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A TWO-FOLD CAPTURE OF COORDINATED EVADERS IN THE PROBLEM OF A SIMPLE PURSUIT ON TIME SCALES[1]

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Abstract: In finite-dimensional Euclidean space, we study the problem of a simple pursuit of two evaders by a group of pursuers in a given time scale. It is assumed that the evaders use the same control and do not move out of a convex polyhedral set. The pursuers use counterstrategies based on information on the initial positions and on the prehistory of the control of evaders. The set of admissible controls of each of the participants is a sphere of unit radius with its center at the origin, and the goal sets are the origin. The goal of the group of pursuers is the capture of at least one evader by two pursuers. In terms of the initial positions and parameters of the game, a sufficient condition for capture is obtained. The study is based on the method of resolving functions, which makes it possible to obtain sufficient conditions for solvability of the pursuit problem in some guaranteed time.

Keywords: Differential game, Group pursuit, Evader, Pursuer, Time scale.

1. Introduction

The modern theory of differential pursuit-evasion games involves the development of methods for solving problems of conflict interaction of groups of pursuers and evaders [3, 6, 7, 10]. In particular, it is concerned with searching for new classes of problems which can be analyzed using the previously developed methods, for example, the method of resolving functions. It was pointed out in [1, 9] that some results obtained separately for the theories of differential and difference equations may be considered from a unified point of view if one admits the possibility of specifying dynamical systems on arbitrary closed subsets $\mathbb{R}^1$ called *time scales*. Time scales find applications in constructing various mathematical models [2, 4]. A nonantagonistic game of N persons in a time scale was considered in [11]. Sufficient conditions for the capture of one evader in the problem of a simple group pursuit in a given time scale were obtained in [15].

Ref. [14] addressed the problem of a simple pursuit of a group of rigidly coordinated evaders by a group of pursuers in a given time scale, where sufficient conditions for the capture of at least one evader were obtained. The problem of a multiple capture of a given number of evaders in time scales, under the condition that the evaders use programmed strategies, each pursuer catches no more than one evader and the motions of the players are simple was treated in [13].

Ref. [17] dealt with the problem of a simple pursuit of rigidly coordinated evaders in a given time scale, under the condition that the evaders do not move out of a convex polyhedral set. The goal of the pursuers was either the capture of one evader by two pursuers or the capture of two evaders. Sufficient conditions for capture were obtained.

In this paper we consider, in a given time scale, the problem of a simple pursuit of two evaders by a group of pursuers who use the same control, under the condition that the evaders do not move

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out of a convex polyhedral set. The goal of the pursuers is the capture of at least one evader by two different pursuers. Sufficient conditions for capture are obtained.

2. Auxiliary definitions and facts

In this section we will outline the basic facts from time scale theory. All results presented below can be found, for example, in [5, 8].

Definition 1. *A nonempty closed subset $\mathbb{T} \subset \mathbb{R}^1$ such that $\sup\limits_{t\in\mathbb{T}} t = +\infty$ is called a time scale.*

Definition 2. *Let $\mathbb{T}$ be a time scale. A function $\sigma : \mathbb{T} \to \mathbb{R}^1$ of the form*

$$\sigma(t) = \inf\big\{s \in \mathbb{T} \mid s > t\big\}$$

is called a translation function.

Definition 3. *A function $f : \mathbb{T} \to \mathbb{R}^1$ is called Δ-differentiable at point $t \in \mathbb{T}$ if there exists a number $\gamma \in \mathbb{R}^1$ such that for any $\varepsilon > 0$ there exists a neighborhood W of point t such that the inequality*

$$|f(\sigma(t)) - f(s) - \gamma(\sigma(t) - s)| < \varepsilon|\sigma(t) - s|$$

holds for all $s \in \mathbb{T} \cap W$.

In this case, the number γ is called the Δ-derivative of the function f at point t. The Δ-derivative of the function f at point t will be denoted by $f^{\Delta}(t) = \gamma$.

Definition 4. *A function $f : \mathbb{T} \to \mathbb{R}^n$, $f(t) = (f_1(t), \ldots, f_n(t))$ is called Δ-differentiable at point $t \in \mathbb{T}$ if all functions $f_1, \ldots, f_n$ are Δ-differentiable at point t.*

Let $\mathbb{T}$ be a time scale, $E \subset \mathbb{T}$. Denote

$$R(E) = \big\{t \in E \mid \sigma(t) > t\big\}.$$

Then the set $R(E)$ is no more than countable.

Definition 5. *The set $E \subset \mathbb{T}$ is called Δ-measurable if the set*

$$\tilde{E} = E \cup \bigcup_{t\in R(E)} (t, \sigma(t))$$

is measurable in the sense of Lebesgue.

