URAL MATHEMATICAL JOURNAL, Vol. 10, No. 1, 2024, pp. 147–158

DOI: 10.15826/umj.2024.1.013

ON G-VERTEX-TRANSITIVE COVERS OF COMPLETE GRAPHS HAVING AT MOST TWO G-ORBITS ON THE ARC SET[1]

Ludmila Yu. Tsiovkina

Krasovskii Institute of Mathematics and Mechanics,
Ural Branch of the Russian Academy of Sciences,
16 S. Kovalevskaya Str., Ekaterinburg, 620108, Russian Federation

tsiovkina@imm.uran.ru

Abstract: We investigate *abelian* (in the sense of Godsil and Hensel) distance-regular covers of complete graphs with the following property: *there is a vertex-transitive group of automorphisms of the cover which possesses at most two orbits in the induced action on its arc set.* We focus on covers whose parameters belong to some known infinite series of feasible parameters. We also complete the classification of arc-transitive covers with a non-solvable automorphism group and show that the automorphism group of any unknown edge-transitive cover induces a one-dimensional affine permutation group on the set of its antipodal classes.

Keywords: Antipodal cover, Distance-regular graph, Vertex-transitive graph, Arc-transitive graph.

1. Introduction

A distance-regular cover of a complete graph with parameters (n, r, μ) (or an (n, r, μ)-*cover*) is a connected graph (by "graph" hereafter we mean an undirected graph without loops or multiple edges) whose vertex set can be partitioned into n blocks (or *antipodal classes*) of equal size $r \geq 2$, such that each block induces an r-coclique, the union of any two different blocks induces a perfect matching, and any two non-adjacent vertices in different blocks have exactly $\mu \geq 1$ common neighbors. For an (n, r, μ)-cover Γ, we denote by $\mathcal{CG}(\Gamma)$ the group of all automorphisms of Γ that fix each of its antipodal classes setwise. If the group $\mathcal{CG}(\Gamma)$ is abelian and acts regularly on each antipodal class, then Γ is called an *abelian* (n, r, μ)-cover (see [9]).

Abelian covers form an intriguing, large subclass of covers that, by a Godsil-Hensel criterion, can be characterized in terms of certain matrices over group algebras [9]. Only a few general techniques for constructing such covers are known, and typically they require the existence of a related object (e.g., a crooked function, a generalized Hadamard matrix) that is again hard to construct or yield only covers with specific parameters. A promising task is to describe covers with "rich" automorphism groups. There are three known families of abelian covers with arc-transitive automorphism groups, namely, distance-transitive Taylor graphs, Thas-Somma covers, collinearity graphs of certain generalized quadrangles with a deleted spread, and related covers that come from the quotient construction due to Godsil and Hensel. A recent study [12, 14] shows that the list is almost complete, i.e., a new arc-transitive abelian (n, r, μ)-cover may be discovered only in a few open subcases. However, the general problem of classification of abelian (n, r, μ)-covers, whose automorphism group is vertex-transitive and has at most two orbits in its induced action on the arc set of the cover, is far from being resolved. In this paper, we will investigate this problem by focusing on covers with the triple of parameters (n, r, μ) belonging to some infinite series of feasible

[1]This work was supported by the Russian Science Foundation (project No. 20-71-00122).

parameters. We will also address some open subcases in classification of arc-transitive (n, r, μ)-covers and show that every arc-transitive (n, r, μ)-cover with a non-solvable automorphism group is known and is indeed a member of one of the above-mentioned families whenever it is abelian.

Note that if Γ is a G-vertex-transitive abelian (n, r, μ)-cover with $\mathcal{CG}(\Gamma) \leq G$ such that G induces a rank s permutation group G^Σ on the set Σ of antipodal classes of Γ, then the arc set of Γ is partitioned into a collection of $s-1$ orbitals of G. Therefore, the arc set of Γ is the union of at most two orbitals of G if and only if $s \leq 3$. This observation allows us to apply the classification of permutation groups of rank at most 3 in our arguments, as well as the classification results on abelian covers with primitive rank 3 groups G^Σ obtained in [15–17].

The first class of covers considered in this paper (see Section 2) is abelian (n, r, μ)-covers Γ with parameters of the form

$$(n, r, \mu) = \Big((t^2-1)^2, r, (t^2+t-1)\frac{(t-1)^2}{r}\Big),$$

where $-t$ is the smallest eigenvalue of the cover, and r is an odd divisor of $t-1$ with $\gcd(r, 3) = 1$. The study of covers with such parameters is motivated by the following result.

Theorem 1 (Coutinho, Godsil, Shirazi, Zhan [6]). *Let X be an abelian (n, r, μ)-cover with eigenvalues $n-1 > \theta > -1 > \tau$, where r is odd and $\delta = \lambda - \mu$. It yields a set of complex equiangular lines for which the absolute bound is attained if and only if either $r = \mu = 3 = \sqrt{n}$ or $t = -\tau \in \mathbb{N}$, $\gcd(3, r) = 1$, r divides $t-1$, and the eigenvalues θ, τ, and the parameters (n, r, μ, δ) are as follows*:

$$\theta = (t^2-2)t, \quad \tau = -t, \quad n = (t^2-1)^2, \quad r\mu = (t-1)^2(t^2+t-1) \quad \delta = (t^2-3)t,$$

where

$$m_\theta = (t^2-1)(r-1), \quad m_\tau = (t^2-2)(t^2-1)(r-1),$$

here m_σ denotes the multiplicity of the eigenvalue $\sigma \in \{\theta, \tau\}$.

In the case $(n, r, \mu) = (9, 3, 3)$, according to the result of Brouwer and Wilbrink (see [3, p. 386]), there exist exactly two non-isomorphic covers from the conclusion of Theorem 1; they can be constructed by removing one of two non-isomorphic spreads from the (unique) generalized quadrangle $\mathrm{GQ}(2, 4)$. One of them is distance-transitive, so its automorphism group induces a rank 2 permutation group on the antipodal classes. The automorphism group of the other acts vertex-transitively but intransitively on the arc set, and induces a rank 3 permutation group on the set of antipodal classes. The existence of covers with parameters

$$(n, r, \mu) = \Big((t^2-1)^2, r, (t^2+t-1)\frac{(t-1)^2}{r}\Big)$$

from the conclusion of this theorem is an open question (see the discussion in [6]). We will describe some basic properties of automorphism groups of covers Γ with these parameters and apply them to investigate G-vertex-transitive covers Γ under restrictions on the rank s of the group G^Σ or on the parameter t: (i) $s \leq 3$ or (ii) $t \leq 11$.

