URAL MATHEMATICAL JOURNAL, Vol. 10, No. 2, 2024, pp. 69–80

DOI: 10.15826/umj.2024.2.007

$\mathcal{I}$-STATISTICAL CONVERGENCE OF COMPLEX UNCERTAIN SEQUENCES IN MEASURE

Amit Halder[†], **Shyamal Debnath**[††*]

Department of Mathematics, Tripura University (A Central University), Suryamaninagar-799022, Agartala, India

[†]mit2905halder@gmail.com, [††]shyamalnitamath@gmail.com

Abstract: The main aim of this paper is to present and explore some of properties of the concept of $\mathcal{I}$-statistical convergence in measure of complex uncertain sequences. Furthermore, we introduce the concept of $\mathcal{I}$-statistical Cauchy sequence in measure and study the relationships between different types of convergencies. We observe that, in complex uncertain space, every $\mathcal{I}$-statistically convergent sequence in measure is $\mathcal{I}$-statistically Cauchy sequence in measure, but the converse is not necessarily true.

Keywords: $\mathcal{I}$-convergence, $\mathcal{I}$-statistical convergence, Uncertainty theory, Complex uncertain variable.

1. Introduction

In the real world, there are different kinds of uncertainty. So, it makes perfect sense to investigate the behavior of uncertain phenomena. To address some aspects of this uncertain phenomena, Liu [12] introduced initially the uncertainty theory in 2007. After that, it has been studied in various fields of mathematics like calculus, set theory, graph theory, sequence and series, etc. In [12] Liu initially proposed the idea of uncertain variables as a functions from measurable space to the set of real numbers ($\mathbb{R}$). Peng [15] later expanded it to include complex uncertain variables.

In the fundamental theory of mathematics, the significance of sequence convergence is highly pivotal which is also one of the most important fields of mathematics. Furthermore, one of the most important aspects of uncertainty theory is the convergence of uncertain variable sequences. For the first time in uncertainty theory, Liu [12] established several convergence notions of uncertain variable sequences, such as convergence almost surely, convergence in measure, convergence in distribution, and convergence in mean.

Following that, by using complex uncertain variables, Chen et al. [1] introduced the concept of convergence of complex uncertain sequences and then numerous researchers have subsequently expanded this idea, including Saha et al. [17], Debnath and Das [2], and You and Yan [23]. The concept of Cauchy convergence in measure and in mean was recently presented by Wu and Xia [24].

On the other hand, in 1951, Fast [8] and Steinhaus[21] extended the concepts of convergence of a real sequence to statistical convergence independently and after that, it was studied by Fridy [9] and many other famous researchers. Later Kostyrko et al. [11] introduced a new concept of convergence namely $\mathcal{I}$-convergence, which is a generalization of statistical convergence.

Savas and Das [19] further expanded the notion of statistical convergence and $\mathcal{I}$-convergence to include $\mathcal{I}$-statistical convergence. This extension prompted further explorations in the field by researchers such as Savas and Das [20], Debnath and Debnath [5], Debnath and Rakshit [6], Mursaleen et al. [13], Savas et al. [18], Esi et al. [7], and numerous others.

[*]Corresponding author

Tripathy and Nath [22] introduced the concept of statistical convergence for complex uncertain sequences within the framework of uncertainty theory. Then many other researchers like Nath and Tripathy [14], Roy et al. [16], Debnath and Das [3, 4], and Kisi [10] have successfully applied the concept of generalized convergence of sequences in uncertainty theory.

Inspired by the above works, in this paper, we introduce the concepts of $\mathcal{I}$-statistical convergence in measure of complex uncertain sequences and study some of its properties. We also introduce the notion of $\mathcal{I}$-statistical Cauchy sequence in measure and identifying the relationship between $\mathcal{I}$-statistical convergence in measure and $\mathcal{I}$-statistical Cauchy sequence in measure.

2. Definitions and preliminaries

The generalized convergence notions and the theory of uncertainty, which will be utilized throughout the study, are defined and findings are presented in this section.

Definition 1 [11]. *Consider a non-empty set S. An ideal on S is defined as a family of subsets $\mathcal{I}$ that satisfies the following conditions*:

(*i*) *The empty set, ϕ, belongs to $\mathcal{I}$.*

(*ii*) *For any $U, V \in \mathcal{I}$, the union of U and V, denoted as $U \cup V$, is also in $\mathcal{I}$.*

(*iii*) *For any $U \in \mathcal{I}$ and any subset $V \subset U$, V is a member of $\mathcal{I}$.*

An ideal $\mathcal{I}$ is called non-trivial if $\mathcal{I} \neq \{\phi\}$ and $S \notin \mathcal{I}$. A non-trivial ideal $\mathcal{I}$ is called an admissible ideal in S if and only if $\{\{s\} : s \in S\} \subset \mathcal{I}$.

Example 1. (i) $\mathcal{I}_f :=$ {The set of all finite subsets of $\mathbb{N}$ forms a non-trivial admissible ideal}. (ii) $\mathcal{I}_d :=$ {The set of all subsets of $\mathbb{N}$ whose natural density is zero forms a non-trivial admissible ideal}.

Definition 2 [11]. *Consider a non-empty set S. A family of subsets $\mathcal{F}$, which is a subset of the power set $P(S)$, is called a filter on S if and only if the following conditions are satisfied*:

(*i*) *The empty set ϕ is not a member of $\mathcal{F}$.*

(*ii*) *For any subsets U and V in $\mathcal{F}$, their intersection $U \cap V$ is also included in $\mathcal{F}$.*

(*iii*) *If U is a member of $\mathcal{F}$ and V is a superset of U, then V is also a member of $\mathcal{F}$.*

Now, let $\mathcal{I}$ be an admissible ideal. The filter $\mathcal{F}(\mathcal{I})$ associated with the ideal $\mathcal{I}$ is defined as

$$\mathcal{F}(\mathcal{I}) = \{S \setminus U : U \in \mathcal{I}\}.$$

Definition 3 [9]. *A real sequence (x_m) is said to be statistically convergent to $\ell \in \mathbb{R}$ provided that for each $\varepsilon > 0$ we have*

$$\lim_{m\to\infty} \frac{1}{m}\big|\{k \leq m : |x_k - \ell| \geq \varepsilon\}\big| = 0, \quad m \in \mathbb{N}.$$

Definition 4 [11]. *A real sequence (x_m) is said to be $\mathcal{I}$-convergent to $\ell \in \mathbb{R}$, if for every $\varepsilon > 0$, we have*

$$\{m \in \mathbb{N} : |x_m - \ell| \geq \varepsilon\} \in \mathcal{I}.$$

The usual convergence of sequences is a special case of $\mathcal{I}$-convergence ($\mathcal{I}$=$\mathcal{I}_f$-the ideal of all fimite subsets of $\mathbb{N}$). The statistical convergence of sequences is also a special case of $\mathcal{I}$-convergence. In this case,

$$\mathcal{I} = \mathcal{I}_d = \Big\{A \subseteq \mathbb{N} : \lim_{m\to\infty} \frac{|\, A \cap \{1, 2, ..., m\} \,|}{m} = 0\Big\},$$

where $|A|$ is the cardinality of the set A.

