 1$, Turàn [14] proved

Theorem 2 [14]. *If $p \in P_n$ and $p(z)$ has all its zeros in $|z| \leq 1$, then*

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \max_{|z|=1} |p(z)| \tag{1.1}$$

with equality for those polynomials, which have all their zeros on $|z| = 1$.

In literature, there exists several generalizations and refinements of inequality (1.1) (see [10–12]). V.K. Jain [6] in 1997 introduced a parameter β and proved the following result which is an interesting generalization of inequality (1.1).

Theorem 3 [6]. *If $p \in P_n$ and $p(z)$ has all its zeros in $|z| \leq 1$, then for $|\beta| \leq 1$*

$$\max_{|z|=1} \left| zp'(z) + \frac{n\beta}{2} p(z) \right| \geq \frac{n}{2} \left\{ 1 + \text{Re}\,(\beta) \right\} \max_{|z|=1} |p(z)|. \tag{1.2}$$

By involving the coefficients of polynomial $p(z)$, Dubinin [4] refined inequality (1.1) and proved the following result.

Theorem 4 [4]. *If $p(z) = \sum_{j=0}^{n} \alpha_j z^j$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then*

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \left\{ 1 + \frac{1}{n} \left(\frac{|\alpha_n| - |\alpha_0|}{|\alpha_n| + |\alpha_0|} \right) \right\} \max_{|z|=1} |p(z)|.$$

As a generalization of Theorem 4, Rather et al. [9] proved the following result.

Theorem 5 [9]. *If $p(z) = \sum_{j=0}^{n} \alpha_j z^j$ is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \leq 1$, then for $|z| = 1$*

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+k} \left\{ 1 + \frac{k}{n} \left(\frac{k^n|\alpha_n| - |\alpha_0|}{k^n|\alpha_n| + |\alpha_0|} \right) \right\} \max_{|z|=1} |p(z)|. \tag{1.3}$$

Li, Mohapatra and Rodriguez [7] extended the inequality (1.1) to the rational functions $r \in \Re_n$ with prescribed poles and replace z^n by Blaschke product $B(z)$. Among other things they proved the following result.

Theorem 6 [7] . *Suppose $r \in \Re_n$, where r has exactly n poles at $a_1, a_2, \ldots, a_n$ and all the zeros of r lie in $|z| \leq 1$, then for $|z| = 1$*

$$|r'(z)| \geq \frac{1}{2} \Big\{ |B'(z)| - (n - m) \Big\} |r(z)|, \tag{1.4}$$

where m is the number of zeros of r.

As a generalization of inequality (1.4), Aziz and Shah [2] proved the following result.

Theorem 7 [2]. *Suppose $r \in \Re_n$, where r has exactly n poles at $a_1, a_2, \dots, a_n$ and all the zeros of r lie in $|z| \le k$, $k \le 1$, then for $|z| = 1$*

$$|r'(z)| \ge \frac{1}{2}\left\{|B'(z)| + \frac{2m - n(1+k)}{1+k}\right\}|r(z)|, \tag{1.5}$$

where m is the number of zeros of r.

Concerning the estimation of the lower bound of $\mathrm{Re}\big(zp'(z)/p(z)\big)$ on $|z| = 1$, Dubinin [4] proved the following result.

Theorem 8 [4] . *If $p(z) = \sum_{j=0}^{n} \alpha_j z^j$ is a polynomial of degree n which has all its zeros in $|z| \le 1$, then for all z on $|z| = 1$ for which $p(z) \ne 0$*

$$\mathrm{Re}\left(\frac{zp'(z)}{p(z)}\right) \ge \frac{n}{2}\left\{1 + \frac{1}{n}\left(\frac{|\alpha_n| - |\alpha_0|}{|\alpha_n| + |\alpha_0|}\right)\right\}.$$

Rather et al. [9] generalized Theorem 8 by proving the following result.

Theorem 9 [9]. *If $p(z) = \sum_{j=0}^{n} \alpha_j z^j$ is a polynomial of degree n and $p(z)$ has all its zeros in $|z| \le k, k \le 1$, then for all z on $|z| = 1$ for which $p(z) \ne 0$,*

$$\mathrm{Re}\left(\frac{zp'(z)}{p(z)}\right) \ge \frac{n}{1+k}\left\{1 + \frac{k}{n}\left(\frac{k^n|\alpha_n| - |\alpha_0|}{k^n|\alpha_n| + |\alpha_0|}\right)\right\}.$$

Concerning the estimation of the lower bound of $\mathrm{Re}\big(zr'(z)/r(z)\big)$ on $|z| = 1$, Dubinin [5] extended Theorem 8 to the rational functions and proved the following result.

Theorem 10 [5]. *Let r be a rational function of the form $r(z) = p(z)/w(z)$, where*

$$p(z) = \alpha_m z^m + \alpha_{m-1} z^{m-1} + \cdots + \alpha_1 z + \alpha_0, \quad \alpha_m \ne 0, \quad m \ge n$$

and the poles c_ν, $\nu = 1, 2, ..., n$ of r are arbitrary with $|c_\nu| \ne 1$ and let all the zeros of the function r lie in the disk $|z| \le 1$. Then, at points of the circle $|z| = 1$, other than the zeros of r, the following inequality holds

$$\mathrm{Re}\left\{\frac{zr'(z)}{r(z)}\right\} \ge \frac{1}{2}\left\{m - n + \frac{zB'(z)}{B(z)} + \frac{|\alpha_m| - |\alpha_0|}{|\alpha_m| + |\alpha_0|}\right\}. \tag{1.6}$$

For $m = n$ inequality (1.6) reduces to

$$\mathrm{Re}\left\{\frac{zr'(z)}{r(z)}\right\} \ge \frac{1}{2}\left\{\frac{zB'(z)}{B(z)} + \frac{|\alpha_m| - |\alpha_0|}{|\alpha_m| + |\alpha_0|}\right\}. \tag{1.7}$$

2. Main results

In this section, we first present the following result, which in particular furnishes a compact generalization of Theorem 10 for the case when all the poles of r lie outside the unit disk and as a consequence of this result, we get various generalizations and refinements of the above mentioned results. More precisely we prove.

Theorem 11. *Suppose $r \in \Re_n$, where r has exactly n poles and all the zeros of*

$$p(z) = \alpha_m z^m + \alpha_{m-1} z^{m-1} + \cdots + \alpha_1 z + \alpha_0, \quad \alpha_m \neq 0,$$

lie in $|z| \leq k$, $k \leq 1$. Then for all z on the circle $|z| = 1$, other than the zeros of r and $|\beta| \leq 1$

$$\begin{aligned} &\operatorname{Re}\left\{\frac{zr'(z)}{r(z)} + \frac{\beta}{1+k}|B'(z)|\right\} \\ &\geq \frac{1}{2}\left[\left\{1 + \frac{2\operatorname{Re}(\beta)}{1+k}\right\}|B'(z)| + \frac{2m - n(1+k)}{(1+k)} + \frac{2k}{1+k}\left\{\frac{k^m|\alpha_m| - |\alpha_0|}{k^m|\alpha_m| + |\alpha_0|}\right\}\right]. \end{aligned} \tag{2.1}$$

The result is best possible in the case $\beta = 0$, and equality holds for

$$r(z) = \frac{(z+k)^m}{(z-a)^n} \quad \text{and} \quad B(z) = \left(\frac{1-az}{z-a}\right)^n, \quad \text{at} \quad z = 1, \quad a > 1 \quad \text{and} \quad \beta = 0.$$

Remark 1. Taking $\beta = 0$, and using the fact that

$$|B'(z)| = \frac{zB'(z)}{B(z)}$$

on $|z| = 1$, inequality (2.1) reduces to the following inequality

$$\operatorname{Re}\left\{\frac{zr'(z)}{r(z)}\right\} \geq \frac{1}{2}\left[\frac{zB'(z)}{B(z)} + \frac{2m - n(1+k)}{(1+k)} + \frac{2k}{1+k}\left\{\frac{k^m|\alpha_m| - |\alpha_0|}{k^m|\alpha_m| + |\alpha_0|}\right\}\right]. \tag{2.2}$$

One can easily note that for $\beta = 0$, Theorem 11 is an extension of Theorem 9 to the rational functions. On the other hand if we take $k = 1$ and $m = n$ in inequality (2.2), we shall obtain inequality (1.7).

Remark 2. Now for the points on the circle $|z| = 1$, other than the zeros of r and $|\beta| \leq 1$, one can easily prove that

$$\left|\frac{zr'(z)}{r(z)} + \frac{\beta}{1+k}|B'(z)|\right| \geq \operatorname{Re}\left\{\frac{zr'(z)}{r(z)} + \frac{\beta}{1+k}|B'(z)|\right\}.$$

In view of this, Theorem 11 reduces to the following result, which contributes a generalization and refinement of inequality (1.5).

Corollary 1. *Suppose $r \in \Re_n$, where r has exactly n poles and all the zeros of r lie in $|z| \leq k$, $k \leq 1$, that is $r(z) = p(z)/w(z)$ with*

$$p(z) = \alpha_m z^m + \alpha_{m-1} z^{m-1} + \cdots + \alpha_1 z + \alpha_0, \quad \alpha_m \neq 0.$$

Then for all z on $|z| = 1$ other than the zeros of r and $|\beta| \leq 1$

$$\left|\frac{zr'(z)}{r(z)} + \frac{\beta}{1+k}|B'(z)|\right| \geq \frac{1}{2}\left[\left\{1 + \frac{2\operatorname{Re}(\beta)}{1+k}\right\}|B'(z)| + \frac{2m - n(1+k)}{(1+k)} + \frac{2k}{1+k}\left\{\frac{k^m|\alpha_m| - |\alpha_0|}{k^m|\alpha_m| + |\alpha_0|}\right\}\right].$$

The result is best possible in the case $\beta = 0$, and equality holds for

$$r(z) = \frac{(z+k)^m}{(z-a)^n} \quad \text{and} \quad B(z) = \left(\frac{1-az}{z-a}\right)^n, \quad \text{at} \quad z = 1, \quad a > 1 \quad \text{and} \quad \beta = 0.$$

Remark 3. For $k = 1$, Corollary 1 reduces to the following result, which yields a generalization as well as refinement of inequality (1.4).

Corollary 2. *Suppose $r \in \Re_n$, where r has exactly n poles and all the zeros of r lie in $|z| \leq 1$, that is $r(z) = p(z)/w(z)$ with*

$$p(z) = \alpha_m z^m + \alpha_{m-1} z^{m-1} + \cdots + \alpha_1 z + \alpha_0, \quad \alpha_m \neq 0.$$

Then for all z on $|z| = 1$ other than the zeros of r and $|\beta| \leq 1$

$$\left|\frac{zr'(z)}{r(z)} + \frac{\beta}{2}|B'(z)|\right| \geq \frac{1}{2}\left[\{1 + \mathrm{Re}\,(\beta)\}\,|B'(z)| - (n-m) + \left\{\frac{|\alpha_m| - |\alpha_0|}{|\alpha_m| + |\alpha_0|}\right\}\right]. \tag{2.3}$$

Inequality (2.3) is sharp in the case $\beta = 0$ and equality holds for

$$r(z) = \frac{(z+1)^m}{(z-a)^n} \quad \text{and} \quad B(z) = \left(\frac{1-az}{z-a}\right)^n, \quad \text{at} \quad z = 1, \quad a > 1 \quad \text{and} \quad \beta = 0.$$

Remark 4. Taking $w(z) = (z-\alpha)^n$, $|\alpha| > 1$, so that

$$B(z) = \left(\frac{1-\alpha z}{z-\alpha}\right)^n$$

with $m = n$ in Corollary 1, we get

$$\begin{aligned} &\left|z\left(\frac{p'(z)}{p(z)} + \frac{n}{z-\alpha}\right) + \frac{\beta}{1+k}|B'(z)|\right| \\ &\geq \frac{1}{2}\left[\left\{1 + \frac{2\mathrm{Re}\,(\beta)}{1+k}\right\}|B'(z)| + \frac{n(1-k)}{1+k} + \frac{2k}{1+k}\left(\frac{k^n|\alpha_n| - |\alpha_0|}{k^n|\alpha_n| + |\alpha_0|}\right)\right]. \end{aligned} \tag{2.4}$$

Letting $|\alpha| \to \infty$ in inequality (2.4) and noting that $|B'(z)| \to n|z|^{n-1} = n$ for $|z| = 1$, we get the following result.

Corollary 3. *If $p(z) = \sum_{j=0}^{n} \alpha_j z^j$ is a polynomial of degree n, having all its zeros in $|z| \leq k$, $k \leq 1$, then for $|\beta| \leq 1$ and $|z| = 1$*

$$\left|zp'(z) + \frac{n\beta}{1+k}p(z)\right| \geq \frac{n}{1+k}\left\{1 + \mathrm{Re}(\beta) + \frac{k}{n}\left(\frac{k^n|\alpha_n| - |\alpha_0|}{k^n|\alpha_n| + |\alpha_0|}\right)\right\}|p(z)|. \tag{2.5}$$

Since $k^n|\alpha_n| \geq |\alpha_0|$, therefore Corollary 3 refines as well as generalizes the well known polynomial inequality (1.2) due to Jain [6].

Remark 5. For $\beta = 0$, inequality (2.5) reduces to inequality (1.3).

Next, we prove the following refinement of Corollary 3.

Theorem 12. *If $p(z) = \sum_{j=0}^{n} \alpha_j z^j$ is a polynomial of degree n, having all its zeros in $|z| \leq k$, $k \leq 1$, then for $|\beta| \leq 1$ and $|z| = 1$*

$$\begin{aligned} \left|zp'(z) + \frac{n\beta}{1+k}p(z)\right| \geq{}& \frac{n}{1+k}\left\{1 + \mathrm{Re}\,(\beta) + \frac{k}{n}\left(\frac{k^n|\alpha_n| - m^* - |\alpha_0|}{k^n|\alpha_n| + m^* + |\alpha_0|}\right)\right\}|p(z)| \\ &+ \frac{nm^*}{1+k}\left\{\left|1 + \mathrm{Re}\,(\beta) + \frac{k}{n}\left(\frac{k^n|\alpha_n| - m^* - |\alpha_0|}{k^n|\alpha_n| + m^* + |\alpha_0|}\right) - |\beta|\right|\right\}, \end{aligned}$$

where $m^* = \min_{|z|=k} |p(z)|$.

Taking $\beta = 0$ in Theorem 12, we get the following result.

Corollary 4. *If $p(z) = \sum_{j=0}^{n} \alpha_j z^j$ is a polynomial of degree n, having all its zeros in $|z| \leq k$, $k \leq 1$, then for $|z| = 1$*

$$|p'(z)| \geq \frac{n}{1+k}\left\{1+\frac{k}{n}\left(\frac{k^n|\alpha_n|-m^*-|\alpha_0|}{k^n|\alpha_n|+m^*+|\alpha_0|}\right)\right\}|p(z)| + \frac{nm^*}{1+k}\left\{\left|1+\frac{k}{n}\left(\frac{k^n|\alpha_n|-m^*-|\alpha_0|}{k^n|\alpha_n|+m^*+|\alpha_0|}\right)\right|\right\},$$

where $m^* = \min_{|z|=k} |p(z)|$.

Remark 6. Since $m^* \geq 0$, hence Corollary 4 is a refinement Theorem 5.

3. Lemmas

For the proof of our results, we need the following lemmas. The first lemma is due to A. Aziz and B.A. Zargar [1].

Lemma 1 [1]. *If $|z| = 1$, then*

$$\mathrm{Re}\left(\frac{zw'(z)}{w(z)}\right) = \frac{n-|B'(z)|}{2},$$

where $w(z) = \prod_{j=1}^{n}(z-a_j)$.

The following lemma is due to Rather et al. [9].

Lemma 2 [9]. *If $\langle\zeta_j\rangle_{j=1}^{m}$ be a finite collection of real numbers such that $0 \leq \zeta_j \leq 1$, $j = 1, 2, \dots, m$, then*

$$\sum_{j=1}^{m}\frac{1-\zeta_j}{1+\zeta_j} \geq \frac{1-\prod_{j=1}^{m}\zeta_j}{1+\prod_{j=1}^{m}\zeta_j}.$$

The next lemma is due to Mezerji et al. [13].

Lemma 3 [13]. *If $p(z)$ is a polynomial of degree n, having all its zeros in $|z| \leq k$, $k \leq 1$, then for any β with $|\beta| \leq 1$,*

$$\min_{|z|=1}\left|zp'(z)+\frac{n\beta}{1+k}p(z)\right| \geq \frac{nm^*}{k^n}\left|1+\frac{\beta}{1+k}\right|,$$

where $m^* = \min_{|z|=k} |p(z)|$.

