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ON ONE ZALCMAN PROBLEM FOR THE MEAN VALUE OPERATOR

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Abstract: Let $\mathcal{D}'(\mathbb{R}^n)$ and $\mathcal{E}'(\mathbb{R}^n)$ be the spaces of distributions and compactly supported distributions on $\mathbb{R}^n$, $n \geq 2$, respectively, let $\mathcal{E}'_{\natural}(\mathbb{R}^n)$ be the space of all radial (invariant under rotations of the space $\mathbb{R}^n$) distributions in $\mathcal{E}'(\mathbb{R}^n)$, let $\widetilde{T}$ be the spherical transform (Fourier–Bessel transform) of a distribution $T \in \mathcal{E}'_{\natural}(\mathbb{R}^n)$, and let $\mathcal{Z}_+(\widetilde{T})$ be the set of all zeros of an even entire function $\widetilde{T}$ lying in the half-plane $\operatorname{Re} z \geq 0$ and not belonging to the negative part of the imaginary axis. Let σ_r be the surface delta function concentrated on the sphere $S_r = \{x \in \mathbb{R}^n : |x| = r\}$. The problem of L. Zalcman on reconstructing a distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ from known convolutions $f * \sigma_{r_1}$ and $f * \sigma_{r_2}$ is studied. This problem is correctly posed only under the condition $r_1/r_2 \notin M_n$, where M_n is the set of all possible ratios of positive zeros of the Bessel function $J_{n/2-1}$. The paper shows that if $r_1/r_2 \notin M_n$, then an arbitrary distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ can be expanded into an unconditionally convergent series

$$f = \sum_{\lambda \in \mathcal{Z}_+(\widetilde{\Omega}_{r_1})} \sum_{\mu \in \mathcal{Z}_+(\widetilde{\Omega}_{r_2})} \frac{4\lambda\mu}{(\lambda^2-\mu^2)\widetilde{\Omega}_{r_1}'(\lambda)\widetilde{\Omega}_{r_2}'(\mu)} \Big(P_{r_2}(\Delta)\big((f*\sigma_{r_2}) * \Omega_{r_1}^{\lambda}\big) - P_{r_1}(\Delta)\big((f*\sigma_{r_1}) * \Omega_{r_2}^{\mu}\big)\Big)$$

in the space $\mathcal{D}'(\mathbb{R}^n)$, where Δ is the Laplace operator in $\mathbb{R}^n$, P_r is an explicitly given polynomial of degree $[(n+5)/4]$, and Ω_r and Ω_r^{λ} are explicitly constructed radial distributions supported in the ball $|x| \leq r$. The proof uses the methods of harmonic analysis, as well as the theory of entire and special functions. By a similar technique, it is possible to obtain inversion formulas for other convolution operators with radial distributions.

Keywords: Compactly supported distributions, Fourier–Bessel transform, Two-radii theorem, Inversion formulas.

1. Introduction

The study of functions $f \in C(\mathbb{R}^2)$ with zero integrals over all sets congruent to a given compact set of positive Lebesgue measure (for example, with zero integrals over all discs of a fixed radius in $\mathbb{R}^2$) goes back to Pompeiu [17, 18]. Motivated by the works of Pompeiu, Nicolesco in his paper [16] presents the following erroneous statement concerning integrals over circles of a fixed radius: if a real-valued function $u(x, y)$ belongs to the class $C^s(\mathbb{R}^2)$ for some $s \in \mathbb{Z}_+$, r is a fixed positive number, and the function

$$v_s(x, y, r) = \int_0^{2\pi} u(x + r\cos\theta,\, y + r\sin\theta) e^{is\theta} d\theta$$

does not depend on (x, y), then $u(x, y)$ is a solution to the equation

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)^s u(x, y) = \text{const}.$$

In particular, if $u \in C(\mathbb{R}^2)$ and u has constant integrals over all circles of fixed radius, then $u = \text{const}$. The impossibility of such a result is shown by the following proposition from a paper by Radon published back in 1917 (see [19, Sect. C]).

Proposition 1. *Let $r > 0$ be fixed, and let λr be an arbitrary positive zero of the Bessel function J_0. Then, for any $k \in \mathbb{Z}$, the function*

$$\mathcal{I}_k(z) = J_k(\lambda\rho)e^{ik\varphi} \quad (\rho \text{ and } \varphi \text{ are the polar coordinates of } z)$$

has zero integrals over all circles of radius r.