Definition 6. *A function $f : \mathbb{T} \to \mathbb{R}^1$ is called Δ-measurable on a Δ-measurable set E if a function $\tilde{f}$ of the form*

$$\tilde{f}(t) = \begin{cases} f(t), & t \in E, \\ f(t_i), & t \in (t_i, \sigma(t_i)), \quad t_i \in R(E) \end{cases}$$

is measurable on the set $\tilde{E}$.

Definition 7. *A function $f : E \to \mathbb{R}^1$, $E \subset \mathbb{T}$ is called Δ-integrable on a Δ-measurable set E if the function $\tilde{f}$ is integrable in the sense of Lebesgue on the set $\tilde{E}$. If f is Δ-integrable on the set E, then we define $\int_E f(s)\Delta s$, assuming*

$$\int_E f(s)\Delta s = \int_{\tilde{E}} f\,d\mu,$$

where μ is the Lebesgue measure.

3. Formulation of the problem

Let a time scale $\mathbb{T}$, $t_0 \in \mathbb{T}$ be given.

In the space $\mathbb{R}^k (k \geqslant 2)$ we consider the differential game $\Gamma(n, 2)$ of $n+2$ persons: n pursuers $P_1, \ldots, P_n$ and two evaders E_1, E_2 with laws of motion of the form

$$x_i^{\Delta} = u_i, \quad x_i(t_0) = x_i^0, \quad u_i \in V, \tag{3.1}$$

$$y_j^{\Delta} = v, \quad y_j(t_0) = y_j^0, \quad v \in V. \tag{3.2}$$

Here $x_i, y_j, x_i^0, y_j^0, u_i, v \in \mathbb{R}^k$, $i \in I = \{1, \ldots, n\}$, $j \in J = \{1, 2\}$, $V = \{v \in \mathbb{R}^k \mid \|v\| \leqslant 1\}$. We assume that $x_i^0 \neq y_j^0$ for all $i \in I, j \in J$. Additionally, we assume that in the process of the game evaders E_1 and E_2 do not move out of a convex set D with a nonempty interior of the form

$$D = \{y \in \mathbb{R}^k \mid (p_l, y) \leqslant \mu_l, \quad l = 1, \ldots, r\},$$

where $p_1, \ldots, p_r$ are unit vectors $\mathbb{R}^k$, $\mu_1, \ldots, \mu_r$ are real numbers, and (a, b) is a scalar product. We also assume that $D = \mathbb{R}^k$ with $r = 0$.

Introduce new variables $z_{ij} = x_i - y_j$. Then instead of the systems (3.1) and (3.2) we obtain the system

$$z_{ij}^{\Delta} = u_i - v, \quad z_{ij}(t_0) = z_{ij}^0 = x_i^0 - y_j^0, \quad u_i, v \in V. \tag{3.3}$$

We will say the Δ-measurable function $v : \mathbb{T} \to \mathbb{R}^k$ is Δ-admissible if $v(t) \in V$, $y_j(t) \in D$ for all $t \in \mathbb{T}, j \in J$. Here $y_j(t)$ is a solution to the Cauchy problem (3.2) with a given function $v(\cdot)$.

We will say that the prehistory $v_t(\cdot)$ of the function v at time $t \in \mathbb{T}$ is a restriction of the function v to $[t_0, t) \cap \mathbb{T}$. Let

$$z^0 = \{z_{ij}^0 \mid i \in I, \quad j \in J\}$$

denote the vector of initial positions.

The actions of the evaders can be interpreted as follows: there is a center which for all evaders E_1 and E_2 chooses the same control $v(t)$.

Definition 8. *We will say that a quasi-strategy $\mathcal{U}_i$ of pursuer P_i is given if a map U_i^0 is defined which associates the Δ-measurable function $u_i(t) = \mathcal{U}_i(z^0, t, v_t(\cdot))$ with values in V to the initial positions z^0, time $t \in \mathbb{T}$ and an arbitrary prehistory of the control $v_t(\cdot)$ of evaders E_1 and E_2.*

Definition 9. *A two-fold capture occurs in the game $\Gamma(n, 2)$ if there exist time $T_0 = T(z^0)$ and quasi-strategies $\mathcal{U}_1, \ldots, \mathcal{U}_n$ of pursuers $P_1, \ldots, P_n$ such that, for any measurable function $v(\cdot)$, $v(t) \in V$, $y(t) \in D$, $t \in [t_0, T_0] \cap \mathbb{T}$, there are numbers $l, m \in I$, $(m \neq l)$, $j \in \{1, 2\}$ and times τ_1, $\tau_2 \in [t_0, T_0] \cap \mathbb{T}$ such that $z_{lj}(\tau_1) = 0$, $z_{mj}(\tau_2) = 0$.*

4. Sufficient conditions for capture

Definition 10 [12]. *The vectors $a_1, a_2, \ldots, a_m$ form a positive basis in $\mathbb{R}^k$ if for any $x \in \mathbb{R}^k$ there exist nonnegative real numbers $\alpha_1, \alpha_2, \ldots, \alpha_m$ such that*

$$x = \alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_m a_m.$$

Let $\operatorname{Int} X$, $\operatorname{co} X$ denote, respectively, the interior and the convex hull of the set $X \subset \mathbb{R}^k$.

