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GRAPHS Γ OF DIAMETER 4 FOR WHICH $\Gamma_{3,4}$ IS A STRONGLY REGULAR GRAPH WITH $\mu = 4, 6$[1]

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Abstract: We consider antipodal graphs Γ of diameter 4 for which $\Gamma_{1,2}$ is a strongly regular graph. A.A. Makhnev and D.V. Paduchikh noticed that, in this case, $\Delta = \Gamma_{3,4}$ is a strongly regular graph without triangles. It is known that in the cases $\mu = \mu(\Delta) \in \{2, 4, 6\}$ there are infinite series of admissible parameters of strongly regular graphs with $k(\Delta) = \mu(r + 1) + r^2$, where r and $s = -(\mu + r)$ are nonprincipal eigenvalues of Δ. This paper studies graphs with $\mu(\Delta) = 4$ and 6. In these cases, Γ has intersection arrays $\{r^2 + 4r + 3, r^2 + 4r, 4, 1; 1, 4, r^2 + 4r, r^2 + 4r + 3\}$ and $\{r^2 + 6r + 5, r^2 + 6r, 6, 1; 1, 6, r^2 + 6r, r^2 + 6r + 5\}$, respectively. It is proved that graphs with such intersection arrays do not exist.

Keywords: Distance-regular graph, Strongly regular graph, Triple intersection numbers.

1. Introduction

We consider undirected graphs without loops or multiple edges.

Let Γ be a connected graph. The *distance* $d(a, b)$ between two vertices a and b of Γ is the length of a shortest path between a and b in Γ. Given a vertex a in a graph Γ, we denote by $\Gamma_i(a)$ the subgraph induced by Γ on the set of all vertices that are at distance i from a. The subgraph $[a] = \Gamma_1(a)$ is called the *neighbourhood of the vertex* a.

Let Γ be a graph and $a, b \in \Gamma$. Then the number of vertices in $[a] \cap [b]$ is denoted by $\mu(a, b)$ (by $\lambda(a, b)$) if a and b are at distance 2 (are adjacent) in Γ. Further, a subgraph induced by $[a] \cap [b]$ is called a μ*-subgraph* (a λ*-subgraph*). Let Γ be a graph of diameter d and $i, j \in \{1, 2, 3, \ldots, d\}$. A graph Γ_i has the same set of vertices as Γ and vertices u and w are adjacent in Γ_i if $d_\Gamma(u, w) = i$. A graph $\Gamma_{i,j}$ has the same set of vertices as Γ and vertices u and w are adjacent in Γ_i if $d_\Gamma(u, w) \in \{i, j\}$.

If vertices u and w are at distance i in Γ, then we denote by $b_i(u, w)$ (by $c_i(u, w)$) the number of vertices in the intersection $\Gamma_{i+1}(u)$ $(\Gamma_{i-1}(u))$ with $[w]$. A graph Γ of diameter d is called *distance-regular with intersection array* $\{b_0, b_1, \ldots, b_{d-1}; c_1, \ldots, c_d\}$ if the values $b_i(u, w)$ and $c_i(u, w)$ are

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independent of the choice of vertices u and w at distance i in Γ for any $i = 0, \ldots, d$ [1]. Let $a_i = k_i - b_i - c_i$. Note that, for a distance-regular graph, b_0 is the degree of the graph and $c_1 = 1$.

Let Γ be a graph of diameter d, and let x and y be vertices of Γ. Denote by $p^l_{ij}(x, y)$ the number of vertices in the subgraph $\Gamma_i(x) \cap \Gamma_j(y)$ if $d(x, y) = l$ in Γ. In a distance-regular graph, the numbers $p^l_{ij}(x, y)$ are independent of the choice of vertices x and y, are denoted by p^l_{ij} and are called the *intersection numbers* of the graph Γ (see [1]).

Let Γ be a distance-regular graph of diameter $d \geq 3$. If Γ is an antipodal graph of diameter 4 with antipodality index r, then, by [1, Proposition 4.2.2], Γ has intersection array $\{k, k - a_1 - 1, (r - 1)c_2, 1; 1, c_2, k - a_1 - 1, k\}$.

