POLYNOMIALS LEAST DEVIATING FROM ZERO IN $L^p(-1;1)$, $0 \le p \le \infty$, WITH A CONSTRAINT ON THE LOCATION OF THEIR ROOTS[1]

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We study Chebyshev's problem on polynomials that deviate least from zero with respect to L^p-means on the interval $[-1;1]$ with a constraint on the location of roots of polynomials. More precisely, we consider the problem on the set $\mathcal{P}_n(D_R)$ of polynomials of degree n that have unit leading coefficient and do not vanish in an open disk of radius $R \ge 1$. An exact solution is obtained for the geometric mean (for $p = 0$) for all $R \ge 1$; and for $0 < p < \infty$ for all $R \ge 1$ in the case of polynomials of even degree. For $0 < p < \infty$ and $R \ge 1$, we obtain two-sided estimates of the value of the least deviation.

Keywords: Algebraic polynomials, Chebyshev polynomials, Constraints on the roots of a polynomial.

1. Statement and discussion of the problem

Let

$$D_R := \{z \in \mathbb{C} : |z| < R\}$$

be an open disk with center at zero and radius $R > 0$. For $R = 1$, denote by D the unit open disk. Let I be the interval $[-1;1]$.

Denote by $\mathcal{P}_n$ the set of algebraic polynomials of (exact) degree n with complex coefficients and leading coefficient equal to one. A polynomial p_n from $\mathcal{P}_n$ is uniquely defined by its roots z_k, $k = \overline{1,n}$, by the equality

$$p_n(z) = \prod_{k=1}^{n}(z - z_k).$$

Denote by $\mathcal{P}_n(D_R)$ the set of algebraic polynomials from $\mathcal{P}_n$ that do not vanish in an open disk of radius $R > 0$:

$$\mathcal{P}_n(D_R) := \{p_n \in \mathcal{P}_n : p_n(z) \ne 0, \ |z| < R\}.$$

We use the following notation:

$$\|p_n\|_\infty = \|p_n\|_{C(I)} := \max\{|p_n(x)| : x \in [-1;1]\};$$

$$\|p_n\|_p := \left(\frac{1}{2}\int_{-1}^{1}|p_n(x)|^p \mathrm{d}x\right)^{1/p}, \quad 0 < p < \infty;$$

$$\|p_n\|_0 := \exp\left(\frac{1}{2}\int_{-1}^{1}\ln|p_n(x)|\mathrm{d}x\right).$$

For $p \geq 1$, this functional is a norm. It is known (see, e.g., [13]) that

$$\|p_n\|_0 = \lim_{p \to 0} \|p_n\|_p, \quad p_n \in \mathcal{P}_n.$$

In this paper, we study polynomials that deviate least from zero with respect to L^p-means on the interval $[-1; 1]$ among all polynomials from the set $\mathcal{P}_n(D_R)$.

Define the *value of the least deviation from zero of polynomials from $\mathcal{P}_n(D_R)$ with respect to L^p-means on the interval $[-1; 1]$* by the equality

$$\tau_n(I, D_R)_p := \min \left\{ \|p_n\|_p \colon p_n \in \mathcal{P}_n(D_R) \right\}. \tag{1.1}$$

The problem is to find quantity (1.1) and polynomials from $\mathcal{P}_n(D_R)$ least deviating from zero on the interval $[-1; 1]$, that is, polynomials for which the minimum in (1.1) is attained. It will follow from the further reasoning that the minimum in (1.1) is attained.

