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REACHABLE SET OF SOME DISCRETE SYSTEM WITH UNCERTAIN LIU DISTURBANCES[1]

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Abstract: The paper considers the problem of finding the reachable set for a linear system with determinate and stochastic Liu's uncertainties. As Liu's uncertainties, we use uniformly distributed ordinary uncertain values defined in some uncertain space and independent of one another. This fact means that the state vector of the system becomes infinite-dimensional. As determinate uncertainties, we consider feedback controls and unknown initial states. Besides, there is a constraint in the form of a sum of uncertain expectations. The initial estimation problem reduces to a determinate multi-step problem for matrices with a fixed constraint at the right end of the trajectory. This reduction requires some information on Liu's theory. We give necessary and sufficient conditions for the finiteness of a target functional in the obtained determinate problem. We provide a numerical example of a two-dimensional two-step system.

Keywords: Uncertainty theory, Uncertain values, Feedback controls, Attainable set, Lagrange multipliers.

1. Introduction

Baoding Liu's uncertainty theory has been widely developed in the last decade [9, 12, 13]. Elements of the theory are used in control theory, mathematical programming, financial mathematics, robotics, and other areas of applied mathematics. Liu notes in his book that "uncertainty theory has become a branch of axiomatic mathematics for modeling belief degrees."

It should be noted that Liu's theory is only one of the possible approaches to describing and accounting for uncertainty. Such approaches include various versions of probability theory, Zadeh's fuzzy set theory, interval analysis, and chaos theory [1, 6, 8, 11]. Possibility theory is actively developed as an alternative to probability in [10]. The theory of guaranteed estimation [7], based on a set-theoretic description of uncertainty, has also gained wide popularity. Of course, Liu's theory overlaps with the theories mentioned above.

This paper presents an extended version of the lecture given at the XIV All-Russian Conference on Control Problems [2]. We consider the estimation problem for discrete time Liu's processes described by the linear equations

$$x_k = (A_k x_{k-1} + B_k v_k)(1 + \lambda_k \xi_k), \quad x_k \in \mathbb{R}^n, \quad k \in 1:m, \tag{1.1}$$

where $|\lambda_k| \le 1$ are real numbers; $v_k = K_k x_{k-1}$ are uncertain feedback controls; ξ_k are ordinary uncertain values uniformly distributed on $[-1,1]$, independent one of another, and defined on the N-space $(\Omega, \mathcal{F}, N)$, where $\mathcal{F}$ is a σ-algebra, and N is the uncertainty measure (function) of the set. The following constraints are also given:

$$J(x_0, \mathbb{K}) = \sum_{k\in 1:m} \mathrm{E}\left(v_k' R_k v_k + x_{k-1}' Q_k x_{k-1}\right) \le 1, \quad x_0 \in \mathbf{X}_0, \tag{1.2}$$

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where E is the uncertain expectation, $v_k = K_k x_{k-1}$, $\mathbb{K} = K_{1:m}$, R_k and Q_k are symmetric matrices of appropriate dimension, and $\mathbf{X}_0$ is a convex compact set in $\mathbb{R}^n$. One can see that the state vector x_k depends on elementary event $\omega \in \Omega$ because of uncertain values $\xi_k(\omega)$. So, estimating the reachable set at the terminal time m becomes the problem of infinite dimension for system (1.1) under constraints (1.2). Note that an estimation problem for a determinate system with an uncertain matrix was studied in [5]. First of all, let us recall some facts on Liu's theory.

2. Necessary facts on Liu's theory

Given a measurable space $(\Omega, \mathcal{F})$, where Ω is an arbitrary set and $\mathcal{F}$ is a σ-algebra of subsets of Ω, the *uncertain measure* N is defined on $\mathcal{F}$ to satisfy the following axioms:

1. Normality: $N(\Omega) = 1$.
2. Duality: $N(A) + N(A^c) = 1$ for any event $A \in \mathcal{F}$, where $A^c = \Omega \setminus A$.
3. Subadditivity: for any sequence $A_i \in \mathcal{F}$,

$$N\Big(\bigcup_{i\in\mathbb{N}} A_i\Big) \le \sum_{i\in\mathbb{N}} N(A_i).$$

Any uncertain measure satisfies the relations $0 \le N(A) \le 1$ and $N(A) \le N(B)$ for all $A, B \in \mathcal{F}$ such that $A \subset B$. But it is not a measure in ordinary sense [4]. Of course, any probability measure P satisfies axioms 1–3 and is, therefore, an uncertainty measure. Every measurable function $\xi : \Omega \to \mathbb{R}$ is called an *uncertain variable*. The independence of the family ξ_t, $t \in T$, of uncertain variables is defined as follows:

$$N\Big(\bigcap_{t\in T} \xi_t^{-1}(B_t)\Big) = \bigwedge_{t\in T} N\left(\xi_t^{-1}(B_t)\right)$$

for any $B_t \in \mathcal{B}$, where $\mathcal{B}$ is the Borelian σ-algebra on $\mathbb{R}$ and $\bigwedge_{t\in T} a_t = \inf_{t\in T} a_t$. The independence of uncertain variables was generalized to an arbitrary set T of indexes t in [3].