Consider an antipodal distance-regular graph Γ of diameter 4 for which $\Gamma_{1,2}$ is a strongly regular graph. Makhnev and Paduchikh noticed in [3] that, in this case, $\Delta = \Gamma_{3,4}$ is a strongly regular graph without triangles and the antipodality index of Γ equals 2. It is known that in the cases $\mu = \mu(\Delta) \in \{2, 4, 6\}$ there arise infinite series of admissible parameters of strongly regular graphs with $k(\Delta) = \mu(r + 1) + r^2$, where r and $s = -(\mu + r)$ are nonprincipal eigenvalues of Δ.

In the present paper, we consider graphs with $\mu(\Delta) = 4$ and 6. In these cases, Γ has intersection arrays

$$\{r^2 + 4r + 3, r^2 + 4r, 4, 1; 1, 4, r^2 + 4r, r^2 + 4r + 3\}$$

and

$$\{r^2 + 6r + 5, r^2 + 6r, 6, 1; 1, 6, r^2 + 6r, r^2 + 6r + 5\},$$

respectively.

If $\mu(\Delta) = 4$, then Δ has parameters $(v, r^2 + 4r + 4, 0, 4)$, where

$$v = 1 + (r^2 + 4r + 4) + \frac{(r^2 + 4r + 4)(r^2 + 4r + 3)}{4}.$$

Further, Δ has nonprincipal eigenvalues r and $-(r + 4)$, and the multiplicity of r is equal to $(r + 3)(r + 2)(r^2 + 5r + 8)/8$.

Theorem 1. *A distance-regular graph with intersection array*

$$\{r^2 + 4r + 3, r^2 + 4r, 4, 1; 1, 4, r^2 + 4r, r^2 + 4r + 3\}$$

does not exist.

If $\mu(\Delta) = 6$, then Δ has parameters $(v, r^2 + 6r + 6, 0, 6)$, where

$$v = 1 + (r^2 + 6r + 6) + (r^2 + 6r + 6)(r^2 + 6r + 5)/6.$$

Further, Δ has nonprincipal eigenvalues r and $-(r + 6)$, and the multiplicity of r is equal to $(r + 5)(r^2 + 6r + 6)(r + 4)/12$. Therefore, r is even or congruent to 3 modulo 4.

Theorem 2. *A distance-regular graph with intersection array*

$$\{r^2 + 6r + 5, r^2 + 6r, 6, 1; 1, 6, r^2 + 6r, r^2 + 6r + 5\}$$

does not exist.

Corollary 1. *Distance-regular graphs with intersection arrays*

$$\{32, 27, 6, 1; 1, 6, 27, 32\}, \quad \{45, 40, 6, 1; 1, 6, 40, 45\}, \quad \{77, 72, 6, 1; 1, 6, 72, 77\},$$
$$\{96, 91, 6, 1; 1, 6, 91, 96\}, \quad \{117, 112, 6, 1; 1, 6, 112, 117\}$$

do not exist.

2. Triple intersection numbers

Let Γ be a distance-regular graph of diameter d. If u_1, u_2, and u_3 are vertices of the graph Γ and r_1, r_2, and r_3 are nonnegative integers not greater than d, then $\left\{ {u_1u_2u_3 \atop r_1r_2r_3} \right\}$ is the set of vertices $w \in \Gamma$ such that

$$d(w, u_i) = r_i, \quad \begin{bmatrix} u_1u_2u_3 \\ r_1r_2r_3 \end{bmatrix} = \left| \begin{Bmatrix} u_1u_2u_3 \\ r_1r_2r_3 \end{Bmatrix} \right|.$$

The numbers $\left[{u_1u_2u_3 \atop r_1r_2r_3} \right]$ are called triple intersection numbers. For a fixed triple u_1, u_2, u_3 of vertices, we will write $[r_1r_2r_3]$ instead of $\left[{u_1u_2u_3 \atop r_1r_2r_3} \right]$.