4. Proof of Theorem 11

P r o o f. Since $r \in \Re_n$ and all the zeros of $r(z)$ lie in $|z| \leq k$, $k \leq 1$, that is $r(z) = p(z)/w(z)$ with

$$p(z) = \alpha_m \prod_{j=1}^{m}(z-b_j) = \alpha_m z^m + \alpha_{m-1}z^{m-1} + \cdots + \alpha_1 z + \alpha_0,$$
$$\alpha_m \neq 0, \quad |b_j| \leq k \leq 1, \quad j = 1, 2, 3, ..., m.$$

Then for $|\beta| \leq 1$ and for all z on $|z| = 1$, where $r(z) \neq 0$, we have

$$\begin{aligned}\operatorname{Re}\left\{\frac{zr'(z)}{r(z)} + \frac{\beta}{1+k}|B'(z)|\right\} &= \operatorname{Re}\left\{\frac{zr'(z)}{r(z)}\right\} + \frac{|B'(z)|}{1+k}\operatorname{Re}\{\beta\} \\ &= \operatorname{Re}\left\{\frac{zp'(z)}{p(z)} - \frac{zw'(z)}{w(z)}\right\} + \frac{|B'(z)|}{1+k}\operatorname{Re}\{\beta\} \\ &= \operatorname{Re}\left\{\frac{zp'(z)}{p(z)}\right\} - \operatorname{Re}\left\{\frac{zw'(z)}{w(z)}\right\} + \frac{|B'(z)|}{1+k}\operatorname{Re}\{\beta\}.\end{aligned}$$

Using Lemma 1, we have for $|\beta| \leq 1$ and for all z on $|z| = 1$, where $r(z) \neq 0$,

$$\begin{aligned}\operatorname{Re}\left\{\frac{zr'(z)}{r(z)} + \frac{\beta}{1+k}|B'(z)|\right\} &= \operatorname{Re}\sum_{j=1}^{m}\left\{\frac{z}{z-b_j}\right\} - \left\{\frac{n-|B'(z)|}{2}\right\} + \frac{|B'(z)|}{1+k}\operatorname{Re}\{\beta\} \\ &= \sum_{j=1}^{m}\operatorname{Re}\left\{\frac{z}{z-b_j}\right\} - \frac{n}{2} + \frac{1}{2}\left\{1 + \frac{2\operatorname{Re}(\beta)}{1+k}\right\}|B'(z)|.\end{aligned} \tag{4.1}$$

Now it can be easily verified that for $|z| = 1$ and $|b_j| \leq k \leq 1$, we have

$$\operatorname{Re}\left\{\frac{z}{z-b_j}\right\} \geq \left\{\frac{1}{1+|b_j|}\right\}.$$

Using this in inequality (4.1), we get for $|\beta| \leq 1$ and for all z on $|z| = 1$, where $r(z) \neq 0$,

$$\begin{aligned}\operatorname{Re}\left\{\frac{zr'(z)}{r(z)} + \frac{\beta}{1+k}|B'(z)|\right\} &\geq \sum_{j=1}^{m}\left\{\frac{1}{1+|b_j|}\right\} - \frac{n}{2} + \frac{1}{2}\left\{1 + \frac{2\operatorname{Re}(\beta)}{1+k}\right\}|B'(z)| \\ &= \frac{1}{2}\left\{1 + \frac{2\operatorname{Re}(\beta)}{1+k}\right\}|B'(z)| + \sum_{j=1}^{m}\left\{\frac{1}{1+|b_j|} - \frac{1}{1+k}\right\} + \frac{m}{1+k} - \frac{n}{2} \\ &= \frac{1}{2}\left\{1 + \frac{2\operatorname{Re}(\beta)}{1+k}\right\}|B'(z)| + \frac{2m-n(1+k)}{2(1+k)} + \frac{k}{1+k}\sum_{j=1}^{m}\left\{\frac{k-|b_j|}{k+k|b_j|}\right\} \\ &\geq \frac{1}{2}\left\{1 + \frac{2\operatorname{Re}(\beta)}{1+k}\right\}|B'(z)| + \frac{2m-n(1+k)}{2(1+k)} + \frac{k}{1+k}\sum_{j=1}^{m}\left\{\frac{k-|b_j|}{k+|b_j|}\right\} \\ &= \frac{1}{2}\left\{1 + \frac{2\operatorname{Re}(\beta)}{1+k}\right\}|B'(z)| + \frac{2m-n(1+k)}{2(1+k)} + \frac{k}{1+k}\sum_{j=1}^{m}\left\{\frac{1-|b_j|/k}{1+|b_j|/k}\right\}.\end{aligned} \tag{4.2}$$

Since $|b_j|/k \leq 1$, therefore by invoking Lemma 2, we conclude from inequality (4.2) that for $|\beta| \leq 1$ and for all z on $|z| = 1$, where $r(z) \neq 0$,

$$\begin{aligned}&\operatorname{Re}\left\{\frac{zr'(z)}{r(z)} + \frac{\beta}{1+k}|B'(z)|\right\} \\ &\geq \frac{1}{2}\left\{1 + \frac{2\operatorname{Re}(\beta)}{1+k}\right\}|B'(z)| + \frac{2m-n(1+k)}{2(1+k)} + \frac{k}{1+k}\left\{\frac{1-\prod_{j=1}^{m} c|b_j|/k}{1+\prod_{j=1}^{m}|b_j|/k}\right\} \\ &= \frac{1}{2}\left\{1 + \frac{2\operatorname{Re}(\beta)}{1+k}\right\}|B'(z)| + \frac{2m-n(1+k)}{2(1+k)} + \frac{k}{1+k}\left\{\frac{k^m|\alpha_m| - |\alpha_0|}{k^m|\alpha_m| + |\alpha_0|}\right\}.\end{aligned}$$

□

5. Proof of Theorem 12

P r o o f. If $p(z)$ has a zero on $|z| = k$, then the result follows from Corollary 3. We assume that all the zeros of $p(z)$ lie in $|z| < k$, $k \leq 1$, so that $m^* > 0$ and we have $m^* \leq |p(z)|$ for $|z| = k$. By Rouche's theorem for every λ with $|\lambda| < 1$, the polynomial $h(z) = p(z) - \lambda m^*$ has all its zeros in $|z| < k$, $k \leq 1$. Applying Corollary 3 to the polynomial $h(z)$, we get for $\lambda, \beta \in \mathbb{C}$ with $|\lambda| < 1$, $|\beta| \leq 1$ and $|z| = 1$,

$$\begin{aligned}&\left|zp'(z) + \frac{n\beta}{1+k}\{p(z) - \lambda m^*\}\right| \\ &\geq \frac{n}{2}\left\{1 + \frac{2\mathrm{Re}(\beta)}{1+k} + \frac{1-k}{1+k} + \frac{2k}{n(1+k)}\left(\frac{k^n|\alpha_n| - |\lambda m^* - \alpha_0|}{k^n|\alpha_n| + |\lambda m^* - \alpha_0|}\right)\right\}|p(z) - \lambda m^*|.\end{aligned}$$

or

$$\begin{aligned}&\left|zp'(z) + \frac{n\beta}{1+k}p(z) - \frac{n\beta}{1+k}\lambda m^*\right| \\ &\geq \frac{n}{2}\left\{1 + \frac{2\mathrm{Re}(\beta)}{1+k} + \frac{1-k}{1+k} + \frac{2k}{n(1+k)}\left(\frac{k^n|\alpha_n| - |\lambda|m^* - |\alpha_0|}{k^n|\alpha_n| + |\lambda|m^* + |\alpha_0|}\right)\right\}|p(z) - \lambda m^*|.\end{aligned} \tag{5.1}$$

Now for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1$ and $k > 0$,

$$k|\beta| \leq |1 + k + \beta|.$$

or,

$$\left|1 + \frac{\beta}{1+k}\right| \geq \frac{k}{1+k}|\beta|, \quad \text{for} \quad |\beta| \leq 1 \quad \text{and} \quad k > 0.$$

Using this in Lemma 3, we have for $|z| = 1$, $|\beta| \leq 1$ and $k \leq 1$,

$$\left|zp'(z) + \frac{n\beta}{1+k}p(z)\right| \geq \frac{nm^*}{k^n}\left|1 + \frac{\beta}{1+k}\right| \geq \frac{nm^*}{k^{n-1}}\frac{|\beta|}{1+k} \geq \left|\frac{n\beta}{1+k}\lambda m^*\right| \quad \text{for} \quad |\lambda| < 1.$$

In view of this, choosing argument of λ in left hand side of (5.1) such that

$$\left|zp'(z) + \frac{n\beta}{1+k}p(z) - \frac{n\beta}{1+k}\lambda m^*\right| = \left|zp'(z) + \frac{n\beta}{1+k}p(z)\right| - \frac{n|\beta|}{1+k}|\lambda|m^*,$$

we obtain from inequality (5.1), for $|\beta| \leq 1$ and $|z| = 1$,

$$\begin{aligned}&\left|zp'(z) + \frac{n\beta}{1+k}p(z)\right| - \frac{n|\beta|}{1+k}|\lambda|m^* \\ &\geq \frac{n}{2}\left\{1 + \frac{2\mathrm{Re}(\beta)}{1+k} + \frac{1-k}{1+k} + \frac{2k}{n(1+k)}\left(\frac{k^n|\alpha_n| - |\lambda|m^* - |\alpha_0|}{k^n|\alpha_n| + |\lambda|m^* + |\alpha_0|}\right)\right\}\{|p(z)| - |\lambda|m^*\}.\end{aligned}$$

or

$$\begin{aligned}&\left|zp'(z) + \frac{n\beta}{1+k}p(z)\right| \\ &\geq \frac{n}{2}\left\{1 + \frac{2\mathrm{Re}(\beta)}{1+k} + \frac{1-k}{1+k} + \frac{2k}{n(1+k)}\left(\frac{k^n|\alpha_n| - |\lambda|m^* - |\alpha_0|}{k^n|\alpha_n| + |\lambda|m^* + |\alpha_0|}\right)\right\}|p(z)| \\ &+\frac{nm^*}{2}|\lambda|\left[\frac{2|\beta|}{1+k} - \left\{1 + \frac{2\mathrm{Re}(\beta)}{1+k} + \frac{1-k}{1+k} + \frac{2k}{n(1+k)}\left(\frac{k^n|\alpha_n| - |\lambda|m^* - |\alpha_0|}{k^n|\alpha_n| + |\lambda|m^* + |\alpha_0|}\right)\right\}\right].\end{aligned} \tag{5.2}$$

Again by inequality (5.1), we have for $|\lambda| < 1$, $|\beta| \leq 1$ and $|z| = 1$,

$$\begin{aligned} &\left|zp'(z) + \frac{n\beta}{1+k}p(z)\right| + \left|\frac{n\beta}{1+k}\lambda m^*\right| \\ \geq \frac{n}{2}&\left\{1 + \frac{2\text{Re}(\beta)}{1+k} + \frac{1-k}{1+k} + \frac{2k}{n(1+k)}\left(\frac{k^n|\alpha_n| - |\lambda|m^* - |\alpha_0|}{k^n|\alpha_n| + |\lambda|m^* + |\alpha_0|}\right)\right\}\{|p(z)| + |\lambda|m^*\}. \end{aligned}$$

or

$$\begin{aligned} &\left|zp'(z) + \frac{n\beta}{1+k}p(z)\right| \\ &\geq \frac{n}{2}\left\{1 + \frac{2\text{Re}(\beta)}{1+k} + \frac{1-k}{1+k} + \frac{2k}{n(1+k)}\left(\frac{k^n|\alpha_n| - |\lambda|m^* - |\alpha_0|}{k^n|\alpha_n| + |\lambda|m^* + |\alpha_0|}\right)\right\}|p(z)| \\ &+\frac{nm^*}{2}|\lambda|\left\{1 + \frac{2\text{Re}(\beta)}{1+k} + \frac{1-k}{1+k} + \frac{2k}{n(1+k)}\left(\frac{k^n|\alpha_n| - |\lambda|m^* - |\alpha_0|}{k^n|\alpha_n| + |\lambda|m^* + |\alpha_0|}\right) - \frac{2|\beta|}{1+k}\right\}. \end{aligned} \tag{5.3}$$

Now from inequality (5.2) and inequality (5.3), we get for $|\beta| \leq 1$ and $|z| = 1$,

$$\begin{aligned} \left|zp'(z) + \frac{n\beta}{1+k}p(z)\right| \geq \frac{n}{2}&\left\{1 + \frac{2\text{Re}(\beta)}{1+k} + \frac{1-k}{1+k} + \frac{2k}{n(1+k)}\left(\frac{k^n|\alpha_n| - |\lambda|m^* - |\alpha_0|}{k^n|\alpha_n| + |\lambda|m^* + |\alpha_0|}\right)\right\}|p(z)| \\ +\frac{nm^*}{2}|\lambda|&\left\{\left|1 + \frac{2\text{Re}(\beta)}{1+k} + \frac{1-k}{1+k} + \frac{2k}{n(1+k)}\left(\frac{k^n|\alpha_n| - |\lambda|m^* - |\alpha_0|}{k^n|\alpha_n| + |\lambda|m^* + |\alpha_0|}\right) - \frac{2|\beta|}{1+k}\right|\right\}. \end{aligned}$$

Letting $|\lambda| \to 1$, we obtain for $|z| = 1$,

$$\begin{aligned} \left|zp'(z) + \frac{n\beta}{1+k}p(z)\right| &\geq \frac{n}{1+k}\left\{1 + \text{Re}(\beta) + \frac{k}{n}\left(\frac{k^n|\alpha_n| - m^* - |\alpha_0|}{k^n|\alpha_n| + m^* + |\alpha_0|}\right)\right\}|p(z)| \\ &+\frac{nm^*}{1+k}\left\{\left|1 + \text{Re}(\beta) + \frac{k}{n}\left(\frac{k^n|\alpha_n| - m^* - |\alpha_0|}{k^n|\alpha_n| + m^* + |\alpha_0|}\right) - |\beta|\right|\right\}, \end{aligned}$$

which proves Theorem 12. □

6. A remark on a recent result concerning rational functions

Recently Idrees Qasim [8] claimed to have proved various results regarding Bernstein-type inequalities for rational functions with prescribed poles and restricted zeros. Among other things he claimed to have proved the following result.

Theorem 13 [8]. *If $r(z) = p(z)/w(z) \in \Re_n$, where $p(z) = \sum_{j=0}^{n} \alpha_j z^j$, $|b|.|\alpha_n| \leq |\alpha_0|$, r has exactly n poles at $a_1, a_2, \ldots, a_n$, and $r(z) \neq 0$ in $|z| > 1$, then for $|z| = 1$,*

$$|r'(z)| \geq \frac{1}{2}\left[|B'(z)| + \frac{\sqrt{|\alpha_n|} - \sqrt{|\alpha_0|}}{\sqrt{|\alpha_n|}}\right](|r(z)| + m^{**}),$$

where $m^{**} = \min_{|z|=1}|r(z)|$ and $b = a_1 a_2 \ldots a_n$.

Since it is assumed throughout the paper that all the poles $(a_1, a_2, ..., a_n)$ of rational function r lie outside unit disk, therefore,

$$|b| = |a_1 \times a_2 \times ... \times a_n| > 1. \tag{6.1}$$

On the other hand, it is also assumed that all the zeros $(z_1, z_2, ..., z_n)$ of r lie in the disc $|z| \leq 1$, implies

$$\frac{|\alpha_0|}{|\alpha_n|} = |z_1 \times z_2 \times ... \times z_n| \leq 1. \tag{6.2}$$

From (6.1) and (6.2), it follows that $|b|.|\alpha_n| > |\alpha_0|$, which is contrary to the hypothesis $|b|.|\alpha_n| \leq |\alpha_0|$ given in the statement of the Theorem 13. Hence the statement of the Theorem 13 is self-contradicting, as such Theorem 13 and its consequences are never applicable.

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A QUADRUPLE INTEGRAL INVOLVING THE EXPONENTIAL LOGARITHM OF QUOTIENT RADICALS IN TERMS OF THE HURWITZ-LERCH ZETA FUNCTION[1]

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Abstract: With a possible connection to integrals used in General Relativity, we used our contour integral method to write a closed form solution for a quadruple integral involving exponential functions and logarithm of quotient radicals. Almost all Hurwitz–Lerch Zeta functions have an asymmetrical zero-distribution. All the results in this work are new.

Keywords: Quadruple integral, Hurwitz–Lerch Zeta function, Catalan's constant, Cauchy integral, Glaisher's constant.

1. Significance statement

Quadruple integrals are broadly utilized in a wide number of disciplines crossing math, science and engineering. Some interesting areas where these integrals are used are in the three-body problem and the equations of dynamics [12], integral solutions to the wave equation [9], path integrals in polymer physics [5], analytical evaluations of double integral expressions related to total variation [6], electrodynamics of moving media [8], and measurements in heat transfer [2].

The authors discovered various uses of quadruple integrals after reviewing the present literature. In some cases these integrals were separable and in some cases asymptotic expansions were used to attain a solution.The authors were unable to uncover quadruple integrals involving exponential functions and the logarithm of quotient radicals generated in terms of a closed form solution. This integral features a kernel with the product of the exponential logarithm of quotient radical functions. The log term mixes the variables so that the integral is not separable except for special values of k.

The book by Prudnikov et al. [13], is structured towards mathematicians, physicists, experts in calculus methods, instructors, graduate students, for all those concerned with integrals, higher transcendental functions and integral transforms and those keen to master the corresponding theories. This book is also of help when dealing with the modern theory of higher functions and integral transformations accessible to undergraduate and graduate students [3].

This famous book contains a vast quantity of mathematical formulae. These formulae are indefinite integrals, definite integrals, multidimensional integrals, finite and infinite sums and and multidimensional finite and infinite sums. In the book of Prudnikov et al. [13] there is a combination of integral examples expressed in terms of fundamental constants and Special functions. Since these types of integral formulae are of such high importance in science, it has encouraged us to contribute to such tables by adding definite quadruple integrals in terms of the Hurwitz–Lerch Zeta function.

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This work represents an illustration of a general approach using contour integration applied to a particular integral in the book of Prudnikov et al. [13].

2. Preliminaries

We proceed by using the contour integral method [14] and the reflection formula for the gamma function given by equation (5.5.3) in [10], applied to equation (3.1.3.9) in [13] to yield the Prudnikov quadruple contour integral representation given by:

$$\int_{\mathbb{R}_+^4} a^w w^{-k-1}(rs)^{(-m-w)/2-1}(r+s)^{(m+w+1)/2}(xy)^{(m+w)/2}(x+y)^{(-m-w-1)/2}e^{-p(r+x)-q(s+y)}dxdydrds$$
$$= -\frac{1}{2\pi i}\int_C \frac{\pi^2 a^w w^{-k-1}\csc\left(\pi(m+w)/2\right)}{pq}dw$$

where $a,k,w,m,p,q \in \mathbb{C}$, $\mathrm{Re}(m+w)>0$, $-1<\mathrm{Re}(m)<0$.

3. Introduction

In this paper the main theorem derived is the quadruple definite integral given by

$$\int_{\mathbb{R}_+^4}(rs)^{-m/2-1}(r+s)^{(m+1)/2}(xy)^{m/2}(x+y)^{(-m-1)/2}e^{-p(r+x)-q(s+y)}\log^k\left(\frac{a\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)dxdydrds$$
$$= \frac{2i\pi^{k+2}e^{i\pi(k+m)/2}\Phi\left(e^{im\pi},-k,1/2-i\log(a)/\pi\right)}{pq},$$

where the parameters k,a,p,q and m are general complex numbers. This integral is derived in terms of the Hurwitz–Lerch Zeta function which is a useful special function. This is a function of three complex variables which is extended by analytic continuation to the complex plane with the exception of a singularity at 1 and a branch cut between one to infinity. The Lerch function is a generalization of several important special functions namely, the geometric series, the natural logarithm, powers and exponentials, polylogarithms, the Riemann zeta function, the alternating Riemann zeta function, and the Hurwitz zeta function. One advantage of the approach in this current work is that it reveals the connection between quadruple integral formulae and classical mathematical functions.

This definite integral will be used to derive special cases in terms of special functions and fundamental constants and we summarize most of the evaluations in Table 7 for easy reading. The derivations follow the method used by us in [14]. This method involves using a form of the generalized Cauchy's integral formula given by

$$\frac{y^k}{\Gamma(k+1)} = \frac{1}{2\pi i}\int_C \frac{e^{wy}}{w^{k+1}}dw, \tag{3.1}$$

where C is in general an open contour in the complex plane where the bilinear concomitant has the same value at the end points of the contour. We then multiply both sides by a function of x, y, z and t, then take a definite quadruple integral of both sides. This yields a definite integral in terms of a contour integral. Then we multiply both sides of equation (3.1) by another function of x, y, r and s and take the infinite sums of both sides such that the contour integral of both equations are the same.