If ξ is an uncertain variable, then its distribution function is defined by the formula $F_\xi(x) = N(\xi \le x)$ for all $x \in \mathbb{R}$, which corresponds to the similar notion in probability theory. It is proved (see [3, 9]) that a nondecreasing function $F : \mathbb{R} \to [0, 1]$ is a distribution function for some uncertain variable if and only if the following properties hold:

(1) $F \not\equiv 1$;

(2) $F \not\equiv 0$;

(3) the condition $F(x) = 1$ for all $x > x^*$ must imply that $F(x^*) = 1$.

For any distribution function satisfying 1–3, one can build a so-called *ordinary* uncertain variable $\xi(x) = x$ on the measure space $(\mathbb{R}, \mathcal{B})$ in the following way. First, consider a family of sets $\mathcal{L}$ consisting of semi-infinite intervals $(-\infty, x]$, their complements (x, ∞), the empty set, and the entire space $\mathbb{R}$. One can define the uncertainty measure N on $\mathcal{L}$ as follows: $N((-\infty, x]) = F(x)$, $N((x, \infty)) = 1 - F(x)$, and $F(\varnothing) = 0$. After that, for $B \in \mathcal{B}$, the uncertainty measure N_ξ is defined on $\mathcal{B}$ by the formula

$$N_\xi(B) = \begin{cases} \inf\limits_{B\subset \bigcup\limits_{i\in\mathbb{N}} A_i} \sum\limits_{i\in\mathbb{N}} N(A_i) & \text{if} \quad \inf\limits_{B\subset \bigcup\limits_{i\in\mathbb{N}} A_i} \sum\limits_{i\in\mathbb{N}} N(A_i) < 0.5, \\ 1 - \inf\limits_{B^c\subset \bigcup\limits_{i\in\mathbb{N}} A_i} \sum\limits_{i\in\mathbb{N}} N(A_i) & \text{if} \quad \inf\limits_{B^c\subset \bigcup\limits_{i\in\mathbb{N}} A_i} \sum\limits_{i\in\mathbb{N}} N(A_i) < 0.5, \\ 0.5 & \text{in other cases.} \end{cases} \tag{2.3}$$

Here, the infimums are taken over all sequences $A_i \in \mathcal{L}$ that cover the corresponding sets. We see that $N_\xi = N$ on $\mathcal{L}$.

An ordinary uncertain variable ξ uniformly distributed on $[-1, 1]$ has its distribution function

$$F_\xi(x) = \begin{cases} 0, & \text{if} \quad x \le -1; \\ (x+1)/2, & \text{if} \quad x \in (-1, 1); \\ 1, & \text{if} \quad x \ge 1. \end{cases}$$

Such uncertain variables belong to the class of regular uncertain variables, for which there is an interval (a, b) where $F_\xi(x)$ is continuous and strictly increasing. Besides, $\lim_{x \to a} F_\xi(x) = 0$ and $\lim_{x \to b} F_\xi(x) = 1$. Here, it is possible that $a = -\infty$ and $b = \infty$.

The mathematical expectation of an uncertain variable ξ is defined by the formula

$$\mathrm{E}\xi = \int_0^\infty N(\xi \ge x) dx - \int_{-\infty}^0 N(\xi \le x) dx$$

if at least one of the integrals is finite. Since the function $1 - F_\xi(x)$ differs from the function $N(\xi \ge x)$ only at countably many points, we have

$$\mathrm{E}\xi = \int_0^\infty (1 - F_\xi(x)) dx - \int_{-\infty}^0 F_\xi(x) dx = \int_0^\infty x dF_\xi(x) + \int_{-\infty}^0 x dF_\xi(x)$$

using integration by parts. For variables with regular distribution functions, we have

$$\int_0^\infty x dF_\xi(x) = \int_{F_\xi(0)}^1 F_\xi^{-1}(\alpha) d\alpha, \quad \int_{-\infty}^0 x dF_\xi(x) = \int_0^{F_\xi(0)} F_\xi^{-1}(\alpha) d\alpha. \tag{2.4}$$

From (2.3), we obtain $\mathrm{E}\xi = 0$ for the ordinary uniformly distributed uncertain variable ξ. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called strictly increasing if $f(u_1, \dots, u_n) \ge f(v_1, \dots, v_n)$ for $u_i \ge v_i$ and $f(u_1, \dots, u_n) > f(v_1, \dots, v_n)$ for $u_i > v_i$.