The problem on polynomials that deviate least from zero is one of the important problems of approximation theory. In the uniform norm, the problem without constraints on the location of roots was posed and solved by Chebyshev in 1854 [5]. The Chebyshev polynomial of the first kind with unit leading coefficient is extremal in this problem. The polynomial that deviates least from zero in the space $L^1(-1; 1)$ was found by E.I. Zolotarev and A.N. Korkin, Chebyshev's disciples, in 1873 (see, for example, [1]). The Chebyshev polynomial of the second kind with unit leading coefficient is extremal. The Legendre polynomials are extremal in the space $L^2(-1; 1)$ (see, for example, [20]). Polynomials that deviate least from zero in the space $L^0(-1; 1)$ were obtained by Glazyrina in 2005 [8]. Although an explicit form of the polynomials least deviating from zero in spaces $L^p(-1; 1)$ for $p \neq 0, 1, 2, \infty$ is unknown, some of their general properties, which can be found in [14, Sects. 2.3–2.4], are useful in studying many important problems of approximation theory.

Note that (as will be seen below), unlike polynomials that deviate least from zero on the interval $[-1; 1]$, an extremal polynomial in (1.1) is, generally speaking, not unique.

Studying extremal properties of algebraic polynomials with restrictions on the location of their roots began apparently in 1939 with paper [21] by Turán devoted to inequalities that give a lower estimate for the norm of the derivative of a polynomial in terms of the norm of the polynomial itself. A detailed history of studies of such inequalities can be found in [9, 10].

In 1947, Lax [15] proved the conjecture of P. Erdős. The statement is that, in the classical Bernstein inequality

$$\|p'_n\|_{C(\overline{D})} \leq n \, \|p_n\|_{C(\overline{D})}, \quad p_n \in \mathcal{P}_n,$$

considered on the set $\mathcal{P}_n(D)$ of polynomials that do not vanish in the unit disk, the exact (smallest) constant is half as large (is equal to $n/2$); i.e., the following inequality holds:

$$\|p'_n\|_{C(\overline{D})} \leq \frac{n}{2} \, \|p_n\|_{C(\overline{D})}, \quad p_n \in \mathcal{P}_n(D).$$

The inequality turns into an equality on an arbitrary polynomial having all its roots on the unit circle.

Akopyan [3, Theorem 2] found polynomials in $\mathcal{P}_n(D_R)$, $R > 0$, that deviate least from zero on the unit circle with respect to L^p-norms, $0 \leq p \leq \infty$ (L^p-means for $0 \leq p < 1$). These are polynomials of the form $z^n + \varepsilon R^n$, $|\varepsilon| = 1$.

The sharp Bernstein inequality on the set of polynomials $\mathcal{P}_n(D)$ with respect to L^p-norms on the unit circle was obtained by Lax [15] ($p = 2, \infty$), de Bruijn [6] ($1 \leq p < \infty$), and Rahman and Schmeisser [18] ($0 \leq p < 1$). Arestov obtained [4] a generalization of the Bernstein inequality on the set of polynomials $\mathcal{P}_n(D)$ for rather wide class of operators. The sharp Bernstein inequality on the set of polynomials $\mathcal{P}_n(D_R)$ in the case $p = \infty$ and $R > 1$ was obtained by Malik [16]. Several results for $p = 2$ can be found in Akopyan's paper [2].

Denote by $M_{n,m}(D_R)_p$ the exact (the smallest) constant in the Markov brothers inequality for polynomials from the class $\mathcal{P}_n(D_R)$ with respect to L^p-means on the interval $I = [-1, 1]$:

$$\|p_n^{(m)}\|_p \le M_{n,m}(D_R)_p \|p_n\|_p, \quad p_n \in \mathcal{P}_n(D_R), \quad 0 \le p \le \infty, \quad m = 0, 1, \dots, n. \qquad (1.2)$$

It is clear that, for $m = n$, the inequality (1.2) is related to problem (1.1); more precisely, the following equality holds:

$$n! = M_{n,n}(D_R)_p \, \tau_n(I, D_R)_p.$$

For the results related to the Markov brothers inequality for $p = \infty$ with constraints on the location of the roots of polynomials, see [7, 12] and the references therein.