Theorem 1 [12]. *The vectors $a_1, a_2, \ldots, a_m$ form a positive basis in $\mathbb{R}^k$ if and only if*

$$0 \in \operatorname{Intco}\{a_1, \ldots, a_m\}.$$

Lemma 1. *Let $m \geqslant 3$, $a_1, \ldots, a_m$, b_1, b_2, $p_1, \ldots, p_r \in \mathbb{R}^k$ be such that*

1) *for each $q \in J_0 = \{1, \ldots, m-2\}$*

$$0 \in \operatorname{Intco}\big\{a_i - b_1, a_i - b_2, i \in J_0 \setminus \{q\}, p_1, \ldots, p_r\big\},$$

2) *$a_{m-1} - b_2 = t_1(b_1 - b_2)$, $a_m - b_2 = t_2(b_1 - b_2)$ for some $t_1 < 0$, $t_2 < 0$.*

Then for each $l \in J = \{1, \ldots, m\}$ the following inclusion holds:

$$0 \in \operatorname{Intco}\big\{a_i - b_2, i \in J \setminus \{l\}, p_1, \ldots, p_r\big\}. \tag{4.1}$$

P r o o f. If $m = 3$, then it follows from condition 1) of the lemma that

$$0 \in \operatorname{Intco}\{p_1, \ldots, p_r\}.$$

Therefore, the condition (4.1) is satisfied automatically.

Let $m \geqslant 4$. Assume that there exists $q \in J$ for which

$$0 \notin \operatorname{Intco}\big\{a_i - b_2, i \in J \setminus \{q\}, p_1, \ldots, p_r\big\}.$$

Then, by the separability theorem, there exists a unit vector $x \in \mathbb{R}^k$ such that

$$(a_i - b_2, x) \leqslant 0 \quad \text{for all} \quad i \in J \setminus \{q\}, \quad (p_j, x) \leqslant 0, \quad \text{for all} \quad j = 1, \ldots, r. \tag{4.2}$$

It follows from condition 2) of the lemma that $(b_1 - b_2, x) \geqslant 0$. Then

$$(a_i - b_1, x) = (a_i - b_2, x) + (b_2 - b_1, x) \leqslant 0 \quad \text{for all} \quad i \in J \setminus \{q\}. \tag{4.3}$$

Inequalities (4.2) and (4.3) contradict condition 1) of the lemma. This proves the lemma. □

Let us introduce the following notation:

$$\lambda(h, v) = \sup\big\{\lambda \geqslant 0 \mid -\lambda h \in V - v\big\},$$

$$K(t) = \int_{t_0}^{t} \Delta s, \quad \Omega(J) = \big\{(i_1, i_2) \big| i_1, i_2 \in J, \ \ i_1 \neq i_2\big\},$$

where J is a finite set of natural numbers.

Lemma 2. *Let $m \geqslant 4$, $a_1, \ldots, a_{m-2}$, c, $p_1 \in \mathbb{R}^k$ be such that for each $q \in J_0 = \{1, \ldots, m-2\}$ the vectors $\{a_i, i \in J_0 \setminus \{q\}, c, p_1\}$ form a positive basis $\mathbb{R}^k$. Then for any $b_1, b_2 \in \mathbb{R}^k$ there exists $\rho_0 > 0$ such that for all $\rho > \rho_0$ the following inequality holds:*

$$\delta(\rho) = \min_{v \in V} \max\big\{\max_{\Lambda \in \Omega^0(J)} \min_{i \in \Lambda} \lambda(w_i, v), (p_1, v)\big\} > 0,$$

where $J = \{1, \ldots, m\}$, $\Omega^0(J) = \Omega(J_0) \cup \{(m-1, m)\}$,

$$w_i = \begin{cases} a_i, & i \in J_0, \\ b_1 + \rho c, & i = m-1, \\ b_2 + \rho c, & i = m. \end{cases}$$

P r o o f. Assume that the statement of the lemma is false. Then there exist $b_1, b_2 \in \mathbb{R}^k$ such that for any $\rho_0 > 0$ there is $\rho > \rho_0$ for which $\delta(\rho) = 0$. It follows from the definition of $\delta(\rho)$ that there exists $v_\rho \in V$ such that $(p_1, v_\rho) \leqslant 0$ and for all $\Lambda \in \Omega^0(J)$

$$\min_{i\in\Lambda} \lambda(w_i, v_\rho) = 0, \quad \text{with} \quad \|v_\rho\| = 1.$$

From the last condition it follows that there exist $J(\rho) \subset J_0$, $|J(\rho)| = m-3$ and $j(\rho) \in \{m-1, m\}$ such that $\lambda(w_i, v_\rho) = 0$ for all $i \in J(\rho) \cup \{j(\rho)\}$.