Definition 5 [18]. *A real sequence (x_m) is said to be $\mathcal{I}$-statistically convergent to $\ell \in \mathbb{R}$, if for every $\varepsilon > 0$, and every $\delta > 0$,*

$$\Big\{m \in \mathbb{N} : \frac{1}{m}\big|\{k \leq m : |x_k - \ell| \geq \varepsilon\}\big| \geq \delta\Big\} \in \mathcal{I}.$$

For $\mathcal{I} = \mathcal{I}_f$, $\mathcal{I}$-statistical convergence coincides with statistical convergence.

Definition 6 [12]. *Let $\mathcal{P}$ be a σ-algebra on a non-empty set Υ. If the set function $\mathcal{X}$ on Υ satisfies the following axioms, it is referred to be an uncertain measure*:

- (i) *The first axiom, which deals with normality, is*: $\mathcal{X}\{\Upsilon\} = 1$;
- (ii) *The second, which deals with duality, is*: $\mathcal{X}\{\Xi\} + \mathcal{X}\{\Xi^c\} = 1$ *for any* $\Xi \in \mathcal{P}$;
- (iii) *The third, which deals with subadditivity, is*: *for every countable sequence of* $\{\Xi_m\} \in \mathcal{P}$,

$$\mathcal{X}\Big\{\bigcup_{m=1}^{\infty} \Xi_m\Big\} \leq \sum_{m=1}^{\infty} \mathcal{X}\{\Xi_m\}.$$

An uncertainty space is denoted by the triplet $(\Upsilon, \mathcal{P}, \mathcal{X})$, and an event is denoted by each member Ξ in $\mathcal{P}$. For an uncertain measure of a compound event, Liu defines a product uncertain measure as follows:

$$\mathcal{X}\Big\{\prod_{r=1}^{\infty} \Xi_r\Big\} = \bigwedge_{r=1}^{\infty} \mathcal{X}\{\Xi_r\}.$$

Definition 7 [15]. *A complex uncertain variable is represented by a variable ζ in the uncertainty space $(\Upsilon, \mathcal{P}, \mathcal{X})$ if and only if both its real part ξ and imaginary part η are uncertain variables. Here, ξ and η correspond to the real and imaginary components of the complex variable $\zeta = \xi + i\eta$, respectively.*

Definition 8 [1]. *A complex uncertain sequence (ζ_m) is said to be convergent in measure to ζ if for every $\varepsilon > 0$,*

$$\lim_{m\to\infty} \mathcal{X}\big\{\|\zeta_m(\varrho) - \zeta(\varrho)\| \geq \varepsilon\big\} = 0.$$

Definition 9 [22]. *A complex uncertain sequence (ζ_m) is said to be statistically convergent in measure to ζ if for any given positive values of ε, δ, we have*

$$\lim_{m\to\infty} \frac{1}{m}\big|\big\{k \leq m : \mathcal{X}\big(\|\zeta_k(\varrho) - \zeta(\varrho)\| \geq \varepsilon\big) \geq \delta\big\}\big| = 0$$

and we write $\zeta_m \xrightarrow{S^{M_s}} \zeta$.

Definition 10. *A complex uncertain sequence (ζ_m) is said to be $\mathcal{I}$-convergent in measure to ζ if for any given positive values of ε, δ, we have*

$$\big\{m \in \mathbb{N} : \mathcal{X}\big(\|\zeta_m(\varrho) - \zeta(\varrho)\| \geq \varepsilon\big) \geq \delta\big\} \in \mathcal{I}.$$

and we write $\zeta_m \xrightarrow{M_s(\mathcal{I})} \zeta$.

In this paper, $\mathcal{I}$ is taken to be an admissible ideal.

3. Main results

Definition 11. *A complex uncertain sequence* (ζ_m) *is considered to be* $\mathcal{I}$*-statistically convergent in measure to* ζ *if, for any given positive values of* $\varepsilon, \delta, \upsilon$*, there exists a set satisfying the condition*

$$\Big\{m \in \mathbb{N} : \frac{1}{m}\big|\big\{k \leq m : \mathcal{X}\big(\|\zeta_k(\varrho) - \zeta(\varrho)\| \geq \varepsilon\big) \geq \delta\big\}\big| \geq \upsilon\Big\} \in \mathcal{I}.$$

This is denoted as $\zeta_m \xrightarrow{S^{M_s}(\mathcal{I})} \zeta$.

Example 2. Consider the uncertainty space $(\Upsilon, \mathcal{P}, \mathcal{X})$ to be $\{\varrho_1, \varrho_2, \cdots\}$ with power set and $\mathcal{X}\{\Upsilon\} = 1$, $\mathcal{X}\{\phi\} = 0$ and

$$\mathcal{X}\{\Xi\} = \begin{cases} \sup\limits_{\varrho_m \in \Xi} \dfrac{3}{(2m+1)}, & \text{if} \quad \sup\limits_{\varrho_m \in \Xi} \dfrac{3}{(2m+1)} < \dfrac{1}{2}, \\ 1 - \sup\limits_{\varrho_m \in \Xi^c} \dfrac{3}{(2m+1)}, & \text{if} \quad \sup\limits_{\varrho_m \in \Xi^c} \dfrac{3}{(2m+1)} < \dfrac{1}{2}, \\ \dfrac{1}{2}, & \text{otherwise} \end{cases} \quad \text{for} \quad m = 1, 2, 3, \ldots,$$

and $\zeta_m(\varrho)$ (the complex uncertain variables) are defined by

$$\zeta_m(\varrho) = \begin{cases} mi, & \text{if} \quad \varrho = \varrho_m, \quad m = 1, 2, 3, \ldots, \\ 0, & \text{otherwise} \end{cases}$$

and $\zeta \equiv 0$. Take $\mathcal{I}$=$\mathcal{I}_d$.