Unfortunately, there are no general formulas for numbers $[r_1r_2r_3]$. However, [2] suggests a method for calculating some numbers $[r_1r_2r_3]$.

Assume that u, v, and w are vertices of the graph Γ, $W = d(u, v)$, $U = d(v, w)$, and $V = d(u, w)$. Since there is exactly one vertex $x = u$ such that $d(x, u) = 0$, then the number $[0jh]$ is 0 or 1. Hence, $[0jh] = \delta_{jW}\delta_{hV}$. Similarly, $[i0h] = \delta_{iW}\delta_{hU}$ and $[ij0] = \delta_{iU}\delta_{jV}$.

Another set of equations can be obtained by fixing the distance between two vertices from $\{u, v, w\}$ and counting the number of vertices located at all possible distances from the third. Then, we get

$$\sum_{l=1}^{d}[ljh] = p^U_{jh} - [0jh], \quad \sum_{l=1}^{d}[ilh] = p^V_{ih} - [i0h], \quad \sum_{l=1}^{d}[ijl] = p^W_{ij} - [ij0]. \tag{2.1}$$

At the same time, some triples disappear. If $|i - j| > W$ or $i + j < W$, then $p^W_{ij} = 0$; therefore, $[ijh] = 0$ for all $h \in \{0, \dots, d\}$. Define

$$S_{ijh}(u, v, w) = \sum_{r,s,t=0}^{d} Q_{ri}Q_{sj}Q_{th} \begin{bmatrix} uvw \\ rst \end{bmatrix}.$$

If Krein's parameter q^h_{ij} is 0, then $S_{ijh}(u, v, w) = 0$.

3. A distance-regular graph with intersection array $\{r^2+4r+3, r^2+4r, 4, 1; 1, 4, r^2+4r, r^2+4r+3\}$

In this section, Γ is a distance-regular graph with intersection array

$$\{r^2 + 4r + 3, r^2 + 4r, 4, 1; 1, 4, r^2 + 4r, r^2 + 4r + 3\}.$$

Then, Γ has

$$1 + (r^2 + 4r + 3) + (r^2 + 4r + 3)(r^2 + 4r)/4 + (r^2 + 4r + 3) + 1$$

vertices and the spectrum

$$(r + 3)(r + 1) \quad \text{of multiplicity} \quad 1,$$

$$r + 3 \quad \text{of multiplicity} \quad \frac{(r^2 + 5r + 8)(r^2 + 3r + 4)(r + 1)}{16(r + 2)},$$

$$r - 1 \quad \text{of multiplicity} \quad \frac{(r^2 + 5r + 8)(r + 4)(r + 3)(r + 1)}{16(r + 2)},$$

$$-(r + 1) \quad \text{of multiplicity} \quad \frac{(r^2 + 5r + 8)(r^2 + 3r + 4)(r + 3)}{16(r + 2)},$$

$$-(r + 5) \quad \text{of multiplicity} \quad \frac{(r^2 + 3r + 4)(r + 3)(r + 1)r}{16(r + 2)}.$$

The multiplicity of $r+3$ is equal to

$$\frac{(r^2+5r+8)(r^2+3r+4)(r+1)}{16(r+2)}.$$

Further,

$$(r^2+5r+8, r+2) = (3r+8, r+2)$$

divides 2 and $(r+2, r^2+3r+4) = (r+2, r+4)$ divides 2; therefore $r+2$ divides 4. Consequently, $r=2$, a contradiction with the fact that the multiplicity of $r+3$ is equal to

$$(r^2+5r+8)(r^2+3r+4)(r+1)/(16(r+2)) = 22 \times 14 \times 3/64.$$

Theorem 1 is proved.