4. Definite integral of the contour integral

We use the method in [14]. The variable of integration in the contour integral is $\alpha = w + m$. The cut and contour are in the first quadrant of the complex α-plane. The cut approaches the origin from the interior of the first quadrant and the contour goes round the origin with zero radius and is on opposite sides of the cut. Using a generalization of Cauchy's integral formula we form the quadruple integral by replacing y by

$$\log\left(\frac{a\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)$$

and multiplying by

$$(rs)^{-m/2-1}(r+s)^{(m+1)/2}(xy)^{m/2}(x+y)^{(-m-1)/2}e^{-p(r+x)-q(s+y)}$$

then taking the definite integral with respect to $x \in [0,\infty)$, $y \in [0,\infty)$, $r \in [0,\infty)$ and $s \in [0,\infty)$ to obtain

$$\begin{aligned}
&\frac{1}{\Gamma(k+1)}\int_{\mathbb{R}_+^4}(rs)^{-m/2-1}(r+s)^{(m+1)/2}(xy)^{m/2}(x+y)^{(-m-1)/2}e^{-p(r+x)-q(s+y)}\\
&\qquad\times\log^k\left(\frac{a\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)dxdydrds\\
&=\frac{1}{2\pi i}\int_{\mathbb{R}_+^4}\int_C a^w w^{-k-1}(rs)^{(-m-w)/2-1}(r+s)^{(m+w+1)/2}(xy)^{(m+w)/2}(x+y)^{(-m-w-1)/2}\\
&\qquad\times e^{-p(r+x)-q(s+y)}dwdxdydrds\\
&=\frac{1}{2\pi i}\int_C\int_{\mathbb{R}_+^4} a^w w^{-k-1}(rs)^{(-m-w)/2-1}(r+s)^{(m+w+1)/2}(xy)^{(m+w)/2}(x+y)^{(-m-w-1)/2}\\
&\qquad\times e^{-p(r+x)-q(s+y)}dxdydrdsdw\\
&=-\frac{1}{2\pi i}\int_C\frac{\pi^2 a^w w^{-k-1}\csc(\pi(m+w)/2)}{pq}dw
\end{aligned} \tag{4.2}$$

from equation (3.1.3.9) in [13] where

$$\mathrm{Re}(w+m) > 0, \quad \mathrm{Re}(p) > 0, \quad \mathrm{Re}(q) > 0, \quad -1 < \mathrm{Re}(m) < 0$$

and using the reflection formula (8.334.3) in [4] for the Gamma function. We are able to switch the order of integration over α, x, y, r and s using Fubini's theorem since the integrand is of bounded measure over the space $\mathbb{C} \times [0,\infty) \times [0,\infty) \times [0,\infty) \times [0,\infty)$.

5. The Hurwitz–Lerch zeta function and infinite sum of the contour integral

5.1. The Hurwitz–Lerch zeta function

The Hurwitz–Lerch Zeta function (see Section 1.11 in [1]) has a series representation given by

$$\Phi(z,s,v) = \sum_{n=0}^{\infty}(v+n)^{-s}z^n,$$

where $|z| < 1, v \neq 0, -1, ...$ and is continued analytically by its integral representation given by

$$\Phi(z,s,v) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-vt}}{1-ze^{-t}}dt = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-(v-1)t}}{e^t - z}dt$$

where $\mathrm{Re}(v) > 0$, and either $|z| \leq 1$, $z \neq 1$, $\mathrm{Re}(s) > 0$, or $z = 1$, $\mathrm{Re}(s) > 1$.

5.2. Derivation of the contour integral

Using equation (3.1) and replacing y by

$$\log(a)+\frac{1}{2}i\pi(2y+1)$$

then multiplying both sides by

$$\frac{2i\pi^2 e^{i\pi m(2y+1)/2}}{pq}$$

taking the infinite sum over $y\in[0,\infty)$ and simplifying in terms of the Hurwitz–Lerch Zeta function we obtain

$$\begin{aligned}\frac{2i\pi^{k+2}e^{i\pi(k+m)/2}\Phi\left(e^{im\pi},-k,1/2-i\log(a)/\pi\right)}{pq\Gamma(k+1)}&=\frac{1}{2\pi i}\sum_{y=0}^{\infty}\int_C\frac{2i\pi^2a^w w^{-k-1}e^{i\pi(2y+1)(m+w)/2}}{pq}dw\\ =\frac{1}{2\pi i}\int_C\sum_{y=0}^{\infty}\frac{2i\pi^2a^w w^{-k-1}e^{i\pi(2y+1)(m+w)/2}}{pq}dw&=-\frac{1}{2\pi i}\int_C\frac{\pi^2a^w w^{-k-1}\csc\left(\pi(m+w)/2\right)}{pq}dw\end{aligned}\tag{5.1}$$

from equation (1.232.2) in [4] where $\mathrm{Im}(w+m)>0$ in order for the sum to converge.

Theorem 1. *For $k,a,p,q,m\in\mathbb{C}$,*

$$\begin{aligned}&\int_{\mathbb{R}_+^4}(rs)^{-m/2-1}(r+s)^{(m+1)/2}(xy)^{m/2}(x+y)^{(-m-1)/2}e^{-p(r+x)-q(s+y)}\\ &\qquad\times\log^k\left(\frac{a\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)dxdydrds\\ &=\frac{2i\pi^{k+2}e^{i\pi(k+m)/2}\Phi\left(e^{im\pi},-k,1/2-i\log(a)/\pi\right)}{pq}.\end{aligned}\tag{5.2}$$

P r o o f. Observe the right-hand sides of (4.2) and (5.1) are the same so we can simplify the gamma function and equate the left-hand sides to yield the stated result. □

6. Main results

In the proceeding section we will evaluate equation (5.2) in terms of special functions and fundamental constants, Hurwitz zeta function $\zeta(s,a)$, given in Section 25.11 in [10], Catalan's constant C, given by equation (25.11.40) in [10], Riemann zeta function $\zeta(s)$, given in Section 25.2 in [10], Glaisher's constant A, given by equation (5.17.6) in [10] and equation (2.2.1.2.7) in [7], and Euler's constant γ, given by equation (5.2.3) in [10].

Example 1.

$$\int_{\mathbb{R}_+^4}\frac{\sqrt[4]{r+s}e^{-3r-2s-3x-2y}\left(\pi^2-4\log^2\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)\right)}{(rs)^{3/4}\sqrt[4]{xy}\sqrt[4]{x+y}\left(4\log^2\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)+\pi^2\right)^2}dxdydrds=\frac{48C+\pi^2}{576\sqrt{2}}$$

and

$$\int_{\mathbb{R}_+^4}\frac{\sqrt[4]{rs}\sqrt[4]{r+s}(xy)^{3/4}e^{-3r-2s-3x-2y}\log\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)}{rsxy\sqrt[4]{x+y}\left(4\log^2\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)+\pi^2\right)^2}dxdydrds=\frac{1}{16\pi}\left(\frac{C}{3\sqrt{2}}-\frac{\pi^2}{144\sqrt{2}}\right).$$

P r o o f. Use equation (5.2) and set $k=-2$, $a=i$, $m=-1/2$, $p=3$, $q=2$, rationalize the denominator and compare real and imaginary parts and simplify in using entry (2) in table (64:12:7) in [11]. □

Example 2.

$$\int_{\mathbb{R}_+^4} \frac{\sqrt{rs}\sqrt{xy}e^{-3r-2s-3x-2y}}{rsxy\left(\log^2\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)+\pi^2\right)}dxdydrds=\frac{4-\pi}{6}$$

and

$$\int_{\mathbb{R}_+^4} \frac{\sqrt{rs}\sqrt{xy}e^{-3r-2s-3x-2y}\log\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)}{rsxy\left(\log^2\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)+\pi^2\right)}dxdydrds=0.$$

P r o o f. Use equation (5.2) and set $k=-1$, $a=-1$, $m=-1$, $p=3$, $q=2$, rationalize the denominator and compare real and imaginary parts and simplify in using entry (1) in table (64:12:7) in [11]. □

Example 3.

$$\int_{\mathbb{R}_+^4} \frac{e^{-2r-3s-2x-3y}\left((r+s)^{3/8}\sqrt[4]{xy}-\sqrt[4]{rs}\sqrt[8]{r+s}\sqrt[4]{x+y}\right)}{(rs)^{7/8}(xy)^{3/8}(x+y)^{3/8}\log\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)}dxdydrds$$
$$=\frac{2}{3}\pi\tanh^{-1}\left(\cos\left(\frac{\pi}{8}\right)-\sin\left(\frac{\pi}{8}\right)\right).$$

P r o o f. Use equation (5.2) and form a second equation by replacing $m\to n$ and take their difference. Next we set $k=-1$, $a=1$, $m=-3/4$, $n=-1/4$, $p=2$, $q=3$ and simplify using equation (9.559) in [4] and entry (3) in table (64:12:7) in [11]. □

Example 4.

$$\int_{\mathbb{R}_+^4} \frac{e^{-r-s-x-y}\left(\sqrt[6]{r+s}\sqrt[24]{xy}-\sqrt[24]{rs}\sqrt[8]{r+s}\sqrt[24]{x+y}\right)}{(rs)^{2/3}(xy)^{3/8}\sqrt[6]{x+y}\log\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)}dxdydrds$$
$$=2\pi\log\left(\sqrt{3}\tan\left(\frac{3\pi}{16}\right)\right).$$

P r o o f. Use equation (5.2) and form a second equation by replacing $m\to n$ and take their difference. Next we set $k=-1$, $a=1$, $m=-3/4$, $n=-2/3$, $p=1$, $q=1$ and simplify using equation (9.559) in [4] and entry (3) in table (64:12:7) in [11]. □

Example 5.

$$\int_{\mathbb{R}_+^4} \frac{e^{-r-2(s+y)-x}}{\sqrt{rs}\sqrt{xy}\left(\log\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)+i\pi\right)^2}dxdydrds=4(C-1).$$

P r o o f. Use equation (5.2) and set $a = e^{ai}$, $m = -1$, $p = 1$, $q = 2$ and simplify in terms of the Hurwitz zeta function using entry (3) in table (64:12:7) in [11]. Next apply l'Hopitals' rule as $k \to -1$ and simplify in terms of the digamma function $\psi^{(0)}(a)$ given by equation (5.15.1) in [10]. Next take the first partial derivative with respect to a and set $a = \pi$ and simplify in terms of Catalan's constant C. □

Example 6.

$$\int_{\mathbb{R}_+^4} \frac{e^{-r-2(s+y)-x}}{\sqrt{rs}\sqrt{xy}\left(\log\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)+i\pi\right)^3} dxdydrds = -\frac{i\left(\pi^3-32\right)}{4\pi}.$$

P r o o f. Use equation (5.2) and set $a = e^{ai}, m = -1, p = 1, q = 2$ and simplify in terms of the Hurwitz zeta function using entry (3) in table (64:12:7) in [11]. Next apply l'Hopitals' rule as $k \to -1$ and simplify in terms of the digamma function $\psi^{(0)}(a)$. Next take the second partial derivative with respect to a and set $a = \pi$ and simplify in terms of π. □

Proposition 1. *For all $a, k, p, q \in \mathbb{C}$ the equality is true*

$$\int_{\mathbb{R}_+^4} \frac{e^{-p(r+x)-q(s+y)} \log^k\left(\frac{a\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)}{\sqrt{rs}\sqrt{xy}} dxdydrds$$
$$= \frac{2ie^{i\pi(k-1)/2}\pi^{k+2}\left(2^k\zeta\left(-k, 1/2\cdot(1/2 - i\log(a)/\pi)\right) - 2^k\zeta\left(-k, 1/2\cdot(3/2 - i\log(a)/\pi)\right)\right)}{pq}. \tag{6.1}$$

P r o o f. Use equation (5.2) and set $m = -1$ and simplify using entry (4) in table (64:12:7) in [11]. □

Proposition 2. *For all $k \in \mathbb{C}$ then,*

$$\int_{\mathbb{R}_+^4} \frac{e^{-r-s-x-y} \log^k\left(\frac{i\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)}{\sqrt{rs}\sqrt{xy}} dxdydrds = -2\left(2^{k+1}-1\right)e^{i\pi k/2}\pi^{k+2}\zeta(-k). \tag{6.2}$$

P r o o f. Use equation (6.2) and set $a = i$, $p = q = 1$ and simplify using entry (2) in table (64:12:7) in [11]. □

Example 7.

$$\int_{\mathbb{R}_+^4} \frac{e^{-r-s-x-y}\sqrt{\log\left(\frac{i\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)}}{\sqrt{rs}\sqrt{xy}} dxdydrds = -2\left(2\sqrt{2}-1\right)e^{i\pi/4}\pi^{5/2}\zeta\left(-\frac{1}{2}\right).$$

P r o o f. Use equation (6.2) and set $k = 1/2$ and simplify. □

Example 8.

$$\int_{\mathbb{R}_+^4} \frac{e^{-3(r+x)-4(s+y)}}{\sqrt{rs}\sqrt{xy}\log\left(\frac{i\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)}dxdydrds = -\frac{1}{6}i\pi\log(2).$$

P r o o f. Use equation (6.1) set $a = i$ and apply l'Hopital's rule as $k \to -1$ and set $q = 3$, $q = 4$ and simplify. □

Example 9.

$$\int_{\mathbb{R}_+^4} \frac{e^{-r-s-x-y}\log\left(\log\left(\frac{i\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)\right)}{\sqrt{rs}\sqrt{xy}}dxdydrds = \frac{1}{2}\pi^2(\log(4) + i\pi).$$

P r o o f. Use equation (6.1) set $a = i$ and take the first partial derivative with respect to k and set $k = 0$, $p = q = 1$ and simplify. □

Example 10.

$$\int_{\mathbb{R}_+^4} \frac{e^{-r-s-x-y}\log\left(\log\left(\frac{i\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)\right)}{\sqrt{rs}\sqrt{xy}\log\left(\frac{i\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)}dxdydrds = \pi\log(2)(2i\gamma + \pi - i(\log(2) + 2\log(\pi))).$$

P r o o f. Use equation (6.1) set $a = i$ and take the first partial derivative with respect to k then apply l'Hopital's rule as $k \to -1$ and set $p = q = 1$ and simplify. □

Example 11.

$$\int_{\mathbb{R}_+^4} \frac{e^{-r-s-x-y}\log\left(\log\left(\frac{i\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)\right)}{\sqrt{rs}\sqrt{xy}\log^2\left(\frac{i\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)}dxdydrds = \frac{1}{12}\pi^2(-24\log(A) + 2\gamma - i\pi + \log(16)).$$

P r o o f. Use equation (6.1) set $a = i$ and take the first partial derivative with respect to k and set $k = -2$, $p = q = 1$ and simplify. □

Proposition 3. *For all $a, p, q \in \mathbb{C}$, $\mathrm{Re}(a) > 0$ then,*

$$\int_{\mathbb{R}_+^4} \frac{\log\left(\log\left(\frac{a\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)\right)e^{-p(r+x)-q(s+y)}}{\sqrt{rs}\sqrt{xy}}dxdydrds = \frac{\pi^2\left(4\log\left(\frac{\sqrt{2\pi}\Gamma(3/4 - i\log(a)/2\pi)}{\Gamma((\pi - 2i\log(a))/4\pi)}\right) + i\pi\right)}{2pq}.$$

P r o o f. Use equation (6.1) and take the first partial derivative with respect to k and set $k = 0$ and simplify using equation (25.11.18) in [10] □

7. Summary table of quadruple integrals involving

$f(x,y,r,s)$	$\int_{\mathbb{R}_+^4} f(x,y,r,s)dudx$
$\frac{\sqrt[4]{r+s}e^{-3r-2s-3x-2y}\left(\pi^2-4\log^2\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)\right)}{(rs)^{3/4}\sqrt[4]{xy}\sqrt[4]{x+y}\left(4\log^2\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)+\pi^2\right)^2}$	$\frac{48C+\pi^2}{576\sqrt{2}}$
$\frac{\sqrt[4]{rs}\sqrt[4]{r+s}(xy)^{3/4}e^{-3r-2s-3x-2y}\log\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)}{rsxy\sqrt[4]{x+y}\left(4\log^2\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)+\pi^2\right)^2}$	$\frac{1}{16\pi}\left(\frac{C}{3\sqrt{2}}-\frac{\pi^2}{144\sqrt{2}}\right)$
$\frac{\sqrt{rs}\sqrt{xy}e^{-3r-2s-3x-2y}}{rsxy\left(\log^2\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)+\pi^2\right)}$	$\frac{4-\pi}{6}$
$\frac{\sqrt{rs}\sqrt{xy}e^{-3r-2s-3x-2y}\log\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)}{rsxy\left(\log^2\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)+\pi^2\right)}$	0
$\frac{e^{-2r-3s-2x-3y}\left((r+s)^{3/8}\sqrt[4]{xy}-\sqrt[4]{rs}\sqrt[8]{r+s}\sqrt[4]{x+y}\right)}{(rs)^{7/8}(xy)^{3/8}(x+y)^{3/8}\log\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)}$	$\frac{2}{3}\pi\tanh^{-1}\left(\cos\left(\frac{\pi}{8}\right)-\sin\left(\frac{\pi}{8}\right)\right)$
$\frac{e^{-r-s-x-y}\left(\sqrt[6]{r+s}\sqrt[24]{xy}-\sqrt[24]{rs}\sqrt[8]{r+s}\sqrt[24]{x+y}\right)}{(rs)^{2/3}(xy)^{3/8}\sqrt[6]{x+y}\log\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)}$	$2\pi\log\left(\sqrt{3}\tan\left(\frac{3\pi}{16}\right)\right)$
$\frac{e^{-r-2(s+y)-x}}{\sqrt{rs}\sqrt{xy}\left(\log\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)+i\pi\right)^2}$	$4(C-1)$
$\frac{e^{-r-2(s+y)-x}}{\sqrt{rs}\sqrt{xy}\left(\log\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)+i\pi\right)^3}$	$-\frac{i\left(\pi^3-32\right)}{4\pi}$
$\frac{e^{-r-s-x-y}\log^k\left(\frac{i\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)}{\sqrt{rs}\sqrt{xy}}$	$-2\left(2^{k+1}-1\right)e^{i\pi k/2}\pi^{k+2}\zeta(-k)$
$\frac{e^{-r-s-x-y}\sqrt{\log\left(\frac{i\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)}}{\sqrt{rs}\sqrt{xy}}$	$-2\left(2\sqrt{2}-1\right)e^{i\pi/4}\pi^{5/2}\zeta\left(-\frac{1}{2}\right)$
$\frac{e^{-3(r+x)-4(s+y)}}{\sqrt{rs}\sqrt{xy}\log\left(\frac{i\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)}$	$-\frac{1}{6}i\pi\log(2)$
$\frac{e^{-r-s-x-y}\log\left(\log\left(\frac{i\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)\right)}{\sqrt{rs}\sqrt{xy}}$	$\frac{1}{2}\pi^2(\log(4)+i\pi)$
$\frac{e^{-r-s-x-y}\log\left(\log\left(\frac{i\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)\right)}{\sqrt{rs}\sqrt{xy}\log\left(\frac{i\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)}$	$\pi\log(2)(2i\gamma+\pi-i(\log(2)+2\log(\pi)))$
$\frac{\log\left(\log\left(\frac{a\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)\right)e^{-p(r+x)-q(s+y)}}{\sqrt{rs}\sqrt{xy}}$	$\frac{\pi^2\left(4\log\left(\frac{\sqrt{2\pi}\Gamma(3/4-i\log(a)/2\pi)}{\Gamma((\pi-2i\log(a))/4\pi)}\right)+i\pi\right)}{2pq}$

8. Discussion

In this work we used our contour integral method to derive a quadruple integral involving the logarithm of quotient radicals in terms of the Hurwitz–Lerch Zeta transcendent. The integrals derived are not easy to numerically evaluate as we suspect the presence of singularities and the integrand maybe highly oscillatory. The importance of this work is that we are able to write down a closed form solution for this integral. This is advantageous as we now have the Hurwitz–Lerch Zeta function with analytic continuation to use in order to evaluate this quadruple integral. We also employed Wolfram Mathematica to assist with numerical computation where needed. We will use our contour method to derive other multiple integrals for future work.