Let $\rho_0 = 1$. Then there are $\rho_1 > \rho_0$, $v_1 \in V$, $J(\rho_1)$ for which

$$(p_1, v_1) \leqslant 0, \quad (w_i, v_1) \leqslant 0 \quad \text{for all} \quad i \in J(\rho_1) \cup \{j(\rho_1)\}, \quad \text{and} \quad \|v_1\| = 1.$$

For $\rho_0 = \rho_1 + 1$ there are $\rho_2 > \rho_0$, $v_2 \in V$, $J(\rho_2)$ for which

$$(p_1, v_2) \leqslant 0, \quad (w_i, v_2) \leqslant 0 \quad \text{for all} \quad i \in J(\rho_2) \cup \{j(\rho_2)\}, \quad \text{and} \quad \|v_2\| = 1.$$

Continuing this process further, we find that there exist sequences $\{\rho_s\}_{s=1}^{\infty}$,

$$\lim_{s\to+\infty} \rho_s = +\infty,$$

$\{v_s\}$, $\{J(\rho_s)\}$, $\{j(\rho_s)\}$ for which

$$(p_1, v_s) \leqslant 0, \quad (w_i, v_s) \leqslant 0 \quad \text{for all} \quad i \in J(\rho_s) \cup \{j(\rho_s)\}, \quad \text{and} \quad \|v_s\| = 1.$$

Consequently, there exists a subsequence $\{\rho_{s_i}\}$, $\lim\limits_{i\to+\infty} \rho_{s_i} = +\infty$ for which there are a subsequence $\{v_{s_i}\}$, a set J^0, $J^0 \subset J_0$, $|J^0| = m-3$, and an index $\hat{j} \in \{m-1, m\}$ such that for all j the following inequalities hold:

$$(p_1, v_{s_j}) \leqslant 0, \quad (w_i, v_{s_j}) \leqslant 0 \quad \text{for all} \quad i \in J^0 \cup \{\hat{j}\}, \quad \text{and} \quad \|v_{s_j}\| = 1.$$

From the sequence $\{v_{s_j}\}$ one can single out a subsequence $\{\overline{v}_l\}$ converging to v_0, with $\|v_0\| = 1$. Therefore, we have

$$(p_1, \overline{v}_l) \leqslant 0, \quad (w_i, \overline{v}_l) \leqslant 0 \quad \text{for all} \quad i \in J^0, \quad \Big(\frac{w_{\hat{j}}}{\rho_l} + c, \overline{v}_l\Big) \leqslant 0.$$

Passing in the last inequalities to the limit as $l \to +\infty$, we obtain

$$(p_1, v_0) \leqslant 0, \quad (w_i, v_0) \leqslant 0 \quad \text{for all} \quad i \in J^0, \quad (c, v_0) \leqslant 0.$$

Therefore, by virtue of Theorem 1 the set of vectors $\{w_i, i \in J^0, c, p_1\}$ does not form the positive basis $\mathbb{R}^k$, which contradicts the condition of the lemma. This proves the lemma. □

Lemma 3. *Let $a_1, \ldots, a_m, p_1 \in \mathbb{R}^k$ be such that*

$$\delta = \min_{v\in V} \max\Big\{ \max_{\Lambda\in\Omega^0(J)} \min_{j\in\Lambda} \lambda(a_j, v), (p_1, v)\Big\} > 0,$$

where $J_0 = \{1, \ldots, m-2\}$, $\Omega^0(J) = \Omega(J_0) \cup \{m-1, m\}$.

Then there exists $T_0 > t_0$, $T_0 \in \mathbb{T}$ such that for any admissible control $v(\cdot)$ of evaders there is $\Lambda = (\alpha, \beta) \in \Omega^0(J)$ such that

$$\int_{t_0}^{T_0} \lambda(a_\alpha, v(s))\Delta s \geqslant 1, \quad \int_{t_0}^{T_0} \lambda(a_\beta, v(s))\Delta s \geqslant 1.$$

P r o o f. Let $v(\cdot)$ be an admissible control of evaders. From [5] it follows that the functions $\lambda(a_j, v(t))$ are Δ-measurable and Δ-integrable. For each $t \in \mathbb{T}$ we define the sets

$$T_1(t) = \{t \in \mathbb{T} \mid (p_1, v(t)) \geqslant \delta\}, \quad T_2(t) = \{t \in \mathbb{T} \mid (p_1, v(t)) < \delta\}.$$