The second class considered (see Section 3) is abelian covers Γ, for which the group G^Σ is an affine group of permutation rank $s = 2$. A classification of such pairs (Γ, G) was given in [14]; however, the question of a complete description of pairs (Γ, G) has remained open in the case when $|\Sigma|$ is even and G^Σ is an affine group of one-dimensional, symplectic or G_2 type. We will show that in this case each cover Γ with $r > 2$ and a non-solvable group G is known and isomorphic to a quotient of a certain distance-transitive Thas-Somma cover (see Theorem 2). As a result, we will complete the classification of arc-transitive (n, r, μ)-covers with a non-solvable automorphism group. We will also show that the automorphism group of any unknown edge-transitive (n, r, μ)-cover must induce a one-dimensional affine permutation group on the set of its antipodal classes (see Theorem 3).

Our terminology and notation are mostly standard and can be found in [1, 3] and [17].

2. Covers with parameters $\big((t^2-1)^2, r, (t^2+t-1)(t-1)^2/r\big)$

In this section, Γ is an abelian cover with parameters

$$\Big((t^2-1)^2, r, (t^2+t-1)(t-1)\frac{(t-1)}{r}\Big),$$

where t is a positive integer, r is an odd divisor of $t-1$ such that $\gcd(3, r) = 1$, and with eigenvalues

$$k = t^2(t^2-2), \quad \theta = (t^2-2)t, \quad -1, \quad \tau = -t$$

of multiplicities

$$1, \quad m_\theta = (t^2-1)(r-1), \quad k, \quad m_\tau = (t^2-2)(t^2-1)(r-1),$$

respectively. Let Σ be the set of antipodal classes of Γ, $n = |\Sigma|$, $v = nr$, $a \in F \in \Sigma$, and suppose that $K = \mathcal{CG}(\Gamma) \le G \le \mathrm{Aut}(\Gamma)$, G acts transitively on the vertices of Γ, and put $M = G_{\{F\}}$ and $C = G_F$.

Note that

$$\mu = (t^3-2t+1)(t-1)/r, \quad \lambda = (t^3-2t+1)(t-1)/r + t(t^2-3),$$

where λ denotes the number of common neighbors for two adjacent vertices.

Further, for an element $g \in G$, we will denote by $\alpha_i(g)$ the number of vertices x of Γ such that $\partial(x, x^g) = i$. By $\pi(l)$ we will denote the set of prime divisors of a positive integer l. For a finite group X, the set $\pi(|X|)$ is called its *prime spectrum* and is briefly denoted by $\pi(X)$. In what follows, if it is clear from the context, for a graph Φ, we denote by $[x]$ the adjacency of x in Φ, that is, $[x] = \Phi_1(x)$.

Lemma 1. *The following statements are true:*

1) $C = C_G(K) \cap G_a$ *and* $M = K : G_a$;
2) $|G : M| = (t^2-1)^2$ *and* $|G : G_a|$ *divides* $(t^2-1)^2(t-1)$;
3) $\pi(G) = \pi(nr) \cup \pi(G_a) \subseteq \pi\big(n! \cdot (t-1)\big)$;
4) $|\mathrm{Fix}(G_a)| = |N_G(G_a) : G_a|$ *divides* nr, *and* $|\mathrm{Fix}_\Sigma(M)| = |N_G(M) : M|$ *divides* n;
5) *if* $p \in \pi(G)$ *and* $p > t+1$, *then* $p \in \pi(G_F)$;
6) $G/C_G(K) \le \mathrm{Aut}(K)$ *and if r is prime, then* $G/C_G(K) \le Z_{r-1}$ *and every element* $g \in G$ *of prime order* $p \notin \pi(r-1)$ *either has no fixed points or* $\mathrm{Fix}(g)$ *is the union of some antipodal classes*;
7) *if g is an element of prime order p in G, $p > r$ and* $p \notin \pi((t^2-1)(t^2-2)t)$, *then* $r_3 = r-1$, $r_2 r_1 > 0$, *and* $|\Omega| \ge r(1+r_1) + r \cdot z$ *for some non-negative integer z which is a multiple of p, where* $r_i = |\Gamma_i(a)| \pmod p$, $i = 1, 2, 3$.

P r o o f straightforwardly follows from the assumptions that Γ is abelian and G is transitive on its vertices. □

Lemma 2. *Let $p \in \pi(G)$. If $p > t-1$ and p does not divide $t+1$, then*

$$\max\{p, |\mathrm{Fix}_\Sigma(g)|\} < \mu;$$

in particular, if $p > \mu/2$, then for any element $g \in G$ of order p, the subgraph $\mathrm{Fix}(g)$ *is an abelian $(|\mathrm{Fix}_\Sigma(g)|, r, \mu')$-cover with*

$$\mu' \equiv \mu \pmod{p}, \quad \mu > |\mathrm{Fix}_\Sigma(g)| > (r-1)\mu', \quad \pi(r) \subseteq \pi(|\mathrm{Fix}_\Sigma(g)|).$$

P r o o f. Let $g \in G$ and $|g| = p > t-1$. Suppose that p does not divide $t+1$. Then $\Omega := \mathrm{Fix}(g) \neq \varnothing$ and $|\Omega| = lr$, where $l = |\mathrm{Fix}_\Sigma(g)|$.

(1) Suppose $\alpha_2(g) > 0$. By definition, this means that $\partial(x, x^g) = 2$ for some vertex $x \in \Gamma - \Omega$. But then $[x] \cap [x^g]$ contains exactly l vertices from Ω, implying $l \leq \mu$.

Assume $p > \mu$. Then $[a] \cap [b] \subset \Omega$ for any non-adjacent vertices a and b from Ω belonging to different antipodal classes. But according to [9, Lemma 3.1], Ω is a (l, r, μ)-cover and thus $l > (r-1)\mu > \mu$, which is clearly impossible. Therefore, $p \leq \mu$ and since the number μ is composite, $p < \mu$. Notice that if $p > \mu/2$, then

$$0 < \mu' := |[a] \cap [b] \cap \Omega| \equiv \mu \pmod{p} < \mu/2 < p$$

for any non-adjacent vertices a and b from Ω, belonging to different antipodal classes. Then by [9, Lemma 3.1], Ω is an abelian (l, r, μ')-cover, implying $\mu > l > (r-1)\mu'$ and $\pi(r) \subseteq \pi(l)$ by [10, Theorem 2.5].