4. A distance-regular graph with intersection array $\{r^2+6r+5, r^2+6r, 6, 1; 1, 6, r^2+6r, r^2+6r+5\}$

In this section, Γ is a distance-regular graph with intersection array

$$\{r^2+6r+5, r^2+6r, 6, 1; 1, 6, r^2+6r, r^2+6r+5\}.$$

Then, Γ has

$$1 + (r^2+6r+5) + (r^2+6r+5)(r^2+6r)/6 + (r^2+6r+5) + 1$$

vertices, the spectrum

$$\begin{gathered}
(r+5)(r+1) \quad \text{of multiplicity} \quad 1,\\
r+5 \quad \text{of multiplicity} \quad f = (r+4)(r+3)(r+2)(r+1)/24,\\
r-1 \quad \text{of multiplicity} \quad (r+6)(r+5)(r+4)(r+1)/24,\\
-(r+1) \quad \text{of multiplicity} \quad (r+5)(r+4)(r+3)(r+2)/24,\\
-(r+7) \quad \text{of multiplicity} \quad (r+5)(r+2)(r+1)r/24,
\end{gathered}$$

and the matrix Q (see [1]) of dual eigenvalues

$$\begin{pmatrix}
1 & f & \dfrac{f(r+6)(r+5)}{(r+2)(r+3)} & \dfrac{f(r+5)}{r+1} & \dfrac{f(r+5)r}{(r+4)(r+3)}\\
1 & \dfrac{f}{r+1} & \dfrac{f(r+6)(r-1)}{(r+2)(r+3)(r+1)} & -\dfrac{f}{r+1} & -\dfrac{f(r+7)r}{(r+4)(r+3)(r+1)}\\
1 & 0 & -r/2-2 & 0 & r/2+1\\
1 & -\dfrac{f}{r+1} & \dfrac{f(r+6)(r-1)}{(r+2)(r+3)(r+1)} & \dfrac{f}{r+1} & -\dfrac{f(r+7)r}{(r+4)(r+3)(r+1)}\\
1 & -f & \dfrac{f(r+6)(r+5)}{(r+2)(r+3)} & -\dfrac{f(r+5)}{r+1} & \dfrac{f(r+5)r}{(r+4)(r+3)}
\end{pmatrix}.$$

Lemma 1. *The intersection numbers are*

$$p^1_{11}=4,\quad p^1_{21}=r^2+6r,\quad p^1_{32}=r^2+6r,\quad p^1_{22}=r^4/6+2r^3+29r^2/6-7r,\quad p^1_{33}=0,\quad p^1_{34}=1;$$
$$p^2_{11}=6,\quad p^2_{12}=r^2+6r-7,\quad p^2_{13}=6,\quad p^2_{22}=r^4/6+2r^3+29r^2/6-7r+12,$$
$$p^2_{23}=r^2+6r-7,\quad p^2_{24}=1,\quad p^2_{33}=2;$$
$$p^3_{12}=r^2+6r,\quad p^3_{13}=4,\quad p^3_{14}=1,\quad p^3_{22}=r^4/6+2r^3+29r^2/6-7r,\quad p^3_{23}=r^2+6r,\quad p^3_{33}=0;$$
$$p^4_{13}=r^2+6r+5,\quad p^4_{22}=r^4/6+2r^3+41r^2/6+5r.$$

P r o o f. Direct calculations using formulas from [1, Lemma 4.1.7]. □

Fix vertices u, v, and w of the graph Γ and define

$$\{ijh\} = \left\{ \begin{matrix} uvw \\ ijh \end{matrix} \right\}, \quad [ijh] = \left[\begin{matrix} uvw \\ ijh \end{matrix} \right].$$

Let $\Delta = \Gamma_2(u)$, and let Λ be a graph with vertices from Δ in which two vertices are adjacent if they are at distance 2 in Γ. Then Λ is a regular graph of degree

$$p^2_{22} = r^4/6 + 2r^3 + 29r^2/6 - 7r + 12$$

on

$$k_2 = (r^2 + 6r + 5)(r^2 + 6r)/6 = r^4/6 + 2r^3 + 41r^2/6 + 5r$$

vertices.