Similar examples related to the zeros of the Bessel function $J_{n/2-1}$ can also be constructed for spherical means in $\mathbb{R}^n$ for $n \geq 2$. This shows that knowing the averages of a function f over all spheres of the same radius is insufficient to reconstruct f uniquely. Subsequently, the class of functions $f \in C(\mathbb{R}^n)$ that have zero integrals over all spheres of fixed radius in $\mathbb{R}^n$ was studied by many authors see [2, 23, 25, 27, 35, 36], and the references therein). A well-known result in this direction is the following analog of Delsarte's famous two-radius theorem [6] for harmonic functions.

Theorem 1 [7, 33]. *Let $r_1, r_2 \in (0, +\infty)$, let $\Upsilon_n = \{\gamma_1, \gamma_2, \ldots\}$ be the sequence of all positive zeros of the function $J_{n/2-1}$ numbered in ascending order, and let M_n be the set of numbers of the form α/β, where $\alpha, \beta \in \Upsilon_n$.*

(1) *If $r_1/r_2 \notin M_n$, $f \in C(\mathbb{R}^n)$, and*

$$\int_{|x-y|=r_1} f(x)d\sigma(x) = \int_{|x-y|=r_2} f(x)d\sigma(x) = 0, \quad y \in \mathbb{R}^n, \tag{1.1}$$

($d\sigma$ is the area element), then $f = 0$.

(2) *If $r_1/r_2 \in M_n$, then there exists a nonzero real analytic function $f : \mathbb{R}^n \to \mathbb{C}$ satisfying the relations in (1.1).*

In terms of convolutions (see formula (2.2) below), Theorem 1 means that the operator

$$\mathcal{P}f = (f * \sigma_{r_1}, f * \sigma_{r_2}), \quad f \in C(\mathbb{R}^n) \tag{1.2}$$

is injective if and only if $r_1/r_2 \notin M_n$. Hereinafter, σ_r is a surface delta function concentrated on the sphere

$$S_r = \{x \in \mathbb{R}^n : |x| = r\},$$

that is,

$$\langle \sigma_r, \varphi \rangle = \int_{S_r} \varphi(x)d\sigma(x), \quad \varphi \in C(\mathbb{R}^n).$$

In this regard, Zalcman [34, Sect. 8] posed the problem of finding an explicit inversion formula for the operator $\mathcal{P}$ under the condition $r_1/r_2 \notin M_n$ (see also [19, Sect. C]). A similar question for ball means values was studied by Berenstein, Yger, Taylor, and others (see [1, 3, 4]). Note that their methods are also applicable in the case of spherical means. In particular, the following local result is valid (see the proof of Theorem 9 in [1]).

Theorem 2. *Let*

$$r_1/r_2 \notin M_n, \quad R > r_1 + r_2, \quad B_R = \{x \in \mathbb{R}^n : |x| < R\},$$

and let $\{\varepsilon_k\}_{k=1}^{\infty}$ *be a strictly increasing sequence of positive numbers with limit*

$$R/(r_1 + r_2) - 1, \quad R_k = (r_1 + r_2)(1 + \varepsilon_k), \quad R_0 = 0.$$

Then, for all $r > 0$, $r \in [R_{k-1}, R_k)$, *and every spherical harmonic* Y *of degree* m *on the unit sphere* $\mathbb{S}^{n-1}$, *one can explicitly construct two sequences* $\mathfrak{C}_l$ *and* $\mathfrak{D}_l$ *of compactly supported distributions in* B_{R-r_1} *and* B_{R-r_2}, *respectively, such that the following estimate holds for* $l \geqslant cm^2$ *and every function* $f \in C^{\infty}(B_R)$:

$$\left| \int_{\mathbb{S}^{n-1}} f(r\sigma)Y(\sigma)d\sigma - \langle \mathfrak{C}_l, f * \sigma_{r_1} \rangle - \langle \mathfrak{D}_l, f * \sigma_{r_2} \rangle \right| \leqslant \frac{\gamma}{l}(R-r)^{-N} r^{-(n-3)/2} \max_{\substack{|\alpha| \leqslant N \\ |x| \leq R'_k}} \left| \frac{\partial^{|\alpha|}}{\partial x^{\alpha}} f(x) \right|, \quad (1.3)$$

where

$$N = [(n+13)/2] + 1, \quad R'_k = (2R + R_k)/3,$$

and γ *and* c *are positive constants depending on* r_1, r_2, R, n, *and* ε_1.