For $m \geq 3$ and small positive values of $\varepsilon, \delta, \upsilon$ we get,

$$\begin{aligned} &\Big\{m \in \mathbb{N} : \frac{1}{m}\big|\{k \leq m : \mathcal{X}\big(\|\zeta_k(\varrho) - \zeta(\varrho)\| \geq \varepsilon\big) \geq \delta\}\big| \geq \upsilon\Big\} \\ &= \Big\{m \in \mathbb{N} : \frac{1}{m}\big|\{k \leq m : \mathcal{X}\big(\varrho : \|\zeta_k(\varrho) - \zeta(\varrho)\| \geq \varepsilon\big) \geq \delta\}\big| \geq \upsilon\Big\} \\ &= \Big\{m \in \mathbb{N} : \frac{1}{m}\big|\{k \leq m : \mathcal{X}\{\varrho_k\} \geq \delta\}\big| \geq \upsilon\Big\} \\ &= \Big\{m \in \mathbb{N} : \frac{1}{m}\Big|\Big\{k \leq m : \frac{3}{2k+1} \geq \delta\Big\}\Big| \geq \upsilon\Big\} \in \mathcal{I}. \end{aligned}$$

Therefore the sequence (ζ_m) is $\mathcal{I}$-statistically convergent in measure to ζ.

Theorem 1. *If* $\zeta_m \xrightarrow{S^{M_s}(\mathcal{I})} \zeta$ *and* $\zeta_m \xrightarrow{S^{M_s}(\mathcal{I})} \zeta^*$ *then* $\mathcal{X}\{\zeta = \zeta^*\} = 1$.

P r o o f. Let $\varepsilon, \delta > 0$ and $0 < \upsilon < 1$, then

$$G = \Big\{m \in \mathbb{N} : \frac{1}{m}\Big|\Big\{k \leq m : \mathcal{X}\Big(\|\zeta_k - \zeta\| \geq \frac{\varepsilon}{2}\Big) \geq \frac{\delta}{2}\Big\}\Big| < \frac{\upsilon}{3}\Big\} \in \mathcal{F}(\mathcal{I}),$$

and

$$H = \Big\{m \in \mathbb{N} : \frac{1}{m}\Big|\Big\{k \leq m : \mathcal{X}\Big(\|\zeta_k - \zeta^*\| \geq \frac{\varepsilon}{2}\Big) \geq \frac{\delta}{2}\Big\}\Big| < \frac{\upsilon}{3}\Big\} \in \mathcal{F}(\mathcal{I}).$$

Since $G \cap H \in \mathcal{F}(\mathcal{I})$ and $\phi \notin \mathcal{F}(\mathcal{I})$ this implies $G \cap H \neq \phi$. Let $m \in G \cap H$. Then

$$\frac{1}{m}\Big|\Big\{k \leq m : \mathcal{X}\Big(\|\zeta_k - \zeta\| \geq \frac{\varepsilon}{2}\Big) \geq \frac{\delta}{2}\Big\}\Big| < \frac{\upsilon}{3}$$

and
$$\frac{1}{m}\Big|\Big\{k \le m : \mathcal{X}\Big(\|\zeta_k - \zeta^*\| \ge \frac{\varepsilon}{2}\Big) \ge \frac{\delta}{2}\Big\}\Big| < \frac{\upsilon}{3}.$$
Therefore
$$\frac{1}{m}\Big|\Big\{k \le m : \mathcal{X}(\|\zeta_k - \zeta\| \ge \frac{\varepsilon}{2}) \ge \frac{\delta}{2} \quad \text{or} \quad \mathcal{X}(\ \| \zeta_k - \zeta^* \| \ge \frac{\varepsilon}{2}) \ge \frac{\delta}{2}\Big\}\Big| < \upsilon < 1.$$
Thus there exists some $k \le m$ such that
$$\mathcal{X}\Big(\|\zeta_k - \zeta\| \ge \frac{\varepsilon}{2}\Big) < \frac{\delta}{2} \quad \text{and} \quad \mathcal{X}\Big(\|\zeta_k - \zeta^*\| \ge \frac{\varepsilon}{2}\Big) < \frac{\delta}{2}.$$
Therefore
$$\mathcal{X}(||\zeta - \zeta^*|| \ge \varepsilon) \le \mathcal{X}\Big(\|\zeta_k - \zeta\| \ge \frac{\varepsilon}{2}\Big) + \mathcal{X}\Big(\|\zeta_k - \zeta^*\| \ge \frac{\varepsilon}{2}\Big) < \delta.$$
Hence we get the result. □

Theorem 2. *Elementary properties are valid*:

(i) $\zeta_m \xrightarrow{S^{M_s}(\mathcal{I})} \zeta \iff \zeta_m - \zeta \xrightarrow{S^{M_s}(\mathcal{I})} 0$;

(ii) $\zeta_m \xrightarrow{S^{M_s}(\mathcal{I})} \zeta \implies c\zeta_m \xrightarrow{S^{M_s}(\mathcal{I})} c\zeta$, *where* $c \in \mathbb{C}$;

(iii) $\zeta_m \xrightarrow{S^{M_s}(\mathcal{I})} \zeta$ *and* $\zeta_m^* \xrightarrow{S^{M_s}(\mathcal{I})} \zeta^* \implies \zeta_m + \zeta_m^* \xrightarrow{S^{M_s}(\mathcal{I})} \zeta + \zeta^*$;

(iv) $\zeta_m \xrightarrow{S^{M_s}(\mathcal{I})} \zeta$ *and* $\zeta_m^* \xrightarrow{S^{M_s}(\mathcal{I})} \zeta^* \implies \zeta_m - \zeta_m^* \xrightarrow{S^{M_s}(\mathcal{I})} \zeta - \zeta^*$.

P r o o f. Let $\varepsilon, \delta, \upsilon$ be any positive real numbers. For (i), (ii), the proofs are straight forward and so omitted.