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ON SOME VERTEX–TRANSITIVE DISTANCE–REGULAR ANTIPODAL COVERS OF COMPLETE GRAPHS[1]

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Abstract: In the present paper, we classify abelian antipodal distance-regular graphs Γ of diameter 3 with the following property: $(*)$ Γ *has a transitive group of automorphisms* $\widetilde{G}$ *that induces a primitive almost simple permutation group* $\widetilde{G}^{\Sigma}$ *on the set* Σ *of its antipodal classes.* There are several infinite families of (arc-transitive) examples in the case when the permutation rank $\mathrm{rk}(\widetilde{G}^{\Sigma})$ of $\widetilde{G}^{\Sigma}$ equals 2; moreover, all such graphs are now known. Here we focus on the case $\mathrm{rk}(\widetilde{G}^{\Sigma})=3$. Under this condition the socle of $\widetilde{G}^{\Sigma}$ turns out to be either a sporadic simple group, or an alternating group, or a simple group of exceptional Lie type, or a classical simple group. Earlier, it was shown that the family of non-bipartite graphs Γ with the property $(*)$ such that $\mathrm{rk}(\widetilde{G}^{\Sigma})=3$ and the socle of $\widetilde{G}^{\Sigma}$ is a sporadic or an alternating group is finite and limited to a small number of potential examples. The present paper is aimed to study the case of classical simple socle for $\widetilde{G}^{\Sigma}$. We follow a classification scheme that is based on a reduction to *minimal* quotients of Γ that inherit the property $(*)$. For each given group $\widetilde{G}^{\Sigma}$ with simple classical socle of degree $|\Sigma|\le 2500$, we determine potential minimal quotients of Γ, applying some previously developed techniques for bounding their spectrum and parameters in combination with the classification of primitive rank 3 groups of the corresponding type and associated rank 3 graphs. This allows us to essentially restrict the sets of feasible parameters of Γ in the case of classical socle for $\widetilde{G}^{\Sigma}$ under condition $|\Sigma|\le 2500$.

Keywords: Distance-regular graph, Antipodal cover, Abelian cover, Vertex-transitive graph, Rank 3 group.

1. Introduction

Let Γ be an antipodal distance-regular graph of diameter three. Then Γ is an antipodal r-cover of a complete graph on $k+1$ vertices, and its intersection array has form $\{k,(r-1)\mu,1;1,\mu,k\}$, where k, r and μ denote the valency of Γ, the size of its antipodal classes and the number of common neighbours for each two vertices at distance two of Γ, respectively (e.g. see [2]); for brevity, we will refer to such a graph as an $(k+1,r,\mu)$-*cover*. We denote by $\mathcal{CG}(\Gamma)$ the group of all automorphisms of Γ fixing setwise each of its antipodal classes. If the group $\mathcal{CG}(\Gamma)$ is abelian and acts regularly on (every) antipodal class of Γ, then Γ is called an *abelian* $(k+1,r,\mu)$-*cover* (see [5]). There are some important links between abelian covers and other combinatorial or geometric objects (we refer to [9] and [5] for more background). The problem of finding new their constructions involves many natural questions on possible structure of such a graph, and one of them is to study vertex-transitive representatives.

In the present paper, we classify abelian $(k+1,r,\mu)$-covers Γ with the following property:

$(*)$ Γ *has a transitive group of automorphisms* $\widetilde{G}$ *that induces a primitive almost simple permutation group* $\widetilde{G}^{\Sigma}$ *on the set* Σ *of its antipodal classes.*

Without loss of generality, we may assume that $\widetilde{G}$ coincides with the full pre-image of $\widetilde{G}^{\Sigma}$ in $\mathrm{Aut}(\Gamma)$. When the permutation rank $\mathrm{rk}(\widetilde{G}^{\Sigma})$ of $\widetilde{G}^{\Sigma}$ equals 2, there are several infinite families of

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(arc-transitive) examples; moreover, all such graphs are now known. Here we focus on the case $\mathrm{rk}(\widetilde{G}^{\Sigma}) = 3$. Under this condition the socle of $\widetilde{G}^{\Sigma}$ turns out to be either a sporadic simple group, or an alternating group, or a simple group of exceptional Lie type, or a classical simple group (see [3, Ch. 11] for an overview on classification of primitive rank 3 permutation groups).

In [16] and [17], it was shown that the family of non-bipartite graphs Γ with the property $(*)$ such that $\mathrm{rk}(\widetilde{G}^{\Sigma}) = 3$ and the socle of $\widetilde{G}^{\Sigma}$ is a sporadic or alternating group is finite and limited to a small number of potential examples. The present paper is aimed to study the case of classical simple socle for $\widetilde{G}^{\Sigma}$. We follow a classification scheme that was proposed in [16] and that is based on a reduction to *minimal* quotients of Γ that inherit the property $(*)$. For each given group $\widetilde{G}^{\Sigma}$, we determine potential minimal quotients of Γ, applying the constraints for their spectrum and parameters obtained in [16] in combination with the classification of primitive rank 3 groups of the corresponding type (see [8], [11], and also [13]) and associated rank 3 graphs (see [3, Ch. 11]). This allows us to essentially restrict the sets of feasible parameters of Γ in the case of classical socle for $\widetilde{G}^{\Sigma}$ with $|\Sigma| \leq 2500$. In particular, we show that for most of these sets Γ must be a covering of a certain distance-transitive Taylor graph.

2. Preliminaries

We keep the notation and terminology from [16] and we refer the reader to [1] and [2] for basic definitions. Next we recall some of them. For a finite group G, we denote by $\mathrm{Soc}(G)$, $Z(G)$ and G' its socle, center and derived subgroup, respectively. If $G = G'$, then $\mathrm{M}(G)$ denotes its Schur multiplier. If $G \neq 1$, then we write "$\mathrm{d}_{\min}(G)$" to denote the number $|G : H|$, where H is a proper subgroup of G of the smallest possible index. Further, if G is a transitive permutation group on a finite set Ω and $\mathrm{O}rb_2(G)$ is the set of G-orbitals on Ω, then the number $|\mathrm{O}rb_2(G)|$, denoted by $\mathrm{rk}(G)$, is called the (*permutation*) *rank* of G. For each $Q \in \mathrm{O}rb_2(G)$, Q^* denotes the orbital paired with Q. If $Q^* = Q$ and $a \in \Omega$, then $Q(a)$ denotes the set of all points $b \in \Omega$ such that $(a, b) \in Q$.

In what follows, we consider only undirected graphs without loops or multiple edges. For a graph Γ by $\mathcal{V}(\Gamma)$ and $\mathcal{A}(\Gamma)$ we denote its vertex set and the arc set, respectively. An (n, r, μ)-cover is equivalently defined as a connected graph, whose vertex set admits a partition into n cells (called antipodal classes or fibres) of the same size $r \geq 2$ such that each cell induces an r-coclique, the union of any two distinct cells induces a perfect matching, and every two non-adjacent vertices that lie in distinct cells have exactly $\mu \geq 1$ common neighbours. Since an (n, r, μ)-cover is bipartite if and only if $r = 2$ and $\mu = n - 2$, and for each $n \geq 3$ there is a unique (abelian) $(n, 2, n - 2)$-cover (see [2, Corollary 1.5.4]), we omit these from further consideration. We will say that the set of parameters (n, r, μ) of a non-bipartite abelian (n, r, μ)-cover Γ is *feasible* if it satisfies the known necessary conditions for the existence of Γ that are collected in [16, Proposition 1] (see [16] for detailed references) and [5, Lemma 3.5, Theorem 5.4]. In view of [5], for every (n, r, μ)-cover Γ and every subgroup N of $\mathcal{CG}(\Gamma)$ of order less than r, the *quotient* Γ^N that is defined as the graph on the set of N-orbits in which two vertices are adjacent if and only if there is an edge of Γ between the corresponding orbits, is a $(n, r/|N|, \mu|N|)$-cover. Hence if Γ is a non-bipartite abelian (n, r, μ)-cover, then, using decomposition $\mathcal{CG}(\Gamma) = O_p(\mathcal{CG}(\Gamma)) \times N$, where p is a prime divisor of r, we obtain that Γ possesses a quotient Γ^N that is a non-bipartite abelian $(n, p^l, \mu|N|)$-cover with $p^l = |O_p(\mathcal{CG}(\Gamma))|$. Clearly, the factor group $\mathrm{Aut}(\Gamma)/N$ acts as a group of automorphisms of Γ^N, and in case $\mathcal{CG}(\Gamma) > M > N$ other quotients Γ^M inherit a similar property when $M \trianglelefteq \mathrm{Aut}(\Gamma)$. Thus parameters of Γ may depend on the structure of $\mathcal{CG}(\Gamma)$. This is also demonstrated by the fact that for each non-bipartite abelian (n, r, μ)-cover, every prime divisor of r is also a divisor of n (see [5, Theorem 9.2] and also [6, Theorem 2.5]). These basic observations are crucial for our following arguments; they will be used further without any additional reference.

The next result from [16] distinguishes several types of quotients that an abelian non-bipartite

$(k+1,r,\mu)$-cover with the property $(*)$ may possess.

Proposition 1 [16, Proposition 2]. *Let Γ be a non-bipartite $(k+1,r,\mu)$-cover and Σ be the set of its antipodal classes. Suppose Γ has a transitive automorphism group G_1 which induces a primitive almost simple permutation group ${G_1}^{\Sigma}$ on Σ and put $T = \mathrm{Soc}({G_1}^{\Sigma})$. Let G be the full pre-image of the group T in G_1 and K be the kernel of the action of the group G on Σ. Then K contains a subgroup N that is normal in G_1 and satisfies one of the following conditions (below the symbol $^-$ denotes factorization with respect to N):*

(T1) *$\overline{K} \simeq E_{p^l}$ is an elementary abelian group of exponent p and either*

- (i) *$\overline{G} = \overline{K} \times \overline{G}'$ and $\overline{G}' \simeq T$, or*
- (ii) *$\overline{G}$ is a quasi-simple group with center $\overline{K}$;*

(T2) *$\overline{K} \simeq E_{p^l}$ is an elementary abelian group of exponent p, T acts faithfully on $\overline{K}$, i.e. $T \leq GL_l(p)$, and $\mathrm{d}_{\min}(T) \leq (p^l-1)/(p-1)$;*

(T3) *$\overline{K} \simeq S^l$, where S is a simple non-abelian group, and either*

- (i) *$\overline{G} = \overline{K} \times C_{\overline{G}}(\overline{K})$ and $C_{\overline{G}}(\overline{K}) \simeq T$, or*
- (ii) *$\overline{G} \leq \mathrm{Aut}(\overline{K})$ and T contains a proper subgroup of index dividing l.*

Each graph Γ that satisfies the hypothesis of Proposition 1 will be referred to as a *minimal $(k+1,r,\mu)$-cover of type* (Tx) with $x = 1,2,3$ and denoted by $\Gamma(G_1,G,K)$ if $|K| = r$, the triple (G_1,G,K) satisfies the condition (Tx) from the conclusion of Proposition 1 and K is a minimal normal subgroup of G_1 (in particular, $N = 1$). Thus, for a minimal $(k+1,r,\mu)$-cover $\Gamma(G_1,G,K)$ the number r is a prime when $G_1 = G$ and $K \leq Z(G)$.

From now on Γ is a non-bipartite abelian $(k+1,r,\mu)$-cover with property $(*)$, Σ is the set of its antipodal classes, $\widetilde{G}$ is a transitive group of automorphisms of Γ which induces a primitive almost simple permutation group $\widetilde{G}^{\Sigma}$ on Σ, $\mathrm{rk}(\widetilde{G}^{\Sigma}) = 3$, k_1 and k_2 are the non-trivial subdegrees of $\widetilde{G}^{\Sigma}$, $K = \mathcal{CG}(\Gamma) \leq \widetilde{G}$ and G is the full pre-image of the group $\mathrm{Soc}(\widetilde{G}^{\Sigma})$ in $\widetilde{G}$.

Now we proceed with final technical definitions. For a vertex x of Γ, by $F(x)$ and $\Gamma_1(x)$ (or $[x]$) we denote, respectively, the antipodal class of Γ containing x, and its neighborhood in Γ. Put $\Omega = \mathcal{V}(\Gamma)$, and fix $a \in \Omega$ and $F = F(a)$. Let $M = \widetilde{G}_{\{F\}}$ and $H = \widetilde{G}_a$ (note $|K| = r$ implies $M = K : H$). Then $\mathcal{A}(\Gamma) = Q_1 \cup Q_2$ for some $Q_1, Q_2 \in \mathrm{O}rb_2(G)$ with $Q_i = Q_i^*$ (see [16]), $|Q_i| = rk_i(k+1)$, and $|H : \widetilde{G}_{a,b_i}| = k_i$ for each arc $(a,b_i) \in Q_i$, so H has exactly two orbits on $\Gamma_1(a)$ (with representatives b_1 and b_2). For $i = 1,2$, let Φ_i denote the (rank 3) graph on Σ in which two vertices $F(x)$ and $F(y)$ are adjacent if and only if $(x,y) \in Q_i$. If $\mathrm{rk}(G^{\Sigma}) = 3$ then the group G^{Σ} is also primitive as $\mu(\Phi_i) \neq 0, k_i$ (see, for example, [1, 16.4]). Moreover, the parameters k_1, k_2 and λ satisfy the following equation (see [16])

$$(\lambda - \lambda_1)k_1 = (\lambda - \lambda_2)k_2,$$

where $\lambda_i = |\Gamma_1(b_i) \cap H(b_i)|$, $i = 1,2$. We will say that Γ admits an *H-uniform edge partition* (*with parameters* (μ_1,μ_2)) (see [16]), if for each $j = 1,2$ and for every two distinct vertices z_1, $z_2 \in F$, the number of edges between $Q_j(z_1)$ and $Q_j(z_2)$ is constant and equal to $k_j\mu_j$, where μ_j is a fixed integer.

Lemma 1 [16, Lemma 1]. *Suppose that $G_{\{F\}} = G_a \times K$ and $\mathrm{rk}(G^{\Sigma}) = 3$. If H acts transitively on $F \setminus \{a\}$ or $r \leq 3$, then Γ admits an H-uniform edge partition.*

Theorem 1 [16, Theorem 1]. *Suppose that* $G_{\{F\}} = G_a \times K$ *and* $\mathrm{rk}(G^\Sigma) = 3$. *Then for each* $x \in F \setminus \{a\}$ *we have*

$$(\mu - \mu_1)k_1 = (\mu - \mu_2)k_2,$$

where $\mu_i = |\Gamma_1(b_i) \cap Q_i(x)|$, $i = 1, 2$. *If, moreover,* Γ *admits an* H*-uniform edge partition with parameters* (μ'_1, μ'_2), *then* $\mu'_i = \mu_i$ *(in particular,* $k_i - 1 = \lambda_i + (r-1)\mu_i$*) for every* $i = 1, 2$ *and* $\gamma = -(\lambda - \lambda_1 - \lambda_2) + (\mu - \mu_1 - \mu_2) = r(\mu - \mu_1 - \mu_2) - 1$ *is an eigenvalue of* Γ.

3. Main results

Theorem 2. *Suppose that* $\Gamma = \Gamma(\widetilde{G}, G, K)$ *is a minimal abelian* $(k+1, r, \mu)$*-cover,* $k+1 \leq 2500$, $\mathrm{rk}(\widetilde{G}^\Sigma) = 3$ *and* $T = \mathrm{Soc}(\widetilde{G}^\Sigma)$ *is a classical simple group, isomorphic to the group* $\widetilde{M}/Z(\widetilde{M})$, *where* $\widetilde{M} = Sp_{2n-2}(q)$, $\Omega^\pm_{2n}(q)$, $\Omega_{2n-1}(q)$ *or* $SU_n(q)$ *for* $n \geq 3$. *Assume* $\widetilde{G} = G$ *whenever* $\mathrm{rk}(T) = 3$. *Then one of the following statements is true*:

(1) $T \simeq PSU_4(4)$, $\mathrm{rk}(T) = 3$, $k+1 = 1105$, $r = 5$ *and* $\mu = 210$;

(2) $T \simeq G' \simeq P\Omega^\pm_{2n}(2)$, $\mathrm{rk}(T) = 3$, $k+1 = (2^{2n-1} - \varepsilon 2^{n-1})$, *where* $\varepsilon = \pm 1$ *and* $n \leq 6$, $2(\lambda(\Phi_1) + \lambda(\Phi_2) + 1) = k - 1$, $r = 2$ *and either* $G = G' \simeq Z_2.P\Omega^+_8(2)$, $\varepsilon = +1$, $k+1 = 120$, *and* $\mu \in \{64, 54\}$, *or the group* G' *is intransitive on* $\mathcal{V}(\Gamma)$;

(3) $T \simeq G' \simeq P\Omega_5(8) \simeq PSp_4(8)$, $\mathrm{rk}(T) = 5$, $2(\lambda(\Phi_1) + \lambda(\Phi_2) + 1) \neq k - 1$, $k+1 = 2016$ *and* $r\mu \in \{2048, 1980\}$, *or* $k+1 = 2080$ *and* $r\mu \in \{2048, 2108\}$, *wherein either* $r = 4$ *and* G' *is intransitive on* $\mathcal{V}(\Gamma)$, *or* $r = 2$ *and* G' *is transitive on* $\mathcal{V}(\Gamma)$;

(4) $T \simeq P\Omega_m(q)$, $\mathrm{rk}(T) = 3$, $2(\lambda(\Phi_1) + \lambda(\Phi_2) + 1) = k - 1$, $r = 2$ *and either*

   (*i*) $m = 5$, $q = 3$, $k+1 = 36$ *and* $\mu \in \{16, 18\}$, *or*

   (*ii*) $m = 5$, $q = 4$, *with* $k+1 = 120$ *and* $\mu \in \{54, 64\}$ *or* $k+1 = 136$ *and* $\mu \in \{64, 70\}$, *or*

   (*iii*) $m = 7$, $q = 4$, *with* $k+1 = 2016$ *and* $\mu \in \{990, 1024\}$ *or* $k+1 = 2080$ *and* $\mu \in \{1024, 1054\}$,

   and in all cases (*i*)–(*iii*) *the group* G' *is intransitive on* $\mathcal{V}(\Gamma)$;

(5) $T \simeq G' \simeq SU_3(3)$, $\mathrm{rk}(T) = 4$, $k+1 = 36$, $2(\lambda(\Phi_1) + \lambda(\Phi_2) + 1) = k - 1$, $r = 2$, $\mu \in \{16, 18\}$ *and* G' *is intransitive on* $\mathcal{V}(\Gamma)$;

(6) $T \simeq G' \simeq PSp_6(2) \simeq P\Omega_7(2)$, $\mathrm{rk}(T) = 3$, $k+1 = 120$, $2(\lambda(\Phi_1) + \lambda(\Phi_2) + 1) = k - 1$, $r = 2$, $\mu \in \{54, 64\}$ *and* G' *is intransitive on* $\mathcal{V}(\Gamma)$.