Here it is appropriate to make a few remarks. The distributions $\mathfrak{C}_l$ and $\mathfrak{D}_l$ have a very complex form and are constructed as inverse Fourier–Bessel transforms to some linear combinations of products of rational and Bessel functions (see the proof of Proposition 8 and Theorem 9 in [1]). Further, every function $f \in C^{\infty}(B_R)$ can be represented as a Fourier series

$$f(x) = \sum_{m=0}^{\infty} \sum_{j=1}^{d_m} f_{m,j}(r) Y_j^{(m)}(\sigma), \quad x = r\sigma, \quad \sigma \in \mathbb{S}^{n-1}, \tag{1.4}$$

converging in the space $C^{\infty}(B_R)$, where $\{Y_j^{(m)}\}_{j=1}^{d_m}$ is a fixed orthonormal basis in the space of spherical harmonics of degree m on $\mathbb{S}^{n-1}$,

$$f_{m,j}(r) = \int_{\mathbb{S}^{n-1}} f(r\sigma)\overline{Y_j^{(m)}(\sigma)}d\sigma$$

(see, for example, [10, Ch. 1, Sect. 2, Proposition 2.7], [24, Sect. 1]). Therefore, estimate (1.3) as $l \to \infty$ and expansion (1.4) imply the reconstruction of a function $f \in C^{\infty}(B_R)$ from its spherical means $f * \sigma_{r_1}$ and $f * \sigma_{r_2}$ in the ball B_R. The transition to the class $C(B_R)$ can be done by smoothing f by convolutions of the form $f * \varphi_{\varepsilon}$, where $\varphi_{\varepsilon} \in C^{\infty}(\mathbb{R}^n)$, $\operatorname{supp} \varphi_{\varepsilon} \subset B_{\varepsilon}$ (see [1, Sect. 3]).

The above remarks and Theorem 2 for $R = \infty$ give a procedure for finding a function from its two spherical means. However, "explicit" inversion formulas for the operator (1.2) were unknown. This work aims to solve this problem.

2. Statement of the main result

In what follows, as usual, $\mathbb{C}^n$ is an n-dimensional complex space with the Hermitian scalar product

$$(\zeta, \varsigma) = \sum_{j=1}^{n} \zeta_j \overline{\varsigma}_j, \quad \zeta = (\zeta_1, \ldots, \zeta_n), \quad \varsigma = (\varsigma_1, \ldots, \varsigma_n),$$

$\mathcal{D}'(\mathbb{R}^n)$ and $\mathcal{E}'(\mathbb{R}^n)$ are the spaces of distributions and compactly supported distributions on $\mathbb{R}^n$, respectively.

The Fourier–Laplace transform of a distribution $T \in \mathcal{E}'(\mathbb{R}^n)$ is the entire function

$$\widehat{T}(\zeta) = \langle T(x), e^{-i(\zeta,x)}\rangle, \quad \zeta \in \mathbb{C}^n.$$

In this case, $\widehat{T}$ grows on $\mathbb{R}^n$ not faster than a polynomial and

$$\langle \widehat{T}, \psi\rangle = \langle T, \widehat{\psi}\,\rangle, \quad \psi \in \mathcal{S}(\mathbb{R}^n), \tag{2.1}$$

where $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of rapidly decreasing functions from $C^\infty(\mathbb{R}^n)$ (see [13, Ch. 7]). If $T_1, T_2 \in \mathcal{D}'(\mathbb{R}^n)$ and at least one of these distributions has compact support, then their convolution $T_1 * T_2$ is a distribution in $\mathcal{D}'(\mathbb{R}^n)$ acting according to the rule

$$\langle T_1 * T_2, \varphi\rangle = \big\langle T_2(y), \langle T_1(x), \varphi(x+y)\rangle\big\rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}^n), \tag{2.2}$$

where $\mathcal{D}(\mathbb{R}^n)$ is the space of finite infinitely differentiable functions on $\mathbb{R}^n$. For $T_1, T_2 \in \mathcal{E}'(\mathbb{R}^n)$, the Borel formula

$$\widehat{T_1 * T_2} = \widehat{T_1}\,\widehat{T_2} \tag{2.3}$$

is valid.

