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SPECTRAL EXPANSION FOR SINGULAR BETA STURM–LIOUVILLE PROBLEMS

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Abstract: In this study, beta Sturm–Liouville problems are discussed. For such equations, the spectral function is established in the singular case. A spectral expansion is given with the help of this function.

Keywords: Sturm–Liouville theory, Fractional derivatives and integrals, Spectral expansion.

1. Introduction

Fractional derivatives are mathematical operations that describe derivatives with non-integer degrees, extending the traditional concept of derivatives with integer degrees. These derivatives are part of a branch of mathematics often referred to as "fractional analysis" or "fractional calculus". The application areas of fractional derivatives are quite wide. For example, mathematical models expressed with fractional derivatives are used in fields such as electromagnetism, diffusion processes, and semiconductor physics. In addition, the concepts of fractional derivatives can be applied during the analysis of some fractal structures or complex systems.

In 2014, Khalil et al. defined conformable fractional derivatives and integrals by using classical derivative methods [9]. Later, Atangana et al. defined the beta fractional derivative and created a model of the famous river blindness disease based on Caputo and beta derivatives [3]. Martinez et al. have created analytical solutions of the space-time generalized nonlinear Schrödinger equation,

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including the beta derivative, using the sub-equation method [13]. Beta derivative has a particular applicability, particularly within the fields of biology and medicine [4, 5].

The spectral expansion of differential equations is a method of converting a given mathematical expression into a simpler and more easily solvable form. This method is particularly useful for solving difficult or complex differential equations. Expansion makes it possible to obtain analytical solutions or to obtain more effective solutions by numerical methods. The expansion of differential equations is important for numerical solutions as well as for obtaining analytical solutions. To solve a differential equation with numerical methods, it may often be possible to take an expanded equation in a simpler way and then solve this simplified equation numerically. The expansion of differential equations has many applications in mathematics, engineering, physics, and other branches of science. It is an indispensable tool, especially for obtaining analytical or numerical solutions to complex and real-world problems. In [1], the authors proved the existence of the spectral function for the singular conformable Sturm–Liouville problem.

The congruent fractional Sturm–Liouville problem is an extended version of the Sturm–Liouville theory and deals with differential equations involving fractional derivatives. While traditional Sturm–Liouville theory determines eigenvalues and eigenfunctions by examining quadratic linear differential equations, the fractional Sturm–Liouville problem includes fractional derivatives in equations. Fractional Sturm–Liouville problems often require eigenvalues and eigenfunctions to be obtained by analytical or semi-analytical expressions. Solving such equations may require spectral analysis methods that are often used for eigenvalue problems. Some researchers examine the solution of differential equations with fractional derivatives by dealing with eigenvalue problems such as fractional Sturm–Liouville problems and standard Sturm–Liouville problems [2, 6–8, 11, 14].

The computation and properties of fractional derivatives are generally more complex compared to integer-order derivatives. They expand the properties of traditional derivatives, and certain rules such as the fractional chain rule apply. Fractional derivatives can yield meaningful and valuable results for specific classes of functions. When calculating fractional derivatives, a method closely related to the integral operation is used. Special fractional derivative operators are used to calculate the fractional derivative of the function. These operators have some special properties and rules.

In this paper, singular beta Sturm–Liouville equations defined as

$$-T_\beta^2 y + v(\zeta)y = \mu y, \quad \zeta \in (0,\infty), \tag{1.1}$$

where μ is a complex eigenvalue parameter, $v(.)$ is a real-valued function defined on $[0,\infty)$, and $v \in L^1_{\beta,loc}(0,\infty)$, were considered. Using Levitan's method [11], the spectral function was established for such equations. A spectral expansion theorem was proved with the help of this function.

2. Preliminaries

Definition 1 [3, 13]. *Let $0 < \beta \le 1$ and $\sigma : [0,\infty) \to \mathbb{R} := (-\infty,\infty)$ be a function. The beta derivative of σ is defined by*

$$T_\beta \sigma(\zeta) = \frac{d^\beta \sigma(\zeta)}{dt^\beta} := \lim_{\varepsilon\to 0} \frac{\sigma(\zeta + \varepsilon(\zeta + 1/\Gamma(\beta))^{1-\beta}) - \sigma(\zeta)}{\varepsilon}.$$

As is known, fractional derivatives do not have the basic properties of the classical derivative (such as the derivative of the product, the derivative of the division). However, the beta derivative has the basic properties of the ordinary derivative and is therefore an extension of the conformable derivative.