In the author's paper [17], the problem on polynomials that deviate least from zero on a compact set K of the complex plane $\mathbb{C}$ with respect to the uniform norm and with a constraint on the location of roots was studied. In particular, a solution to problem (1.1) was found for $p = \infty$ (see Theorem A below). Moreover, the existence of an extremal polynomial was proved, and the problem was reduced to polynomials with roots on the boundary of the domain which gives the constraints.

Similar statements are valid for the more general case of problem (1.1) for $0 \le p \le \infty$. In the following statement, we prove that an extremal polynomial exists for $0 \le p \le \infty$.

Assertion 1. *For $0 \le p \le \infty$, the minimum in problem (1.1) is attained.*

P r o o f of Assertion 1 is performed by the scheme of the proof of Theorem 1 from [17]. Let $q_{n,k}$, $k \in \mathbb{N}$, be an extremal sequence in (1.1), i.e., $\lim_{k \to \infty} \|q_{n,k}\|_p = \tau_n(I, D_R)_p$. Using the different metrics inequality, we get

$$\|q_{n,k}\|_\infty \le c(n)_p \|q_{n,k}\|_p,$$

where the constant $c(n)_p$ is independent of k. The existence of $c(n)_p$ in the case $p \ge 1$ is a well-known fact (the equivalence of norms in finite-dimensional spaces). In the case $0 \le p < 1$, a finite constant also exists, see [9, Lemma 1] for $0 < p < 1$ and [8] for $p = 0$. Then the sequence $q_{n,k}$ is uniformly bounded on $[-1; 1]$. Hence, using the Lagrange interpolation formula, we get its uniformly boundedness on an arbitrary compact set from $\mathbb{C}$.

By the principle of compactness (condensation) in analytic function theory, there exists a subsequence that uniformly converges inside $\mathbb{C}$. It follows from the convergence of coefficients of polynomials of the subsequence that the limiting analytic function is a polynomial. Taking into account the continuity of roots of polynomials as functions of their coefficients and the closedness of $\overline{\mathbb{C}} \setminus D_R$, we conclude that zeros of the limiting polynomial do not belong to D_R. At the same time, the roots of polynomials of the extremal sequence do not tend to infinity, because we get $\tau_n(I, D_R)_p = \infty$ if even one root tends to infinity. Thus, we conclude that the limiting polynomial belongs to $\mathcal{P}_n(D_R)$. The assertion is proved. $\qquad \square$

The following statement on the reduction of problem (1.1) to a similar problem for polynomials with roots on a circle is a consequence of a more general Theorem 2 from [17] (see Remark 1). In the particular case considered in the present paper, the proof is simplified. We will give it for the completeness.

Assertion 2. *For $0 \le p \le \infty$ and $R \ge 1$, every extremal polynomial in problem (1.1) has all n roots on the circle of radius R centered at the origin.*

P r o o f. Assume that at least one root of a polynomial $p_n \in \mathcal{P}_n(D_R)$ does not lie on the circle of radius R. Denote it by $z_0 = \rho e^{it}$, where $\rho > R$. Then the polynomial p_n can be represented in the form

$$p_n(x) = p_{n-1}(x)(x - z_0), \quad p_{n-1} \in \mathcal{P}_{n-1}(D_R).$$

Consider the polynomial $q_n(x) = p_{n-1}(x)(x - \widetilde{z_0})$, where $\widetilde{z_0} = Re^{it}$. It is clear that $q_n \in \mathcal{P}_n(D_R)$. Since $R < \rho$, we have $|x - \widetilde{z_0}| < |x - z_0|$ for all $x \in [-1; 1]$, and hence the following pointwise inequality holds: $|q_n(x)| < |p_n(x)|$, $x \in [-1; 1]$. Taking into account the monotonicity of L^p-means, we obtain the inequality $\|q_n\|_p < \|p_n\|_p$. Consequently, the polynomial p_n is not a polynomial from $\mathcal{P}_n(D_R)$ that deviates least from zero on the interval $[-1; 1]$ with respect to L^p-means. The assertion is proved. □

The further scheme of presentation in the paper is as follows. In the next two sections, we give a solution to the problem in the two extreme cases $p = \infty$ and $p = 0$. In the last section, we estimate quantity (1.1) from below and above for $0 < p < \infty$. These estimates coincide for polynomials of even degrees, which makes it possible to find an exact value of (1.1) and extremal polynomials.