The following theorem is often used in applications.

Theorem 1 [12, Theorem 2.6]. *Let $[u; v] = [u_1; \dots; u_m; v_1; \dots; v_n]$ be independent uncertain variables with regular distribution functions $[F_u; F_v] = [F_{u_1}; \dots; F_{u_m}; F_{v_1}; \dots; F_{v_n}]$, respectively. If the function $f(u, v)$ is strictly increasing in u and strictly decreasing in v, then the uncertain variable $\xi = f(u, v)$ has the inverse distribution function*

$$\begin{aligned} &F_\xi^{-1}(a) = f\left(F_u^{-1}(a), F_v^{-1}(1-a)\right), \quad a \in (0, 1), \\ &F_u^{-1} = [F_{u_1}^{-1}; \dots; F_{u_m}^{-1}], \quad F_v^{-1} = [F_{v_1}^{-1}; \dots; F_{v_n}^{-1}]. \end{aligned} \tag{2.5}$$

Corollary 1. *For any uncertain variable ξ, $\mathrm{E}(a\xi) = a\mathrm{E}\xi$ for all $a \in \mathbb{R}$. For any regular and independent uncertain variables ξ and η with finite mathematical expectations, $\mathrm{E}(a\xi + b\eta) = a\mathrm{E}\xi + b\mathrm{E}\eta$ for all $a, b \in \mathbb{R}$.*

Indeed,

$$N(a\xi \le x) = N(\xi \le x/a) = F_\xi(x/a), \quad a > 0,$$

and

$$\mathrm{E}(a\xi) = \int_{-\infty}^\infty x dF_\xi(x/a) = a\mathrm{E}\xi.$$

If $a < 0$, then $N(a\xi \le x) = N(\xi \ge x/a) = 1 - F_\xi(x/a)$ N-almost everywhere and

$$\mathrm{E}(a\xi) = \int_{-\infty}^{\infty} x d(1 - F_\xi(x/a)) = a\mathrm{E}\xi.$$

Moreover,

$$F^{-1}_{\xi+\eta} = F^{-1}_\xi + F^{-1}_\eta, \quad F^{-1}_{-\xi}(a) = F^{-1}_\xi(1-a)$$

if ξ and η are regular. Unfortunately, the linear property of mathematical expectation is not valid for arbitrary uncertain variables.

It must be kept in mind that uncertain variables ξ and η with identical distribution functions $F_\xi \equiv F_\eta \equiv F$ may have different distributions N_ξ and N_η on $\mathbb{R}$. For example, the ordinary uncertain variable ξ uniformly distributed on $[-1,1]$ has the uncertainty measure $N_\xi(\xi = x) = F(x)\bigwedge(1 - F(x)) \ne 0$ for all $x \in (-1,1)$ by (2.3). On the other hand, the uniformly distributed uncertain variable η with probability $N_\eta((a,b]) = (b-a)/2$, $a, b \in [-1,1]$, has $N_\eta(\eta = x) = 0$ for all $x \in (-1,1)$.

In contrast to probability theory, the distribution N_ξ of an ordinary uncertain variable cannot be analytically expressed in terms of the distribution function $F_\xi(x) = N(\xi \le x)$. Additionally, a function identically equal to a constant on $\mathbb{R}$ cannot be a distribution function in probability theory but in Liu's theory (see [3]). The results of the following lemma were proved in [13, Example 1.6] but for completeness, we present a proof, which is, moreover, simpler.

Lemma 1. *For an ordinary uncertain variable ξ uniformly distributed on* $[-1,1]$, $\mathrm{E}(\xi^2 + b\xi) = 1/3$ *for all* $|b| \ge 2$ *and* $\mathrm{E}\xi^2 = 7/24$.

P r o o f. Let $b \ge 2$. The function $f(x) = x^2 + bx$ strictly increases on $[-1,1]$ from $1-b$ to $1+b$. If $\eta = f(\xi)$, then $F^{-1}_\eta = f(F^{-1}_\xi)$ by (2.4). Therefore,

$$\mathrm{E}f(\xi) = \int_0^1 F^{-1}_\eta(a)da = \int_0^1 \left((2a-1)^2 + b(2a-1)\right) da = (2a-1)^3/6 + b(2a-1)^2/4\Big|_0^1 = 1/3$$

by (2.4). If $b \le -2$, then the function $f(x) = x^2 + bx$ strictly decreases on $[-1,1]$ from $1-b$ to $1+b$. We have $\eta = f(\xi)$ and $F^{-1}_\eta(a) = f(F^{-1}_\xi(1-a))$ by Theorem 1. So, $\mathrm{E}\eta = 1/3$ as well.