(iii) It is obvious from the inequality
$$\mathcal{X}\Big(\|(\zeta_k + \zeta_k^*) - (\zeta + \zeta^*)\| \ge \varepsilon\Big) \le \mathcal{X}\Big(\|\zeta_k - \zeta\| \ge \frac{\varepsilon}{2}\Big) + \mathcal{X}\Big(\|\zeta_k^* - \zeta^*\| \ge \frac{\varepsilon}{2}\Big).$$
We have
$$\Big\{k \le m : \mathcal{X}(\|(\zeta_k - \zeta) + (\zeta_k^* - \zeta^*)\| \ge \varepsilon) \ge \delta\Big\}$$
$$\subseteq \Big\{k \le m : \mathcal{X}\Big(\|\zeta_k - \zeta\| \ge \frac{\varepsilon}{2}\Big) \ge \frac{\delta}{2}\Big\} \cup \Big\{k \le m : \mathcal{X}\Big(\|\zeta_k^* - \zeta^*\| \ge \frac{\varepsilon}{2}\Big) \ge \frac{\delta}{2}\Big\}.$$
Therefore
$$\Big\{m \in \mathbb{N} : \frac{1}{m}\big|\big\{k \le m : \mathcal{X}(\|(\zeta_k + \zeta_k^*) - (\zeta + \zeta^*)\| \ge \varepsilon) \ge \delta\big\}\big| \ge \upsilon\Big\}$$
$$\subseteq \Big\{m \in \mathbb{N} : \frac{1}{m}\Big|\Big\{k \le m : \mathcal{X}\Big(\|\zeta_k - \zeta\| \ge \frac{\varepsilon}{2}\Big) \ge \frac{\delta}{2}\Big\}\Big| \ge \frac{\upsilon}{2}\Big\}$$
$$\cup\Big\{m \in \mathbb{N} : \frac{1}{m}\Big|\Big\{k \le m : \mathcal{X}\Big(\|\zeta_k^* - \zeta^*\| \ge \frac{\varepsilon}{2}\Big) \ge \frac{\delta}{2}\Big\}\Big| \ge \frac{\upsilon}{2}\Big\} \in \mathcal{I}.$$
This implies
$$\zeta_m + \zeta_m^* \xrightarrow{S^{M_s}(\mathcal{I})} \zeta + \zeta^*.$$

(iv) The reason it was left out was because it was equivalent to the proof of (iii) above. □

Theorem 3. *If the complex uncertain sequences* (ζ_m), (ζ_m^*) *are* $\mathcal{I}$*-statistically convergent in measure to* ζ *and* ζ^**, respectively, and there exist positive numbers* $p_1, p, q_1,$ *and* q *such that* $p_1 \leq \|\zeta_m\|$, $\|\zeta\| \leq p$ *and* $q_1 \leq \|\zeta_m^*\|$, $\|\zeta^*\| \leq q$ *for any* n*, then*

(*i*) $(\zeta_m \zeta_m^*)$ *is* $\mathcal{I}$*-statistically convergent in measure to* $\zeta\zeta^*$.
(*ii*) (ζ_m / ζ_m^*) *is* $\mathcal{I}$*-statistically convergent in measure to* ζ/ζ^*.

P r o o f. Let (ζ_m), (ζ_m^*) are $\mathcal{I}$-statistically convergent in measure to ζ and ζ^*, respectively, where (ζ_m), (ζ_m^*) both are complex uncertain sequences. For $p, q > 0$ and any given positive values of $\varepsilon, \delta, \upsilon$, we obtain

$$\Big\{m \in \mathbb{N} : \frac{1}{m}\Big|\Big\{k \leq m : \ \mathfrak{X}\Big(\|\zeta_k - \zeta\| \geq \frac{\varepsilon}{2q}\Big) \geq \frac{\delta}{2}\Big\}\Big| \geq \upsilon\Big\} \in \mathcal{I},$$
$$\Big\{m \in \mathbb{N} : \frac{1}{m}\Big|\Big\{k \leq m : \ \mathfrak{X}\Big(\|\zeta_k^* - \zeta^*\| \geq \frac{\varepsilon}{2p}\Big) \geq \frac{\delta}{2}\Big\}\Big| \geq \upsilon\Big\} \in \mathcal{I}.$$

Now

$$\begin{aligned}\mathfrak{X}\Big(\|\zeta_m\zeta_m^* - \zeta\zeta^*\| \geq \varepsilon\Big) &= \mathfrak{X}\Big(\|\zeta_m\zeta_m^* - \zeta_m\zeta^* + \zeta_m\zeta^* - \zeta\zeta^*\| \geq \varepsilon\Big)\\ &\leq \mathfrak{X}\Big(\|\zeta_m\zeta_m^* - \zeta_m\zeta^*\| \geq \frac{\varepsilon}{2}\Big) + \mathfrak{X}\Big(\|\zeta_m\zeta^* - \zeta\zeta^*\| \geq \frac{\varepsilon}{2}\Big)\\ &\leq \mathfrak{X}\Big(p\|\zeta_m^* - \zeta^*\| \geq \frac{\varepsilon}{2}\Big) + \mathfrak{X}\Big(q||\zeta_m - \zeta|| \geq \frac{\varepsilon}{2}\Big)\\ \implies \mathfrak{X}\Big(\|\zeta_m\zeta_m^* - \zeta\zeta^*\| \geq \varepsilon\Big) &\leq \mathfrak{X}\Big(\|\zeta_m^* - \zeta^*\| \geq \frac{\varepsilon}{2p}\Big) + \mathfrak{X}\Big(||\zeta_m - \zeta|| \geq \frac{\varepsilon}{2q}\Big).\end{aligned}$$

Then for small number $\delta > 0$,

$$\{k \leq m : \mathfrak{X}(\|\zeta_k\zeta_k^* - \zeta\zeta^*\| \geq \varepsilon) \geq \delta\}$$
$$\subseteq \Big\{k \leq m : \mathfrak{X}\Big(\|\zeta_k^* - \zeta^*\| \geq \frac{\varepsilon}{2p}\Big) \geq \frac{\delta}{2}\Big\} \cup \Big\{k \leq m : \mathfrak{X}\Big(\|\zeta_k - \zeta\| \geq \frac{\varepsilon}{2q}\Big) \geq \frac{\delta}{2}\Big\}$$
$$\implies \frac{1}{m}\big|\{k \leq m : \mathfrak{X}(\|\zeta_k\zeta_k^* - \zeta\zeta^*\| \geq \varepsilon) \geq \delta\}\big|$$
$$\leq \frac{1}{m}\Big|\Big\{k \leq m : \mathfrak{X}\big(\|\zeta_k^* - \zeta^*\| \geq \frac{\varepsilon}{2p}\big) \geq \frac{\delta}{2}\Big\}\Big| + \frac{1}{m}\Big|\Big\{k \leq m : \mathfrak{X}\Big(\|\zeta_k - \zeta\| \geq \frac{\varepsilon}{2q}\Big) \geq \frac{\delta}{2}\Big\}\Big|.$$