Since $(y_j(t), p_1) \leqslant \mu_1$ for all $t \in \mathbb{T}$, $j = 1, 2$, the following inequality holds:

$$\int_{t_0}^{t} (p_1, v(s))\Delta s \leqslant \mu_0 = \min\{\mu_1 - (p_1, y_1^0), \mu_1 - (p_1, y_2^0)\}.$$

Therefore,

$$\mu_0 \geqslant \int_{t_0}^{t} (p_1, v(s))\Delta s \geqslant \delta \int_{T_1(t)} \Delta s - \int_{T_2(t)} \Delta s, \quad K(t) = \int_{T_1(t)} \Delta s + \int_{T_2(t)} \Delta s.$$

The last two relations imply the validity of the inequality

$$\int_{T_2(t)} \Delta s \geqslant \frac{\delta K(t) - \mu_0}{1 + \delta}. \tag{4.4}$$

Next, we have

$$\max_{\Lambda \in \Omega^0(J)} \min_{j \in \Lambda} \int_{t_0}^{t} \lambda(a_j, v(s))\Delta s \geqslant \max_{\Lambda \in \Omega^0(J)} \int_{t_0}^{t} \min_{j \in \Lambda} \lambda(a_j, v(s))\Delta s. \tag{4.5}$$

For any nonnegative numbers $\gamma_\Lambda (\Lambda \in \Omega^0(J))$ we have

$$\max_{\Lambda \in \Omega^0(J)} \gamma_\Lambda \geqslant \frac{1}{N_0} \sum_{\Lambda \in \Omega^0(J)} \gamma_\lambda, \quad \text{where} \quad N_0 = 1 + \frac{(m-2)(m-3)}{2}.$$

Therefore,

$$\max_{\Lambda \in \Omega^0(J)} \int_{t_0}^{t} \min_{j \in \Lambda} \lambda(a_j, v(s))\Delta s \geqslant \frac{1}{N_0} \int_{t_0}^{t} \sum_{\Lambda \in \Omega^0(J)} \min_{j \in \Lambda} \lambda(a_j, v(s))\Delta s$$

$$\geqslant \frac{1}{N_0} \int_{t_0}^{t} \max_{\Lambda \in \Omega^0(J)} \min_{j \in \Lambda} \lambda(a_j, v(s))\Delta s \geqslant \frac{1}{N_0} \int_{T_2(t)} \max_{\Lambda \in \Omega^0(J)} \min_{j \in \Lambda} \lambda(a_j, v(s))\Delta s.$$

Hence, from (4.4) and (4.5) we obtain

$$\max_{\Lambda \in \Omega^0(J)} \min_{j \in \Lambda} \int_{t_0}^{t} \lambda(a_j, v(s))\Delta s \geqslant \frac{\delta}{N_0} \int_{T_2(t)} \Delta s \geqslant \frac{\delta}{N_0} \cdot \frac{\delta K(t) - \mu_0}{1 + \delta}.$$

Since

$$\lim_{t \in \mathbb{T}, t \to +\infty} K(t) = +\infty,$$

it follows from the last inequality that there exists $T_0 \in \mathbb{T}$ for which the following inequality holds:

$$\max_{\Lambda \in \Omega^0(J)} \min_{j \in \Lambda} \int\limits_{t_0}^{T_0} \lambda(a_j, v(s))\Delta s \geqslant 1,$$

which implies the validity of the statement of the lemma. This proves the lemma. □

Lemma 4 [17]. *Let $a_1, \ldots, a_m, p_1 \in \mathbb{R}^k$ be such that for each $q \in J = \{1, \ldots, m\}$ the vectors $\{a_i, i \in J \setminus \{q\}, p_1\}$ form a positive basis $\mathbb{R}^k$. Then*

$$\delta = \min_{v \in V} \max\big\{ \max_{\Lambda \in \Omega(J)} \min_{i \in \Lambda} \lambda(a_i, v), (p_1, v)\big\} > 0.$$

Lemma 5 [17]. *Let $a_1, \ldots a_m, p_1 \in \mathbb{R}^k$ be such that for each $q \in J = \{1, \ldots, m\}$ the vectors $\{a_i, i \in J \setminus \{q\}, p_1\}$ form a positive basis $\mathbb{R}^k$. Then there exists $T_0 > t_0$, $T_0 \in \mathbb{T}$ such that for any admissible control $v(\cdot)$ of evaders there is $\Lambda = (\alpha, \beta) \in \Omega(J)$,*

$$\int\limits_{t_0}^{T_0} \lambda(a_\alpha, v(s))\Delta s \geqslant 1, \quad \int\limits_{t_0}^{T_0} \lambda(a_\beta, v(s))\Delta s \geqslant 1.$$

Theorem 2. *Let $r = 1$ and suppose that there exists $j \in \{1, 2\}$ such that for any $q \in I$*

$$0 \in \operatorname{Intco}\big\{z_{ij}^0, i \in I \setminus \{q\}, p_1\big\}.$$

Then a two-fold capture occurs in the game $\Gamma(n, 2)$.