Now let $\eta = \xi^2$. Compute F_η via (2.3). Define $x_1 = -\sqrt{x}$ and $x_2 = \sqrt{x}$. We have

$$F_\eta(x) = N_\xi(x_1 \le \xi \le x_2).$$

Since $[x_1, x_2] = (-\infty, x_2]\bigcap[x_1, \infty)$, we have

$$F_\xi(x_2)\bigwedge(1 - F_\xi(x_1)) = (x_2 + 1)/2 \ge 1/2.$$

For the complement of $[x_1, x_2]$, we obtain $[x_1, x_2]^c = (-\infty, x_1)\bigcup(x_2, \infty)$ and

$$F_\xi(x_1) + 1 - F_\xi(x_2) = 1 - \sqrt{x}.$$

Therefore,

$$F_\eta(x) = \begin{cases} 0 & \text{if} \quad x < 0, \\ 0.5 & \text{if} \quad \sqrt{x} \in [0, 1/2], \\ \sqrt{x} & \text{if} \quad \sqrt{x} \in (1/2, 1], \\ 1 & \text{if} \quad \sqrt{x} > 1. \end{cases}$$

Finally,

$$\mathrm{E}\eta = \int_0^1 x dF_\eta(x) = \int_{1/4}^1 \sqrt{x} dx/2 = 7/24.$$

□

In what follows, we need the distribution function of $\eta = (1+\lambda\xi)^2$, $|\lambda| \le 1$. It can be found by Theorem 1.10 from [13]. Define

$$x_1 = -(\sqrt{x}+1)/\lambda, \quad x_2 = (\sqrt{x}-1)/\lambda.$$

Using the theorem, we obtain

$$F_\eta(x) = \begin{cases} 0 & \text{if} \quad x < (1-\lambda)^2, \\ F_\xi(x_2) \bigwedge (1 - F_\xi(x_1)) & \text{if} \quad F_\xi(x_2) \bigwedge (1 - F_\xi(x_1)) < 0.5, \\ F_\xi(x_2) - F_\xi(x_1) & \text{if} \quad F_\xi(x_2) - F_\xi(x_1) > 0.5, \\ 0.5 & \text{otherwise,} \end{cases}$$

for $0 < \lambda \le 1$. By the formula for $F_\xi(x)$, we have

$$F_\eta(x) = \begin{cases} 0 & \text{if} \quad x < (1-|\lambda|)^2, \\ (\sqrt{x}-1)/(2|\lambda|) + 1/2 & \text{if} \quad x \in [(1-|\lambda|)^2, (1+|\lambda|)^2], \\ 1, & \text{if} \quad x > (1+|\lambda|)^2. \end{cases} \tag{2.6}$$

If $-1 \le \lambda < 0$, then $x_2 < x_1$ and $\eta = (|\lambda|\xi - 1)^2$. We come to formula (2.6) as well. The inverse function for the regular uncertain variable η has the form

$$F_\eta^{-1}(x) = (|\lambda|(2x-1)+1)^2, \quad x \in [0,1].$$

3. Statement of the problem

Definition 1. *A set $\mathcal{X}_m$ is called the reachable set for system (1.1) under constraints (1.2) if it consists of uncertain variables $\eta \in \mathbb{R}^n$ for which there exists a family $(x_0, \mathbb{K})$ satisfying (1.2) and such that $x_m = \eta$ with equations (1.1) satisfied. The equalities are considered N-almost everywhere.*

The *problem* is to find the reachable set $\mathcal{X}_m$. Let

$$V_m(\eta) = \min\left\{J(x_0, \mathbb{K}) : x_m = \eta,\, x_0 \in \mathbf{X}_0,\, K_k \in \mathbb{R}^{q\times n}\right\}. \tag{3.1}$$

Then

$$\mathcal{X}_m = \{\eta : V_m(\eta) \le 1\}.$$

Note that if $0 \in \mathbf{X}_0$, then always $0 \in \mathcal{X}_m$. Further, we exclude the uncertain variables η for which $V_m(\eta) = -\infty$. This can be so because the matrices R_k and Q_k in (1.2) can be nonpositive definite.

4. Main results

First, we transform the computation of $V_m(\eta)$ into an equivalent deterministic optimal control problem.

Let $X_k = \mathrm{E}(x_k x_k')$. Since $x_k \in \mathbb{R}^n$, the matrix $x_k x_k'$ belongs to $\mathbb{R}^{n\times n}$ and its elements are uncertain variables. Therefore, X_k is a symmetric matrix for all $k \in 1:m$.