For small number $\upsilon > 0$,

$$\Big\{m \in \mathbb{N} : \frac{1}{m}\big|\{k \leq m : \mathfrak{X}(\|\zeta_k\zeta_k^* - \zeta\zeta^*\| \geq \varepsilon) \geq \delta\}\big| \geq \upsilon\Big\}$$
$$\subseteq \Big\{m \in \mathbb{N} :\leq \frac{1}{m}\Big|\Big\{k \leq m : \mathfrak{X}\Big(\|\zeta_k^* - \zeta^*\| \geq \frac{\varepsilon}{2p}\Big) \geq \frac{\delta}{2}\Big\}\Big| \geq \upsilon\Big\}$$
$$\cup\Big\{m \in \mathbb{N} : \frac{1}{m}\Big|\Big\{k \leq m : \mathfrak{X}\Big(\|\zeta_k - \zeta\| \geq \frac{\varepsilon}{2q}\Big) \geq \frac{\delta}{2}\Big\}\Big| \geq \upsilon\Big\} \in \mathcal{I}.$$

Hence the sequence $(\zeta_m\zeta_m^*)$ converges $\mathcal{I}$-statistical in measure to $\zeta\zeta^*$.

(*ii*) It was left out because it is similar to the (*i*) proof above. □

Theorem 4. *If a sequence* (ζ_m) *is* $\zeta_m \xrightarrow{M_s(\mathcal{I})} \zeta$ *then it is* $\zeta_m \xrightarrow{S^{M_s}(\mathcal{I})} \zeta$.

P r o o f. It is evidently true. □

But in general, the converse may not be true.

Example 3. Consider the uncertainty space $(\Upsilon, \mathcal{P}, \mathcal{X})$ to be $\{\varrho_1, \varrho_2, ...\}$ with power set and $\mathcal{X}\{\Upsilon\} = 1$, $\mathcal{X}\{\phi\} = 0$ and

$$\mathcal{X}\{\Xi\} = \begin{cases} \sup\limits_{\varrho_m \in \Xi} \dfrac{m\beta_m}{2m+1}, & \text{if} \quad \sup\limits_{\varrho_m \in \Xi} \dfrac{m\beta_m}{2m+1} < \dfrac{1}{2}, \\ 1 - \sup\limits_{\varrho_m \in \Xi^c} \dfrac{m\beta_m}{2m+1}, & \text{if} \quad \sup\limits_{\varrho_m \in \Xi^c} \dfrac{m\beta_m}{2m+1} < \dfrac{1}{2}, \\ \dfrac{1}{2}, & \text{otherwise}, \end{cases} \quad \text{for} \quad m = 1, 2, 3, \dots,$$

where

$$\beta_m = \begin{cases} 1, & \text{if} \quad m = k^2, \quad k \in \mathbb{N}, \\ 0, & \text{otherwise}, \end{cases} \quad \text{for} \quad m = 1, 2, 3, \dots .$$

Furthermore, $\zeta_m(\varrho)$ (the complex uncertain variables) are defined by

$$\zeta_m(\varrho) = \begin{cases} (m+1)i, & \text{if} \quad \varrho = \varrho_m, \\ 0, & \text{otherwise}, \end{cases} \quad \text{for} \quad m = 1, 2, 3, \dots,$$

and $\zeta \equiv 0$. Take $\mathcal{I}$=$\mathcal{I}_f$.

For every positive value of ε, we obtain

$$\mathcal{X}\big(\{\varrho \in \Upsilon : \|\zeta_m(\varrho) - \zeta(\varrho)\| \geq \varepsilon\}\big) = \mathcal{X}(\varrho_m) = \frac{m\beta_m}{2m+1}.$$

Then

$$\{m \in \mathbb{N} : \mathcal{X}\big(\|\zeta_m - \zeta\| \geq \varepsilon\big) \geq \delta\} = \Big\{m \in \mathbb{N} : \frac{m\beta_m}{2m+1} \geq \delta\Big\} \notin \mathcal{I}_f.$$

Now

$$\frac{1}{m}\big|\{k \leq m : \mathcal{X}(\|\zeta_k - \zeta\| \geq \varepsilon) \geq \delta\}\big| \leq \frac{\sqrt{m}}{m} = \frac{1}{\sqrt{m}}.$$

Then

$$\Big\{m \in \mathbb{N} : \frac{1}{m}\big|\{k \leq m : \mathcal{X}\big(\|\zeta_k - \zeta\| \geq \varepsilon\big) \geq \delta\}\big| \geq \upsilon\Big\} \subseteq \Big\{m \in \mathbb{N} : \frac{1}{\sqrt{m}} \geq \upsilon\Big\} \in \mathcal{I}_f.$$

Hence the sequence (ζ_m) is not $\mathcal{I}$-convergent in measure to $\zeta \equiv 0$ but it is $\mathcal{I}$-statistically convergent in measure to $\zeta \equiv 0$.

Theorem 5. *For any sequence(ζ_m), $\zeta_m \xrightarrow{S^{M_s}} \zeta$ implies $\zeta_m \xrightarrow{S^{M_s}(\mathcal{I})} \zeta$.*

P r o o f. Let $\zeta_m \xrightarrow{S^{M_s}} \zeta$. Then for each $\varepsilon, \delta > 0$

$$\lim_{m \to \infty} \frac{1}{m}\big|\{k \leq m : \mathcal{X}\big(\|\zeta_k - \zeta\| \geq \varepsilon\big) \geq \delta\}\big| = 0.$$

So for every $\upsilon > 0$,

$$\Big\{m \in \mathbb{N} : \frac{1}{m}\big|\{k \leq m : \mathcal{X}\big(\|\zeta_k - \zeta\| \geq \varepsilon\big) \geq \delta\}\big| \geq \upsilon\Big\}$$

is a finite set and since $\mathcal{I}$ is an admissible ideal, it must belong to $\mathcal{I}$. Hence

$$\zeta_m \xrightarrow{S^{M_s}(\mathcal{I})} \zeta.$$

But in general, the converse may not hold. □

Example 4. Let

$$\mathbb{N} = \bigcup_{j=1}^{\infty} D_j,$$

where

$$D_j = \{2^{j-1}k : 2 \text{ does not divide } k,\ k \in \mathbb{N}\}$$

be the decomposition of $\mathbb{N}$ such that each D_j is infimite and $D_j \cap D_k = \phi$, for $j \neq k$. Let $\mathcal{I}$ be the class of all subsets of $\mathbb{N}$ that can intersect only fimite number of $D_j's$. Then $\mathcal{I}$ is a nontrivial admissible ideal of $\mathbb{N}$ [11].