Theorem 1 [13]. *Let σ, ω be beta differentiable functions for $\zeta > 0$ and $(0 < \beta \le 1)$. The following relations hold*:

(i)
$$T_\beta(\lambda\sigma+\delta\omega)=\lambda T_\beta\sigma+\delta T_\beta\omega,\quad \text{for all } \mu,\delta\in\mathbb{R},$$
(ii)
$$T_\beta(\sigma\omega)=\sigma T_\beta(\omega)+\omega T_\beta(\sigma),$$
(iii)
$$T_\beta(\frac{\sigma}{\omega})=\frac{\omega T_\beta(\sigma)-\sigma T_\beta(\omega)}{\omega^2},$$
(iv)
$$T_\beta(\sigma(\zeta))=\left(\zeta+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\frac{df(\zeta)}{dt},$$
(v)
$$T_\beta(\zeta^n)=(\zeta+\frac{1}{\Gamma(\beta)})^{1-\beta}n\zeta^{n-1},\quad \mathbb{N}:=\{1,2,3,...\}.$$

P r o o f. The proof is clear, so we omit it. □

Definition 2. *Let $\sigma:[a,\infty)\to\mathbb{R}$, be a given function, then the beta-integral of σ is:*
$$_aI^\beta(\sigma(\zeta))=\int_a^\zeta\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}\sigma(t)dt,$$
where $0<\beta\leq 1$ and
$$(^bT_\beta\sigma)(\zeta)=\lim_{\zeta\to b^-}(^bT_\beta\sigma)(\zeta).$$

Theorem 2. *Let σ,ω be beta-differentiable functions. Then, the following relation holds*
$$\int_a^b\sigma(\zeta)T_\beta(\omega)(\zeta)d_\beta\zeta=\sigma(\zeta)\omega(\zeta)\big|_a^b-\int_a^b\omega(\zeta)T_\beta(\sigma)(\zeta)d_\beta\zeta.$$

P r o o f. By Theorem 1 the proof is clear. □

Let
$$L^2_\beta(0,b):=\left\{\sigma:\Big(\int_0^b|\sigma(\zeta)|^2d_\beta(\zeta)\Big)^{1/2}<\infty\right\}.$$
Then $L^2_\beta(0,b)$ is a Hilbert space endowed with the inner product
$$\langle\sigma,\omega\rangle:=\int_0^b\sigma\left(\zeta\right)\overline{\omega\left(\zeta\right)}d_\beta\zeta,\quad \sigma,\omega\in L^2_\beta(0,b).$$
The β-*Wronskian* of σ and ω is defined by
$$W_\beta(\sigma,\omega)(\zeta)=p(\zeta)\big[\sigma(\zeta)T_\beta\omega(\zeta)-\sigma(\zeta)T_\beta\omega(\zeta)\big],\quad \zeta\in[0,b].$$

Theorem 3. *Let A be an operator defined as $A\left\{t_i\right\}=\left\{y_i\right\}$, where*
$$y_i=\sum_{k=1}^\infty a_{ik}t_k\quad \textit{and}\quad i\in\mathbb{N}.$$
If
$$\sum_{i,k=1}^\infty|a_{ik}|^2<+\infty \tag{2.1}$$
then A is a compact operator in l^2 [12].