Theorem 2. *Let $\Xi = E\eta\eta'$. If the minimization problem* (3.1) *has a finite value $V_m(\eta)$, then it is equivalent to the following deterministic optimal control problem*:

$$\mathcal{V}_m(\Xi) = \min \left\{ \mathbf{J}(x_0, \mathbb{K}) : X_m = \Xi,\ x_0 \in \mathbf{X}_0,\ K_k \in \mathbb{R}^{q\times n} \right\}, \\ \mathbf{J}(x_0, \mathbb{K}) = \sum_{k\in 1:m} \operatorname{tr}\left(\left(K_k' R_k K_k + Q_k \right) X_{k-1} \right), \tag{4.1}$$

where tr means the trace of a matrix, under the recurrent relations

$$X_k = \nu_k U_k(K_k, X_{k-1}), \quad X_0 = x_0 x_0', \quad \nu_k = \mu_k/\mu_{k-1}, \\ U_k(K_k, X) = (A_k + B_k K_k) X (A_k + B_k K_k)', \\ \mu_k = \int_0^1 \left(|\lambda_k|(2x-1)+1 \right)^2 \dots \left(|\lambda_1|(2x-1)+1 \right)^2 dx. \tag{4.2}$$

The functionals $J(x_0, \mathbb{K})$ and $\mathbf{J}(x_0, \mathbb{K})$ coincide and therefore $\mathcal{V}_m(\Xi) = V_m(\eta)$.

P r o o f. Using equality (1.1), we have

$$x_k x_k' = U_k\left(K_k, x_{k-1} x_{k-1}' \right) (1 + \lambda_k \xi_k)^2 \\ = (A_k + B_k K_k) \dots (A_1 + B_1 K_1) x_0 x_0' (A_1 + B_1 K_1)' \dots (A_k + B_k K_k)' \\ \times (1+\lambda_k\xi_k)^2 \dots (1+\lambda_1\xi_1)^2, \text{ and, therefore,} \\ X_k = (A_k + B_k K_k) \dots (A_1 + B_1 K_1) x_0 x_0' (A_1 + B_1 K_1)' \dots (A_k + B_k K_k)' \\ \times \mathrm{E}((1+\lambda_k\xi_k)^2 \dots (1+\lambda_1\xi_1)^2). \tag{4.3}$$

By Theorem 1 and formulas (2.4), (2.5), and (2.6), we see that

$$\mathrm{E}((1+\lambda_k\xi_k)^2 \dots (1+\lambda_1\xi_1)^2) = \mu_k$$

in (4.2). Thus, $X_0 = x_0 x_0'$ and $X_k = \nu_k U_k(K_k, X_{k-1})$ for $k \in 1:m$. The coincidence of $J(x_0, \mathbb{K})$ and $\mathbf{J}(x_0, \mathbb{K})$ follows from (4.3) and the equalities

$$J(x_0, \mathbb{K}) = \sum_{k\in 1:m} \mathrm{E} x_{k-1}' \left(K_k' R_k K_k + Q_k \right) x_{k-1} \\ = \sum_{k\in 1:m} \mathrm{E} \operatorname{tr}\left(\left(K_k' R_k K_k + Q_k \right) x_{k-1} x_{k-1}' \right) = \sum_{k\in 1:m} \operatorname{tr}\left(\left(K_k' R_k K_k + Q_k \right) X_{k-1} \right) = \mathbf{J}(x_0, \mathbb{K}).$$

□

Remark 1. We can compute $\mathrm{E}(1+\lambda\xi)^2 = 1 + \lambda^2 \mathrm{E}(2/\lambda\xi + \xi^2) = 1 + \lambda^2/3$ by Lemma 1. Suppose that $|\lambda_k| \equiv 1$. Then $\mu_k = 4^k \int_0^1 x^{2k} dx = 4^k/(2k+1)$. Therefore, $\lim_{k\to\infty} \nu_k = 4$ in this case.

Remark 2. Note that the properties $R_k \geq 0$ and $Q_k \geq 0$ were not used in the proof of Theorem 2. If these properties hold, then we have $V_m(\eta) = \mathcal{V}_m(X_m) \geq 0$.

Corollary 2. *The reachable set is $\mathcal{X}_m = \{\eta : \mathcal{V}_m(E\eta\eta') \leq 1\}$.*

Problem (4.1) is determinate. So, we seek the minimum of the smooth functional $\mathbf{J}(x_0, \mathbb{K})$ in (4.1) under the equality conditions (4.2) with given $X_m = \Xi$. According to the Kuhn–Tucker theorem, we form the Lagrange function

$$\mathcal{L} = \mathbf{J}(x_0, \mathbb{K}) + \sum_{k\in 1:m} \operatorname{tr}\left(H_k(\nu_k U_k(K_k, X_{k-1}) - X_k) \right) + \operatorname{tr}(\Gamma(X_m - \Xi)), \quad X_0 = x_0 x_0',$$

where the symmetric matrices $H_{1:m}$ and Γ are the Lagrange multipliers. Let us write necessary optimality conditions (x_0 is fixed):