(2) Suppose $\alpha_2(g) = 0$. Then $\alpha_0(g) = lr = \alpha_0(g^i)$ for all $1 \leq i \leq p-1$ and $\alpha_1(g) = r(n-l)$. If $\alpha_2(g^s) > 0$ for some $1 < s \leq p-1$, then the desired result is obtained by reasoning as in part (1). Suppose that $\alpha_2(g) = \alpha_2(g^i) = 0$ for all $1 \leq i \leq p-1$. Then each $\langle g \rangle$-orbit in $\Gamma - \Omega$ is an $(n-l)$-clique. Since $n - l \leq \lambda + 2$, we have $l \geq (r-1)\mu \geq \lambda$. Since $|[x] \cap [x^g] \cap \Omega| = l$ for any vertices x and x^g from $\Gamma - \Omega$, we obtain $n - l = 2$, which means that each $\langle g \rangle$-orbit in $\Gamma - \Omega$ is an edge and $g^2 \in G_x$, so $|g| = 2$, a contradiction. The lemma is proven. □

Lemma 3. *Suppose g is an involution of G and $\Omega = \mathrm{Fix}(g) \neq \varnothing$. Let $f = |\Omega \cap F(z)|$ for $z \in \Omega$, $l = |\mathrm{Fix}_\Sigma(g)|$, and*

$$X = \{x \in \Gamma \setminus \Omega |\ [x] \cap \Omega \neq \varnothing\}.$$

Then Ω is a regular graph of degree $l-1$ on lf vertices and the following statements hold:

1) *if $l = 1$, then t is even, $\alpha_3(g) = 0$, $\alpha_1(g) + \alpha_2(g) = (n-1)r$, and Ω coincides with the antipodal class $F(z)$;*
2) *if $l > 1$, then $\alpha_3(g) = (r-f)l$, $\alpha_1(g) + \alpha_2(g) = (n-l)r$, each vertex in X is "on average" adjacent to $fl(n-l)/|X|$ vertices in Ω, and the number of such vertices in Ω does not exceed l (in particular, if $|X| = f(n-l)$, then $l \leq \lambda$) and*

$$f \leq \frac{|X|}{n-l} \leq |\Omega| \leq \frac{(\lambda - \mu)\alpha_1(g)}{n-l} + r\mu \leq r\lambda,$$

   furthermore, if $F(z) \subset \Omega$, then $X = \Gamma \setminus \Omega$ and either (i) $\alpha_1(g) = (n-l)r$, $l \leq \lambda$, and each $\langle g \rangle$-orbit on $\Gamma \setminus \Omega$ is an edge, or (ii) $\alpha_2(g) > 0$ and $l \leq \mu$;
3) *if t is even, then $X = \{x \in \Gamma |\ \partial(x, x^g) \in \{1, 2\}\}$;*
4) *if $f = 1$ and $l > 1$, then Ω is an l-clique and $l \leq r\mu/(t-1) \leq \mu$;*
5) *if $f > 1$, t is even, and $l > 1$, then the diameter of the graph Ω is 3 and $|\Omega| \leq l + (l-1)^2(l-2)$.*

P r o o f. Let $f = |\Omega \cap F(z)|$ and assume $\Omega \neq \varnothing$ (which is automatically satisfied for even t). Then $|\Omega| = f \cdot l > 0$ and for any two vertices a and b in Ω, we have

$$l - 1 = |[a] \cap \Omega| = |[b] \cap \Omega|,$$

meaning that the graph Ω is regular of degree $l - 1$. Recall that

$$\mu = (t^2 + t - 1)\frac{(t-1)^2}{r}, \quad \lambda = (t^2 - 1)^2 - 2 - (r-1)\mu.$$

Since r is odd and divides $t - 1$, we have

$$t - 1 \equiv l \equiv \lambda \equiv \mu \pmod 2.$$

Let Λ be the set of all vertices from antipodal classes that intersect Ω.

Note that if $\partial(x, x^g) = 3$, then $x \in \Lambda$, $|F(x) \cap \Omega| = f$, and $[x] \subset \Gamma \setminus \Omega$. And if $x \neq x^g$ and $\partial(x, x^g) < 3$, then $x \in \Gamma \setminus \Lambda$ and $|[x] \cap [x^g] \cap \Omega| \equiv \lambda \equiv \mu \pmod 2$. Therefore,

$$\Lambda = \Omega \cup \{x \in \Gamma |\ \partial(x, x^g) = 3\}, \quad \Gamma \setminus \Lambda = \{x \in \Gamma |\ \partial(x, x^g) \in \{1, 2\}\}.$$

From the above, we have

$$\alpha_3(g) = (r - f)l, \quad \alpha_1(g) + \alpha_2(g) = (n - l)r,$$

and for each vertex $y \in \Gamma \setminus \Lambda$, $|[y] \cap \Omega| \leq l$, so that either $|[y] \cap \Omega| \leq \mu$ and $\partial(y, y^g) = 2$, or $|[y] \cap \Omega| \leq \lambda$ and $\partial(y, y^g) = 1$. On the other hand, each vertex in Λ is adjacent exactly to $n - l$ vertices in $\Gamma \setminus \Lambda$. Let

$$X = \big\{x \in \Gamma \setminus \Omega |\ [x] \cap \Omega \neq \varnothing\big\}.$$

Then

$$\bigcup_{b \in F(z)} [b] \cap (\Gamma \setminus \Lambda) \subseteq X,$$

implying $|X| \geq (n - l)f$, and $X = \Gamma \setminus \Lambda$ for even t. Estimating the number of edges between X and Ω gives

$$|\Omega|(n - l) \leq \min\big\{l(\alpha_1(g) + \alpha_2(g)), \lambda\alpha_1(g) + \mu\alpha_2(g)\big\}.$$

And since

$$\lambda\alpha_1(g) + \mu\alpha_2(g) = (\lambda - \mu)\alpha_1(g) + \mu(n - l)r$$

and the number of vertices in X does not exceed the number of edges from Ω to X, we have $|X| \leq |\Omega|(n - l)$ and

$$\frac{|X|}{n - l} \leq |\Omega| = f \cdot l \leq \frac{(\lambda - \mu)\alpha_1(g)}{n - l} + r\mu \leq r \max\{\mu, \lambda\} = r\lambda.$$

Hence for even t we have $|\Omega| \geq r$, and in particular, for $l = 1$, we have $\Omega = \Lambda$.