Let $\mathcal{E}'_\natural(\mathbb{R}^n)$ be the space of radial (invariant under rotations of the space $\mathbb{R}^n$) distributions in $\mathcal{E}'(\mathbb{R}^n)$, $n \geq 2$. The simplest example of distribution in the class $\mathcal{E}'_\natural(\mathbb{R}^n)$ is the Dirac delta function δ with support at zero. We set

$$\mathbf{I}_\nu(z) = \frac{J_\nu(z)}{z^\nu}, \quad \nu \in \mathbb{C}.$$

The spherical transform $\widetilde{T}$ of a distribution $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$ is defined as

$$\widetilde{T}(z) = \langle T, \varphi_z\rangle, \quad z \in \mathbb{C}, \tag{2.4}$$

where φ_z is a spherical function on $\mathbb{R}^n$, i.e.,

$$\varphi_z(x) = 2^{n/2-1}\Gamma\left(\frac{n}{2}\right)\mathbf{I}_{n/2-1}(z|x|), \quad x \in \mathbb{R}^n$$

(see [9, Ch. 4]). The function φ_z is uniquely determined by the following conditions:

(1) φ_z is radial and $\varphi_z(0) = 1$;

(2) φ_z satisfies the Helmholtz differential equation

$$\Delta(\varphi_z) + z^2\varphi_z = 0. \tag{2.5}$$

We note that $\widetilde{T}$ is an even entire function of exponential type and the Fourier transform $\widehat{T}$ is expressed in terms of $\widetilde{T}$ as

$$\widehat{T}(\zeta) = \widetilde{T}\big(\sqrt{\zeta_1^2 + \ldots + \zeta_n^2}\,\big), \quad \zeta \in \mathbb{C}^n. \tag{2.6}$$

The set of all zeros of the function $\widetilde{T}$ that lie in the half-plane $\operatorname{Re} z \geq 0$ and do not belong to the negative part of the imaginary axis will be denoted by $\mathcal{Z}_+(\widetilde{T})$.

For $T = \sigma_r$, we have (see [27, Part 2, Ch. 3, formula (3.90)])

$$\widetilde{\sigma}_r(z) = (2\pi)^{n/2} r^{n-1}\mathbf{I}_{n/2-1}(rz). \tag{2.7}$$

Hence, by the formula

$$\mathbf{I}'_\nu(z) = -z\mathbf{I}_{\nu+1}(z) \tag{2.8}$$

(see [12, Ch. 7, Sect. 7.2.8, formula (51)]), we find

$$\widetilde{\sigma}'_r(z) = -(2\pi)^{n/2} r^{n+1} z\mathbf{I}_{n/2}(rz). \tag{2.9}$$

Using the well-known properties of zeros of Bessel functions (see, for example, [12, Ch. 7, Sect. 7.9]), one can obtain the corresponding information about the set $\mathcal{Z}_+(\widetilde{\sigma}_r)$. In particular, all zeros of $\widetilde{\sigma}_r$ are simple, belong to $\mathbb{R}\backslash\{0\}$, and

$$\mathcal{Z}_+(\widetilde{\sigma}_r) = \left\{\frac{\gamma_1}{r}, \frac{\gamma_2}{r}, \dots\right\}. \tag{2.10}$$

In addition, since the functions $J_{n/2-1}$ and $J_{n/2}$ do not have common zeros on $\mathbb{R}\backslash\{0\}$, the function

$$\sigma_r^\lambda(x) = -\frac{1}{r\lambda^2}\frac{\mathbf{I}_{n/2-1}(\lambda|x|)}{\mathbf{I}_{n/2}(\lambda r)}\chi_r(x), \quad \lambda \in \mathcal{Z}_+(\widetilde{\sigma}_r),$$

is well defined, where χ_r is the indicator of the ball B_r.

Let

$$P_r(z) = \prod_{j=1}^{m}\left(z - \left(\frac{\gamma_j}{r}\right)^2\right), \quad m = \left[\frac{n+5}{4}\right], \tag{2.11}$$

$$\Omega_r = P_r(\Delta)\sigma_r. \tag{2.12}$$

Then, by the formula

$$\widetilde{p(\Delta)T}(z) = p(-z^2)\widetilde{T}(z) \quad (p \text{ is an algebraic polynomial}), \tag{2.13}$$

we have

$$\widetilde{\Omega}_r(z) = P_r(-z^2)\widetilde{\sigma}_r(z), \tag{2.14}$$

$$\mathcal{Z}_+\big(\widetilde{\Omega}_r\big) = \left\{\frac{\gamma_1}{r}, \frac{\gamma_2}{r}, \dots\right\} \cup \left\{\frac{i\gamma_1}{r}, \frac{i\gamma_2}{r}, \dots, \frac{i\gamma_m}{r}\right\}, \tag{2.15}$$

and all zeros of $\widetilde{\Omega}_r$ are simple. Besides,

$$\mathcal{Z}_+\big(\widetilde{\Omega}_{r_1}\big) \cap \mathcal{Z}_+\big(\widetilde{\Omega}_{r_2}\big) = \varnothing \quad \Leftrightarrow \quad \frac{r_1}{r_2} \notin M_n. \tag{2.16}$$