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial K_k} &= 2R_k K_k X_{k-1} + 2\nu_k B_k' H_k A_k X_{k-1} + 2\nu_k B_k' H_k B_k K_k X_{k-1} = 0, \quad k \in 1:m, \\
\frac{\partial \mathcal{L}}{\partial X_{k-1}} &= K_k' R_k K_k + \nu_k (A_k + B_k K_k)' H_k (A_k + B_k K_k) - H_{k-1} + Q_k = 0, \quad k \in 2:m, \\
\frac{\partial \mathcal{L}}{\partial X_m} &= -H_m + \Gamma = 0.
\end{aligned} \tag{4.4}$$

Here, we use the formula of differentiation

$$\frac{\partial \operatorname{tr}(A'B)}{\partial A} = B$$

for matrices of appropriate dimensions. Define

$$L_k = R_k + \nu_k B_k' H_k B_k, \quad M_k = \nu_k B_k' H_k A_k. \tag{4.5}$$

To resolve equalities equalities (4.4), we set

$$K_k = -L_k^+ M_k + Y_k - L_k^+ L_k Y_k.$$

This expression satisfies the equation $L_k K_k = -M_k$ if and only if $L_k L_k^+ M_k = M_k$. Here A^+ is the pseudoinverse matrix and $Y_k \in \mathbb{R}^{q\times n}$ is an arbitrary matrix. Substituting the expression for K_k into the second row of (4.4), we obtain the equation

$$H_{k-1} = \nu_k A_k' H_k A_k - M_k' L_k^+ M_k + Q_k, \quad H_m = \Gamma, \quad k \in 1:m. \tag{4.6}$$

This equation determines H_0 for $k = 1$. We come to the conclusion.

Theorem 3. *Let the value $V_m(\eta) = \mathcal{V}_m(\Xi)$ be finite and be reached at the pair $(x_0, \mathbb{K}^0)$. Then there exist symmetric matrices $H_{1:m}$ and Γ satisfying equations (4.6) with the matrix coefficients given in (4.5) and*

$$L_k L_k^+ M_k = M_k \quad \textit{and} \quad L_k \geq 0. \tag{4.7}$$

Moreover, optimal matrices have the form

$$K_k^0 = -L_k^+ M_k + Y_k - L_k^+ L_k Y_k,$$

where the matrices $Y_k \in \mathbb{R}^{q\times n}$ are arbitrary. The optimal value is

$$V_m(\eta) = \mathcal{V}_m(\Xi) = \min\left\{-\operatorname{tr}(\Gamma\Xi) + x_0' H_0 x_0 : x_0 \in \mathbf{X}_0,\, X_m = \Xi\right\}.$$

P r o o f. From (4.1), we know that

$$\begin{aligned}
\mathbf{J}(x_0, \mathbb{K}) &= \sum_{k\in 1:m} \operatorname{tr}\left(\left(K_k' R_k K_k + Q_k\right) X_{k-1}\right) \\
&= \sum_{k\in 1:m} \left\{\operatorname{tr}\left(\left(K_k' R_k K_k + Q_k\right) X_{k-1}\right) + \operatorname{tr}\left(H_k X_k\right) - \operatorname{tr}\left(H_{k-1} X_{k-1}\right)\right\} - \operatorname{tr}(\Gamma\Xi) + x_0' H_0 x_0.
\end{aligned} \tag{4.8}$$

Substituting X_k and H_{k-1} from (4.2) and (4.6) into (4.8), we can write the cost functional as follows:

$$\begin{aligned}\mathbf{J}(x_0,\mathbb{K}) &= \sum_{k\in 1:m} \operatorname{tr}\left(\left(K_k' L_k K_k + M_k' K_k + K_k' M_k + M_k' L_k^+ M_k\right) X_{k-1}\right) - \operatorname{tr}(\Gamma\Xi) + x_0' H_0 x_0 \\ &= \sum_{k\in 1:m} \operatorname{tr}\Big(\left(K_k + L_k^+ M_k\right)' L_k \left(K_k + L_k^+ M_k\right) X_{k-1}\Big) - \operatorname{tr}(\Gamma\Xi) + x_0' H_0 x_0.\end{aligned} \tag{4.9}$$

Here $L_k \geq 0$. If $L_p \ngeq 0$ for some $p \in 1:m$, then there exist a vector h and a number $\alpha < 0$ such that $L_p h = \alpha h$. Let $N = [\underbrace{h, \dots, h}_{\text{n vectors}}]$. Then $L_p N = \alpha N$. If

$$K_k = -L_k^+ M_k, \quad k \neq p, \quad K_p = -L_p^+ M_p + \delta N/\sqrt{|\alpha|},$$

then

$$\lim_{\delta\to\infty} \mathbf{J}(x_0,\mathbb{K}) = -\infty.$$

□

These relations are sufficient for optimality.