Lemma 2. *Let $d(u,v) = d(u,w) = 2$ and $d(v,w) = 1$. Then, the triple intersection numbers are*

$$[111] = r_4, \quad [112] = [121] = -r_4 + 6, \quad [122] = r_3 + r_4 + r^2 + 6r - 19, \quad [123] = [132] = -r_3 + 6;$$
$$[211] = -r_3 - r_4 + 4, \quad [212] = [221] = r_3 + r_4 + r^2 + 6r - 12,$$
$$[222] = r^4/6 + 2r^3 + 17r^2/6 - 19r + 36,$$
$$[223] = [232] = r_3 + r_4 + r^2 + 6r - 12, \quad [233] = -r_3 - r_4 + 4, \quad [234] = [243] = 1;$$
$$[311] = r_3, \quad [312] = [321] = -r_3 + 6, \quad [322] = r_3 + r_4 + r^2 + 6r - 19, \quad [323] = [332] = -r_4 + 6;$$
$$[333] = r_4, \quad [422] = 1,$$

where $r_3 + r_4 \leq 4$.

P r o o f. Simplification of formulas (2.1). □

By Lemma 2, we have

$$r^4/6 + 2r^3 + 17r^2/6 - 19r + 28$$
$$\leq [222] = -2r_3 - 2r_4 + r^4/6 + 2r^3 + 17r^2/6 - 19r + 36 \leq r^4/6 + 2r^3 + 17r^2/6 - 19r + 36.$$

Lemma 3. *Let $d(u,v) = d(u,w) = 2$ and $d(v,w) = 3$. Then, the triple intersection numbers are*

$$[112] = -r_{11} + 6, \quad [113] = r_{11},$$
$$[121] = -r_{12} + 6, \quad [122] = r_{11} + r_{12} + r^2 + 6r - 19, \quad [123] = -r_{11} + 6, \quad [132] = -r_{12} + 6;$$
$$[212] = [221] = r_{11} + r_{12} + r^2 + 6r - 12, \quad [213] = [231] = -r_{11} - r_{12} + 4, \quad [214] = [241] = 1,$$
$$[222] = -2r_3 - 2r_4 + r^4/6 + 2r^3 + 17r^2/6 - 19r + 36, \quad [223] = [232] = r_{11} + r_{12} + r^2 + 6r - 12;$$
$$[312] = -r_{12} + 6, \quad [313] = r_{12}, \quad [321] = -r_{11} + 6, \quad [322] = r_{11} + r_{12} + r^2 + 6r - 19,$$
$$[323] = -r_{12} + 6, \quad [331] = r_{11}, \quad [332] = -r_{11} + 6; \quad [422] = 1,$$

where $r_{11} + r_{12} \leq 4$.

P r o o f. Simplification of formulas (2.1). □

By Lemma 3, we have

$$r^4/6 + 2r^3 + 17r^2/6 - 19r + 28$$
$$\leq [222] = -2r_3 - 2r_4 + r^4/6 + 2r^3 + 17r^2/6 - 19r + 36 \leq r^4/6 + 2r^3 + 17r^2/6 - 19r + 36.$$

Lemma 4. *Let $d(u,v) = d(u,w) = 2$ and $d(v,w) = 4$. Then, the triple intersection numbers are*

$$[113] = [131] = 6, \quad [122] = r^2 + 6r - 7;$$
$$[213] = [231] = r^2 + 6r - 7, \quad [222] = r^4/6 + 2r^3 + 29r^2/6 - 7r + 12;$$
$$[313] = [331] = 6, \quad [322] = r^2 + 6r - 7;$$
$$[422] = 1.$$

P r o o f. Simplification of formulas (2.1). □

By Lemma 4, we have

$$[222] = r^4/6 + 2r^3 + 29r^2/6 - 7r + 12.$$

Recall that

$$p^2_{12} = r^2 + 6r - 7, \quad p^2_{22} = r^4/6 + 2r^3 + 29r^2/6 - 7r + 12, \quad p^2_{23} = r^2 + 6r - 7, \quad p^2_{24} = 1.$$