Moreover, if $r = 2$ *and* $G' \simeq T$, *then for any given pair of parameters* k *and* μ, Γ *is a unique (up to isomorphism) distance-transitive* $(k+1, 2, \mu)$*-cover.*

P r o o f. Let $k+1 \leq 2500$. Under this condition $\mathrm{rk}(T) = 3$ for all k except the following cases (a)–(d) (note that in [16, Example] the case (d) is missing, and the subdegrees k_1, k_2 for the case (c) are mistyped):

(a) $k+1 = 36$, $k_1 = 14$, $k_2 = 21$, $T \simeq PSL_2(8)$, $\mathrm{rk}(T) = 5$, $\widetilde{G}^\Sigma \simeq P\Gamma L_2(8) = T.3$;

(b) $k+1 = 36$, $k_1 = 14$, $k_2 = 21$, $T \simeq PSU_3(3)$, $\mathrm{rk}(T) = 4$, $\widetilde{G}^\Sigma \simeq P\Gamma U_3(3) = T.2$;

(c) $k+1 = 2016$, $k_1 = 455$, $k_2 = 1560$, $G^\Sigma \simeq Sp_4(8)$, $\mathrm{rk}(G^\Sigma) = 5$ and $\widetilde{G}^\Sigma \simeq Sp_4(8).Z_3$

(*d*) $k+1=2080$, $k_1=567$, $k_2=1512$, $G^\Sigma \simeq Sp_4(8)$, $\mathrm{rk}(G^\Sigma)=5$ and $\widetilde{G}^\Sigma \simeq Sp_4(8).Z_3$.

Then, by [16, Propositions 2, 3], if $\mathrm{rk}(T)=3$, $T \not\leq \mathrm{Aut}(K)$ and $2(\lambda(\Phi_1)+\lambda(\Phi_2)+1) \neq k-1$, then either $G' \simeq T$ acts transitively on $\mathcal{V}(\Gamma)$, or G is a quasisimple group. Therefore, in order to find some necessary conditions for Γ to exist (as well as for its covers with property $(*)$), in case $\mathrm{rk}(T)=3$ it suffices to consider the case $\widetilde{G}=G$, and if, moreover, $K \leq Z(G)$, then one may assume that r is prime. Taking this into account, we further specify the possible structure of G for each potential pair $(\widetilde{G}^\Sigma, \Phi_1)$.

Throughout the rest of the proof, we put $N=G'$ and denote by θ and $-\tau$, respectively, the positive and negative eigenvalues of Γ, other than k and -1. We will consider the following possible combinations for T and complementary rank 3 graphs Φ_1 and Φ_2 associated with $\widetilde{G}^\Sigma$, applying their description from [8] and [3, Theorem 11.3.2].

(A) Let $k_1=q(q^{n-1}-1)(tq^{n-1}+1)/(q-1)$ and suppose the graph Φ_1 has parameters

$$\Big(\frac{q^n-1}{q-1}(tq^{n-1}+1), q\frac{q^{n-1}-1}{q-1}(tq^{n-2}+1), q^2\frac{q^{n-2}-1}{q-1}(tq^{n-3}+1)+q-1, \frac{q^{n-1}-1}{q-1}(tq^{n-2}+1)\Big),$$

where $t=q,1,q,q^2,q^{1/2},q^{3/2}$ for $\widetilde{M}=Sp_{2n}(q)$, $\Omega^+_{2n}(q)$, $\Omega_{2n+1}(q)$, $\Omega^-_{2n+2}(q)$, $SU_{2n}(\sqrt{q})$ or $SU_{2n+1}(\sqrt{q})$, respectively (see [3, Theorem 11.3.2(i)]).

By condition $k+1 \leq 2500$, hence the equality $2(\lambda(\Phi_1)+\lambda(\Phi_2)+1)=k-1$ holds if and only if $t=1$, $q=3$, $n=2$ and $(v,k_1,\lambda(\Phi_1),\mu(\Phi_1))=(16,6,2,2)$, which contradicts the constraint $n \geq 3$ for $t=1$. If r is a power of a prime p, say $r=p^l$, then feasible sets of parameters k, r, and μ are described by Table 1, and Γ has no H-uniform edge partitions in the cases $t=1,q,q^2$ (this can be easily checked by complete enumeration in GAP [14], based on Theorem 1, [16, Proposition 1] and [5, Lemma 3.5, Theorem 5.4]).

(A1) Let $T \simeq PSp_{2n}(q)$ and $k+1=(q^{2n}-1)/(q-1)$. Then $\mathrm{rk}(T)=3$, while $\mathrm{d_{min}}(T)=k+1$, except for the cases when $q=2$, $2n \geq 6$ and $\mathrm{d_{min}}(T)=2^{n-1}(2^n-1)$ or $2n=4, q=3$ and $\mathrm{d_{min}}(T)=27$ (see [12, Theorem 2]). Moreover, $\mathrm{M}(T)=Z_{\mathrm{gcd}(2,q-1)}$ for $(q;n) \neq (2;2),(2;3)$ and $\mathrm{M}(T)=Z_2$ for $(q;n)=(2;2),(2;3)$, $\mathrm{Out}(T)=Z_{\mathrm{gcd}(2,q-1)} \cdot Z_e$, where $q=p^e$, p is a prime.

According to Table 1 $(q;n) \notin \{(2;3),(3;2)\}$. Hence $\mathrm{d_{min}}(T)=k+1$. It follows that $K \leq Z(G)$ and, as noted above, it suffices to consider the case of prime r.

Since Γ has no H-uniform edge partitions, we have $r \geq 5$. Also, $2(\lambda(\Phi_1)+\lambda(\Phi_2)+1) \neq k-1$. Hence, due to [16, Proposition 3] $N=G' \simeq T$ acts transitively on $\mathcal{V}(\Gamma)$. But then $G_a \simeq N_{\{F\}}$ contains a subgroup of index r and $G_{\{F\}}=G_aN_{\{F\}}$.

If $n=3=q$, then $(|N|)_5=5$ and hence $|N_{\{F\}}|$ is not divisible by 5, a contradiction.

Let $n=2$. Then $N_{\{F\}}$ is an extension of a group of order q^3 by a group of the form $((q-1)/2 \times L_2(q)).2$ or $((q-1) \cdot L_2(q))$ (see, for example, [4] or [13]). In any case, $N_{\{F\}}$ does not contain subgroups of index 5, a contradiction.

(A2) Let $T \simeq O_{2n+1}(q)$, $t=q$ and $k+1=(q^{2n}-1)/(q-1)$. Recall that $PSp_4(q) \simeq O_5(q)$ for $n=2$, and also that $O_{2n+1}(q) \simeq PSp_{2n}(q)$ for even q (see, for example, [18]). Since the corresponding cases are considered in case (A1), we will further assume that $n \geq 3$ and q is odd. Then $\mathrm{rk}(T)=3$ and by [18, Theorem] $\mathrm{d_{min}}(T)=k+1$, except for the case $q=3$, in which $\mathrm{d_{min}}(T)=3^n(3^n-1)/2$. Moreover, $\mathrm{M}(T)=Z_{(2,q-1)}$ for $(q;n) \neq (3;3)$ and $\mathrm{M}(T)=Z_2 \times Z_2 \times Z_3$ for $q=3=n$ (e.g. see [7]).

As in case (A1) we have $q=n=3$ and since $\mathrm{d_{min}}(T)>r$ we conclude $K \leq Z(G)$. Hence we may assume that r is prime. But then Table 1 gives $r=2$ and hence by Lemma 1 and Theorem 1 Γ admits an H-uniform edge partition, a contradiction.

Table 1. Feasible parameters of Γ with $r = p^l$ in case (A)

	q, n	$k+1$	k_1, k_2	θ	$-\tau$	$r\mu$	r
Type $t = q$:							
	7, 2	400	56, 343	19	-21	400	$2, 4, 5, 8, 25$
				21	-19	396	2
	9, 2	820	90, 729	21	-39	836	2
				39	-21	800	$2, 4, 5, 8, 16, 25$
	3, 3	364	120, 243	11	-33	384	$2, 4, 8, 16$
				33	-11	340	2
Type $t = 1$:	$\varnothing$						
Type $t = q^2$:							
	4, 2	325	68, 256	9	-36	350	5
				12	-27	338	13
	3, 3	1066	336, 729	$\sqrt{1065}$	$-\sqrt{1065}$	1064	$2, 4$
	2, 4	495	238, 256	19	-26	500	$5, 25$
				26	-19	486	$3, 9, 27, 81, 243$
Type $t = \sqrt{q}$:							
	4, 2	45	12, 32	4	-11	50	5
				11	-4	36	$3, 9$
	9, 2	280	36, 243	9	-31	300	$2, 5$
				31	-9	256	$2, 4, 8, 16, 32, 64, 128$
	16, 2	1105	80, 1024	16	-69	1156	17
				69	-16	1050	$5, 25$
Type $t = \sqrt{q}^3$:	$\varnothing$						

(A3) Let $T \simeq O^+_{2n}(q)$, where $n \geq 3$, $t = 1$ and $k+1 = (q^n - 1)(q^{n-1}+1)/(q-1)$. Then condition $k+1 \leq 2500$ implies either $n = 3$ and $q \leq 5$, or $n = 4$ and $q \leq 3$, or $n = 5, 6$ and $q = 2$. According to Table 1 none of these cases is possible.

(A4) Let $T \simeq O^-_{2n}(q)$, where $n \geq 2$, $t = q^2$ and $k+1 = (q^n - 1)(q^{n+1}+1)/(q-1)$. Then $\mathrm{rk}(T) = 3$ and in view of Table 1 $n = 2, 3, 4$. Recall that $O^-_4(q) \simeq L_2(q^2)$ and $O^-_6(q) \simeq U_4(q)$ (e.g. see [18]).

If $n = 4$ and $q = 2$, then by [4] $\mathrm{d_{min}}(T) = 119$. By [12, Theorem 1, Theorem 3] $\mathrm{d_{min}}(T) = q^2 + 1 = 17$ for $n = 2$ and $\mathrm{d_{min}}(T) = (q^3 + 1)(q + 1) = 112$ for $n = 3 = q$. In each case $\mathrm{d_{min}}(T) > r$ and hence $K \leq Z(G)$. Arguing as in case (A3), we obtain that $r = 5$ and $N = G'$ acts transitively on $\mathcal{V}(\Gamma)$. But then $N \simeq L_2(16)$ or $O^-_8(2)$, and $|N|$ is not divisible by 25. This contradicts the fact that $N_{\{F\}}$ must contain a subgroup of index r.

(A5) Let $T \simeq PSU_{2n}(\sqrt{q})$. In view of Table 1 either $T \simeq PSU_4(2) \simeq PSp_4(3)$ and $\mathrm{d_{min}}(T) = 27$ or $T \simeq PSU_4(3) \simeq O^-_6(3) \not\leq GL_7(2)$ and $\mathrm{d_{min}}(T) = 112$, or $T \simeq PSU_4(4)$ and $\mathrm{d_{min}}(T) = 325$ (see [4] and [12, Theorem 3]). Hence $K \leq Z(G)$ and we may assume that r is a prime. If G is a quasi-simple group, then r divides $|\mathrm{M}(T)|$ and so by [7] $r = 2$ and $q = 9$. If $N \simeq T$ acts transitively on $\mathcal{V}(\Gamma)$, then r^2 divides $|N|$ and so $r = 5$ for $q = 16$ and $r \leq 3$ for $q \leq 9$.

Suppose $q = 9$. Then $r = 2$, Γ admits an H-uniform edge partition with parameters (μ_1, μ_2) and $\{(\lambda_1, \lambda_2), (\mu_1, \mu_2)\} = \{(15, 130), (20, 112)\}$. Enumeration of orbital graphs in GAP [14] shows that Γ does not exist when $N \simeq T$. But for $G = N$ the groups $(G_a)^{[a]}$ and $(G_{\{F\}})^{\Sigma - \{F\}}$ are permutation isomorphic. Moreover, for the vertex $b_1 \in Q_1(a)$ the group G_{a,b_1} has exactly two orbits of length 4 and one orbit of length 27 on $[a]$, which contradicts the fact $\lambda_1 \in \{15, 20\}$.

Suppose $q = 4$. Then $r = 3$, $N \simeq T$ acts transitively on $\mathcal{V}(\Gamma)$, $(\lambda_1, \lambda_2) = (3, 13)$ and Γ admits an H-uniform edge partition with parameters $(\mu_1, \mu_2) = (4, 9)$. A complete enumeration of orbital graphs in GAP [14] shows that this case cannot occur.

(A6) In the case of $T \simeq PSU_{2n+1}(\sqrt{q})$ and $t = \sqrt{q}^3$ we have $n = 2$ and $q = 4, 9$, but according to Table 1 none of the cases gives a feasible parameter set.

(B) Let us consider the cases $T \simeq \widetilde{M}/Z(\widetilde{M})$, where $\widetilde{M} = Sp_4(q)$, $SU_4(q)$, $SU_5(q)$, $\Omega_6^-(q)$, $\Omega_8^+(q)$ or $\Omega_{10}^+(q)$ from [3, Theorem 11.3.2(ii)] (see also [8]).

(B1) Let $k_1 = t(q+1)$ and the graph Φ_1 have parameters

$$\big((t+1)(tq+1), t(q+1), t-1, q+1\big),$$

where $t = q$, q^2, $q^{1/2}$, $q^{3/2}$ for $\widetilde{M} = Sp_4(q)$, $\Omega_6^-(q)$, $SU_4(\sqrt{q})$ or $SU_5(\sqrt{q})$, respectively. If $r = p^l$ is a power of a prime p, then feasible sets of parameters k, r, and μ are described by Table 2, and Γ does not admit H-uniform edge partitions when $t = q, \sqrt{q}$ (this can be easily checked in GAP [14], applying Theorem 1, [16, Proposition 1] and [5, Lemma 3.5, Theorem 5.4]). Moreover, cases $t = q$, q^2, $\sqrt{q}$ correspond to the above cases (A1), (A5) and (A4), respectively.

Table 2. Feasible parameters of Γ with $r = p^l$ in case (B)

	q	$k+1$	k_1, k_2	θ	$-\tau$	$r\mu$	r
(B1), type $t = q$:							
	7	400	56, 343	19	-21	400	$2, 4, 5, 8, 25$
				21	-19	396	2
	9	820	90, 729	21	-39	836	2
				39	-21	800	$2, 4, 5, 8, 16, 25$
(B1), type $t = q^2$:							
	2	45	12, 32	4	-11	50	5
				11	-4	36	$3, 9$
	3	280	36, 243	9	-31	300	$2, 5$
				31	-9	256	$2, 4, 8, 16, 32, 64, 128$
	4	1105	80, 1024	16	-69	1156	17
				69	-16	1050	$5, 25$
(B1), type $t = \sqrt{q}$:							
	16	325	68, 256	9	-36	350	5
				12	-27	338	13
(B1), type $t = \sqrt{q}^3$:	$\varnothing$						
(B2),(B3):	$\varnothing$						

(B2)&(B3) Let $T = \Omega_8^+(q)$, $k_1 = q(q^2+1)(q^3-1)/(q-1)$ and the graph Φ_1 have parameters

$$\big(1 + q(q^2+1)\frac{q^3-1}{q-1} + q^6, q(q^2+1)\frac{q^3-1}{q-1}, q(q^2+1)\frac{q^3-1}{q-1} - q^5 - 1, (q^2+1)\frac{q^3-1}{q-1}\big),$$

(see [3, Theorem 2.2.17, Proposition 3.2.3]) or let $T = \Omega_{10}^+(q)$, $k_1 = q(q^2+1)(q^5-1)/(q-1)$ and the graph Φ_1 have parameters

$$\big((q^4+1)(q^3+1)(q^2+1)(q+1), q(q^2+1)\frac{q^5-1}{q-1}, q-1+q^2(q+1)(q^2+q+1), (q^2+1)(q^2+q+1)\big).$$

As $k+1 \leq 2500$, it follows that $q \leq 3$ and hence either $k+1 = 135, 1120$ and $T = O_8^+(q)$ for $q = 2, 3$ respectively, or $q = 2$, $k+1 = 2295$ and $T = O_{10}^+(2)$. According to Table 2 in any case, none of the parameter sets k, r and μ is feasible (this was checked in GAP [14] using [16, Proposition 1]) and [5, Lemma 3.5, Theorem 5.4]).

(C) Now let us consider the cases $T \simeq \widetilde{M}/Z(\widetilde{M})$, where $\widetilde{M} = SU_m(2)$, $\Omega_{2m}^{\pm}(2)$, $\Omega_{2m}^{\pm}(3)$, $\Omega_{2m-1}(3)$, $\Omega_{2m-1}(4)$ or $\Omega_{2m-1}(8)$ for $m \geq 3$, from [3, Theorem 11.3.2 (iii,iv)] (see also [8]).

(C1) Let $T = U_n(2)$ (see [3, § 3.1.6]) and the graph $\Phi_1 = \mathrm{NU}_n(2)$ have parameters

$$\big(2^{n-1}(2^n - \varepsilon)/3, (2^{n-1} + \varepsilon)(2^{n-2} - \varepsilon), 2^{2n-5}3 - \varepsilon 2^{n-2} - 2, 2^{n-3}3(2^{n-2} - \varepsilon)\big),$$

where $\varepsilon = (-1)^n$.

In view of Table 3 we have $n = 5$ and $k+1 = 176$, i.e. $T \simeq U_5(2)$. Since

$$2(\lambda(\Phi_1) + \lambda(\Phi_2) + 1) \neq k - 1$$

and r divides 4, then either $N \simeq T$ acts transitively on $\mathcal{V}(\Gamma)$, or G is a quasisimple group and by [7] $K \leq \mathrm{M}(T) = Z_2$. But in the first case, by [4], $L = N_{\{F\}} \simeq Z_3 \times U_4(2)$ has no subgroups of index r, a contradiction. In the second case $r = 2$ and Γ admits an H-uniform edge partition with parameters (μ_1, μ_2), and $\{(\mu_1, \mu_2), (\lambda_1, \lambda_2)\} = \{(78, 21), (56, 18)\}$. But then subdegrees of the group G_a on $Q_1(a)$ (recall that $|Q_1(a)| = k_1$) are as follows: $1^1, 6^1, 32^4, 36^1$ (the upper indices denote the multiplicities of the corresponding subdegrees). This contradicts the fact $\lambda_1 \in \{78, 56\}$.

(C2) Let $T = P\Omega_{2n}^{\pm}(2)$ (see [3, § 3.1.2]) and the graph $\Phi_1 = \mathrm{NO}_{2n}^{\varepsilon}(2)$ have parameters

$$\big(2^{2n-1} - \varepsilon 2^{n-1}, 2^{2n-2} - 1, 2^{2n-3} - 2, 2^{2n-3} + \varepsilon 2^{n-2}\big),$$

where $\varepsilon = \pm 1$. Since $k+1 \leq 2500$, $n \leq 6$. Then

$$2(\lambda(\Phi_1) + \lambda(\Phi_2) + 1) = k - 1$$

for all n and ε (see also [17, Example 1]).

Suppose $n = 3$.