Theorem 4. *Equations (4.6) along with relations (4.2), (4.5), (4.7) are sufficient for finiteness of values $\mathcal{V}_m(\Xi) = V_m(\eta) > -\infty$, and optimal values K_k^0 with corresponding minimum are specified in Theorem 3. The system contains $2mn(n+1)/2$ equations with the same quantity of variables, namely, $mn(n+1)/2$ variables $H_{0:m-1}$, $n(n+1)/2$ variables Γ, and $(m-1)n(n+1)/2$ variables $X_{1:m-1}$.*

Indeed, if relations (4.2), (4.5), (4.7) are valid, then

$$\mathbf{J}(x_0,\mathbb{K}) \geq -\operatorname{tr}(\Gamma\Xi) + x_0' H_0 x_0$$

according to (4.9).

Corollary 3. *Consider the matrix*

$$\mathbf{H}_0 = \prod_{k\in 0:m-1} (A_{m-k} + B_{m-k} K_{m-k}^0).$$

It follows from (4.3) that $X_m = \mu_m \mathbf{H}_0 X_0 \mathbf{H}_0'$. Therefore, $\operatorname{tr}(\Gamma X_m) = \mu_m x_0' \mathbf{H}_0' \Gamma \mathbf{H}_0 x_0$. The reachable set is

$$\mathcal{X}_m = \{\eta : \min\{x_0'(H_0 - \mu_m \mathbf{H}_0' \Gamma \mathbf{H}_0) x_0 : x_0 \in \mathbf{X}_0,\ \mu_m \mathbf{H}_0 X_0 \mathbf{H}_0' = \Xi = E\eta\eta'\} \leq 1\}.$$

The minimization is provided here under inequalities (4.7).

5. Example

Consider the 2-dimensional system (1.1) in which

$$n = m = 2, \quad q = 1, \quad \lambda_1 = 0.2, \quad \lambda_2 = -0.1,$$
$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Under constraints (1.2), we have

$$R_1 = p, \quad R_2 = 4, \quad Q_1 = p\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad Q_2 = p\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},$$

where $p \geq -1$ is a numeric parameter. There are 12 variables and the same number of equations. Let

$$X_1 = (x_{1ij}), \quad H_k = (h_{kij}), \quad \Gamma = (\gamma_{ij}), \quad x_0 = (x_{0i}), \quad K_k = (k_{ki}), \quad M_k = (m_{ki}).$$

For $k = 1$, we obtain

$$h_{011} = p\nu_1 h_{111}/(p + \nu_1 h_{111}) + p, \quad h_{012} = 0, \quad h_{022} = p, \quad k_{11} = -\nu_1 h_{111}/(p + \nu_1 h_{111}), \quad k_{12} = 0,$$

where

$$\nu_1 = \mu_1 = 1 + \lambda_1^2/3 = 76/75 = 1.0133.$$

From equation (4.2), we have $x_{111} = \nu_1(1 + k_{11})^2 x_{01}^2$and $x_{1i2} = 0$. For $k = 2$, we have $\xi_{ij} = \nu_2(1 + k_{21})^2 x_{111}$ for any i and j. Let $\xi_{ij} = \xi$ and $k_{22} = 0$. On the other hand, since $H_2 = \Gamma$, we have $k_{21} = -\nu_2\gamma/(4 + \nu_2\gamma)$, where $\gamma = \gamma_{11} + \gamma_{22} + 2\gamma_{12}$. Finally, from equation (4.6) for $k = 2$, we obtain

$$h_{111} = \nu_2\gamma + p - \nu_2^2\gamma^2/(4 + \nu_2\gamma), \quad h_{112} = h_{122} = 0.$$

Here

$$\mu_2 = \int_0^1 (|\lambda_1|(2x-1)+1)^2(|\lambda_2|(2x-1)+1)^2 dx = 1.0434, \quad \nu_2 = \mu_2/\mu_1 = 1.0297.$$

These equations imply that an uncertain variable $\eta = (\eta_1, \eta_2)$ belongs to $\mathcal{X}_2$ if and only if $E\eta_1^2 = E\eta_2^2 = E\eta_1\eta_2$. A family of uncertain variables of the form $\eta_1 = \eta_2 = \alpha$, where α is some uncertain variable, satisfies these conditions.