Let v and w be vertices from Λ. Then the number d of edges between $\Lambda(v)$ and $\Lambda - (\{v\} \cup \Lambda(v))$ is

$$d = p^2_{12} \begin{bmatrix} uvx \\ 221 \end{bmatrix} + p^2_{32} \begin{bmatrix} uvy \\ 223 \end{bmatrix} + p^2_{42} \begin{bmatrix} uvz \\ 224 \end{bmatrix},$$

where $x, y,$ and z are vertices from $\left\{ {uv \atop 2i} \right\}$ for $i = 1, 3,$ and 4, respectively. Now, d satisfies the inequalities

$$(r^2 + 6r - 7)(r^4/3 + 4r^3 + 17r^2/3 - 38r + 56) + r^4/6 + 2r^3 + 29r^2/6 - 7r + 12 \leq d$$
$$\leq (r^2 + 6r - 7)(r^4/3 + 4r^3 + 17r^2/3 - 38r + 72) + r^4/6 + 2r^3 + 29r^2/6 - 7r + 12.$$

On the other hand,

$$d = \sum_{w \in \Lambda(v)} (p^2_{22} - 1 - \lambda_\Lambda(v,w)) = k_\Lambda \Big(p^2_{22} - 1 - \frac{\sum_{w \in \Lambda(v)} \lambda_\Lambda(v,w)}{k_\Lambda} \Big).$$

So,

$$d = (r^4/6 + 2r^3 + 29r^2/6 - 7r + 12)(r^4/6 + 2r^3 + 29r^2/6 - 7r + 11 - \lambda),$$

where λ is the average value of degree of the vertex w in the graph Λ. Consequently,

$$\frac{(r^2 + 6r - 7)(r^4/3 + 4r^3 + 17r^2/3 - 38r + 56)}{r^4/6 + 2r^3 + 29r^2/6 - 7r + 12} + 1 \leq \frac{r^4}{6} + 2r^3 + \frac{29r^2}{6} - 7r + 11 - \lambda$$
$$\leq \frac{(r^2 + 6r - 7)(r^4/3 + 4r^3 + 17r^2/3 - 38r + 72)}{r^4/6 + 2r^3 + 29r^2/6 - 7r + 12} + 1$$

and

$$\frac{r^4}{6} + 2r^3 + \frac{29r^2}{6} - 7r + 10 - \frac{(r^2 + 6r - 7)(r^4/3 + 4r^3 + 17r^2/3 - 38r + 72)}{r^4/6 + 2r^3 + 29r^2/6 - 7r + 12} \leq \lambda$$
$$\leq \frac{r^4}{6} + 2r^3 + \frac{29r^2}{6} - 7r + 10 - \frac{(r^2 + 6r - 7)(r^4/3 + 4r^3 + 17r^2/3 - 38r + 56)}{r^4/6 + 2r^3 + 29r^2/6 - 7r + 12}.$$

Lemma 5. *Let $d(u,v)=d(u,w)=d(v,w)=2$. Then, the triple intersection numbers are*

$$[111]=r_9,\quad [112]=-r_7-r_9+6,\quad [113]=r_7,\quad [121]=-r_{10}-r_9+6,$$
$$[122]=r_7+r_8+r_9+r_{10}+r^2+6r-19,\quad [123]=-r_7-r_8+6,$$
$$[131]=r_{10},\quad [132]=-r_{10}-r_8+6,\quad [133]=r_8;$$
$$[211]=-r_8-r_9+6,\quad [212]=[221]=r_7+r_8+r_9+r_{10}+r^2+6r-19,$$
$$[213]=[231]=-r_{10}-r_7+6,\quad [222]=-2r_7-2r_8-2r_9-2r_{10}+r^4/6+2r^3+17r^2/6-19r+48,$$
$$[223]=[232]=r_7+r_8+r_9+r_{10}+r^2+6r-19,\quad [224]=[242]=1,\quad [233]=-r_8-r_8+6;$$
$$[311]=r_8,\quad [312]=-r_{10}-r_8+6,\quad [313]=r_{10},\quad [321]=-r_7-r_8+6,$$
$$[322]=r_7+r_8+r_9+r_{10}+r^2+6r-19,\quad [323]=-r_{10}-r_9+6,$$
$$[331]=r_7,\quad [332]=-r_7-r_9+6,\quad [333]=r_9;\quad [422]=1,$$