2. Solution to problem (1.1) in the case $p = \infty$

Let ϱ_n be equal to $1/\sqrt{2}$ if $n = 2m$ and to the unique root of the equation

$$(\varrho^2 - 1)^{2m}(\varrho^2 + 1) = \varrho^{4m+2}$$

in the interval $(1/\sqrt{2}, 1/\sqrt[4]{2})$ if $n = 2m + 1$, $m \geq 1$.

Theorem A. [17, Theorem 3] *The following equality holds:*

$$\tau_n(I, D_R)_\infty = \begin{cases} \sqrt{1 + R^2}, & n = 1, \quad R \geq 0, \\ R^n, & n > 1, \quad R \geq \varrho_n. \end{cases} \tag{2.1}$$

The minimum in (1.1) *is attained on the polynomials*

$$p_n^*(x) = (x^2 - R^2)^m \quad \text{for} \quad n = 2m;$$
$$p_n^*(x) = (x^2 - R^2)^m(x \pm iR) \quad \text{for} \quad n = 2m + 1.$$

The polynomials from $\mathcal{P}_n(D_R)$, given in the theorem, that deviate least from zero on $[-1, 1]$ are not unique. For example, the polynomials

$$p_{2mk}^{**}(x) = (x^{2k} - R^{2k})^m, \quad k, m \in \mathbb{N},$$

are extremal for even n and $R \geq 1/\sqrt[2k]{2}$.

3. Solution to problem (1.1) in the case $p = 0$

In this section, we find an exact solution to problem (1.1) in the case $p = 0$ for $R \geq 1$.

Theorem 1. *The following equality holds for $R \geq 1$:*

$$\tau_n(I, D_R)_0 = \|x + R\|_0^n = \begin{cases} 2^n e^{-n}, & R = 1, \\ e^{-n}\left((R+1)^{(R+1)}/(R-1)^{(R-1)}\right)^{n/2}, & R > 1. \end{cases} \tag{3.1}$$

The polynomials

$$p_n^*(x) = (x - R)^k(x + R)^{n-k}, \quad 0 \leq k \leq n,$$

are unique extremal polynomials.

P r o o f. According to Assertion 2, it suffices to consider polynomials with roots on the circle. First, we study the case of polynomials of the first degree ($n = 1$). Consider the polynomials $p(x) = x - z_0$, where $|z_0| = |x_0 + iy_0| = R$. The following equalities hold:

$$\ln \|p\|_0 = \frac{1}{2} \int_{-1}^{1} \ln |p(x)| \mathrm{d}x = \frac{1}{2} \int_{0}^{1} (\ln |p(x)| + \ln |p(-x)|) \mathrm{d}x = \frac{1}{2} \int_{0}^{1} \ln((x^2 + R^2)^2 - 4x^2 x_0^2) \mathrm{d}x.$$

It is clear that, under the condition $x_0 \in [-R; R]$, the quantity $\|p\|_0$ attains its minimal value only for $x_0 = R$ and $-R$. Thus, in the case $n = 1$, the polynomials $p_1^*(x) = x \pm R$ are extremal. It is not difficult to verify the equalities

$$\|p_1^*\|_0 = e^{-1} \left((R+1)^{(R+1)}/(R-1)^{(R-1)} \right)^{1/2} \quad \text{for} \quad R > 1,$$

$$\|p_1^*\|_0 = 2e^{-1} \quad \text{for} \quad R = 1.$$

Now, let $n > 1$. In view of the multiplicativity of L^0-means, for the polynomial

$$p_n(x) = \prod_{k=1}^{n} (x - z_k),$$

we have

$$\|p_n\|_0 = \exp \left(\frac{1}{2} \int_{-1}^{1} \sum_{k=1}^{n} \ln |x - z_k| \mathrm{d}x \right) = \prod_{k=1}^{n} \exp \left(\frac{1}{2} \int_{-1}^{1} \ln |x - z_k| \mathrm{d}x \right) = \prod_{k=1}^{n} \|x - z_k\|_0.$$