Let $l > 1$ and Φ be a connected component of the graph Ω. Then Φ is an f-covering of the complete graph on $m = |\Phi|/f$ vertices, in particular, $\Phi = \Omega$ for even t. Since the number of edges from Ω to X is $|\Omega|(n - l)$, each vertex in X is adjacent "on average" to $fl(n - l)/|X|$ vertices in Ω. Recall that $\mu < \lambda$. Therefore, if $|X| = f(n - l)$, then $l \leq \lambda$.

If $f = 1$, then Ω is an l-clique and $l \leq r\mu/(t - 1)$. And if $f > 1$ and t is even, then the diameter of Ω is 3 and, due to the Moore bound, we have

$$|\Omega| \leq l + (l - 1)^2(l - 2).$$

Suppose $f = r$. If $\alpha_1(g) = (n - l)r$, then $l \leq \lambda$ and each $\langle g \rangle$-orbit on $\Gamma \setminus \Lambda$ is an edge. If $\alpha_1(g) < (n - l)r$ (which is equivalent to $\alpha_2(g) > 0$), then $l \leq \mu$. The lemma is proven. □

Lemma 4. *Suppose that G^Σ has permutation rank* 3 *with subdegrees* $1 \le k_1 \le k_2$. *Then*

1) *G_a has exactly two orbits on $[a]$, denoted X_1 and X_2, with lengths k_1 and k_2 respectively, satisfying*
$$k_1(\lambda - \lambda_1) = k_2(\lambda - \lambda_2), \tag{2.1}$$
where $\lambda_i = |[x_i] \cap X_i|$ for vertex $x_i \in X_i$, $i = 1, 2$;
2) *if G_a fixes a vertex $a^* \in F(a) - \{a\}$, then*
$$k_1(\mu - \mu_1) = k_2(\mu - \mu_2), \tag{2.2}$$
where $\mu_1 = |[y_1] \cap X_1|$ for vertex $y_1 \in X_1^*$, $\mu_2 = |[y_2] \cap X_2^*|$ for vertex $y_2 \in X_2$, and X_i^* is a G_a-orbit on $[a^*]$ of length k_i, $i = 1, 2$.

P r o o f. As the groups ${G_a}^{[a]}$ and ${G_{\{F\}}}^{\Sigma - \{F\}}$ are permutation isomorphic, (2.1) follows by counting the number of edges between X_1 and X_2 in two different ways, and (2.2) is obtained in a similar way by counting the number of edges between X_1^* and X_2.

□

Proposition 1. *The permutation rank s of the group G^Σ is not equal to* 2 (*so* Γ *cannot be arc-transitive*).

P r o o f. On the contrary, suppose $s = 2$. Then, since Γ is abelian and $K \le G$, we conclude that G acts transitively on the arcs of Γ. As n is not a prime power, by Burnside's theorem it follows that G^Σ is an almost simple group. But then the number $k = t^2(t^2 - 2)$ is a power of a prime (see [12]), a contradiction. □

Proposition 2. *If G^Σ is a primitive group of rank* 3, *then the group G^Σ is either almost simple and its socle cannot be an alternating group or a sporadic simple group, or $T \times T \trianglelefteq G^\Sigma \le T_0 \wr 2$, where T_0 is a* 2*-transitive group of degree $n_0 = t^2 - 1$ with a simple non-abelian socle T and T cannot be an alternating group of degree n_0.*

P r o o f. Suppose that G^Σ is a primitive group of rank 3 with nontrivial subdegrees k_1 and k_2, where $k_1 \le k_2$. Under this condition, according to [5, Theorem 2.6.14] and [2], the group G^Σ of degree $(t^2 - 1)^2$ has two self-paired orbitals on Σ.

Since $n = (t^2 - 1)^2$ is not a prime power, by the classification of primitive groups of rank 3 (e.g., see [4, Chapter 11, Theorem 11.1.1]), the group G^Σ is either almost simple or is of wreath product type. Furthermore, according to [15, 16], the socle $\mathrm{Soc}(G^\Sigma)$ of the group G^Σ cannot be an alternating or a sporadic simple group for all t.

Suppose that the group G^Σ is of wreath product type, that is, $P = T \times T \trianglelefteq G^\Sigma \le T_0 \wr 2$, where T_0 is a 2-transitive group of degree n_0 with a simple non-abelian socle T and $n = n_0^2$. Then nontrivial subdegrees of the group G^Σ are $k_1 = 2(n_0 - 1)$ and $k_2 = (n_0 - 1)^2$. Simplifying the equation (2.1), we get
$$2(\lambda - \lambda_1) = (n_0 - 1)(\lambda - \lambda_2).$$
Since the group G_a contains a subgroup $A = S \times S$, where S is the stabilizer of a point in T, and A has exactly two orbits on X_1, say Y_1 and Y_2, of lengths $|Y_1| = |Y_2| = n_0 - 1$, and $A^{Y_i} \simeq S$, λ_1 must be a sum of subdegrees of A^{X_1}. Hence in the case $T \simeq \mathrm{Alt}_{n_0}$ we have $S \simeq \mathrm{Alt}_{n_0 - 1}$ and $\lambda_1 \in \{0, n_0 - 2, n_0 - 1, 2n_0 - 3\}$. But then the above equation implies that $n_0 - 1 = t^2 - 2$ divides

Table 1. Parameters of Γ with $t \leq 11$ (see [6]).

n	r	μ	$\delta = \lambda - \mu$	θ	$\tau = -t$	m_θ	m_τ
1225	5	205	198	204	-6	140	4760
3969	7	497	488	496	-8	378	23436
14400	5	2620	1298	1309	-11	480	57120

one of 2λ or $2(\lambda - 1)$, which is impossible. So $T \not\simeq \mathrm{Alt}_{n_0}$ for all t. □

Next we will consider the case $t \leq 11$, in which the parameters of Γ are given in Table 1.

Proposition 3. *If $t \leq 11$ and G^Σ is a primitive group of rank s, then $s > 3$.*

P r o o f. Suppose that G^Σ is a primitive group of rank 3 with subdegrees $1, k_1$ and k_2, where $k_1 \leq k_2$. Let us consider the case $t \leq 11$.

Case $t = 6$. Suppose $t = 6$ (so $n = 1225 = 5^2 7^2$). According to [16, 17], it suffices to consider the case where the group G^Σ is of wreath product type with degree $n = n_0^2$. But then $n_0 = 35$, T_0 is a 2-transitive group of degree 35, and by the classification of finite 2-transitive permutation groups (e.g., see [4, Theorem 11.2.1]), $T \simeq \mathrm{Alt}_{n_0}$, which contradicts to Proposition 2.