where

$$r_9+r_7,\ r_9+r_{10},\ r_7+r_8,\ r_{10}+r_8,\ r_8+r_9,\ r_7+r_{10}\le 6.$$

P r o o f. Simplification of formulas (2.1). □

By Lemma 5, we have

$$\frac{r^4}{6}+2r^3+\frac{17r^2}{6}-19r+24\le[222]=-2r_7-2r_8-2r_9-2r_{10}+\frac{r^4}{6}+2r^3+\frac{17r^2}{6}-19r+48$$
$$\le\frac{r^4}{6}+2r^3+\frac{17r^2}{6}-19r+48.$$

Let $d(u,v)=2$.

Let us count the number e_2 of pairs of vertices (s,t) at distance 2, where $s\in\left\{{uv\atop 21}\right\}$ and $t\in\left\{{uv\atop 22}\right\}$. On the one hand, by Lemma 2, we have

$$r^4/6+2r^3+17r^2/6-19r+28\le[222]\le r^4/6+2r^3+17r^2/6-19r+36,$$

so,

$$(r^2+6r-7)\Big(\frac{r^4}{6}+2r^3+\frac{17r^2}{6}-19r+28\Big)\le e_2\le(r^2+6r-7)\Big(\frac{r^4}{6}+2r^3+\frac{17r^2}{6}-19r+36\Big).$$

On the other hand, by Lemma 5, we have

$$[212]=r_7+r_8+r_9+r_{10}+r^2+6r-19$$

and

$$(r^2+6r-7)\Big(\frac{r^4}{6}+2r^3+\frac{17r^2}{6}-19r+28\Big)\le e_2$$
$$=-\sum_i(r_7^i+r_8^i+r_9^i+r_{10}^i)+(r^2+6r-19)\Big(\frac{r^4}{6}+2r^3+\frac{29r^2}{6}-7r+12\Big)$$
$$\le(r^2+6r-7)\Big(\frac{r^4}{6}+2r^3+\frac{17r^2}{6}-19r+36\Big).$$

In this way,

$$(r^2+6r-19)\Big(\frac{r^4}{6}+2r^3+\frac{29r^2}{6}-7r+12\Big)-(r^2+6r-7)\Big(\frac{r^4}{6}+2r^3+\frac{17r^2}{6}-19r+36\Big)$$
$$\le (r_7^i+r_8^i+r_9^i+r_{10}^i)$$
$$\le (r^2+6r-19)\Big(\frac{r^4}{6}+2r^3+\frac{29r^2}{6}-7r+12\Big)-(r^2+6r-7)\Big(\frac{r^4}{6}+2r^3+\frac{17r^2}{6}-19r+28\Big).$$

Consequently,

$$(r_7^i+r_8^i+r_9^i+r_{10}^i)\le -145r^3/6-16r^2-96r-12,$$

a contradiction.

Theorem 2 is proved. □

The corollary follows from Theorems 1 and 2.

So, we have shown the nonexistence of graphs with intersection arrays

$$\{r^2+4r+3, r^2+4r, 4, 1;\ 1, 4, r^2+4r, r^2+4r+3\}$$

and

$$\{r^2+6r+5, r^2+6r, 6, 1;\ 1, 6, r^2+6r, r^2+6r+5\}.$$

In particular, distance-regular graphs with intersection arrays

$$\{32,27,6,1;\ 1,6,27,32\},\quad \{45,40,6,1;\ 1,6,40,45\},\quad \{77,72,6,1;1,6,72,77\},$$
$$\{96,91,6,1;\ 1,6,91,96\},\quad \{117,112,6,1;\ 1,6,112,117\}$$

do not exist.

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