P r o o f. By virtue of Lemma 5

$$T^0 = \min\Big\{t \in \mathbb{T} \mid t > t_0, \inf_{v(\cdot)} \max_{\Lambda \in \Omega(J)} \min_{i \in \Lambda} \int\limits_{t_0}^{t} \lambda(z_{ij}^0, v(s))\Delta s \geqslant 1\Big\}$$

is finite. Let $v(\cdot)$ be an admissible control of evaders. Define the functions

$$h_i(t) = 1 - \int_{t_0}^{t} \lambda(z_{ij}^0, v(s))\Delta s.$$

Let pursuer P_i construct a control as follows. If the inequality $h_i(t) \geqslant 0$ is satisfied at time $t \in \mathbb{T}$, then we assume

$$u_i(t) = v(t) - \lambda(z_{ij}^0, v(t))z_{ij}^0.$$

If $\tau \in \mathbb{T}$ is the first time instant for which $h_i(\tau) = 0$, we assume that $\lambda(z_{ij}^0, v(t)) = 0$ for all $t \geqslant \tau$.

Let $\tau \in \mathbb{T}$ be the first time instant for which $h_i(\tau) < 0$, and let the inequality $h_i(t) > 0$ be satisfied for all $t \in \mathbb{T}$, $t < \tau$. Define the number

$$\tau_i^* = \sup\{t \in \mathbb{T} \mid h_i(t) > 0\}.$$

Then $(\tau_i^*, \tau) \cap \mathbb{T} = \emptyset$. Indeed, if there existed a time instant $t \in (\tau_i^*, \tau) \cap \mathbb{T}$, then the inequality $h_i(t) > 0$ would be satisfied, which is impossible by virtue of the definition of the number τ_i^*. In this case, we assume

$$u_i(\tau) = v(\tau) - \lambda^*(z_{ij}^0, v(\tau)) z_{ij}^0, \quad \text{where} \quad \lambda^*(z_{ij}^0, v(\tau)) = \frac{h_i(\tau_i^*)}{\sigma(\tau_i^*) - \tau_i^*} = \frac{h_i(\tau_i^*)}{\tau - \tau_i^*}.$$

We note that in this case $\lambda^*(z_{ij}^0, v(\tau)) \leqslant \lambda(z_{ij}^0, v(\tau))$ and therefore $u_i(\tau) \in V$. Then

$$1 - \int_{t_0}^{\tau_i^*} \lambda(z_{ij}^0, v(s))\Delta s - \int_{\tau_i^*}^{\tau} \lambda^*(z_{ij}^0, v(s))\Delta s = h_i(\tau_i^*) - \int_{\tau_i^*}^{\tau} \frac{h_i(\tau_i^*)}{\tau - \tau_i^*}\Delta s = 0.$$

Then from the definition of the controls of the pursuers and the system (3.3) it follows that for all $t \in [t_0, T^0] \cap \mathbb{T}$ the equalities $z_{ij}(t) = z_{ij}^0 h_i(t), i \in I$, hold.

From Lemma 5 and the definition of the controls of the pursuers it follows that there exist numbers $l, m \in I$ such that $h_l(T^0) = 0$, $h_m(T^0) = 0$. This implies that pursuers P_l and P_m perform a capture of evader E_j. Consequently, a two-fold capture occurs in the game $\Gamma(n, 2)$. This proves the theorem. □

Theorem 3. *Let $r = 1$ and suppose that there exists a set $I_0 \subset I$, $|I_0| = n - 2$ such that for all $l \in I_0$*

$$0 \in \operatorname{Intco}\{z_{i1}^0, z_{i2}^0, i \in I_0 \setminus \{l\}, p_1\}. \tag{4.6}$$

Then a two-fold capture occurs in the game $\Gamma(n, 2)$.

P r o o f. By virtue of Theorem 1, it follows from condition (4.6) that for all $l \in I_0$ the set $\{z_{i1}^0, z_{i2}^0, i \in I_0 \setminus \{l\}, p_1\}$ forms a positive basis $\mathbb{R}^k$. Denote $c = y_1^0 - y_2^0$. Since

$$z_{i2}^0 = x_i^0 - y_2^0 = x_i^0 - y_1^0 + c = z_{i1}^0 + c,$$

for all $l \in I_0$ the positive basis $\mathbb{R}^k$ forms a set $\{z_{i1}^0, i \in I_0 \setminus \{l\}, c, p_1\}$.