Case $t = 8$. Suppose $t = 8$ (so $n = 3969 = 3^4 7^2$). The case of an almost simple group G^Σ is impossible, since according to the classification of primitive almost simple groups of rank 3 (e.g., see [4, Chapter 11] or [7, Tables 4–6]), $\mathrm{Soc}(G^\Sigma)$ cannot be an exceptional simple group or a classical simple group of degree n.

Suppose that the group G^Σ is of wreath product type with degree $n = n_0^2$. Then $n_0 = 63$ and T_0 is a 2-transitive group of degree 63. By Proposition 2, $T \not\simeq \mathrm{Alt}_{n_0}$, so the classification of 2-transitive groups (e.g., see [4, Theorem 11.2.1]) gives $T \simeq PSL_6(2)$. But if $T \simeq PSL_6(2)$, then the subdegrees of A^{Y_i} on Y_i are 1, 1 and 60. Therefore,

$$\lambda_1 \pmod{n_0 - 1} \in \{0, 1, n_0 - 3, n_0 - 2\},$$

and the equation (2.1) has no solution, a contradiction.

Case $t = 11$. Suppose $t = 11$ (so $n = 14400 = 2^6 3^2 5^2$). The case of an almost simple group G^Σ is impossible, since according to the classification of primitive almost simple groups of rank 3 (e.g., see [4, Chapter 11]), $\mathrm{Soc}(G^\Sigma)$ cannot be an exceptional simple group or a classical simple group of degree n.

Suppose the group G^Σ is of wreath product type of degree $n = n_0^2$. By Proposition 2, $T \not\simeq \mathrm{Alt}_{n_0}$, so by [4, Theorem 11.2.1] we have $T \simeq PSp_8(2)$. But if $T \simeq PSp_8(2)$, then the subdegrees of A^{Y_i} on Y_i are $1, 54$ and 64. Therefore,

$$\lambda_1 \pmod{n_0 - 1} \in \{0, 54, 64, n_0 - 1, n_0 - 2, 2n_0 - 3\}.$$

But $\lambda = \mu + \delta = 3918$, and the equation (2.1) has no solution, a contradiction.

The proposition is proven. □

Proposition 4. *If $t = 6$, $\overline{G} := G^\Sigma$ is an imprimitive group of rank s and $\mathcal{B}$ is its imprimitivity system, then either $s > 3$ or $s = 3$ and $\{|B|, |\mathcal{B}|\} = \{25, 49\}$, where $B \in \mathcal{B}$, $\overline{G}_B^B$ and $\overline{G}^{\mathcal{B}}$ are affine 2-transitive groups.*

P r o o f. Suppose that $\overline{G}$ is a rank 3 imprimitive group with subdegrees 1, k_1, and k_2, where $k_1 \le k_2$. Let $\mathcal{B}$ be the unique nontrivial system of imprimitivity of the group $\overline{G}$ (see [8, Lemma 3.3]), and fix arbitrarily its block B (of size k_1+1). Then k_1+1 divides

$$\gcd(k_2,(t^2-1)^2), \quad \overline{G} \le \overline{G}_B{}^B \wr \mathrm{Sym}(\mathcal{B}),$$

$\overline{G}_B$ acts 2-transitively on B, and G acts 2-transitively on $\mathcal{B}$ (e.g., see [8, Lemma 3.1]). Let $T_0 = \overline{G}_B{}^B$ and $T = \mathrm{Soc}(T_0)$. Let S be the kernel of the action of G on $\mathcal{B}$, W be the full pre-image of $\overline{G}_B$ in G and N be the kernel of the action of W on B. Without loss of generality, we assume that $a \in F \in B$. Then $K \le N \cap S \trianglelefteq W$ and $N = K : N_a$.

Since $n = 35^2$, then $|B| \in \{5, 7, 25, 35, 49, 175, 245\}$. If $|B| = 35, 175$ or 245, then applying the classification of finite 2-transitive groups (e.g., see [4, Theorem 11.2.1]) we obtain $T \simeq \mathrm{Alt}(B)$. But Lemma 2 implies $\max(\pi(G)) \le 101$, so $|B| = 35 = |\mathcal{B}|$ and $T \simeq \mathrm{Soc}(G^{\mathcal{B}}) \simeq \mathrm{Alt}(\mathcal{B})$. In this case $\mathrm{Alt}_{34} \le W/S \trianglelefteq \mathrm{Sym}_{34}$. On the other hand, $W/N \trianglelefteq \mathrm{Sym}_{35}$, in particular, W/N contains an element of order 35. Since the group NS is normal in W, then either $S \le N$ or $NS/N \ge \mathrm{Soc}(W/N) \simeq T$. If $S \le N$, then $\mathrm{Alt}_{35} \le (W/S)/(N/S) \simeq W/N$, a contradiction. Hence $|W : NS| \le 2$ and the group NS acts 3-transitively on B, so $|G_{\{F\}} : NS_{\{F\}}| \le 2$. But then, simplifying the equation (2.1), we get

$$(\lambda - \lambda_1) = (k_1+1)(\lambda - \lambda_2),$$

which contradicts the fact that $\lambda_1 \in \{0, k_1-1\}$ in this case.

If $|B| = 5$ or 7, then $|\mathcal{B}| = 175$ or 245, and in view of [4, Theorem 11.2.1] we obtain $\mathrm{Soc}(G^{\mathcal{B}}) \simeq \mathrm{Alt}(\mathcal{B})$. But $\max(\pi(G)) \le 101$ by Lemma 2, a contradiction. So $\{|B|, |\mathcal{B}|\} = \{25, 49\}$.

1. Let $|B| = 25$. Then simplifying the equation (2.1), we get

$$(\lambda - \lambda_1) = 50(\lambda - \lambda_2) > 0, \quad \lambda_1\lambda_2 > 0.$$

   In view of [4, Theorem 11.2.1] we have either $T \simeq \mathrm{Alt}(B)$ or $T \simeq E_{25}$ and W^B is an affine group. But in the first case G_a is 2-transitive on X_1, hence $\lambda_1 \in \{0, k_1-1\}$ and the above equation has no solution, a contradiction. Hence $T \simeq E_{25}$. Applying [4, Theorem 11.2.1] again, we obtain that either $\mathrm{Soc}(G^{\mathcal{B}}) \simeq \mathrm{Alt}(\mathcal{B})$ or $\mathrm{Soc}(G^{\mathcal{B}}) \simeq E_{49}$ and $G^{\mathcal{B}}$ is an affine group.