If $T \simeq P\Omega_6^+(2) \simeq L_4(2) \simeq \mathrm{Alt}_8$, then $r = 2$ and N is intransitive on $\mathcal{V}(\Gamma)$ (note that Γ is a graph from [17, Theorem 2(ii)]).

Let $T \simeq P\Omega_6^-(2) \simeq U_4(2) \simeq PSp_4(3)$. Then $k+1 = 36$, $\mathrm{M}(T) = Z_2$ and $\mathrm{rk}(T) = 3$. Since $\mathrm{d_{min}}(U_4(2)) = 27$ (see [4]), we get $K \leq Z(G)$.

Assume that N is transitive on $\mathcal{V}(\Gamma)$. Then $r = 2$, $N = G \simeq Sp_4(3)$ or $PSp_4(3)$. Consequently, $G_a \simeq SL_2(9)$ or $G_a \simeq \mathrm{Alt}_6$. In the first case $K = Z(G) \leq G_a$, and in the second case the rank of the transitive representation N on $\mathcal{V}(\Gamma)$ is equal to 5. Both cases are impossible.

Let $n > 3$. Since $\mathrm{d_{min}}(T) = 2^{n-1}(2^n - 1)$ (see [18]) for $\varepsilon = +1$, $\mathrm{d_{min}}(T) = 119$ (see [4]) for $\varepsilon = -1$ and $n = 4$, $\mathrm{d_{min}}(T) = 495$ (see [4]) for $\varepsilon = -1$ and $n = 5$, and $\mathrm{d_{min}}(T) = 2015$ (see [13]) for $\varepsilon = -1$ and $n = 6$, we get $K \leq Z(G)$. Then, by [16, Proposition 3], either $N \simeq T$ is intransitive on $\mathcal{V}(\Gamma)$, or N is transitive on $\mathcal{V}(\Gamma)$. Let us consider the second case. Recall that $\mathrm{M}(T) = Z_2 \times Z_2$ for $n = 4$, $\varepsilon = +1$ and $\mathrm{M}(T) = 1$ otherwise (e.g. see [7]). Further, the group $T_{\{F\}}$ is isomorphic to the group $PSp_{2n}(2)$ (see [13]) and it has no subgroup of index r from the corresponding case in Table 3. Hence $N = G$, $n = 4$, $\varepsilon = +1$ and $r = 2$. By Lemma 1 and Theorem 1 Γ admits an H-uniform edge partition with parameters (μ_1, μ_2) and $\{(\lambda_1, \lambda_2), (\mu_1, \mu_2)\} = \{(32, 28), (30, 27)\}$, and $\mu \in \{64, 54\}$.

Table 3. Feasible parameters of Γ with $r = p^l$ in cases (C1)–(C3)

	n	$k+1$	$k_1,\ k_2$	θ	$-\tau$	$r\mu$	r
(C1)	5	176	135, 40	5	-35	204	2
				35	-5	144	$2, 4$
(C2), $\varepsilon = -1$:	3	36	15, 20	5	-7	36	$2, 3, 9$
				7	-5	32	$2, 4, 8, 16$
	4	136	63, 72	9	-15	140	2
				15	-9	128	$2, 4, 8, 16, 32, 64$
	5	528	255, 272	17	-31	540	$2, 3, 9, 27$
				31	-17	512	$2, 4, 8, 16, 32, 64, 128, 256$
	6	2080	1023, 1056	9	-231	2300	2
				21	-99	2156	2
				27	-77	2128	$2, 4, 8$
				33	-63	2108	2
				63	-33	2048	$r = 2^l, l \leq 10$
				77	-27	2028	$2, 13, 169$
				99	-21	2000	$2, 4, 5, 8, 25, 125$
				231	-9	1856	$2, 4, 8, 32$
(C2), $\varepsilon = +1$:	3	28	15, 12	3	-9	32	$2, 4$
				9	-3	20	2
	4	120	63, 56	7	-17	128	$2, 4, 8$
				17	-7	108	$2, 3, 9, 27$
	5	496	255, 240	15	-33	512	$2, 4, 8, 16$
				33	-15	476	2
	6	2016	1023, 992	13	-155	2156	$2, 7$
				31	-65	2048	$2, 4, 8, 16, 32$
				65	-31	1980	$2, 3, 9$
				155	-13	1872	$2, 3, 4$
(C3), $\varepsilon = -1$:	3	126	45, 80	5	-25	144	$2, 3$
				25	-5	104	$2, 4$
				$\sqrt{125}$	$-\sqrt{125}$	124	2
(C3), $\varepsilon = 1$:	$\varnothing$						

Table 4. Feasible parameters of Γ with $r = p^l$ in case (C4)

	ε, q, n	$k+1$	$k_1,\ k_2$	θ	$-\tau$	$r\mu$	r
(C4)	$-1, 3, 2$	36	20, 15	5	-7	36	$2, 3, 9$
				7	-5	32	$2, 4, 8, 16$
	$-1, 3, 3$	351	224, 126	14	-25	360	$3, 9$
				25	-14	338	$13, 169$
				35	-10	324	$3, 9, 27, 81$
	$1, 3, 2$	45	32, 12	4	-11	50	5
				11	-4	36	$3, 9$
	$1, 3, 3$	378	260, 117	13	-29	392	$2, 7$
				29	-13	360	$2, 3, 4, 9$
				$\sqrt{377}$	$-\sqrt{377}$	376	$2, 4$
	$-1, 4, 2$	120	51, 68	7	-17	128	$2, 4, 8$
				17	-7	108	$2, 3, 9, 27$
	$-1, 4, 3$	2016	975, 1040	13	-155	2156	$2, 7$
				31	-65	2048	$2, 4, 8, 16, 32$
				65	-31	1980	$2, 3, 9$
				155	-13	1872	$2, 3, 4$
	$1, 4, 2$	136	75, 60	9	-15	140	2
				15	-9	128	$2, 4, 8, 16, 32, 64$
	$1, 4, 3$	2080	1071, 1008	9	-231	2300	2
				21	-99	2156	2
				27	-77	2128	$2, 4, 8$
				33	-63	2108	2
				63	-33	2048	$r = 2^l, l \leq 10$
				77	-27	2028	$2, 13, 169$
				99	-21	2000	$2, 4, 5, 8, 25, 125$
				231	-9	1856	$2, 4, 8, 32$
	$-1, 8, 2$	2016	455, 1560	13	-155	2156	$2, 7$
				31	-65	2048	$2, 4, 8, 16, 32$
				65	-31	1980	$2, 3, 9$
				155	-13	1872	$2, 3, 4$
	$1, 8, 2$	2080	567, 1512	9	-231	2300	2
				21	-99	2156	2
				27	-77	2128	$2, 4, 8$
				33	-63	2108	2
				63	-33	2048	$r = 2^l, l \leq 10$
				77	-27	2028	$2, 13, 169$
				99	-21	2000	$2, 4, 5, 8, 25, 125$
				231	-9	1856	$2, 4, 8, 32$

A computer check in GAP [14] shows that in the case when $r = 2$, $N \simeq T$ and N is intransitive on $\mathcal{V}(\Gamma)$, Γ exists and it is unique distance-transitive $(k+1, 2, \mu)$-cover (note it can be also constructed using [17, Theorem 1] or appears in [17, Example 1]).

(C3) Let $T = P\Omega^{\pm}_{2n}(3)$ (see [3, § 3.1.3]) and the graph $\Phi_1 = \mathrm{NO}^{\varepsilon}_{2n}(3)$ have parameters

$$\big(\frac{1}{2}3^{n-1}(3^n-\varepsilon), \frac{1}{2}3^{n-1}(3^{n-1}-\varepsilon), \frac{1}{2}3^{n-2}(3^{n-1}+\varepsilon), \frac{1}{2}3^{n-1}(3^{n-2}-\varepsilon)\big)\big),$$

where $\varepsilon = \pm 1$.

In view of Table 3 we have $k+1 = 126$, $\varepsilon = -1$ and $r \le 4$. Then $T \simeq U_4(3)$ and $\mathrm{d_{min}}(T) = 112$ (see [4]). Hence $K \le Z(G)$. Enumeration of feasible parameters in GAP [14] shows that Γ does not admit H-uniform edge partitions when $\lambda = \mu$, a contradiction with Lemma 1 and Theorem 1.

If $N \simeq T$ acts transitively on $\mathcal{V}(\Gamma)$, then $N_{\{F\}} \simeq U_4(2)$ contains a subgroup of index $r \le 4$, a contradiction. Therefore $G = N$ is a quasi-simple group and, by [7], $r = 2$. Hence, by Lemma 1 and Theorem 1, Γ admits an H-uniform edge partition with parameters (μ_1, μ_2) and $\{(\lambda_1, \lambda_2), (\mu_1, \mu_2)\} = \{(24, 45), (20, 34)\}$. Since $G = N$, the groups $(G_a)^{[a]}$ and $(G_{\{F\}})^{\Sigma-\{F\}}$ are permutation isomorphic. Moreover, for the vertex $b_1 \in Q_1(a)$ the group G_{a,b_1} has exactly two non-single-point orbits on $Q_1(a)$: one orbit of length 12 and one orbit of length 32. This is impossible, since $\lambda_1 \in \{20, 24\}$.

(C4) Let $T = P\Omega_{2n+1}(q)$ (see [3, § 3.1.4]) and the graph $\Phi_1 = \mathrm{NO}_{2n+1}(q)$ have parameters

$$\big(\frac{1}{2}q^n(q^n+\varepsilon), (q^{n-1}+\varepsilon)(q^n-\varepsilon), 2(q^{2n-2}-1)+\varepsilon q^{n-1}(q-1), 2q^{n-1}(q^{n-1}+\varepsilon)\big),$$

where $\varepsilon = \pm 1$, $q = 3, 4, 8$ and $n \ge 2$. According to Table 4, the equality $2(\lambda(\Phi_1)+\lambda(\Phi_2)+1) = k-1$ holds only when either $k+1 = 36$ and $q = 3$ or $q = 4$.

For $q = 3$ we have either $n = 2$ and $\mathrm{d_{min}}(T) = 27$, or $n = 3$ and $\mathrm{d_{min}}(T) = 351$ (see [4]). For even q we have $P\Omega_{2n+1}(q) \simeq PSp_{2n}(q)$ and, by [12, Theorem 2], $\mathrm{d_{min}}(T) = (q^{2n}-1)/(q-1)$, i.e. $\mathrm{d_{min}}(T) = 85$ for $2n = q = 4$, $\mathrm{d_{min}}(T) = 585$ for $4n = q = 8$ and $\mathrm{d_{min}}(T) = 1365$ for $n = 3$ and $q = 4$. Moreover, $r = p^l \ge \mathrm{d_{min}}(T)$ is possible only for $4n = q = 8$. Together with the fact that $PSp_4(8) \not\le GL_{10}(2)$, this implies $K \le Z(G)$.

First we consider the cases when $2(\lambda(\Phi_1)+\lambda(\Phi_2)+1) \ne k-1$.

If $\varepsilon = +1$, $q = 3$ and $n = 2$ then $T \simeq P\Omega_5(3) \simeq PSp_4(3)$ and $\mathrm{rk}(T) = 3$. This possibility was treated in case (A5).

Let $q = n = 3$. Then $T \simeq P\Omega_7(3)$, $\mathrm{rk}(T) = 3$ and $k+1$ is equal to 351 (for $\varepsilon = -1$) or 378 (for $\varepsilon = +1$). In any case by [4] L has no subgroup of index 3, 7 or 13.

Hence if $N \simeq T$ is transitive on $\mathcal{V}(\Gamma)$ then $r = 2$, $\varepsilon = +1$ and $N_a = N_F \simeq L_4(3)$ has two orbits on $[a]$. Moreover, for the vertex $b_2 \in Q_2(a)$ the group N_{a,b_2} has exactly two non-single-point orbits on $Q_2(a)$ (recall that $k_2 = |Q_2(a)| = 117$), and the lengths of these orbits are 80 and 36. This contradicts the fact that by Lemma 1 and Theorem 1 Γ admits an H-uniform edge partition with parameters (μ_1, μ_2) and $\{(\lambda_1, \lambda_2), (\mu_1, \mu_2)\} = \{(133, 56), (126, 60)\}$.

Hence $G = N$ and, by [7], $\mathrm{M}(T) = Z_2 \times Z_3$, which together with Table 4 implies $r \le 3$ for $k+1 = 378$ and $r = 3$ for $k+1 = 351$. Then, by Lemma 1 and Theorem 1, Γ admits an H-uniform edge partition with parameters (μ_1, μ_2). More precisely, if $k+1 = 378$, then $\{(\lambda_1, \lambda_2), (\mu_1, \mu_2)\} = \{(133, 56), (126, 60)\}$ for $r = 2$ and $\{(\lambda_1, \lambda_2), (\mu_1, \mu_2)\} = \{(84, 40), (91, 36)\}$ for $r = 3$, and if $k+1 = 351$, then $\{(\lambda_1, \lambda_2), (\mu_1, \mu_2)\} = \{(75, 40), (73, 45)\}$ and $r = 3$. Since the groups $(G_a)^{[a]}$ and $(G_{\{F\}})^{\Sigma-\{F\}}$ are permutation isomorphic, in the case $r = 2$ a contradiction is achieved in a similar way as above. Let $r = 3$. For $k+1 = 351$ the group G_{a,b_1}, where $b_1 \in Q_1(a)$, has five orbits on $Q_1(a)$ (recall that $k_1 = |Q_1(a)| = 224$): two orbits of length 81, one orbit of length 60 and two single-point orbits. This is impossible, since $\lambda_1 = 73$ or 75. Let $k+1 = 378$. Since for the vertex $b_2 \in Q_2(a)$ the group G_{a,b_2} has exactly two non-single-point orbits on $Q_2(a)$ (recall that $k_2 = |Q_2(a)| = 117$), and the lengths of these orbits are 80 and 36, then $\lambda_2 = 36$. But then $\mu_2 = 40$, which is impossible, since $G_a = G_F$ and the group G_{a,b_2} moves 36 or 80 vertices from $Q_2(a^*) \cap [b_2]$ for some vertex $a^* \in F(a)$.

Let $q = 8$. According to Table 4 $T \simeq P\Omega_5(8) \simeq PSp_4(8)$ and as noted above $\mathrm{rk}(T) = 5$. Further, the group $(\widetilde{G}_{\{F\}})^{\Sigma-\{F\}}$ has the form $L_2(64).Z_3.Z_2$ for $k+1 = 2016$ and $(L_2(8) \times L_2(8)).Z_6$ for $k+1 = 2080$. Hence, by [16, Proposition 3] and taking into account that $\mathrm{M}(T) = 1$, we obtain either $r = 4$, one of -65 or 63 is an eigenvalue of Γ and N is intransitive on $\mathcal{V}(\Gamma)$, or $N \simeq T$ acts transitively on $\mathcal{V}(\Gamma)$. Let us consider the second case. If $k+1 = 2080$ then for a subgroup of index r in $N_{\{F\}}$ we have either $p = 3$ and r divides 3^5, or $r = p = 2$. If $k+1 = 2016$ then for a subgroup of index r in $N_{\{F\}}$ we have $r = p \le 3$. Enumeration of the orbital graphs of N in GAP [14] shows that the case $r = 3$ is impossible, while for $r = 2$ the graph Γ exists: for $k+1 = 2016$ the parameter μ equals to 1024 or 990, and for $k+1 = 2080$ the parameter μ equals to 1024 or 1054. More precisely, for each feasible set of parameters k, μ, it turns out to be the unique (up to isomorphism) distance-transitive $(k+1, 2, \mu)$-cover.

Now let $2(\lambda(\Phi_1) + \lambda(\Phi_2) + 1) = k - 1$.

Let us consider the case when N is transitive on $\mathcal{V}(\Gamma)$.

For transitive N, the case $\varepsilon = -1$, $q = 3$ and $n = 2$ was excluded earlier in (C2).

Let $q = 4$. Then $\mathrm{rk}(T) = 3$ and by [7] $\mathrm{M}(T) = 1$. If $n = 2$ then by [4] $N_{\{F\}} \simeq L_2(16)$ (for $k+1 = 120$) or $(\mathrm{Alt}_5 \times \mathrm{Alt}_5) : Z_2$ (for $k+1 = 136$) has no subgroup of index 3, so $r = 2$. If $n = 3$, then $N_{\{F\}} \simeq P\Omega_6^{\varepsilon}(4) : Z_2$ (see [13]) has no subgroup of index $3, 5, 7$ or 13, so $r = 2$ again. Enumeration of the orbital graphs of $PSp_{2n}(q)$ on $r(k+1)$ points in GAP [14] shows that none of these cases is realized.

A computer check in GAP [14] shows that in the case when $r = 2$, $N \simeq T$ and N is intransitive on $\mathcal{V}(\Gamma)$, Γ exists and it is unique distance-transitive $(k+1, 2, \mu)$-cover (note it can be also constructed using [17, Theorem 1]).

(D) Finally, let the pair $(\widetilde{M}, Y)$, where Y is the pre-image in $\widetilde{M}$ of a point stabilizer in T, be one of the following (up to conjugacy in $\mathrm{Aut}(\widetilde{M})$ (see [3, § 11.3.2, Theorem 11.3.2(v)–(x)])):

$$(SU_3(3), PSL_3(2)),\ (SU_3(5), 3.\mathrm{Alt}_7),\ (SU_4(3), 4.PSL_3(4)),\ (Sp_6(2), G_2(2)),\ (\Omega_7(3), G_2(3)),\ (SU_6(2), 3.PSU_4(3).2);$$

let further the graph Φ_1 have parameters

$$(36, 14, 4, 6), \quad (50, 7, 0, 1), \quad (162, 56, 10, 24), \quad (120, 56, 28, 24), \quad (1080, 351, 126, 108) \quad \text{or} \quad (1408, 567, 246, 216),$$

respectively (for their detailed description, see [3, §-§ 10.14, 10.19, 10.48, 10.39, 10.78, 10.81]). Then feasible parameters of Γ are described by Table 5, which, in particular, shows the cases $k+1 = 56$, 1080 are impossible.

Let $T \simeq SU_3(3)$. Then $\mathrm{rk}(T) = 4$, $\mathrm{M}(T) = 1$, and by [4] $\mathrm{d}_{\min}(T) = 28 > r$. Hence $K \le Z(G)$ and $N \simeq T$. Suppose N is intransitive on $\mathcal{V}(\Gamma)$. Then by [16, Proposition 3] we have either $r = 4$ and 7 is an eigenvalue of Γ, or $r = 2$ and $\gamma = -2(\lambda(\Phi_i) + k_j \mu(\Phi_i)/k_i + 1) + k$ is an eigenvalue of Γ. In the second case $\gamma \in \{\pm 7\}$, which in view of Table 5 implies $\mu \in \{16, 18\}$. Computer check in GAP [14] shows that for $r = 2$ and each μ, Γ exists and it is the only (up to isomorphism) distance-transitive $(36, 2, \mu)$-cover.

Suppose $N \simeq T$ is transitive on $\mathcal{V}(\Gamma)$. Then $N_{\{F\}} \simeq L_3(2)$ must contain a subgroup of index r. But in view of [4] the index of a proper subgroup in $L_3(2)$ must be divisible by 7 or 8, which implies $r = 8$. Enumeration of the orbital graphs of the group $SU_3(3)$ on $36r$ points in GAP [14] shows that this is impossible.