We assume that $I_0 = \{1, \ldots, n-2\}$. It follows from Lemma 2 that there exists a number $\rho > 0$ such that for all $l \in I$ the vectors $\{w_i^0, i \in I \setminus \{l\}, p_1\}$ form a positive basis $\mathbb{R}^k$, where

$$w_i^0 = \begin{cases} z_{i1}^0, & \text{if} \quad i \in I_0, \\ z_{n-12}^0 + \rho c, & \text{if} \quad i = n-1, \\ z_{n2}^0 + \rho c, & \text{if} \quad i = n. \end{cases}$$

Hence, by virtue of Theorem 1, we find that for all $l \in I$

$$0 \in \operatorname{Intco}\{w_i^0, \ i \in I \setminus \{l\}, p_1\}.$$

It follows from Lemmas 2 and 3 that the number

$$T_0 = \min\Big\{t \,\Big|\, t > t_0, t \in \mathbb{T}, \inf_{v(\cdot)} \max_{\Lambda \in \Omega^0(I)} \min_{j \in \Lambda} \int_{t_0}^{t} \lambda(w_j^0, v(s))\Delta s \geqslant 1\Big\}$$

is finite. Let $v(\cdot)$ be an admissible control of the evaders. Define the functions

$$h_i(t) = 1 - \int_{t_0}^{t} \lambda(w_j^0, v(s))\Delta s.$$

Let pursuer P_i construct a control as follows. If the inequality $h_i(t) \geqslant 0$ is satisfied at time $t \in \mathbb{T}$, then we assume

$$u_i(t) = v(t) - \lambda(w_i^0, v(t))w_i^0.$$

If $\tau \in \mathbb{T}$ is the first time instant for which $h_i(\tau) = 0$, then we assume that $\lambda(w_i^0, v(t)) = 0$ for all $t \geqslant \tau$.

Let $\tau \in \mathbb{T}$ be the first time instant for which $h_i(\tau) < 0$, and let the inequality $h_i(t) > 0$ be satisfied for all $t \in \mathbb{T}$, $t < \tau$. Define the number

$$\tau_i^* = \sup\{t \in \mathbb{T} \mid h_i(t) > 0\}.$$

Then $(\tau_i^*, \tau) \cap \mathbb{T} = \emptyset$. Indeed, if there existed a time instant $t \in (\tau_i^*, \tau) \cap \mathbb{T}$, then the inequality $h_i(t) > 0$ would be satisfied, which is impossible by virtue of the definition of the number τ_i^*. In this case, we assume

$$u_i(\tau) = v(\tau) - \lambda^*(w_i^0, v(\tau))w_i^0, \quad \text{where} \quad \lambda^*(w_i^0, v(\tau)) = \frac{h_i(\tau_i^*)}{\sigma(\tau_i^*) - \tau_i^*} = \frac{h_i(\tau_i^*)}{\tau - \tau_i^*}.$$

We note that in this case $\lambda^*(w_i^0, v(\tau)) \leqslant \lambda(w_i^0, v(\tau))$ and therefore $u_i(\tau) \in V$. Then

$$1 - \int_{t_0}^{\tau_i^*} \lambda(w_i^0, v(s))\Delta s - \int_{\tau_i^*}^{\tau} \lambda^*(w_i^0, v(s))\Delta s = h_i(\tau_i^*) - \int_{\tau_i^*}^{\tau} \frac{h_i(\tau_i^*)}{\tau - \tau_i^*}\Delta s = 0.$$

Then from the definition of the controls of the pursuers and the system (3.3) it follows that for all $t \in [t_0, \hat{T}] \cap \mathbb{T}$ the following equalities hold:

$$z_{i1}(t) = z_{i1}^0 h_i(t), \quad i \in I_0,$$
$$z_{n-12}(t) = z_{n-12}^0 h_{n-1}(t) - \rho c(1 - h_{n-1}(t)),$$
$$z_{n2}(t) = z_{n2}^0 h_n(t) - \rho c(1 - h_n(t)).$$

From Lemma 3 and the definition of the controls of the pursuers it follows that there exist numbers $l, m \in I$ such that

$$h_l(T_0) = 0, \quad h_m(T_0) = 0. \tag{4.7}$$

Also, the following cases are possible.

1. $l, m \in I_0$. In this case, pursuers P_l and P_m perform a capture of evader E_1, which implies that a two-fold capture occurs in the game $\Gamma(n, 2)$.