3. Regular beta Sturm–Liouville problem

Consider the following regular problem

$$-T_\beta^2 y(\zeta) + v(\zeta)y(\zeta) = \mu y(\zeta), \quad 0< \zeta < b < \infty, \tag{3.1}$$

$$y(0,\mu)\cos\theta + T_\beta y(0,\mu)\sin\theta = 0, \tag{3.2}$$

$$y(b,\mu)\cos\gamma + T_\beta y(b,\mu)\sin\gamma = 0, \quad \gamma,\theta \in \mathbb{R}, \tag{3.3}$$

where $v(\cdot)$ is a real-valued function defined on $[0,\infty)$, μ is a complex eigenvalue parameter, and $v \in L^1_{\beta,loc}(0,\infty)$, where

$$L^1_{\beta,loc}(0,\infty) := \left\{\sigma : [0,\infty) \to \mathbb{C} : \int_0^b |\sigma(\zeta) e| \, d_\beta(\zeta) < \infty, \ \forall b \in (0,\infty)\right\}.$$

We denote by $\phi(\zeta,\mu)$ and $\psi(\zeta,\mu)$ two solutions of (3.1) satisfying

$$\phi(0,\mu) = \sin\theta, \quad T_\beta\phi(0,\mu) = -\cos\theta, \tag{3.4}$$

$$\psi(b,\mu) = \sin\gamma, \quad T_\beta\psi(b,\mu) = -\cos\gamma. \tag{3.5}$$

Then *Green's function of* (3.1)–(3.3) is defined as

$$G(\zeta,t,\mu) = \frac{1}{W(\phi,\psi)}\begin{cases} \psi(\zeta,\mu)\phi(t,\mu), & 0 \le t < \zeta, \\ \phi(\zeta,\mu)\psi(t,\mu), & \zeta < t < b. \end{cases} \tag{3.6}$$

Without loss of generality we can assume that $\mu = 0$ is not an eigenvalue of (3.1)–(3.3). From (3.6), we find

$$G(\zeta,t) = G(\zeta,t,0) = \frac{1}{W(\phi,\psi)}\begin{cases} \psi(\zeta)\phi(t), & 0 \le t < \zeta, \\ \phi(\zeta)\psi(t), & \zeta < t < b. \end{cases}$$

Theorem 4. *$G(\zeta,t)$ is a beta Hilbert–Schmidt kernel, i.e.,*

$$\int_0^b \int_0^b |G(\zeta,t)|^2 d_\beta(\zeta) d_\beta(t) < +\infty.$$

P r o o f. From (3.6), we infer that

$$\int_0^b d_\beta(\zeta) \int_0^\zeta |G(\zeta,t)|^2 d_\beta(t) < +\infty,$$

and

$$\int_0^b d_\beta(\zeta) \int_\zeta^b |G(\zeta,t)|^2 d_\beta(t) < +\infty$$

since $\phi,\psi \in L^2_\beta(0,b)$. Hence we obtain

$$\int_0^b \int_0^b |G(\zeta,t)|^2 d_\beta(\zeta) d_\beta(t) < +\infty. \tag{3.7}$$

□

Theorem 5. *The operator $\digamma$ defined as*

$$(\digamma\sigma)(\zeta) = \int_0^b G(\zeta,t)\sigma(t)d_\beta(t)$$

is compact and self-adjoint on $L^2_\beta(0,b)$.

P r o o f. Let $\varphi_i = \varphi_i(t)$ ($i \in \mathbb{N}$) be an orthonormal basis of $L^2_\beta(0,b)$. Define

$$t_i = (\sigma,\varphi_i) = \int_0^b \sigma(t)\overline{\varphi_i(t)}d_\beta(t),$$

$$y_i = (\omega,\varphi_i) = \int_0^b \omega(t)\overline{\varphi_i(t)}d_\beta(t),$$

$$a_{ik} = \int_0^b\int_0^b G(\zeta,t)\varphi_i(\zeta)\overline{\varphi_k(t)}d_\beta(\zeta)d_\beta(t) \quad (i,k \in \mathbb{N}).$$

Then, $L^2_\beta(0,b)$ is mapped isometrically onto l^2. By this mapping, $\digamma$ transforms into A on l^2, (3.7) is translated into (2.1). By Theorems 3 and 4, the operator A is compact. Therefore the operator $\digamma$ is compact.