For $r = 4$ enumeration of the orbital graphs of the group $K \times SU_3(3)$ on 144 points in GAP [14] shows this case is also impossible.

In all other cases $\mathrm{rk}(T) = 3$ and $\mathrm{d}_{\min}(T) > r$. Hence $K \leq Z(G)$ and, by the remark after Proposition 1, we will assume that r is prime.

For $T \simeq PSU_3(5)$ we have $2(\lambda(\Phi_1) + \lambda(\Phi_2) + 1) \neq k - 1$ and by [7] $\mathrm{M}(T) = Z_3$. In view of Table 5 $r = 2$ and hence $N = G' \simeq T$. Enumeration of the orbital graphs of the group $Z_2 \times SU_3(5)$ on 100 points in GAP [14] shows this case is impossible.

For $T \simeq Sp_6(2)$, we have $2(\lambda(\Phi_1) + \lambda(\Phi_2) + 1) = k - 1$ and, by [7] $\mathrm{M}(T) = 1$, so $N = G' \simeq T$. Since the rank of the representation of the group $Sp_6(2)$ on cosets by its subgroup isomorphic to the group $G_2(2)'$, equals 5, we obtain that N is intransitive on $\mathcal{V}(\Gamma)$. Further, in view of Lemma 1 and Theorem 1 Γ admits an H-uniform edge partition with parameters (μ_1, μ_2) and either $\{(\lambda_1, \lambda_2), (\mu_1, \mu_2)\} = \{(28, 32), (27, 30)\}$ and $r = 2$, or $\{(\lambda_1, \lambda_2), (\mu_1, \mu_2)\} = \{(18, 20), (19, 22)\}$ and $r = 3$. Since the groups $(G_a)^{[a]}$ and $(G_{\{F\}})^{\Sigma-\{F\}}$ are permutation isomorphic, for $b_1 \in Q_1(a)$ G_{a,b_1}-orbits on $Q_1(a)$ have lengths 1, 1, 27 and 27. For $r = 3$ this is impossible, since $\lambda_1 = 18$ or 19. Hence $r = 2$. Enumeration of the orbital graphs of the group $Z_r \times Sp_6(2)$ on 240 points in GAP [14] shows that Γ exists and it is distance-transitive with $\mu = 54$ or 64.

For $T \simeq PSU_6(2)$ we have $2(\lambda(\Phi_1) + \lambda(\Phi_2) + 1) \neq k - 1$ and, by [7], $\mathrm{M}(T) = Z_3 \times Z_2 \times Z_2$. Since the rank of transitive representation of $PSU_6(2)$ on its right X-cosets with $X \simeq U_4(3)$ equals 5, then G is a quasi-simple group and $r = 2$. In view of Lemma 1 and Theorem 1 Γ admits an H-uniform edge partition with parameters (μ_1, μ_2) and $\{(\lambda_1, \lambda_2), (\mu_1, \mu_2)\} = \{(286, 429), (280, 410)\}$. Since the groups $(G_a)^{[a]}$ and $(G_{\{F\}})^{\Sigma-\{F\}}$ are permutation isomorphic, for $b_1 \in Q_1(a)$ G_{a,b_1}-orbits on $Q_1(a)$ have lengths 1, 320, 30, 96 and 120. This is a contradiction, since $\lambda_1 = 286$ or 280.

Table 5. Feasible parameters of Γ with $r = p^l$ in case (D)

$(\widetilde{M}, Y)$	$k+1$	k_1, k_2	θ	$-\tau$	$r\mu$	r
$(SU_3(3), PSL_3(2))$	36	14, 21	5	-7	36	$2, 3, 9$
			7	-5	32	$2, 4, 8, 16$
$(SU_3(5), 3.\mathrm{Alt}_7)$	50	7, 42	7	-7	48	$2, 4, 8$
$(Sp_6(2), G_2(2))$	120	56, 63	7	-17	128	$2, 4, 8$
			17	-7	108	$2, 3, 9, 27$
$(SU_6(2), 3.PSU_4(3).2)$	1408	567, 840	21	-67	1452	$2, 11$
			67	-21	1360	$2, 4, 8$

□

Theorem 3. *Suppose that* $\Gamma = \Gamma(\widetilde{G}, G, K)$ *is a minimal abelian* $(k+1, r, \mu)$*-cover,* $k+1 \leq 2500$*,* $\mathrm{rk}(\widetilde{G}^{\Sigma}) = 3$ *and* $T = \mathrm{Soc}(\widetilde{G}^{\Sigma}) \simeq PSL_d(q)$*. Assume* $\widetilde{G} = G$ *whenever* $\mathrm{rk}(T) = 3$*. Suppose further that* $(T, k+1) \neq (\mathrm{Alt}_s, \binom{s}{2})$*. Then* $\widetilde{G}^{\Sigma} \simeq P\Gamma L_2(8)$*,* $k+1 = 36$*,* $r = 2$*,* $\mu \in \{16, 18\}$*,* $G' \simeq T$*,* G' *is transitive on* $\mathcal{V}(\Gamma)$*, and* Γ *is a unique (up to isomorphism) distance-transitive* $(36, 2, \mu)$*-cover.*

P r o o f. Let $T \simeq PSL_d(q)$. Next we consider potential combinations for T and the complementary rank 3 graphs Φ_1 and Φ_2 associated with $\widetilde{G}^{\Sigma}$, applying their description from [8] and [3, Theorem 11.3.3]. Since $k + 1 \leq 2500$, we are left with the following two cases (E) and (H).

(E) Let either $T = PSL_2(4) \simeq PSL_2(5) \simeq \mathrm{Alt}_5$, $k+1 = \binom{5}{2}$, or $T = PSL_2(9) \simeq \mathrm{Alt}_6$, $k+1 = \binom{6}{2}$, or $T = PSL_4(2) \simeq \mathrm{Alt}_8$, $k+1 = \binom{8}{2}$, or $G = P\Gamma L_2(8)$, $k+1 = \binom{9}{2}$ (see [8] and also [3, Theorem 11.3.3(ii)]). Then $\Phi_1 \simeq T(m)$ and $m = 5, 6, 8, 9$, respectively. The cases $m \le 8$ were considered in [17, Theorem 2]. Below we treat the remaining case $m = 9$.

Let $k+1 = 36$ and $\widetilde{G}^{\Sigma} \simeq P\Gamma L_2(8)$. Then $T \simeq L_2(8)$, $\mathrm{rk}(T) = 4$, $\mathrm{M}(T) = 1$, the graph Φ_1 has parameters $(36, 14, 7, 4)$ and $2(\lambda(\Phi_1) + \lambda(\Phi_2) + 1) \ne k - 1$. If $r = p^l$, p is prime, then $p \le 3$. Note that $L_2(8) \not\le GL_l(3)$ for $l < 4$ and $L_2(8) \not\le GL_l(2)$ for $l < 5$. Hence $K \le Z(G)$. By [16, Proposition 3] $r \ne 3$ and if $r \le 16$, then by [16, Proposition 3] $G' \simeq T$ is transitive on $\mathcal{V}(\Gamma)$, which, in view of [4], implies $r = 2$. Enumeration of the orbital graphs of the group $Z_2 \times L_2(8)$ on 72 points in GAP [14] shows that $\mu = 16$ or 18, and Γ is a unique distance-transitive $(36, 2, \mu)$-cover (see also [16, Example]).

(H) If $T = PSL_3(4)$, $T_{\{F\}} \simeq \mathrm{Alt}_6$ and Φ_1 is the Gewirtz graph (with parameters $(56, 10, 0, 2)$) or $T = PSL_4(3)$, $T_{\{F\}} \simeq PSp_4(3)$ and $\Phi_1 \simeq \mathrm{NO}_6^{+}(3)$ (with parameters $(117, 36, 15, 9)$), then there is no feasible set of parameters. □

Remark 1. In proofs of Theorems 2 and 3, in a computer search for distance-regular orbital graphs we used GAP packages GRAPE [15] and coco2p [10].

Remark 2. An explicit construction of covers with $r = 2$ and intransitive group G' from the conclusions of Theorem 2 can be found in [17, Theorem 1, Example 1].

Corollary 1. *Suppose that Ψ is a non-bipartite abelian (n, r', μ')-cover with a transitive group of automorphisms X that induces a primitive almost simple permutation group X^{Ξ} on the set Ξ of its antipodal classes such that $\mathrm{rk}(X^{\Xi}) = 3$ and the pair (X^{Ξ}, n) satisfies conditions of Theorem 2 or 3. Then Ψ has a minimal quotient $\Gamma(\widetilde{G}, G, K)$ that is an (n, r, μ)-cover from the conclusion of the respective theorem with $\mathrm{Soc}(X^{\Xi}) \simeq G/K$ and $r'\mu' = r\mu$.*

4. Concluding remarks

In this paper, we continued studying abelian antipodal distance-regular graphs Γ of diameter 3 with the property $(*)$: *Γ has a transitive group of automorphisms $\widetilde{G}$ that induces a primitive almost simple permutation group $\widetilde{G}^{\Sigma}$ on the set Σ of its antipodal classes.* As in [16], we focused on the case $\mathrm{rk}(\widetilde{G}^{\Sigma}) = 3$. In [16] and [17], it was shown that in the alternating and sporadic cases for $\widetilde{G}^{\Sigma}$ the family of non-bipartite graphs Γ with the property $(*)$ and $\mathrm{rk}(\widetilde{G}^{\Sigma}) = 3$ is finite and limited to a small number of potential examples with $|\Sigma| \in \{10, 28, 120, 176, 3510\}$. Here we assumed that the socle of $\widetilde{G}^{\Sigma}$ is a classical simple group. The case of classical simple socle seems to be both most interesting and complicated, since, on one hand, there is an infinite family of non-bipartite representatives Γ (see [17, Example 1]), and on the other hand, its study requires a profound inspection of $\widetilde{G}^{\Sigma}$. So we started classification of graphs Γ with "small" $|\Sigma|$. In order to describe minimal quotients of Γ, we used the technique for bounding their spectrum that is based on analysis of their local properties and the structure of $\widetilde{G}^{\Sigma}$, which was developed in [16] and applied in [16] and [17] for the cases of sporadic, alternating and exceptional socle (the latter was investigated under condition $|\Sigma| \le 2500$). As a result, we significantly refined the sets of feasible parameters of Γ with $|\Sigma| \le 2500$ in the case of classical socle, showing, in particular, that for most of these sets Γ must be a covering of a certain distance-transitive Taylor graph.

We also wish to mention two more challenging examples of graphs with the property $(*)$, namely, abelian $(n, 3, 12)$-covers with $n = 36$ or 45 and $\mathrm{rk}(\widetilde{G}^{\Sigma}) = 4$ or 5, respectively (for their constructions, see [9]). A computer assisted inspection shows that they are the only minimal abelian (n, r, μ)-covers $\Gamma(\widetilde{G}, G, K)$ such that $3 \leq \mathrm{rk}(\widetilde{G}^{\Sigma}) \leq 5$, $r > 2$, $n \leq 2500$ and $G = G'$ is a quasi-simple group.

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BIHARMONIC GREEN FUNCTION AND BISUPERMEDIAN ON INFINITE NETWORKS

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Abstract: In this article, we have discussed Biharmonic Green function on an infinite network and bimedian functions. We have proved some standard results in terms of supermedian and bimedian. Also, we have proved the Discrete Riquier problem in the setting of bimedian functions.

Keywords: Biharmonic Green function, Bimedian function, Dirichlet problem, Discrete Riquier problem, Hyperbolic networks.

1. Introduction

An electrical network X is a finite graph consisting of a finite number of nodes and branches, each branch connecting some two nodes. There is a certain resistance $r(a,b)$ on each branch $[a,b]$ connecting the nodes a and b in X; the reciprocal of the resistance is called the conductance $c(a,b)$. Thus, an electrical network X can be considered as a graph $\{X, c(a,b)\}$ with finite number of vertices (nodes) and a finite number of edges (branches); the non-negative conductance $c(a,b)$ is positive if and only if a and b are neighbors, that is $[a,b]$ is an edge in X. A vertex e in X is called a terminal vertex if e has only one neighbor in X. If a and b are neighbors, we write $b \sim a$. We assume also that if $a \sim b$, then there is only one branch $[a,b]$ connecting a and b and there is no self-loops in X; there is no edge of the form $[a,a]$ so that $c(a,b) = 0$ for all a in X. We also assume that there is always a path $\{a = a_0, a_1, a_2, \ldots, a_n = b\}$ connecting two vertices a and b in X where $a_i \sim a_{i+1}$ for $0 \le i \le n-1$.

An electrical current regime voltage is considered on the finite network $\{X, c(a,b)\}$ assuming the Ohm–Kirchhoff laws: if ψ is the potential function on X, when extremal currents are applied on X, then the voltage on the branch $[a,b]$ is $[\psi(b) - \psi(a)]$ and the current is $c(a,b)[\psi(b) - \psi(a)]$ so that the total current at the node a is $\sum\limits_{b\sim a} c(a,b)[\psi(a) - \psi(b)]$. Based on these basic notions, the condenser principle, the equilibrium principle, the minimum principle, etc. are proved on X in [4].

In an abstract sense, can we consider these principles on an infinite network in a meaningful manner? Is it possible to think of an infinite electrical network with Ohm–Kirchhoff laws suitably modified by Nash-Williams [10] in his remarkable paper on random walks and electrical currents in networks, where it is shown that the random walk in probability theory has features analogous to electrical networks. A random walk considered as an irreducible, reversible Markov chain will serve as a model to develop a function theory on infinite graphs analogous to that of electrical networks (Abodayeh and Anandam [1, 2], Woess [12] and Zemanian [13]). Let $\{X, p(a,b)\}$ stand for a countably infinite state X with the transition probabilities $p(a,b)$. Assume that $\{X, p(a,b)\}$ is irreducible, i.e., is it possible to move along a path from any state a to any other state b in X; it is also reversible, i.e., there is a function $\varphi(a) > 0$ on X such that $\varphi(a)p(a,b) = \varphi(b)p(b,a)$ for any two states a and b in X. Then, as an example of the analogy between random walks and electrical currents, consider two disjoint subsets A and B. Denote by $\psi(a)$ the probability that the walker starting at the state a reaches A before meeting any state in B. Then $\psi(a) = 1$ for $a \in A$ and $\psi(a) = 0$ for $a \in B$; if $a \notin A \cup B$, then $\varphi(x) = \sum_{b \sim a} p(a,b)\psi(a)$ and, since $\sum_{b \sim a} p(a,b) = 1$, we have $\sum_{b \sim a} p(a,b)[\psi(b) - \psi(a)] = 0$, which is equality analogous to the situation where the total current at the node a is 0.

In a random walk, the Green function $G(a,b)$ represents the expected number of visits that the walker starting at a makes to reach b. The function $G(a,b)$ takes the value ∞ if $\{X, p(a,b)\}$ is recurrent, i.e., the walker starting at any state a comes back to a infinitely often; $G(a,b) < \infty$ for all pairs a, b if $\{X, p(a,b)\}$ is transient, i.e., the walker starting at a vertex a definitely wanders off from a. A situation similar to this occurs in the study of Riemann surfaces. If a Riemann surface R is parabolic, there is no Green potential on R. If R is hyperbolic, then there is a Green kernel on R.

When this analogy between random walks and functions on Riemann surfaces is properly developed, a successful application of function-theoretic methods on Riemann surfaces to solve problems in random walks on an irreducible, reversible $\{X, p(a,b)\}$ is possible. For this case, we can define the Dirichlet norm on a, and then the functional analysis methods enable us to establish a correspondence between some function-theoretic problems on a Riemann surface and problems connected with a random walk on X. For example, Lyons [9], modifying a Royden criterion on Riemann surfaces, gives a necessary and sufficient condition for a reversible Markov chain to be transient. However, these arguments establishing relations between random walks and Riemann surfaces are valid only when it is assumed that the random walk is reversible. Intending to develop a function theory on infinite networks that will be applicable even in the case of non-reversible Markov chains, we adopt here potential theoretic methods on locally compact spaces. The basic result is the solution to a generalized Dirichlet problem in infinite networks; using which we introduce the analogous of balayage, maximum principle, equilibrium principle, condenser principle, the classifications based on the notions of transient and recurrent random walks, etc. in infinite networks.

2. Preliminaries

Let N be an infinite graph that is connected and locally finite but without self-loops [4, 7]. Let $\varphi(a,b) \geq 0$ be a nonnegative number associated with each pair of vertices a and b in N such that $\varphi(a,b) > 0$ iff $b \sim a$. Then $\{N, \varphi(a,b)\}$ is called an infinite network. We do not assume that $\varphi(a,b)$ is symmetric. Given a set B in N, say that a vertex a is an interior vertex of B if a and all its neighbors are in B; denote by $\mathring{B}$ the set of all interior vertices of B, and let $\partial B = B \setminus \mathring{B}$. If $f(a)$ is a real-valued function on B, write

$$\Delta f(a) = \sum_{b \sim a} \varphi(a,b)\,[f(b) - f(a)]$$

for any $a \in \overset{\circ}{B}$. Say that $f(a)$ is superharmonic on B if $\Delta f(a) \leq 0$ for any $a \in \overset{\circ}{B}$; and $f(a)$ is said to be harmonic on B if $\Delta f(a) = 0$ for any $x \in \overset{\circ}{B}$. A function $f(a)$ on B is subharmonic if $-f(a)$ is superharmonic on B. The following statements are valid.

(1) If $\{f_n(a)\}$ is a sequence of superharmonic functions on B and $f(a) = \lim_n f_n(a)$ is real-valued on B, then $f(a)$ is superharmonic on B; consequently, if $\{g_n(a)\}$ is a sequence of superharmonic functions on B such that $g(a) = \sum_n g_n(a)$ is finite for each a in B, then $g(a)$ is superharmonic on B.

(2) **Minimum Principle:** If $s(a) \geq 0$ is superharmonic on N and $s(a_0) = 0$ for some vertex a_0, then $s \equiv 0$.

(3) **Greatest harmonic minorant:** Let f be superharmonic on B and g be subharmonic on B such that $f \geq g$ on B. Let $u(a)$ be the sequence $\mathfrak{F}$ of all subharmonic functions on B such that $u \leq f$. Let

$$\lambda(a) = \sup_{u \in \mathfrak{F}} u(a)$$

for $x \in B$. Then $\lambda(a)$ is a harmonic function on B such that if $\lambda'(a)$ is another harmonic function and $\lambda' \leq f$ on B, then $\lambda' \leq \lambda$. We call $\lambda(a)$ the greatest harmonic minorant of f on B. Similarly, we define the least harmonic majorant of g on B.

(4) **Generalised Dirichlet Solution:** Let F be an arbitrary set in N and $B \subset \overset{\circ}{F}$. Suppose that $u(a)$ is a real-valued function on $F \setminus B$ such that there exist a superharonic function f and a subharmonic function g on F such that $f \geq g$ on F and $f \geq u \geq g$ on $F \setminus B$. Then there exists a function λ on F such that $\lambda = u$ on $F \setminus B$ and $\Delta\lambda(a) = 0$ for $a \in B$. This generalised Dirichlet solution λ on F is uniquely determined if F is a finite set.