2. Condition (4.7) is satisfied for $\Lambda = \{n-1, n\}$. Then

$$z_{n-12}(T_0) = -\rho c, \quad z_{n2}(T_0) = -\rho c. \tag{4.8}$$

We prove that in this case the following inclusion holds for any $l \in I_0$:

$$0 \in \operatorname{Intco}\{z_{i1}(T_0), z_{i2}(T_0), i \in I_0 \setminus \{l\}, p_1\}. \tag{4.9}$$

Let $l \in I_0$. We have

$$z_{i1}(T_0) = z_{i1}^0 h_i(T_0), \quad z_{i2}(T_0) = z_{i1}(T_0) + c = z_{i1}(T_0)h_i(T_0) + z_{i2}^0 - z_{i1}^0.$$

Therefore,

$$z_{i1}^0 = \frac{z_{i1}(T_0)}{h_i(T_0)}, \quad z_{i2}^0 = z_{i2}(T_0) + \frac{z_{i1}(T_0)(1 - h_i(T_0))}{h_i(T_0)}.$$

Since the set $\{z_{i1}^0, z_{i2}^0, i \in I_0 \setminus \{l\}, p_1\}$ forms a positive basis $\mathbb{R}^k$, the positive basis $\mathbb{R}^k$ is formed by the vectors

$$\left\{ \frac{z_{i1}(T_0)}{h_i(T_0)},\ z_{i2}(T_0) + \frac{z_{i1}(T_0)(1 - h_i(T_0))}{h_i(T_0)}, i \in I_0 \setminus \{l\}, p_1 \right\}.$$

From the condition $h_i(T_0) \in (0, 1]$, for all $i \in I_0$ we find that the positive basis $\mathbb{R}^k$ forms a set

$$\{z_{i1}(T_0), z_{i2}(T_0), i \in I_0 \setminus \{l\}, p_1\}.$$

By virtue of Theorem 1, the last relation implies the validity of (4.9).

From equations (4.8) we obtain

$$z_{n-12}(T_0) = -\rho(y_1(T_0) - y_2(T_0)), \quad z_{n2}(T_0) = -\rho(y_1(T_0) - y_2(T_0)).$$

By virtue of Lemma 1, we find that

$$0 \in \operatorname{Intco}\{z_{i2}(T_0), i \in I_0 \setminus \{l\}, p_1\}.$$

Taking T_0 to be the initial time and using Theorem 2, we find that there are pursuers P_r and P_q, $r \neq q$, that perform a capture of evader E_2. This proves the theorem. □

Example 1. Let $k = 2$, $x_1^0 = (3; 1)$, $x_2^0 = (1; -2)$, $x_3^0 = (5; -2)$, $x_4^0 = (1; 3)$, $x_5^0 = (2; -3)$, $y_1^0 = (0; 0)$, $y_2^0 = (6; 0)$, $p_1 = (0; 1)$, $\mu_1 = 100$.

Then the condition for capture from Theorem 2 is not satisfied, and the condition for capture from Theorem 3 is satisfied for $I_0 = \{1, 2, 3\}$.

Theorem 4. *Let $D = \mathbb{R}^k$ and suppose that there exists $j \in \{1, 2\}$ such that for any $q \in I$*

$$0 \in \operatorname{Intco}\{z_{ij}^0, i \in I \setminus \{q\}\}.$$

Then a two-fold capture occurs in the game $\Gamma(n, 2)$.

This theorem is proved along the same lines as Theorem 2 using the results of [16].

Theorem 5. *Let $D = \mathbb{R}^k$ and suppose that there exists a set $I_0 \subset I$, $|I_0| = n - 2$ such that for all $l \in I_0$*

$$0 \in \operatorname{Intco}\{z_{i1}^0, z_{i2}^0, i \in I_0 \setminus \{l\}\}.$$

Then a two-fold capture occurs in the game $\Gamma(n, 2)$.

This theorem is proved along the same lines as Theorem 3 using the results of [16].

Theorem 6. *Let $r > 1$ and suppose that there exist $p \in \mathbb{R}^k$, $\mu \in \mathbb{R}^1$, $I_0 \subset I$, $|I_0| = n - 2$ such that $D \subset \{x \in \mathbb{R}^k \mid (p, x) \leqslant \mu\}$ and*

$$0 \in \operatorname{Intco}\{z_{i1}^0, z_{i2}^0, i \in I_0 \setminus \{l\}, p\}.$$

Then a two-fold capture occurs in the game $\Gamma(n, 2)$.

The validity of this theorem immediately follows from Theorem 3.

5. Conclusion

In the problem of a simple pursuit by a group of pursuers of two coordinated evaders on a given time scale, we obtained sufficient conditions for a two-fold capture, provided that the evaders didn't move out of a convex polyhedral set. To solve the problem, we used the method of resolving functions. The results obtained can be used in the study of new problems of conflict interaction between groups of pursuers and evaders on time scales.

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