Let $\sigma,\omega \in L^2_\beta(0,b)$. Then we see that

$$(\digamma\sigma,\omega) = \int_0^b (\digamma\sigma)(\zeta)\overline{\omega(\zeta)}d_\beta(\zeta) = \int_0^b\int_0^b G(\zeta,t)\sigma(t)\overline{\omega(\zeta)}d_\beta(\zeta)d_\beta(t)$$

$$= \int_0^b \sigma(\zeta)\overline{\left(\int_0^b G(t,\zeta)\omega(t)d_\beta(t)\right)}d_\beta(\zeta) = \int_0^b \sigma(\zeta)\overline{\left(\int_0^b G(\zeta,t)\omega(t)d_\beta(t)\right)}d_\beta(\zeta) = (\sigma,\digamma\omega),$$

due to $G(\zeta,t) = G(t,\zeta)$. □

4. Eigenfunction expansion

Let $\mu_{m,b}$ ($m \in \mathbb{N}$) denote the eigenvalues of (3.1)–(3.3) and $\phi_{m,b}(\zeta) = \phi(\zeta,\mu_{m,b})$ are the corresponding eigenfunctions. By virtue of Theorem 5 and the Hilbert–Schmidt theorem [10], we infer that

$$\int_0^b |\sigma(\zeta)|^2 d_\beta(\zeta) = \sum_{m=1}^{\infty}\frac{1}{\gamma^2_{m,b}}\int_0^b |\sigma(\zeta)\phi_{m,b}(\zeta)|^2 d_\beta(\zeta),$$

where $\sigma(.) \in L^2_\beta(0,b)$ and

$$\gamma^2_{m,b} = \int_0^b \phi^2_{m,b}(\zeta)d_\beta(\zeta).$$

Set

$$\rho_b(\mu) = \begin{cases} -\sum\limits_{\mu<\mu_{m,b}<0}\dfrac{1}{\gamma^2_{m,b}}, & \text{for } \mu \le 0,\\ \sum\limits_{\mu<\mu_{m,b}<0}\dfrac{1}{\gamma^2_{m,b}}, & \text{for } \mu \ge 0.\end{cases}$$

Then we obtain

$$\int_0^b |\sigma(\zeta)|^2 d_\beta(\zeta) = \int_{-\infty}^{\infty} |\Upsilon(\mu)|^2 d_{\rho_b}(\mu), \tag{4.1}$$

which is called the *Parseval equality*, where

$$\Upsilon(\mu) = \int_0^b \sigma(\zeta)\phi(\zeta,\mu)d_\beta(\zeta).$$

Lemma 1. *For any $\tau > 0$, there exists a positive constant $P = P(s)$ not depending on b such that*

$$\bigvee_{-R}^{R} \{\rho_b(\mu)\} = \sum_{-R \le \mu_{m,b} < R} \frac{1}{\gamma_{m,b}^2} = \rho_b(R) - \rho_b(-R) < P, \tag{4.2}$$

where $\bigvee$ *denotes the total variation.*

P r o o f. Let $\sin\theta \neq 0$. By (3.4), there exists a positive number k nearby 0 such that

$$\Big(\frac{1}{k}\int_0^k \phi(\zeta,\mu) d_\beta \zeta\Big)^2 > \frac{1}{2}\sin^2\theta \tag{4.3}$$

due to $\phi(\zeta,\mu)$ is continuous at 0. Let us define $\sigma_k(t)$ by

$$\sigma_k(\zeta) = \begin{cases} \dfrac{1}{k}, & 0 \le \zeta \le k, \\ 0, & \zeta > k. \end{cases}$$

Combining (4.1), (4.2) and (4.3), we conclude that

$$\int_0^k \sigma_k^2(\zeta) d_\beta\zeta = \frac{1}{k^2\beta}(k + \frac{1}{\Gamma(\beta)})^\beta = \int_{-\infty}^{\infty} \Big(\frac{1}{k}\int_0^k \phi(\zeta,\mu) d_\alpha\zeta\Big)^2 d_{\rho_b}(\mu)$$
$$\ge \int_{-R}^{R} \Big(\frac{1}{k}\int_0^k \phi(\zeta,\mu)\, d_\beta\zeta\Big)^2 d_{\rho_b}(\mu) > \frac{1}{2}\sin^2\theta \{\rho_b(R) - \rho_b(-R)\}.$$

If $\sin\theta = 0$, then $\sigma_k(\zeta)$ is defined by

$$\sigma_k(t) = \begin{cases} \Big(\dfrac{1}{k}\Big)^2, & 0 \le \zeta \le k, \\ 0, & \zeta > k. \end{cases}$$