Then the following equality holds for the value of the least deviation:

$$\tau_n(I, D_R)_0 = \prod_{k=1}^{n} \tau_1(I, D_R)_0 = \|x + R\|_0^n.$$

The uniqueness of extremal polynomials of degree n follows from the uniqueness of polynomials for $n = 1$. This proves equality (3.1). $\qquad\square$

4. Studying of problem (1.1) in the case $0 < p < \infty$

In this section, we find estimates of quantity (1.1) from below and above for $0 < p < \infty$. These estimates coincide for polynomials of even degrees; hence, we find an exact value of (1.1).

Lemma 1. *The following inequality holds for arbitrary $0 < p < \infty$, $R \ge 1$, and positive integer n:*

$$\tau_n(I, D_R)_p \ge \left(\int_0^1 (R^2 - x^2)^{np/2} \mathrm{d}x \right)^{1/p}. \tag{4.1}$$

P r o o f. The following chain of relations holds for an arbitrary polynomial $p_n \in \mathcal{P}_n(D_R)$:

$$\|p_n\|_p^p = \frac{1}{2} \int_{-1}^{1} |p_n(x)|^p \mathrm{d}x = \frac{1}{2} \int_{0}^{1} (|p_n(-x)|^p + |p_n(x)|^p) \mathrm{d}x \ge \frac{1}{2} \int_{0}^{1} \psi_n(x) \mathrm{d}x,$$

where

$$\psi_n(x) = \min \left\{ (|p_n(-x)|^p + |p_n(x)|^p) : p_n \in \mathcal{P}_n(D_R) \right\}.$$

Using the inequality of means, we obtain the inequality

$$|p_n(-x)|^p + |p_n(x)|^p \geq 2|p_n(-x)\,p_n(x)|^{p/2}.$$

Consider the absolute value of the product:

$$|p_n(-x)\,p_n(x)| = \Big|\prod_{k=1}^{n}(x^2 - z_k^2)\Big| = |q_n(x^2)|,$$

where $q_n(x) \in \mathcal{P}_n(D_{R^2})$. It follows the inequality

$$\psi_n(x) \geq 2\min\big\{|q_n(x^2)|^{p/2} : q_n \in \mathcal{P}_n(D_{R^2})\big\}.$$

The following equality holds for an arbitrary point $z_0 \in D_{R^2}$:

$$\min\big\{|q_n(z_0)| : q_n \in \mathcal{P}_n(D_{R^2})\big\} = \min\big\{|z_0 - z|^n : |z| = R^2\big\}.$$

Using this equality, we obtain
$$\psi_n(x) \geq 2(R^2 - x^2)^{np/2}.$$

Consequently, the inequality

$$\|p_n\|_p^p \geq \int_0^1 (R^2 - x^2)^{np/2}\mathrm{d}x$$

holds for an arbitrary polynomial $p_n \in \mathcal{P}_n(D_R)$. This implies estimate (4.1). The lemma is proved.
□

Now, we pass to obtaining an upper estimate.

Lemma 2. *The following inequality holds for arbitrary $0 < p < \infty$, $R \geq 1$, and positive integer n:*

$$\tau_n(I, D_R)_p \leq \begin{cases} \left(\dfrac{1}{2}\displaystyle\int_{-1}^{1}(R^2 - x^2)^{mp}\mathrm{d}x\right)^{1/p}, & n = 2m, \\[4mm] \left(\dfrac{1}{2}\displaystyle\int_{-1}^{1}(R^2 - x^2)^{mp}\cdot(R^2 + x^2)^{p/2}\mathrm{d}x\right)^{1/p}, & n = 2m + 1. \end{cases} \tag{4.2}$$

P r o o f. We obtain an upper estimate directly from the definition of the value of the least deviation by means of the polynomials

$$p_n(x) = (R^2 - x^2)^m \quad \text{for} \quad n = 2m,$$
$$p_n(x) = (R^2 - x^2)^m(x + iR) \quad \text{for} \quad n = 2m + 1.$$

□

For polynomials of even degrees, the lower and upper estimates coincide, therefore, we obtain an exact solution to problem (1.1) for all $0 < p < \infty$.