For $\lambda \in \mathcal{Z}_+\big(\widetilde{\Omega}_r\big)$, we set

$$\Omega_r^\lambda = P_r(\Delta)\sigma_r^\lambda \tag{2.17}$$

if $\lambda \in \mathcal{Z}_+(\widetilde{\sigma}_r)$ and

$$\Omega_r^\lambda = Q_{r,\lambda}(\Delta)\sigma_r \tag{2.18}$$

if $P_r(-\lambda^2) = 0$, where

$$Q_{r,\lambda}(z) = -\frac{P_r(z)}{z+\lambda^2}. \tag{2.19}$$

The main result of this work is the following theorem.

Theorem 3. *Let*

$$\frac{r_1}{r_2} \notin M_n, \quad f \in \mathcal{D}'(\mathbb{R}^n), \quad n \geq 2.$$

Then

$$f = \sum_{\lambda \in \mathcal{Z}_+(\widetilde{\Omega}_{r_1})} \sum_{\mu \in \mathcal{Z}_+(\widetilde{\Omega}_{r_2})} \frac{4\lambda\mu}{(\lambda^2-\mu^2)\widetilde{\Omega}_{r_1}'(\lambda)\widetilde{\Omega}_{r_2}'(\mu)} \Big(P_{r_2}(\Delta)\big((f * \sigma_{r_2}) * \Omega_{r_1}^{\lambda}\big) - P_{r_1}(\Delta)\big((f * \sigma_{r_1}) * \Omega_{r_2}^{\mu}\big)\Big), \tag{2.20}$$

where the series (2.20) *converges unconditionally in the space* $\mathcal{D}'(\mathbb{R}^n)$.

Equality (2.20) reconstruct a distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ from its known convolutions $f * \sigma_{r_1}$ and $f * \sigma_{r_2}$ (see (2.11), (2.14), (2.15), and (2.17)–(2.19)). Thus, Theorem 3 gives a solution to the Zalcman problem formulated above. Note that there is great arbitrariness in the choice of polynomials P_{r_1} and P_{r_2} in formula (2.20) (see the proof of Corollary 1 and Lemma 5 in Section 3). In particular, they can be defined fully explicitly without using the zeros of the function $J_{n/2-1}$. For other results related to the inversion of the spherical mean operator, see [5, 8, 11, 20, 21, 26, 28–32].

3. Auxiliary statements

Let us first describe the properties of the functions $\mathbf{I}_\nu$, which we will need later.

Lemma 1. (1) *The following inequality holds for* $\nu > -1/2$ *and* $z \in \mathbb{C}$:

$$|\mathbf{I}_\nu(z)| \leq \frac{e^{|\operatorname{Im} z|}}{2^\nu \Gamma(\nu+1)}. \tag{3.1}$$

(2) *If* $\nu \in \mathbb{R}$, *then*

$$|\mathbf{I}_\nu(z)| \sim \frac{1}{\sqrt{2\pi}} \frac{e^{|\operatorname{Im} z|}}{|z|^{\nu+1/2}}, \quad \operatorname{Im} z \to \infty. \tag{3.2}$$

(3) *Let* $\nu > -1$ *and let* $\{\gamma_{\nu,j}\}_{j=1}^{\infty}$ *be the sequence of all positive zeros of the function* $\mathbf{I}_\nu$ *numbered in ascending order. Then*

$$\gamma_{\nu,j} = \pi\left(j + \frac{\nu}{2} - \frac{1}{4}\right) + O\left(\frac{1}{j}\right), \quad j \to \infty. \tag{3.3}$$

In addition,

$$\lim_{j\to\infty} \left(\gamma_{\nu,j}\right)^{\nu+3/2} |\mathbf{I}_{\nu+1}(\gamma_{\nu,j})| = \sqrt{\frac{2}{\pi}}. \tag{3.4}$$

P r o o f. (1) By the Poisson integral representation [12, Ch. 7, Sect. 7.12, formula (8)], we have

$$\mathbf{I}_\nu(z) = \frac{2^{1-\nu}}{\sqrt{\pi}\Gamma(\nu+1/2)} \int_0^1 \cos(uz)(1-u^2)^{\nu-1/2} du.$$

Hence,

$$|\mathbf{I}_\nu(z)| \leq \frac{2^{1-\nu}}{\sqrt{\pi}\Gamma(\nu+1/2)} \int_0^1 e^{u|\operatorname{Im} z|}(1-u^2)^{\nu-1/2} du$$

$$\leq \frac{2^{1-\nu}}{\sqrt{\pi}\Gamma(\nu+1/2)} \frac{1}{2}\mathrm{B}\left(\frac{1}{2}, \nu+\frac{1}{2}\right) e^{|\mathrm{Im}\, z|} = \frac{e^{|\mathrm{Im}\, z|}}{2^{\nu}\Gamma(\nu+1)},$$

which is required.