The proof of the lemma follows from Parseval's equality. □

Let ρ be any nondecreasing function on $-\infty < \mu < \infty$. We will denote by $L^2_\rho(\mathbb{R})$ the Hilbert space of all functions $\sigma : \mathbb{R} \to \mathbb{R}$ measurable with respect to the Lebesgue–Stieltjes measure defined by ρ, with the condition

$$\int_{-\infty}^{\infty} \sigma^2(\mu)\, d_\rho(\mu) < \infty$$

and with the inner product

$$(\sigma,\omega)_\rho := \int_{-\infty}^{\infty} \sigma(\mu)\,\omega(\mu)\, d_\rho(\mu).$$

Theorem 6. *For Problem (3.1)–(3.2), there exists a nondecreasing function $\rho(\mu)$ $(-\infty < \mu < \infty)$ with the following properties.*

(i) *If σ is a real-valued function and $\sigma \in L^2_\beta(0,\infty)$, then there exists a function $\Upsilon \in L^2_\rho(\mathbb{R})$ satisfying*

$$\lim_{b\to\infty} \int_{-\infty}^{\infty} \Big\{\Upsilon(\mu) - \int_0^b \sigma(\zeta)\phi(\zeta,\mu) d_\beta(\zeta)\Big\} d_\rho(\mu) = 0, \tag{4.4}$$

and the Parseval equality

$$\int_0^\infty \sigma^2(\zeta) d_\beta(\zeta) = \int_{-\infty}^{\infty} \Upsilon^2(\mu) d_\rho(\mu) \tag{4.5}$$

holds.

(ii) *The integral*

$$\int_{-\infty}^{\infty} \Upsilon(\mu)\phi(\zeta,\mu)d_{\rho}(\mu),$$

converges to σ *in* $L_{\beta}^{2}(0,\infty)$. *That is,*

$$\lim_{n\to\infty}\int_{0}^{\infty}\left\{\sigma(\zeta)-\int_{-n}^{n}\Upsilon(\mu)\phi(\zeta,\mu)d_{\rho}(\mu)\right\}^{2}d_{\beta}(\zeta)=0.$$

P r o o f. (i) Suppose that:

1) The real-valued function $\sigma_{\xi}(.)$ vanishes outside the interval $[0,\xi]$, where $\xi<b$.

2) $\sigma_{\xi}(\zeta)$ and $T_{\beta}\sigma_{\xi}(\zeta)$ are continuous.

3) $\sigma_{\xi}(\zeta)$ satisfies (3.2).

By (4.1), we deduce that

$$\int_{0}^{\xi}\sigma_{\xi}^{2}(\zeta)d_{\rho}(\zeta)=\int_{-\infty}^{\infty}\Upsilon_{\xi}^{2}(\mu)d_{\rho}(\mu), \tag{4.6}$$

where

$$\Upsilon_{\xi}(\mu)=\int_{0}^{\xi}\sigma_{\xi}(\zeta)\phi(\zeta,\mu)d_{\beta}(\zeta). \tag{4.7}$$

Since $\phi(t,\mu)$ satisfies the equation (3.1), we see that

$$\phi(\zeta,\mu)=\frac{1}{\mu}\left[-T_{\beta}^{2}\phi(\zeta,\mu)+v(\zeta)\phi(\zeta,\mu)\right].$$

By (4.7), we get

$$\Upsilon_{\xi}(\mu)=\frac{1}{\mu}\int_{0}^{\xi}\sigma_{\xi}(\zeta)\left[-T_{\beta}^{2}\phi(\zeta,\mu)+v(\zeta)\phi(\zeta,\mu)\right]d_{\beta}(\zeta).$$

Since $\sigma_{\xi}(\zeta)$ and $\phi(\zeta,\mu)$ satisfy the boundary condition (3.4) and $\sigma_{\xi}(\zeta)$ vanishes in a neighborhood of the point ξ, we get

$$\Upsilon_{\xi}(\mu)=\frac{1}{\mu}\int_{0}^{b}\phi(\zeta,\mu)\left[-T_{\beta}^{2}\sigma_{\xi}(\zeta)+v(\zeta)\sigma_{\xi}(\zeta)\right]d_{\beta}(\zeta),$$

via the integration by parts.