Theorem 2. *The following equality holds for $0 < p < \infty$ and $R \geq 1$ in the case of even $n = 2m$:*

$$\tau_n(I, D_R)_p = \Big(\frac{1}{2}\int_{-1}^{1}(R^2 - x^2)^{mp}\mathrm{d}x\Big)^{1/p}. \tag{4.3}$$

The polynomials $p_{2m}^*(x) = (x^2 - R^2)^m$ are extremal.

In the case $p = 2$, we write out a solution for polynomials of small odd degrees.

Theorem 3. *The following equalities hold for $R \ge 1$:*

$$\tau_1(I, D_R)_2 = \left(R^2 + \frac{1}{3}\right)^{1/2}, \tag{4.4}$$

$$\tau_3(I, D_R)_2 = \left(R^6 - \frac{R^4}{3} - \frac{R^2}{5} + \frac{1}{7}\right)^{1/2}. \tag{4.5}$$

Any polynomial with a root on a circle of radius R, i.e., any polynomial of the form $(x + Re^{it})$, $t \in [0; 2\pi]$, is extremal for $n = 1$. Any polynomial of the form $(x^2 - R^2)(x + Re^{it})$, $t \in [0; 2\pi]$, is extremal for $n = 3$.

P r o o f. According to Assertion 2, we may consider polynomials with roots on the circle. In the case $n = 1$, all polynomials with roots on the circle have the same norm; this implies equality (4.4).

Consider polynomials of the third degree. Let

$$p_3(x) = \prod_{k=1}^{3}(x - z_k),$$

where $|z_k| = |x_k + iy_k| = R$, $k = 1, 2, 3$. Calculating the norm of p_3, we obtain the relation

$$\tau_3(I, D_R)_2^2 = \min_{x_k \in [-R;R]} \left(R^6 + R^4 + \frac{3R^2}{5} + \frac{1}{7} + \left(\frac{4}{5} + \frac{4R^2}{3}\right)(x_1x_2 + x_2x_3 + x_1x_3)\right).$$

Minimizing the function $\sigma(x_1, x_2, x_3) = x_1x_2 + x_2x_3 + x_1x_3$ in $x_k \in [-R; R]$, we get equality (4.5). The theorem is proved. □

In conclusion, let us give explicit values of the quantity $\tau_{2m}(I, D_R)_p$ for $0 < p < \infty$.

(1) The following equality holds for $R = 1$ and $0 < p < \infty$:

$$\tau_{2m}(I, D_1)_p = \left(\frac{1}{2}\int_{-1}^{1}(1 - x^2)^{mp}dx\right)^{1/p} = \left(\frac{\sqrt{\pi}}{2}\Gamma\left(\frac{np + 3}{2}\right)\right)^{1/p}.$$

(2) The relation

$$\tau_{2m}(I, D_R)_p = \left(\frac{1}{2}R^{2mp}\int_{-1}^{1}\left(1 - \frac{x^2}{R^2}\right)^{mp}dx\right)^{1/p} = \left(R^{2mp} + \sum_{k=1}^{\infty}\frac{(-1)^k C_{mp}^k R^{2(mp-k)}}{2k + 1}\right)^{1/p},$$

where

$$C_{mp}^k = \prod_{l=1}^{k}\frac{mp - l + 1}{l},$$

holds for arbitrary $0 < p < \infty$ and $R > 1$.

Acknowledgements

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