   Suppose that $\mathrm{Soc}(G^{\mathcal{B}}) \simeq \mathrm{Alt}(\mathcal{B})$. Then $\mathrm{Alt}_{48} \le W/S \le \mathrm{Sym}_{48}$ and $W - S$ contains an element g of order 37, which fixes pointwise 11 blocks in $\mathcal{B} \setminus \{B\}$. In view of Lemma 1 $g \in G_F \cap N$ and $|\mathrm{Fix}_\Sigma(g)| \ge 25 \cdot 12 = 300$, which is a contradiction to Lemma 2.

2. Let $|B| = 49$. Then $k_1 = 48$ and $k_2 = 49 \cdot 24$, so the equation (2.1) implies

$$2(\lambda - \lambda_1) = 49(\lambda - \lambda_2) > 0, \quad \lambda_1\lambda_2 > 0.$$

   In view of the classification of 2-transitive groups (e.g., see [4, Theorem 11.2.1]) we obtain either $T \simeq \mathrm{Alt}(B)$ or $T \simeq E_{49}$ and W^B is an affine group. But in the first case G_a is 2-transitive on X_1, so $\lambda_1 \in \{0, k_1-1\}$ and the equation above has no solution, a contradiction. Hence $T \simeq E_{49}$. Applying again [4, Theorem 11.2.1], we obtain either $\mathrm{Soc}(G^{\mathcal{B}}) \simeq \mathrm{Alt}(\mathcal{B})$ or $\mathrm{Soc}(G^{\mathcal{B}}) \simeq E_{25}$ and $G^{\mathcal{B}}$ is an affine group.

   Suppose $\mathrm{Soc}(G^{\mathcal{B}}) \simeq \mathrm{Alt}(\mathcal{B})$. Then $\mathrm{Alt}_{24} \le W/S \le \mathrm{Sym}_{24}$. On the other hand, $W/N \le AGL_2(7)$ and $G_{\{F\}}/N \le GL_2(7)$. Since $|GL_2(7)| = 2^5 3^2 7$ and $|B| = 49$, N contains an element g of order 13, which fixes pointwise 11 blocks in $\mathcal{B} \setminus \{B\}$. In view of Lemma 1 $g \in G_F \cap N$ and $|\mathrm{Fix}_\Sigma(g)| \ge 49 \cdot 12 = 588$, which is a contradiction to Lemma 2.

The proposition is proven. □

3. Arc-transitive affine covers

Throughout this section, Γ is an arc-transitive $\big(2^e, r, 2^e/r\big)$-cover, where $r > 2$, with eigenvalues $k = 2^e - 1, \theta, -1, \tau$ of multiplicities $1, m_\theta, k, m_\tau$ respectively. Let Σ be the set of antipodal classes of the graph Γ, $2^e = n = |\Sigma| = r\mu$, $v = nr$, $a \in F \in \Sigma$, and $K = \mathcal{CG}(\Gamma)$. Suppose $|K| = r$, the group $G \le \mathrm{Aut}(\Gamma)$ acts transitively on the arc set of Γ and induces an affine 2-transitive permutation group G^Σ on Σ. Let $C = G_F$, T be the full preimage of the socle of the group G^Σ in G and $L = {G_a}^\infty$. In [14], it was shown that under these assumptions, the pair (Γ, G) satisfies one of the following conditions:

1) $e = 2dc$, $d = 3$, r divides 2^c, T is an elementary abelian group of order $2^e r$, not containing any subgroup of order 2^e that is normal in G, and $G_2(2^c) \simeq L \trianglelefteq G_a$;
2) $e = 2dc$, $d \ge 1$, r divides 2^c, T is an elementary abelian group of order $2^e r$, not containing any subgroup of order 2^e that is normal in G, and $Sp_{2d}(2^c) \simeq L \trianglelefteq G_a$;
3) $G_a \le \Gamma L_1(2^e)$.

In the corresponding cases, we will say that the group G_a is of type G_2, of symplectic type, or of one-dimensional type. Moreover, by [14], $L \le C$ whenever the group G_a is of symplectic or of G_2 type.

Recall that for each subgroup $1 < N < K$ the *quotient graph* Γ^N, that is the graph on the set of N-orbits where two orbits are adjacent if and only if there is an edge of Γ joining them, is a $(n, r/|N|, \mu|N|)$-cover [9].

Theorem 2. *Suppose $r > 2$ and the group G_a is of symplectic or of G_2 type. Then the $\big(2^e, r, 2^e/r\big)$-cover Γ is isomorphic to a quotient Φ^U of a distance-transitive Thas–Somma $\big(2^e, 2^c, 2^{c(2d-1)}\big)$-cover Φ for some subgroup $U \le \mathcal{CG}(\Phi)$ of index r. Furthermore, if $r = 2^c$, then $\Gamma \simeq \Phi$ and Γ is characterized by its parameters in the class of arc-transitive distance-regular covers of complete graphs with a non-solvable automorphism group.*

P r o o f. Let $L \simeq G_2(2^c)$ or $Sp_{2d}(2^c)$, and $r = 2^s \le 2^c$. First of all, we will determine possible complements to T in TL, then we will find how many classes of conjugate complements to T the group TL can contain, and to which one the group L can belong. From the conditions, it follows that TL centralizes K and $(TL/K)' = TL/K$. This means that the group TL is a central extension of the group TL/K (e.g., see [1, Ch. 33]).

Let us consider the group T as a $\mathbb{GF}(2)L$-module and K as its L-invariant subspace. Then, with respect to a suitable $\mathbb{GF}(2)$-basis in T (e.g., see [1, 13.2]), the matrices of elements g of the group L, considered as a subgroup of $\mathrm{Aut}(T) \simeq GL_{e+s}(2)$, have the following form:

$$\begin{pmatrix} \varphi(g) & 0 \\ \psi(g) & I_s \end{pmatrix},$$

where $\varphi : L \to GL_e(2)$ is a homomorphism and I_s is the identity matrix of order $s \times s$. Since T is irreducible as a $\mathbb{GF}(2)L$-module, we have $\psi \not\equiv 0$. Let $\widetilde{V} = T/K$ and let us fix a nonzero vector $w \in K$. Define the map $\gamma_w : L \to \widetilde{V}$ by the rule $g\gamma_w = w\psi(g) + K$ for all $g \in L$. Then for all $g, h \in L$, we have

$$(gh)\gamma_w = (g\gamma_w)^h + h\gamma_w,$$

which means γ_w is a cocycle (e.g., see [1, Ch. 6]). Recall that according to [11], the first cohomology group $\mathrm{H}^1(L, \widetilde{V})$ of the group L in $\widetilde{V}$ is one-dimensional as a $\mathbb{GF}(2^c)$-space. Therefore (e.g., see [1, 17.7]), multiplication by a scalar $f \in \mathbb{GF}(2^c)$ gives a set of functions $(fI_s)\psi(g)$, which determines representatives R of all classes of conjugate complements to T in TL.