For any finite $R>0$, by using (4.1), we get

$$\begin{aligned}\int_{|\mu|>R}\Upsilon_{\xi}^{2}(\mu)d_{\rho_{b}}(\mu) &\leq \frac{1}{R^{2}}\int_{|\mu|>R}\left\{\int_{0}^{b}\left[\phi(\zeta,\mu)\left[-T_{\beta}^{2}\sigma_{\xi}(\zeta)+v(\zeta)\sigma_{\xi}(\zeta)\right]\right]d_{\beta}(\zeta)\right\}^{2}d_{\rho_{b}}(\mu)\\ &\leq \frac{1}{R^{2}}\int_{-\infty}^{\infty}\left\{\int_{0}^{b}\left[\phi(\zeta,\mu)\left[-T_{\beta}^{2}\sigma_{\xi}(\zeta)+v(\zeta)\sigma_{\xi}(\zeta)\right]\right]d_{\beta}(\zeta)\right\}^{2}d_{\rho_{b}}(\mu)\\ &= \frac{1}{R^{2}}\int_{0}^{\xi}\left[-T_{\beta}^{2}\sigma_{\xi}(\zeta)+v(\zeta)\sigma_{\xi}(\zeta)\right]^{2}d_{\beta}(\zeta).\end{aligned}$$

From (4.6), we see that

$$\left|\int_{0}^{\xi}\sigma_{\xi}^{2}(\zeta)d_{\beta}(\zeta)-\int_{-R}^{R}\Upsilon_{\xi}^{2}(\mu)d_{\rho_{b}}(\mu)\right|\leq\frac{1}{R^{2}}\int_{0}^{\xi}\left[-T_{\beta}^{2}\sigma_{\xi}(\zeta)+v(\zeta)\sigma_{\xi}(\zeta)\right]^{2}d_{\beta}(\zeta). \tag{4.8}$$

By Lemma 1, we see that $\{\rho_b(\mu)\}$ is bounded. By Helly's theorems [10], we can find a sequence $\{b_{n_k}\}$ such that the sequence $\rho_{b_{n_k}}(\mu)$ converges $(b_{n_k} \to \infty)$ to a monotone function $\rho(\mu)$. Passing to the limit as $b_{n_k} \to \infty$ in (4.8), we get

$$\left|\int_0^{\xi} \sigma_\xi^2(\zeta) d_\beta(\zeta) - \int_{-R}^{R} \Upsilon_\xi^2(\mu) d_\beta(\mu)\right| \leq \frac{1}{R^2}\int_0^{\xi}\left[-T_\beta^2 \sigma_\xi(\zeta) + \phi(\zeta)\sigma_\xi(\zeta)\right]^2 d_\beta(\zeta).$$

Hence, letting $R \to \infty$, we obtain

$$\int_0^{\xi} \sigma_\xi^2(\zeta) d_\beta(\zeta) = \int_{-\infty}^{\infty} \Upsilon_\xi^2(\mu) d_\rho(\mu).$$

Assume that σ is an arbitrary real-valued function on $L_\beta^2(a,\infty)$. Then there exists a sequence $\{\sigma_s(\zeta)\}$ satisfying the conditions 1)–3) and such that

$$\lim_{s\to\infty} \int_0^{\infty} (\sigma(\zeta) - \sigma_s(\zeta))^2 d_\beta(\zeta) = 0. \tag{4.9}$$

Let

$$\Upsilon_\tau(\mu) = \int_0^{\infty} \sigma_\tau(\zeta)\phi(\zeta,\mu) d_\beta(\zeta).$$

Then, we have

$$\int_0^{\infty} \sigma_\tau^2(\zeta) d_\beta(\zeta) = \int_{-\infty}^{\infty} \Upsilon_\tau^2(\mu) d_\rho(\mu).$$