Now we consider the uncertainty space $(\Upsilon, \mathcal{P}, \mathcal{X})$ to be $\{\varrho_1, \varrho_2, ...\}$ with power set and $\mathcal{X}\{\Upsilon\} = 1$, $\mathcal{X}\{\phi\} = 0$ and

$$\mathcal{X}\{\Xi\} = \begin{cases} \sup\limits_{\varrho_m \in \Xi} \beta_m, & \text{if } \sup\limits_{\varrho_m \in \Xi} \beta_m < \dfrac{1}{2}, \\ 1 - \sup\limits_{\varrho_m \in \Xi^c} \beta_m, & \text{if } \sup\limits_{\varrho_m \in \Xi^c} \beta_m < \dfrac{1}{2}, \\ \dfrac{1}{2}, & \text{otherwise}, \end{cases}$$

where

$$\beta_m = \frac{1}{j+1}, \quad \text{if} \quad m \in D_j \quad \text{for} \quad m = 1, 2, 3, \ldots.$$

Furthermore, $\zeta_m(\varrho)$ (the complex uncertain variables) are defined by

$$\zeta_m(\varrho) = \begin{cases} (m+1)i, & \text{if } \varrho = \varrho_m, \\ 0, & \text{otherwise}, \end{cases} \quad \text{for} \quad m = 1, 2, 3, \ldots,$$

and $\zeta \equiv 0$.
For $m \in \mathbb{N} \setminus D_1$ and any positive number ε, we have

$$\mathcal{X}\big(\{\varrho \in \Upsilon : \|\zeta_m(\varrho) - \zeta(\varrho)\| \geq \varepsilon\}\big) = \mathcal{X}(\varrho_m) = \beta_m.$$

Then

$$\lim_{m\to\infty} \frac{1}{m} \big|\{k \leq m : \mathcal{X}(\|\zeta_k - \zeta\| \geq \varepsilon) \geq \delta\}\big| \neq 0.$$

Thus the sequence (ζ_m) is not statistically convergent in measure to $\zeta \equiv 0$.
Again

$$\{m \in \mathbb{N} : \mathcal{X}\big(\|\zeta_m - \zeta\| \geq \varepsilon\big) \geq \delta\} = \{m \in \mathbb{N} : \beta_m \geq \delta\} \in \mathcal{I}.$$

Therefore the sequence (ζ_m) is $\mathcal{I}$-convergent in measure to $\zeta \equiv 0$. By Theorem 4, the sequence (ζ_m) is $\mathcal{I}$-statistically convergent in measure to $\zeta \equiv 0$.

Theorem 6. *(ζ_m) is $\mathcal{I}$-statistically convergent in measure to ζ if each of its subsequences is $\mathcal{I}$-statistically convergent in measure to ζ.*

P r o o f. Assume that (ζ_m) does not $\mathcal{I}$-statistically convergent in measure to ζ. Consequently, there are positive constants $\varepsilon, \delta, \upsilon$ such that

$$A = \Big\{m \in \mathbb{N} : \frac{1}{m}\big|\{k \leq m : \mathcal{X}\big(\|\zeta_k(\varrho) - \zeta(\varrho)\| \geq \varepsilon\big) \geq \delta\}\big| \geq \upsilon\Big\} \notin \mathcal{I}.$$

As $\mathcal{I}$ is an admissible ideal, it implies that the set A must be infinite.

Let $A = \{m_1 < m_1 < \cdots < m_k < \cdots\}$. Let $\zeta_k^* = \zeta_{m_k}$, $k \in \mathbb{N}$. Then $(\zeta_k^*)_{k\in\mathbb{N}}$ is a subsequence of (ζ_m) which is not $\mathcal{I}$-statistical convergent in measure to ζ, we have got a contradiction.

But in general, the converse may not hold.

Example 5. Consider the uncertainty space $(\Upsilon, \mathcal{P}, \mathcal{X})$ to be $\{\varrho_1, \varrho_2, \cdots\}$ with power set and $\mathcal{X}\{\Upsilon\} = 1$, $\mathcal{X}\{\phi\} = 0$ and

$$\mathcal{X}\{\Xi\} = \begin{cases} \sup\limits_{\varrho_m \in \Xi} \dfrac{1}{m}, & \text{if} \quad \sup\limits_{\varrho_m \in \Xi} \dfrac{1}{m} < \dfrac{1}{2}, \\ 1 - \sup\limits_{\varrho_m \in \Xi^c} \dfrac{1}{m}, & \text{if} \quad \sup\limits_{\varrho_m \in \Xi^c} \dfrac{1}{m} < \dfrac{1}{2}, \\ \dfrac{1}{2}, & \text{otherwise}, \end{cases} \quad \text{for} \quad m = 1, 2, 3, \ldots,$$

and $\zeta_m(\varrho)$ (the complex uncertain variables) are defined by

$$\zeta_m(\varrho) = \begin{cases} (m+1)i, & \text{if} \quad \varrho = \varrho_{m=k^2}, \quad k \in \mathbb{N}, \\ 0, & \text{otherwise}, \end{cases} \quad \text{for} \quad m = 1, 2, 3, \ldots,$$

and $\zeta \equiv 0$. Take $\mathcal{I}$=$\mathcal{I}_d$.

Clearly, the sequence (ζ_m) is $\mathcal{I}$-statistically convergent in measure to $\zeta \equiv 0$. But the subsequence $(\zeta_{m=k^2})$, $k \in \mathbb{N}$ is not $\mathcal{I}$-statistically convergent in measure to $\zeta \equiv 0$.