Consider the value $\xi = Fx_{01}^2$, where $F = \nu_1\nu_2(1 + k_{11})^2(1 + k_{21})^2$. Let x_0 be fixed and $f(\gamma) = h_{011}/F - \gamma$. Then

$$V_2(\eta) = \mathcal{V}_2(\Xi) = \xi \min\{f(\gamma) : L_i \geq 0\} + px_{02}^2$$

for $p > 0$, where $L_2 = 4 + \nu_2\gamma$ is the increasing linear function of γ. We see that the expression $f(\gamma) = h_{011}/F - \gamma$ in the braces depends only on γ, but the entire problem is to minimize the function $\mathcal{V}_2(\Xi)$ of two variables γ and x_{02} under nonlinear constraints and the equality condition $\xi = Fx_{01}^2$. Let $x_{02} = 1$ and $|x_{01}| \leq 1$ for simplicity. We set $p = 0.5$, for example. Computing the minimum of the smooth convex function $f(\gamma)$, we have $\min f(\gamma) = 0.4316$, and it is achieved at $\gamma^0 = -0.4316$. All the constraints are satisfied. So, $\mathcal{V}_2(\Xi) = 0.4316\xi + 0.5 \leq 1$ or $\xi \in [0, 1.1584]$. As matrix Γ, it is possible to take any symmetric matrix with $\gamma^0 = -0.4316$. If $p \downarrow 0$, then $\xi \in [0, b(p)]$, where $b(p) \to \infty$. If $-1 \leq p < 0$, then the value $V_m(\eta) = \mathcal{V}_m(\Xi) \leq 0$ is also finite for all $\xi \geq 0$. This means that $\xi \in [0, \infty)$.

6. Conclusion

- The discrete-time estimation problem has been considered for one class of uncertain Liu processes whose equations include unknown deterministic parameters subject to a priori constraints.
- The initial value problem is reduced to a deterministic multi-step problem for matrices with a fixed constraint at the right end of the trajectory.
- Necessary and sufficient conditions for the finiteness of the objective functional in the deterministic problem are obtained.

• A numerical solution of the initial value problem is considered with an example.

• In the general case, since the expectation has no, generally speaking, property of additivity, the reduction of problems with uncertain Liu disturbances to determinate ones is difficult. The received determinate problem is also unusual because it deals with implicit matrix equations.

• The ordinary uniformly distributed uncertain variables in this paper can be easily replaced by any regular and independent Liu variables.

• A similar problem for continuous systems will be considered in the future.

REFERENCES

1. Ananyev B. I. A state estimation of Liu equations. *AIP Conf. Proc.*, 2015. Vol. 1690. Art. no. 040005. DOI: 10.1063/1.4936712
2. Ananyev B. I. State estimation of some class of discrete Liu's processes. In: *Proc. XIV All-Russian Conference on Control Problems (VSPU-2024), Moscow, June 17–20, 2024*. Moscow: IPU RAS, 2024. P. 604–608. URL: https://vspu2024.ipu.ru/prcdngs (in Russian)
3. Ananyev B. I. On some complements to Liu's theory. *Proc. Steklov Inst. Math.*, 2024. Vol. 325, No. Suppl. 1. P. S1–S16. DOI: 10.1134/S0081543824030015
4. Bulinskii A. V., Shiryaev A. N. *Teoriya sluchainikh processov* [The Theory of Random Processes]. Moscow: Fizmatlit, 2003. 400 p. (in Russian)
5. Chernousko F. L. An estimate for the reachable sets of linear systems with an indefinite matrix. *Dokl. Akad. Nauk*, 1996. Vol. 349, No. 1. P. 32–34. (in Russian)
6. Hou Y., Yang Sh. Uncertain programming with recourse. *J. Chem. Pharmaceutical Res.*, 2015. Vol. 7, No. 3. P. 2366–2372.
7. Kurzhanski A., Varaiya P. *Dynamics and Control of Trajectory Tubes: Theory and Computation.* SCFA. Boston: Birkhäuser, 2014. 445 p. DOI: 10.1007/978-3-319-10277-1
8. Liu B. Toward uncertain finance theory. *J. Uncertain. Anal. Appl.*, 2013. Vol. 1. Art. no. 1. 15 p. DOI: 10.1186/2195-5468-1-1
9. Liu B. *Uncertainty Theory.* 4-th ed. Berlin, Heidelberg: Springer, 2015. 487 p. DOI: 10.1007/978-3-662-44354-5
10. Pytyev Yu. P. *Vozmozhnost kak alternativa veroyatnosti. Matematicheskiye i empiricheskiye osnovy, prilozheniya. 2-e izd.* [Possibility as the Alternative of Probability. Mathematical and Emperical Grounds, Applications. 2-nd ed.]. Moscow: Fizmatlit, 2016. 600 p. (in Russian)
11. Sheng L., Zhu Y., Yan H., Wang K. Uncertain optimal control approach for CO_2 mitigation problem. *Asian J. Control*, 2017. Vol. 19, No. 6. P. 1931–1942. DOI: 10.1002/asjc.1524
12. Yao K. *Uncertain Differential Equations.* Berlin, Heidelberg: Springer, 2016. 158 p. DOI: 10.1007/978-3-662-52729-0
13. Zhu Y. *Uncertain Optimal Control.* Berlin, Heidelberg: Springer, 2019. 208 p. DOI: 10.1007/978-981-13-2134-4