By (4.9), we see that $\sigma_s(\zeta)$ is a Cauchy sequence, i.e.,

$$\int_0^{\infty} (\sigma_{\tau_1}(\zeta) - \sigma_{\tau_2}(\zeta))^2 d_\beta(\zeta) \to 0$$

as $\tau_1, \tau_2 \to \infty$. Thus we have

$$\int_{-\infty}^{\infty} (\Upsilon_{\tau_1}(\mu) - \Upsilon_{\tau_2}(\mu))^2 d_\rho(\mu) = \int_0^{\infty} (\sigma_{\tau_1}(t) - \sigma_{\tau_2}(t))^2 d_\beta(\zeta) \to 0$$

as $\tau_1, \tau_2 \to \infty$. Therefore, there exists a limit function Υ satisfying

$$\int_0^{\infty} \sigma^2(\zeta) d_\beta(\zeta) = \int_{-\infty}^{\infty} \Upsilon^2(\mu) d_\rho(\mu),$$

by the completeness of the space $L_\rho^2(\mathbb{R})$.

Now, we show that K_τ defined as

$$K_\tau(\mu) = \int_0^{\tau} \sigma(\zeta)\phi(\zeta,\mu) d_\beta(\zeta)$$

converges to Υ as $\tau \to \infty$. Assume that ω is another function in $L_\beta^2(0,\infty)$. Similarly, $\Omega(\mu)$ can be defined by ω. Then we have

$$\int_0^{\infty} (\sigma(\zeta) - \omega(\zeta))^2 d_\beta(\zeta) = \int_{-\infty}^{\infty} \{\Upsilon(\mu) - \Omega(\mu)\}^2 d_\rho(\mu).$$

Now set

$$\omega(\zeta) = \begin{cases} \sigma(\zeta), & \zeta \in [0,\tau], \\ 0, & \zeta \in (\tau,\infty). \end{cases}$$

Then we have

$$\int_{-\infty}^{\infty}\{\Upsilon(\mu)-K_{\tau}(\mu)\}^{2}\,d_{\rho}(\mu)=\int_{\tau}^{\infty}\sigma^{2}(\zeta)d_{\beta}(\zeta)\to 0\ (\tau\to\infty).$$

(ii) Suppose that $\sigma,\omega\in L_{\beta}^{2}(0,\infty)$ and $\Upsilon(\mu),\Omega(\mu)$ are their Fourier transforms, respectively. Then $\Upsilon\mp\Omega$ are the transforms of $\sigma\mp\omega$. From (4.5), we obtain

$$\int_{0}^{\infty}[\sigma(\zeta)+\omega(\zeta)]^{2}\,d_{\beta}(\zeta)=\int_{-\infty}^{\infty}(\Upsilon(\mu)+\Omega(\mu))^{2}\,d_{\rho}(\mu), \tag{4.10}$$

$$\int_{0}^{\infty}[\sigma(\zeta)-\omega(\zeta)]^{2}\,d_{\beta}(\zeta)=\int_{-\infty}^{\infty}(\Upsilon(\mu)-\Omega(\mu))^{2}\,d_{\rho}(\mu). \tag{4.11}$$

Combining (4.10) and (4.11), we conclude that

$$\int_{0}^{\infty}\sigma(\zeta)\omega(\zeta)d_{\beta}(\zeta)=\int_{-\infty}^{\infty}\Upsilon(\mu)\Omega(\mu)d_{\varrho}(\mu). \tag{4.12}$$

Define

$$\sigma_{\varsigma}(\zeta)=\int_{-\varsigma}^{\varsigma}\Upsilon(\mu)\phi(\zeta,\mu)d_{\rho}(\mu),$$

where Υ is defined in (4.4) and ς is a positive number. Let $\omega(.)$ be a function which is equal to zero outside the finite interval $[0,\tau]$. Hence

$$\begin{aligned}\int_{0}^{\tau}\sigma_{\varsigma}(\zeta)\omega(\zeta)d_{\beta}(\zeta)&=\int_{0}^{\tau}\Big\{\int_{-\varsigma}^{\varsigma}\Upsilon(\mu)\phi(\zeta,\mu)d_{\rho}(\mu)\Big\}\omega(\zeta)d_{\beta}(\zeta)\\ =\int_{-\varsigma}^{\varsigma}\Upsilon(\mu)\Big\{\int_{0}^{\tau}\phi(\zeta,\mu)\omega(\zeta)d_{\beta}(\zeta)\Big\}d_{\rho}(\mu)&=\int_{-\varsigma}^{\varsigma}\Upsilon(\mu)\Omega(\mu)d_{\rho}(\mu).\end{aligned} \tag{4.13}$$