Definition 12. *A complex uncertain sequence, denoted as (ζ_m), is called $\mathcal{I}$-statistically Cauchy sequence in measure if, for any given ε and δ (both greater than zero), there exists a natural number m_0 such that, for any $\upsilon > 0$, we have*

$$\Big\{m \in \mathbb{N} : \frac{1}{m}\big|\{k \leq m : \mathcal{X}\big(\|\zeta_k(\varrho) - \zeta_{m_0}(\varrho)\| \geq \varepsilon\big) \geq \delta\}\big| \geq \upsilon\Big\} \in \mathcal{I}.$$

Theorem 7. *A complex uncertain sequence (ζ_m) is $\mathcal{I}$-statistically Cauchy sequence in measure if it is $\mathcal{I}$-statistically convergent in measure to ζ.*

P r o o f. Let the complex uncertain sequence (ζ_m) be $\mathcal{I}$-statistically convergent in measure to ζ. Then for $0 < \upsilon < 1$ and every positive number ε, δ, we have

$$\Big\{m \in \mathbb{N} : \frac{1}{m}\Big|\Big\{k \leq m : \mathcal{X}\Big(\|\zeta_k - \zeta\| \geq \frac{\varepsilon}{2}\Big) \geq \frac{\delta}{2}\Big\}\Big| \geq \upsilon\Big\} \in \mathcal{I}.$$

Then for $0 < \upsilon < 1$,

$$G = \Big\{m \in \mathbb{N} : \frac{1}{m}\Big|\Big\{k \leq m : \mathcal{X}\Big(\|\zeta_k - \zeta\| \geq \frac{\varepsilon}{2}\Big) \geq \frac{\delta}{2}\Big\}\Big| < \upsilon\Big\} \in \mathcal{F}(\mathcal{I}).$$

Since $G \in \mathcal{F}(\mathcal{I})$ and $\phi \notin \mathcal{F}(\mathcal{I})$, so $G \neq \phi$. Let $m \in G$. Then

$$\frac{1}{m}\Big|\Big\{k \leq m : \mathcal{X}\Big(\|\zeta_k - \zeta\| \geq \frac{\varepsilon}{2}\Big) \geq \frac{\delta}{2}\Big\}\Big| < \upsilon < 1.$$

So, there exists some $m_0 \leq m$ such that

$$\mathcal{X}\Big(\|\zeta_{m_0} - \zeta\| \geq \frac{\varepsilon}{2}\Big) < \frac{\delta}{2}.$$

We have

$$\big\{k \leq m : \mathcal{X}\big(\|\zeta_k - \zeta_{m_0}\| \geq \varepsilon\big) \geq \delta\big\} \subset \Big\{k \leq m : \mathcal{X}\Big(\|\zeta_k - \zeta\| \geq \frac{\varepsilon}{2}\Big) \geq \frac{\delta}{2}\Big\}$$

which implies

$$\Big\{m \in \mathbb{N} : \frac{1}{m}\Big|\Big\{k \leq m : \mathcal{X}\big(\|\zeta_k - \zeta_{m_0}\| \geq \varepsilon\big) \geq \delta\Big\}\Big| \geq \upsilon\Big\}$$
$$\subset \Big\{m \in \mathbb{N} : \frac{1}{m}\Big|\Big\{k \leq m : \mathcal{X}\Big(\|\zeta_k - \zeta\| \geq \frac{\varepsilon}{2}\Big) \geq \frac{\delta}{2}\Big\}\Big| \geq \upsilon\Big\} \in \mathcal{I}.$$

Hence the sequence (ζ_m) is $\mathcal{I}$-statistically Cauchy sequence in measure. □

But in general, the converse may not hold.

Example 6. Consider the uncertainty space $(\Upsilon, \mathcal{P}, \mathcal{X})$ to be equivalent to set of real number $\mathbb{R}$ with $\Xi_m = (m, \infty)$, for $m = 1, 2, 3, \ldots$ and

$$\mathcal{X}\{\Xi\} = \begin{cases} 0, & \text{if } \Xi = \phi \text{ or } \Xi \text{ is upper bounded,} \\ \frac{1}{2}, & \text{if both } \Xi \text{ and } \Xi^c \text{ are upper unbounded,} \\ 1, & \text{if } \Xi = \Upsilon \text{ or } \Xi^c \text{ is upper bounded.} \end{cases}$$

Furthermore, $\zeta_m(\varrho)$ (the complex uncertain variables) are defined by

$$\zeta_m(\varrho) = \begin{cases} i, & \text{if } \varrho \in \Xi_m, \\ 0, & \text{if } \varrho \notin \Xi_m, \end{cases} \quad \text{for} \quad m = 1, 2, 3, \ldots,$$

and $\zeta \equiv 0$. Take $\mathcal{I}$=$\mathcal{I}_d$. Now

$$\big\{\varrho \in \Upsilon : \|\zeta_m(\varrho) - \zeta_{m_0}(\varrho)\| \geq \varepsilon\big\} = \begin{cases} (m_0, m], & \text{if } 0 < \varepsilon \leq 1, \\ \phi, & \text{if } \varepsilon > 1. \end{cases}$$
$$\implies \mathcal{X}\big(\big\{\varrho \in \Upsilon : \|\zeta_m(\varrho) - \zeta_{m_0}(\varrho)\| \geq \varepsilon\big\}\big) = 0$$
$$\implies \lim_{m \to \infty} \frac{1}{m}\big|\big\{k \leq m : \mathcal{X}(\|\zeta_k - \zeta_{m_0}\| \geq \varepsilon) \geq \delta\big\}\big| = 0.$$

Therefore

$$\Big\{m \in \mathbb{N} : \frac{1}{m}\big|\big\{k \leq m : \mathcal{X}(\|\zeta_k - \zeta_{m_0}\| \geq \varepsilon) \geq \delta\big\}\big| \geq \upsilon\Big\} \in \mathcal{I}.$$

Again,

$$\mathcal{X}\big(\big\{\varrho \in \Upsilon : \|\zeta_m(\varrho) - \zeta(\varrho)\| \geq \varepsilon\big\}\big) = \mathcal{X}(\Xi_m) = 1.$$

So,

$$\lim_{m \to \infty} \frac{1}{m}\big|\big\{k \leq m : \mathcal{X}(\|\zeta_k - \zeta\| \geq \varepsilon) \geq \delta\big\}\big| \neq 0.$$

Therefore

$$\Big\{m \in \mathbb{N} : \frac{1}{m}\big|\big\{k \leq m : \mathcal{X}(\|\zeta_k - \zeta\| \geq \varepsilon) \geq \delta\big\}\big| \geq \upsilon\Big\} \notin \mathcal{I}.$$

Hence the sequence (ζ_m) is not $\mathcal{I}$-statistically convergent in measure to $\zeta \equiv 0$ but it is $\mathcal{I}$-statistically Cauchy sequence in measure.