Case $r = 2^c$. Since $[T, L] = T$ and $K \leq Z(TL)$, by [1, 17.12] when $r = 2^c$, the $\mathbb{GF}(2)L$-module T is the largest among $\mathbb{GF}(2)L$ extensions V of the module K by $\widetilde{V}$ with the property $[V, L] = V$ and $K \leq C_V(G)$. Representatives R_1 and R_2 of any two classes of complements to T in TL are conjugate in $\mathrm{Aut}(T)$ by matrices of the form $diag(I_e, fI_s)$, where $f \in \mathbb{GF}(2^c)$. This implies that the set of orbital $(2^e, r, 2^e/r)$-covers of the group TL with vertex stabilizer R_1 gives the set of orbital $(2^e, r, 2^e/r)$-covers of the group TL with vertex stabilizer R_2, where each graph from the first collection is isomorphic to some graph from the second collection and vice versa. Therefore, it is enough to find the set of all orbital $(2^e, r, 2^e/r)$-covers of the group TL with vertex stabilizer L. One of such covers is the distance-transitive $(2^e, r, 2^e/r)$-cover Φ from the Thas–Somma construction [10, Example 3.6, Proposition 6.2]. According to [10], it is characterized by its parameters in the class of distance-transitive covers. Next we will show that it is also characterized by its parameters in the class of arc-transitive covers with a non-solvable group of automorphisms. Note that $G_2(2^c) \leq Sp_6(2^c)$ (e.g., see [18, Theorem 3.7]), and only in the case when $d = 3$ two different types are permissible for the group G_a.

Let $L \simeq Sp_{2d}(2^c)$. Since the groups $L^{[a]}$ and $L^{\Sigma-\{F\}}$ are permutation isomorphic, in order to determine the subdegrees of the group G_a on $[a]$, it suffices to consider the action of the group $Sp_{2d}(2^c)$ on its natural module W. Recall that it acts transitively on the set of all hyperbolic pairs in the space W, and for each non-zero vector w in W, there are exactly q^{2d-1} hyperbolic pairs of the form (w, u). Therefore, the stabilizer of the vector w fixes pointwise $q-1$ non-zero vectors from $\langle w \rangle$ and has $q-1$ orbits on W of length q^{2d-1}. If a vertex $b \in [a]$ is associated with the vector w, then, since each vertex b^* from $F(b) - \{b\}$ is adjacent to exactly $\mu = q^{2d-1}$ vertices from $[a] - [b]$ and $\lambda - \mu = -2$, the last statement implies that the group $L_{a,b} = L_{a,b^*}$ acts transitively on each μ-subgraph of the form $[a] \cap [b^*]$.

Suppose $\Gamma \neq \Phi$. Let Q and S be the self-paired orbitals of the group TL, corresponding to the arc sets of the graphs Φ and Γ, respectively. Then $Q(a) \neq S(a)$ and $\Gamma_1(a^*) = S(a^*) = Q(a) = \Phi_1(a)$ for some vertex $a^* \in F(a)$. Since T is an elementary abelian group, regular on vertices, for some involutions $g \in T - K$ and $h \in K$, we have $a^g \in Q(a)$ and $a^{gh} \in S(a)$. Without loss of generality, we may assume that $a^* = a^h$, so that $(a^*)^g = a^{gh}$. Then $\Gamma_1(a) = S(a) = Q(a^*) = \Phi_1(a^*)$. Since $|Q(a^{gh}) \cap Q(a^*)| = |S(a^{gh}) \cap S(a)| = \lambda$ and $\lambda < \mu$, from the action of the group $L_{a,a^g} = L_{a^*,(a^*)^g}$ on Σ, we obtain $Q(a^{gh}) \cap Q(a^*) = S(a^{gh}) \cap S(a)$, which implies $Q \cap S \neq \varnothing$, a contradiction.

Let $L \simeq G_2(2^c)$. Then, to determine the subdegrees of the group G_a on $[a]$, it suffices to consider the action of the group $G_2(2^c)$ as a subgroup of $Sp_6(2^c)$ on its natural module W (e.g., see [18, 4.3]). Recall that the stabilizer of a 1-dimensional subspace in L has exactly four orbits on the 1-dimensional subspaces of W, and their lengths are 1, $q(q+1)$, $q^3(q+1)$, and q^5. Therefore, the stabilizer of a non-zero vector w in L fixes pointwise $q-1$ non-zero vectors from $\langle w \rangle$ and has $q-1$ orbits on W of length q^5. By reasoning as above, we obtain the required statement.

Further, according to [9, Theorem 6.2], for any subgroup U in K of index $2^l := |K : U| > 1$, the quotient graph Φ^U is a $(2^e, 2^l, 2^{c-l}\mu)$-cover, and the group TL/U acts as an arc-transitive group of automorphisms of this graph. Next we will show that these covers exhaust all possibilities for Γ when $s < c$.

Case $r < 2^c$. In this case, $s < c$. Let us show that the only candidates for Γ are the quotient graphs Φ^U of the distance-transitive Thas-Somma $(2^e, 2^c, 2^{c(2d-1)})$-cover Φ defined for subgroups $U \leq \mathcal{CG}(\Phi)$ of index 2^s. Let $\Psi = \Phi^U$ be such a quotient graph, and let Q and S be the self-paired orbitals of the group TL, corresponding to the arc sets of the graphs Ψ and Γ, respectively.

Suppose $\Gamma \neq \Psi$. A contradiction is obtained by a similar argument as before. Indeed, then $Q(a) \neq S(a)$ and $\Gamma_1(a^*) = S(a^*) = Q(a) = \Psi_1(a)$ for some vertex $a^* \in F(a)$. Since T is an elementary abelian group, regular on vertices, for some involutions $g \in T - K$ and $h \in K$, we have $a^g \in Q(a)$ and $a^{gh} \in S(a)$. Without loss of generality, we may assume that $a^* = a^h$, so that

$(a^*)^g = a^{gh}$. Then $\Gamma_1(a) = S(a) = Q(a^*) = \Psi_1(a^*)$. Since

$$|Q(a^{gh}) \cap Q(a^*)| = |S(a^{gh}) \cap S(a)| = \lambda = \mu - 2 = q^{2d-1}2^{c-s} - 2,$$

and λ is not divisible by q^{2d-1}, from the action of the group $L_{a,a^g} = L_{a^*,(a^*)^g}$ on Σ, we obtain

$$\big(Q(a^{gh}) \cap Q(a^*)\big) \cap \big(S(a^{gh}) \cap S(a)\big) \neq \varnothing,$$

which implies $Q \cap S \neq \varnothing$, a contradiction.