3. Biharmonic Green Function

Definition 1 (Potential). *A nonnegative superharmonic function p defined on a subset B is said to be potential if and only if the greatest harmonic minorant of p on B is* 0.

Definition 2 (Bipotential). *A potential u in (N, p) is said to be a bipotential if and only if $(-\Delta)u = p$, where p is a potential in N. We say that N is a bipotential network if there exists a positive bipotential on N.*

Definition 3 (Biharmonic Green function). *For a fixed vertex z in N, a potential $u_z(a)$ in (N, p) is said to be the biharmonic Green function with biharmonic support $\{z\}$ if and only if $(-\Delta)u_z(a) = G_z(a)$, where $G_z(a)$ is the harmonic Green function with harmonic support z.*

Proposition 1. *The biharmonic Green function exists on (N, p) if and only if there is a positive bipotential on (N, p).*

P r o o f. Clearly, the binarmonic Green function is a bipotential. Conversely, let u be a positive bipotential, $(-\Delta)u = p$. Then,

$$u(a) = \sum_b G(a, b)p(b).$$

Let z be a fixed vertex. Then, for some $\lambda > 0$, $G_z(b) \leq \lambda p(b)$ for any $b \in N$ (Domination Principle). Hence,

$$Q_z(a) = \sum_b G(a, b)G_z(b)$$

is a well-defined potential such that $(-\Delta)u_z(a) = G_z(a)$. □

Theorem 1. *Let (N,p) be a bipotential infinite network, and let $u_y(a)$ be the biharmonic Green potential on (N,p). If $\sum_b f(b)u_b(a)$ is finite at some vertex a_0 for some $f > 0$, then*

$$u(a) = \sum_b f(b)u_b(a)$$

is a bipotential on (N,p). Conversely, every bipotential $u(a)$ can be represented as

$$u(a) = \sum_b f(b)u_b(a),$$

where $f(a) = (-\Delta)^2 u(a)$.

Proof. Let $(-\Delta)u = p$ on (N,p). For a finite set E in (N,p), write

$$s(a) = u(a) - \sum_{b\in E}(-\Delta)p(b)u_b(a).$$

Then,

$$(-\Delta)s(a) = p(a) - \sum_{b\in E}(-\Delta_q)p(b)G_b(a) = \sum_{b\notin E}(-\Delta)p(b)G_b(a) \geq 0.$$

Hence, s is superharmonic on (N,p), and since

$$-s(a) \leq \sum_{b\in E}(-\Delta)p(b)u_b(a),$$

we conclude that $-s \leq 0$. Hence,

$$u(a) \geq \sum_{b\in E}(-\Delta)p(b)u_b(a).$$

Allow E to grow into (N,p), to conclude that

$$u(a) \geq \sum_{b\in E}(-\Delta)p(b)u_b(a).$$

Write

$$\varphi(a) = q(a) - \sum_{b\in(N,p)}(-\Delta)p(b)u_b(a).$$

Then,

$$(-\Delta)\varphi(a) = p(a) - \sum_{b\in(N,p)}(-\Delta)p(b)G_b(a) = 0.$$

Hence, $\varphi(a)$ is a nonnegative harmonic function majorized by the potential $u(a)$. Hence, $\varphi = 0$ so that

$$u(a) = \sum_{b\in(N,p)}(-\Delta)p(b)u_b(a)$$

for any $a \in (N,p)$. If $f(a) = (-\Delta)p(a)$, then $f \geq 0$ and $f(a) = (-\Delta)^2 u(a)$. Conversely, suppose that

$$u(a) = \sum_b f(b)u_b(a),$$

which is a convergent sum of potentials if $u(a_0)$ is finite at some vertex a_0. Then, $u(a)$ is a potential in (N,p) and

$$(-\Delta)u(a) = \sum_b f(b)(-\Delta)u_b(a) = \sum_b f(b)G_b(a),$$

being finite at each a, defines a potential $p(a)$. Thus, $(-\Delta)u(a) = p(a)$. □

Proposition 2. *Let (N,p) be a bipotential infinite network. For $z \in (N,p)$, if $u_z(a)$ and $G_z(a)$ are the biharmonic and harmonic Green potentials, then $u_z(a) > G_z(a)$ for any $a \in (N,p)$.*

P r o o f. Since $(-\Delta)u_z(a) = G_z(a)$, $(-\Delta)G_z(a) = \delta_z(a)$, and $G_z(z) \geq G_z(a)$ for all a (Domination principle), we have

$$(-\Delta)\left[\frac{u_z(a)}{G_z(z)}\right] = \frac{G_z(a)}{G_z(z)} \geq \delta_z(a) = (-\Delta)G_z(a).$$

Hence,

$$u(a) = \frac{u_z(a)}{G_z(z)} - G_z(a)$$

is a superharmonic function such that $-u(a) \leq G_z(a)$; hence, $-u \leq 0$ on (N,p). Consequently,

$$u_z(a) \geq G_z(z)G_z(a) > G_z(a)$$

since $G_z(z) > 1$. □

Proposition 3. *Let u be a potential in (N,p), $(-\Delta)u = p$. Suppose that $0 \leq f \leq p$. Then, there exists a potential v, $v \leq u$, such that $(-\Delta)v = f$ on (N,p).*

P r o o f. Let

$$u(a) = \sum_b G(a,b)p(b) \geq \sum_b G(a,b)f(b) = v(a),$$

then $v(a)$ is a potential, $v \leq u$ and $(-\Delta)v(a) = f(a)$ for all $a \in (N,p)$. □

Corollary 1. *Let (N,p) be a bipotential infinite network. If u is a potentials with finite harmonic support in (N,p), then there exist is a bipotential v on (N,p) such that $(-\Delta)v = u$ on (N,p).*

P r o o f. By hypothesis, there are postive potentials p and q such that $(-\Delta)u = p$ on (N,p). Since u has finite harmonic support, $u \leq \lambda p$ on (N,p) for some $\lambda > 0$ (Domination Principle). Hence use the above Proposition , there is a potential $v \leq \lambda q$ such that $(-\Delta)v = u$ on (N,p). □

Lemma 1. *Let F be a finite subset in any infinite network (N,p). Let $E \subset \overset{\circ}{F}$ and $f \geq 0$ be a real-valued function on E. Then there exist a potential u on F such that $(-\Delta)u(a) = f(a)$ for any $a \in E$.*

P r o o f. Assume that f is defined on F by giving it values 0 in $F \setminus E$. Let $G_b^F(a)$ be the Green function in F with point harmonic support $b \in \overset{\circ}{F}$ such that $G_b^F(a) = 0$ if $a \in \partial F$. Let

$$u(a) = \sum_{b \in F} f(b)G_b^F(a).$$

Then, u is a potential in F such that $(-\Delta)u(a) = f(a)$ if $a \in E$. □

4. Transient and hyperbolic networks

Let $\{N, p(a,b)\}$ be a countable set of state space N with transition probabilities $\{p(a,b)\}$. Let E be a fixed vertex in N. If a walker starting at e comes back to the state e infinitely often, i.e., with probability 1, then $\{N, p(a,b)\}$ is said to be recurrent; otherwise, it is transient (see [5, 6]).

In classical potential theory (Brelot [8] and Al-Gwaiz M.A., Anandam V. [3]), a superharmonic function $s(a) \geq 0$ is called a potential if its greatest harmonic minorant is 0. In the discrete case, we can show that the superharmoinc function $s(a) \geq 0$ on an infinite network $\{N, \varphi(a,b)\}$ is a potential if its greatest harmonic minorant is 0. If there exists a potential $p(a) > 0$ on N, then we say that $\{N, \varphi(a,b)\}$ is a hyperbolic network; otherwise, it is called a parabolic network.

Let $\{N, \varphi(a,b)\}$ be an infinite network. Write

$$\varphi(a) = \sum_{b \sim a} \varphi(a,b).$$

Then $0 < \varphi(a) < \infty$. Write

$$p(a,b) = \frac{\varphi(a,b)}{\varphi(a)}.$$

Then $\{N, p(a,b)\}$ becomes a probability space, which need not be reversible. Therefore, we can say that $\{N, \varphi(a,b)\}$ is transient when the associated probability space $\{N, p(a,b)\}$ is transient.

Theorem 2. *The infinite network* $\{N, \varphi(a,b)\}$ *is transient if and only if it is hyperbolic.*

P r o o f. Let e be a fixed vertex in N. Consider a sequence of finite subsets $\{F_n\}$ such that $e \in \mathring{F}_1$, $F_n \subset \mathring{F}_{n+1}$, and $N = \bigcup_n F_n$. For a vertex a in N, let $\psi_n(a)$ denote the probability that the walker starting at a reaches the vertex e before contacting any vertex in F_n^C. Then $\psi_n(e) = 1$, $\psi_n(a) = 0$ for $a \notin F_n$, and

$$\psi_n(a) = \sum_b p(a,b)\psi_n(b)$$

for $a \notin \{e\} \cup \{F_n^C\}$. Since

$$\sum_b p(a,b) = 1$$

for all a, we have

$$\sum_b p(a,b)[\psi_n(b) - \psi_n(a)] = 0;$$

that is $\Delta\psi_n(a) = 0$ if $a \notin \{e\} \cup \{F_n^C\}$. Since $\{\psi_n(a)\}$ is an increasing sequence, $\psi(a) = \lim_n \psi_n(a)$ exists and $0 \leq \psi(a) \leq 1$ for all a in N. Clearly, $\psi(a)$ denotes the probability that the walker starting at $\{e\}$ returns to $\{e\}$. Consequently, $\psi \equiv 1$ if and only if N is recurrent. Hence, $\{N, \psi(a,b)\}$ is transient if and only if ψ is not the constant 1.

Now, another interpretation of $\psi_n(a)$ is that it is the Dirichlet solution with boundary values $\psi_n(e) = 1$ and $\psi_n(a) = 0$ if $a \notin \mathring{F}_n$. Hence, if we extend ψ_n to the whole space N assuming it equal to 0 on F_n, then $\psi_n(a)$ becomes subharmonic at each vertex other than e, harmonic at each vertex in $\mathring{F}_n \setminus \{e\}$, and superharmonic at e. Hence, in the limit, we find that $\psi(a)$ is a nonnegative superharmonic function on N that is harmonic outside the vertex e. Consequently, if ψ is not the constant 1, then ψ is a positive superharmonic function that is not harmonic on N. Let $h(a)$ be the greatest harmonic minorant of $\psi(a)$ on N. Then, $p(a) = \psi(a) - h(a)$ is a positive superharmonic function that is a potential on N. That is, $\{N, \psi(a,b)\}$ is hyperbolic. Then, the following statements are equivalent:

(1) the function φ is not the constant 1;

(2) the probability space $\{N, p(a,b)\}$ is transient;

(3) the infinite network $\{N, \varphi(a,b)\}$ is hyperbolic.

□

5. Bimedian functions on infinite networks

In this section, we assume that $\{N, \varphi(a,b)\}$ is an infinite network that is a tree without terminal vertices. Write

$$Au(a) = u(a) - \sum_{b\sim a} \varphi(a,b)u(b)$$

for a real-valued function $u(a)$ on N. Note that A is the Lapalcian operator $-\Delta$ if

$$\varphi(a) = \sum_{b\sim a} \varphi(a,b) = 1$$

for all a in N.

Definition 4. *A real-valued function $u(a)$ on N is said to be supermedian if*

$$u(a) \geq \sum_{b\sim a} \varphi(a,b)u(b)$$

for all a in N; $u(a)$ is said to be median if

$$u(a) = \sum_{b\sim a} \varphi(a,b)u(b)$$

for all a in N.

Remark 1.

(1) A supermedian function is the same as superharmonic if and only if $\varphi(a) = \sum_{b\sim a} \varphi(a,b) = 1$ for all a in N.

(2) A solution to the Schrödinger equation corresponds to a median function if and only if $\varphi(a) \leq 1$ for all a in N and $\varphi(a_0) < 1$ for at least one vertex a_0 in N.
We can develop a theory of supermedian functions exactly in the same way as the theory of discrete superharmonic functions. For example, we have the following.

(a) If $u(a)$ is supermedian and $v(a)$ is submedian such that $u(a) \geq v(a)$ on N, then there exists a median function $h(a)$ on N such that $u(a) \geq h(a) \geq v(a)$; and if $h^{'}(a)$ is another median function such that $u(a) \geq h^{'} \geq v(a)$, then $h^{'}(a) \geq h(a)$.

(b) If $u(a) \geq 0$ is supermedian, then there exists a unique decomposition $u(a) = p(a) + h(a)$, where $p(a)$ is a superpotential (i.e., a nonnegative supermedian function whose greatest median minorant is 0) and $h(a) \geq 0$ is a median function. Recall that a finite or infinite graph is known as a tree if there is no closed path of the form $\{a_0, a_1, \ldots, a_n = a_0\}$ with more than 2 distinct vertices.

Corollary 2. *Let $\{N, \varphi(a,b)\}$ be an infinite tree without terminal vertex. Then, for any vertex e in N, there exists a supermedian function $\varphi_e(a)$ on N such that $\varphi_e(a)$ is a median function at each vertex in $N \setminus \{e\}$, i.e., $\varphi_e(a)$ is not median at e.*

P r o o f. Let F be the set consisting of $\{e\}$ and all its neighbors. Define a function $u(a)$ on F such that $u(a) = 1$ and $u(a) = 0$ at each neighbor of e. Then extend $v(a)$ to N as in the above theorem to get the function $v(a)$ which equal to $u(a)$ on F and is a median function at each vertex $a \neq e$. Note that $u(a)$, and hence $v(a)$, is superharmonic at e but not median. Denote the function $v(x)$ by $\varphi_e(a)$ to prove the statement in the corollary. □

Remark 2. This function $\varphi_e(a)$ is an analog of the Newtonian potential function $1/|x|$ in $\mathbb{R}^3$ if there is a positive superpotential on N; otherwise, $\varphi_e(a)$ is an analog of the logarithmic function $\log(1/|x|)$ in $\mathbb{R}^2$.

Theorem 3. *Let $\{N, \varphi(a,b)\}$ be an infinite tree without terminal vertices. Let F be a connected subset of N, and let $u(a)$ be a real-valued function on F. Then, there exists a real-valued function $v(a)$ on N such that $v(a) = u(a)$ if $a \in F$ and $v(a)$ is a median function at each vertex not in $\mathring{F}$.*

P r o o f. Let $a_0 \in \partial F$. Let $\{a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_m\}$ be the neighbors of a_0, where $\{a_1, \ldots, a_k\}$ are in F and $\{b_1, \ldots, b_m\}$ are outside F. Note that the latter subset $\{b_1, b_2, \ldots, b_m\}$ is non-empty since $a_0 \in \partial F$. Choose a constant λ and define a function $v(a)$ on $F \cup \{$all neighbors of $a_0\}$ such that

$$v(a) = \begin{cases} u(a) & \text{for} \quad a \in F, \\ \lambda & \text{for} \quad a \notin F. \end{cases}$$

Now, if the constant λ is chosen so that

$$v(a_0) = \sum_{i=1}^{k} \varphi(a_0, a_i) u(a_i) + \lambda \sum_{j=1}^{m} t(a_0, b_j),$$

then $v(a)$ is a median function at the vertex a_0.

This procedure can be adopted with respect to each vertex on ∂F. Denoting this extended function also by $v(a)$, we get a function $v(a)$ defined on $Nbr(F)$, which consists of F and all neighbors of each vertex in F such that $v(x) = u(x)$ if $a \in F$ and $v(a)$ is a median function at each vertex in ∂F.

Repeat this procedure with respect to $v(x)$ as $Nbr(f)$. Since N is a connected network,

$$N = \ldots Nbr[Nbr[Nbr(f)]]$$

so that $v(a)$ is a function defined on N such that $v(a) = u(a)$ if $a \in F$ and $v(a)$ is a median function at each vertex not in $\mathring{F}$.

Theorem 4. *Let $f(a)$ be a real-valued function on N. Then there exists a function $u(a)$ on N such that $Au(a) = f(a)$ for every a in N.*

P r o o f. From Theorem 3, we have a function $\varphi_e(a)$ such that $A\varphi_e(a) = \lambda\delta_e(a)$, where $\lambda > 0$ is a constant and $\delta_e(a)$ is the Dirac function. Write $q_e(a) = 1/\lambda \cdot \varphi_e(a)$. Thus, we conclude that given any vertex e in N, there exists a real-valued function $q_e(a)$ on N such that $Aq_e(a) = \delta_e(a)$.

Take a finite exhaustion $\{E_n\}$ of N, i.e., E_n is a non-empty finite set, $E_n \subset \overset{\circ}{E}_{n+1}$, and $N = \cup E_n$. For $n > 1$, let

$$u_n(a) = \sum_{e \in E_{n+1} \setminus E_n} q_e(a) f(e).$$

Then, $Au_n(a) = 0$ for $x \notin E_{n+1} \setminus E_n$ and $Au_n(a) = f(a)$ for $x \in E_{n+1} \setminus E_n$. Define

$$u(a) = \sum_{n=1}^{\infty} u_n(a),$$

where

$$u_1(a) = \sum_{e \in E_1} q_e(a) f(e)$$

is such that $Au_1(a) = 0$ for $x \notin E_1$ and $Au_1(a) = f(a)$ for $a \in E_1$. Note that the infinite sum is well-defined. For, if a_0 is any vertex in N, then $a_0 \in \overset{\circ}{E}_m$ for some m and $\sum_{n=m}^{\infty} u_n(a)$ is a convergent series consisting of functions that are median at the vertex a_0. Consequently, $u(a)$ is a well-defined function on N such that $Au(a) = f(a)$ for all $a \in N$.

Definition 5 (Bimedian) [11]. *A real-valued function $v(a)$ on N is said to be bimedian if there exists a median function $u(a)$ on N such that $Av(a) = u(a)$ for all $a \in N$. If A is the Laplacian operator, then $v(a)$ is called a biharmonic function on N.*

Theorem 5 (Discrete Riquier problem). *Let E be a finite subset of N. Let f and g be two real-valued functions on ∂E. Then, there exists a unique bimedian function v on E such that $Av(a) = f(a)$ and $v(a) = g(a)$ for $a \in \partial E$.*

P r o o f. Let $h_1(a)$ be the unique Dirichlet solution on E such that $Ah_1(a) = 0$ for $a \in \overset{\circ}{E}$ and $h_1(a) = f(a)$ for $a \in \partial E$. By Theorem 4, we can choose a function $s(a)$ on E such that $As(a) = h_1(a)$ on E.

Let $h_2(a)$ be the unique Dirichlete solution on E such that $Ah_2(a) = 0$ for $a \in \overset{\circ}{E}$ and $h_2(a) = g(a) - s(a)$ on ∂E. Take $v(a) = s(a) + h_2(a)$. Then, $v(a) = g(a)$ on ∂E and $Av(a) = As(a) = h_1(a)$ for $a \in \overset{\circ}{E}$, so that $A[Av(a)] = Ah_1(a) = 0$ for $a \in \overset{\circ}{E}$; further, $Av(a) = f(a)$ for $a \in \partial E$. Thus, $v(a)$ is the unique bimedian function on E such that $Av(a) = f(a)$ and $v(a) = g(a)$ for $a \in \partial E$. □

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