4. Conclusion

This paper mainly contributes to the study of $\mathcal{I}$-statistical convergence in measure of complex uncertain sequences, by establishing some of its properties. Also, we define $\mathcal{I}$-statistical Cauchy sequence in measure and study the relationship among them. It is possible to generalize and apply these concepts and results to future research in this area.

Acknowledgment

The first author wishes to acknowledge the *Council of Scientific and Industrial Research, India* for their fellowships funding granted under the *CSIR-JRF* scheme (File No: 09/0714(11674)/2021-EMR-I) for the preparation of this manuscript.

REFERENCES

1. Chen X., Ning Y., Wang X. Convergence of complex uncertain sequences. *J. Intell. Fuzzy Syst.*, 2016. Vol. 30, No. 6. P. 3357–3366. DOI: 10.3233/IFS-152083
2. Debnath S., Das B. Statistical convergence of order α for complex uncertain sequences. *J. Uncertain Syst.*, 2021. Vol. 14, No. 2. Art. no. 2150012. DOI: 10.1142/S1752890921500124
3. Debnath S., Das B. On λ-statistical convergence of order α for complex uncertain sequences. *Int. J. Gen. Syst.*, 2022. Vol. 52, No. 2. P. 191–202. DOI: 10.1080/03081079.2022.2132490
4. Debnath S., Das B. On rough statistical convergence of complex uncertain sequences. *New Math. Nat. Comput.*, 2023. Vol. 19, No. 1. P. 1–17. DOI: 10.1142/S1793005722500454
5. Debnath S., Debnath J. On $\mathcal{I}$-statistically convergent sequence spaces defined by sequence of Orlicz functions using matrix transformation. *Proyecciones J. Math.*, 2014. Vol. 33, No. 3. P. 277–285. DOI: 10.4067/S0716-09172014000300004
6. Debnath S., Rakshit D. On $\mathcal{I}$-statistical convergence. *Iranian J. Math. Sci. Inform.*, 2018. Vol. 13, No. 2. P. 101–109. DOI: 10.7508/ijmsi.2018.13.009
7. Esi A., Debnath S., Saha S. Asymptotically double λ_2-statistically equivalent sequences of interval numbers. *Mathematica*, 2020. Vol. 62(85), No. 1. P. 39–46. DOI: 10.24193/mathcluj.2020.1.05
8. Fast H. Sur la convergence statistique. *Colloq. Math.*, 1951. Vol. 2, No. 3–4. P. 241–244. (in French)
9. Fridy J. A. On statistically convergence. *Analysis*, 1985. Vol. 5, No. 4. P. 301–313. DOI: 10.1524/anly.1985.5.4.301
10. Kişi Ö. $S_\lambda(\mathcal{I})$-convergence of complex uncertain sequences. *Mat. Stud.*, 2019. Vol. 51, No. 2. P. 183–194.
11. Kostyrko P., Wilczyński W., Šalát T. $\mathcal{I}$-convergence. *Real Anal. Exchange*, 2000/2001. Vol. 26, No. 2. P. 669–686.
12. Liu B. *Uncertainty Theory*, 4th ed. Berlin, Heidelberg: Springer-Verlag, 2015. 487 p. DOI: 10.1007/978-3-662-44354-5
13. Mursaleen M., Debnath S., Rakshit D. $\mathcal{I}$-statistical limit superior and $\mathcal{I}$-statistical limit inferior. *Filomat*, 2017. Vol. 31, No. 7. P. 2103–2108. DOI: 10.2298/FIL1707103M
14. Nath P. K., Tripathy B. C. Convergent complex uncertain sequences defined by Orlicz function. *Ann. Univ. Craiova-Math. Comput. Sci. Ser.*, 2019. Vol. 46, No. 1. P. 139–149.
15. Peng Z. *Complex Uncertain Variable.* PhD, Tsinghua University, 2012.
16. Roy S., Tripathy B. C., Saha S. Some results on p-distance and sequence of complex uncertain variables. *Commun. Korean. Math. Soc.*, 2020. Vol. 35, No. 3. P. 907–916. DOI: 10.4134/CKMS.c200026
17. Saha S., Tripathy B. C., Roy S. On almost convergence of complex uncertain sequences. *New Math. Nat. Comput.*, 2016. Vol. 16, No. 03. P. 573–580. DOI: 10.1142/S1793005720500349
18. Savas E., Das P. A generalized statistical convergence via ideals. *App. Math. Lett.*, 2011. Vol. 24, No. 6. P. 826–830. DOI: 10.1016/j.aml.2010.12.022
19. Savas E., Das P. On $\mathcal{I}$-statistically pre-Cauchy sequences. *Taiwanese J. Math.*, 2014. Vol. 18, No. 1. P. 115–126. DOI: 10.11650/tjm.18.2014.3157
20. Savas E., Debnath S., Rakshit D. On $\mathcal{I}$-statistically rough convergence. *Publ. Inst. Math.*, 2019. Vol. 105, No. 119. P. 145–150. DOI: 10.2298/PIM1919145S

21. Steinhaus H. Sur la convergence ordinaire et la convergence asymptotique. *Colloq. Math.*, 1951. Vol. 2, No. 1. P. 73–74. (in French)
22. Tripathy B. C., Nath P. K. Statistical convergence of complex uncertain sequences. *New Math. Nat. Comput.*, 2017. Vol. 13, No. 3. P. 359–374. DOI: 10.1142/S1793005717500090
23. You C., Yan L. Relationships among convergence concepts of uncertain sequences. *Comput. Model. New Technol.*, 2016. Vol. 20, No. 3. P. 12–16.
24. Wu J., Xia Y. Relationships among convergence concepts of uncertain sequences. *Inform. Sci.*, 2012. Vol. 198. P. 177–185. DOI: 10.1016/j.ins.2012.02.048