The rest statements of theorem follow from the classification of edge-transitive covers in the almost simple case (see [12]) and in the affine case (see [14]). □

Theorem 2 together with the results of [13, 14] and [12] yields the main theorem of this section.

Theorem 3. *If Φ is an unknown edge-transitive (n, r, μ)-cover, then the group* $\mathrm{Aut}(\Phi)$ *induces a one-dimensional affine permutation group on the set of its antipodal classes. Moreover, if the automorphism group of the cover Φ acts transitively on its arc set, then it is solvable, $\mu > 1$, $n = r\mu$ is a prime power, and $|\mathcal{CG}(\Phi)| = r$.*

4. Concluding remarks

This paper finalizes a major part of the project of classifying edge-transitive distance-regular covers of complete graphs, providing a complete classification in the case when the automorphism group of such cover is non-solvable and arc-transitive.

In general, a larger class of G-vertex-transitive covers having at most two G-orbits in the induced action of G on the arc set is still not well understood, and its study remains relevant in the context of finding new constructions of covers with various parameters of interest, e.g., of abelian covers with parameters $((t^2-1)^2, r, (t^2+t-1)(t-1)^2/r)$ whose study is motivated by important problems in discrete geometry [6]. In this paper, we established some basic properties of automorphism groups of abelian covers Γ with these parameters and applied them to investigate G-vertex-transitive covers Γ under restrictions on the rank s of the group G^Σ or the parameter t: $s \leq 3$ or $t \leq 11$. This work will be extended in a forthcoming publication of the author for covers with arbitrary values of t.

Next we list a few open questions:

1) Classify arc-transitive covers with a solvable automorphism group.
2) Classify half-transitive covers.
3) Is there any G-vertex-transitive cover with parameters $((t^2-1)^2, r, (t^2+t-1)(t-1)^2/r)$ having precisely two G-orbits in the induced action of G on the arc set?

REFERENCES

1. Aschbacher M. *Finite Group Theory*, 2nd ed. Cambridge: Cambridge University Press, 2000. 305 p. DOI: 10.1017/CBO9781139175319
2. Berggren J. L. An algebraic characterization of finite symmetric tournaments. *Bull. Austral. Math. Soc.*, 1972. Vol. 6, No. 1. P. 53–59. DOI: 10.1017/S0004972700044257
3. Brouwer A. E., Cohen A. M., Neumaier A. *Distance–Regular Graphs.* Berlin etc: Springer–Verlag, 1989. 494 p. DOI: 10.1007/978-3-642-74341-2
4. Brouwer A. E., Van Maldeghem H. *Strongly Regular Graphs.* Cambridge: Cambridge University Press, 2022. 462 p. DOI: 10.1017/9781009057226
5. Chen G., Ponomarenko I. *Lecture Notes on Coherent Configurations.* 2023. 356 p. URL: https://www.pdmi.ras.ru/ inp/ccNOTES.pdf

6. Coutinho G., Godsil C., Shirazi M., Zhan H. Equiangular lines and covers of the complete graph. *Lin. Alg. Appl.*, 2016. Vol. 488. P. 264–283. DOI: 10.1016/j.laa.2015.09.029
7. Coutts H. J., Quick M. R., Roney-Dougal C. M. The primitive groups of degree less than 4096. *Comm. Algebra*, 2011. Vol. 39. P. 3526–3546. DOI: 10.1080/00927872.2010.515521
8. Devillers A., Giudici M., Li C.H., Pearce G., Praeger Ch. E. On imprimitive rank 3 permutation groups. *J. London Math. Soc.*, 2011. Vol. 84. P. 649–669. DOI: 10.1112/jlms/jdr009
9. Godsil C. D., Hensel A. D. Distance regular covers of the complete graph. *J. Comb. Theory Ser. B.*, 1992. Vol. 56, No. 1. P. 205–238. DOI: 10.1016/0095-8956(92)90019-T
10. Godsil C. D., Liebler R. A., Praeger C. E. Antipodal distance transitive covers of complete graphs. *Europ. J. Comb.*, 1998. Vol. 19, No. 4. P. 455–478. DOI: 10.1006/eujc.1997.0190
11. Jones W., Parshall B. On the 1-cohomology of finite groups of Lie type. In: *Proc. Conf. on Finite Groups, eds: W.R. Scott and F. Gross*. New York: Academic Press, 1976. P. 313–327. DOI: 10.1016/B978-0-12-633650-4.50022-9
12. Tsiovkina L. Yu. Covers of complete graphs and related association schemes. *J. Comb. Theory Ser. A.*, 2022. Vol. 191. Art. no. 105646. DOI: 10.1016/j.jcta.2022.105646
13. Tsiovkina L. Yu. On a class of edge-transitive distance-regular antipodal covers of complete graphs. *Ural Math. J.*, 2021. Vol. 7, No. 2. P. 136–158. DOI: 10.15826/umj.2021.2.010
14. Tsiovkina L. Yu. Arc-transitive groups of automorphisms of antipodal distance-regular graphs of diameter 3 in affine case. *Sib. Èlektron. Mat. Izv.*, 2020. Vol. 17. P. 445–495. DOI: 10.33048/semi.2020.17.029 (in Russian)
15. Tsiovkina L. Yu. On a class of vertex-transitive distance-regular covers of complete graphs. *Sib. Èlektron. Mat. Izv.*, 2021. Vol. 18, No. 2. P. 758–781. DOI: 10.33048/semi.2021.18.056 (in Russian)
16. Tsiovkina L. Yu. On a class of vertex-transitive distance-regular covers of complete graphs II. *Sib. Èlektron. Mat. Izv.*, 2022. Vol. 19, No. 1. P. 348–359. DOI: 10.33048/semi.2022.19.030 (in Russian)
17. Tsiovkina L. Yu. On some vertex-transitive distance-regular antipodal covers of complete graphs. *Ural Math. J.*, 2022. Vol. 8, No. 2. P. 162–176. DOI: 10.15826/umj.2022.2.014
18. Wilson R. A. *The Finite Simple Groups*. Grad. Texts in Math., vol. 251. London: Springer–Verlag, 2009. 298 p. DOI: 10.1007/978-1-84800-988-2