From (4.12), we get

$$\int_{0}^{\infty}\sigma_{\varsigma}(\zeta)\omega(\zeta)d_{\beta}(\zeta)=\int_{-\infty}^{\infty}\Upsilon(\mu)\Omega(\mu)d_{\rho}(\mu). \tag{4.14}$$

By (4.13) and (4.14), we have

$$\int_{0}^{\infty}(\sigma(\zeta)-\sigma_{\varsigma}(\zeta))\,\omega(\zeta)d_{\beta}(\zeta)=\int_{|\mu|>\varsigma}\Upsilon(\mu)\Omega(\mu)d_{\rho}(\mu).$$

From the Cauchy–Schwarz inequality, we see that

$$\begin{aligned}\left|\int_{0}^{\infty}(\sigma(\zeta)-\sigma_{\varsigma}(\zeta))\,\omega(\zeta)d_{\beta}(\zeta)\right|^{2}&\leq\int_{|\mu|>\varsigma}\Upsilon^{2}(\mu)d_{\rho}(\mu)\int_{|\mu|>\varsigma}\Omega^{2}(\mu)d_{\rho}(\mu)\\ &\leq\int_{|\mu|>\varsigma}\Upsilon^{2}(\mu)d_{\rho}(\mu)\int_{-\infty}^{\infty}\Omega^{2}(\mu)d_{\rho}(\mu).\end{aligned} \tag{4.15}$$

Let

$$\omega(\zeta)=\begin{cases}\sigma(\zeta)-\sigma_{\varsigma}(\zeta), & \zeta\in[0,\tau]\,,\\ 0, & \zeta\in(\tau,\infty).\end{cases}$$

From (4.15), we obtain

$$\int_{0}^{\infty}(\sigma(\zeta)-\sigma_{\varsigma}(\zeta))^{2}\,d_{\beta}(\zeta)\leq\int_{|\mu|>\varsigma}\Upsilon^{2}(\mu)d_{\rho}(\mu).$$

Letting $\varsigma\to\infty$ gives the desired result due to the right-hand side does not depend on τ. □

Example 1. If we take $\beta = 1$ in (1.1), then we obtain the ordinary Sturm–Liouville problem defined by

$$-y'' + v(\zeta)y = \mu y, \quad \zeta \in (0, \infty),$$

where μ is a complex eigenvalue parameter, $v(.)$ is a real-valued function defined on $[0, \infty)$, and $v \in L^1_{loc}(0, \infty)$. Then Theorem 6 gives the spectral expansion for this problem (see [11]).

Example 2. Consider the following problem

$$\begin{gathered} -T^2_\beta y(\zeta) - ky(\zeta) = \mu y(\zeta), \quad 0 < \zeta < \infty, \\ y(0) = 0, \end{gathered} \tag{4.16}$$

where k is a constant. It is clear that

$$\phi(\zeta, \mu) = \frac{\sin\left(\int_0^\zeta \sqrt{\mu + k}\, d_\beta \zeta\right)}{\sqrt{\mu + k}}$$

is the solution of (4.16). By Theorem 6, we obtain

$$\Upsilon(\mu) = \int_0^\infty \sigma(\zeta) \frac{\sin\left(\int_0^\zeta \sqrt{\mu + k}\, d_\beta \zeta\right)}{\sqrt{\mu + k}} d_\beta(\zeta),$$

and

$$\sigma(\zeta) = \int_{-\infty}^\infty \Upsilon(\mu) \frac{\sin\left(\int_0^\zeta \sqrt{\mu + k}\, d_\beta \zeta\right)}{\sqrt{\mu + k}} d_\rho(\mu).$$

5. Conclusion

The present study is devoted to the discussion of beta Sturm–Liouville problems. In the context of such equations, the spectral function was established in the singular case. A spectral expansion was derived with the aid of this function. The Titchmarh–Weyl theory for this type of equations may be the subject of future research.

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