(2) The asymptotic expansion of Bessel functions [12, Ch. 7, Sect. 7.13.1, formula (3)] implies the equality

$$\mathbf{I}_\nu(z) = \sqrt{\frac{2}{\pi}}\, z^{-\nu-1/2}\left(\cos\left(z - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) + O\left(\frac{e^{|\mathrm{Im}\, z|}}{|z|}\right)\right), \quad z \to \infty, \quad -\pi < \arg z < \pi. \tag{3.5}$$

Considering that

$$|\cos w| \sim \frac{e^{|\mathrm{Im}\, w|}}{2}, \quad \mathrm{Im}\, w \to \infty,$$

by (3.5), we obtain (3.2).

(3) The asymptotic behavior (3.3) for the zeros of $\mathbf{I}_\nu$ is well known (see, for example, [25, Ch. 7, formula (7.9)]). Then

$$\cos\left(\gamma_{\nu,j} - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) = \cos\left(\pi j - \frac{\pi}{2} + O\left(\frac{1}{j}\right)\right) = O\left(\frac{1}{j}\right), \quad j \to \infty.$$

It follows that

$$\lim_{j\to\infty} \left|\sin\left(\gamma_{\nu,j} - \frac{\pi\nu}{2} - \frac{\pi}{4}\right)\right| = 1.$$

Using this relation and the equality

$$\mathbf{I}_{\nu+1}(z) = \sqrt{\frac{2}{\pi}}\, z^{-\nu-3/2}\left(\sin\left(z - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) + O\left(\frac{e^{|\mathrm{Im}\, z|}}{|z|}\right)\right), \quad z \to \infty, \quad -\pi < \arg z < \pi,$$

(see (3.5)), we arrive at (3.4). □

Corollary 1. *For all* $r > 0$,

$$\sum_{\lambda\in\mathcal{Z}_+(\widetilde{\Omega}_r)} \frac{1}{|\widetilde{\Omega}_r{}'(\lambda)|} < +\infty. \tag{3.6}$$

P r o o f. Using (2.14) and (2.9), we find

$$\widetilde{\Omega}_r{}'(\lambda) = P_r(-\lambda^2)\widetilde{\sigma}_r'(\lambda) - 2\lambda P_r'(-\lambda^2)\widetilde{\sigma}_r(\lambda) = -(2\pi)^{n/2} r^{n+1}\lambda P_r(-\lambda^2)\mathbf{I}_{n/2}(r\lambda) - 2\lambda P_r'(-\lambda^2)\widetilde{\sigma}_r(\lambda).$$

Now, from (2.10) and (2.15), we have

$$\sum_{\lambda\in\mathcal{Z}_+(\widetilde{\Omega}_r)} \frac{1}{|\widetilde{\Omega}_r{}'(\lambda)|} = \sum_{j=1}^{m} \frac{1}{|\widetilde{\Omega}_r{}'(i\gamma_j/r)|} + \frac{1}{(2\pi)^{n/2} r^n} \sum_{j=1}^{\infty} \frac{1}{\gamma_j |P_r(-\gamma_j^2/r^2)||\mathbf{I}_{n/2}(\gamma_j)|}.$$

This series is comparable with the convergent series

$$\sum_{j=1}^{\infty} \frac{1}{j^{2m-(n-1)/2}}$$

(see (2.11), (3.3), and (3.4)). Hence, we obtain the required assertion. □

Lemma 2. *Let $g : \mathbb{C} \to \mathbb{C}$ be an even entire function, and let $g(\lambda) = 0$ for some $\lambda \in \mathbb{C}$. Then*

$$\left|\frac{\lambda g(z)}{z^2-\lambda^2}\right| \leq \max_{|\zeta - z|\leq 2} |g(\zeta)|, \quad z \in \mathbb{C}; \tag{3.7}$$

the left-hand side in (3.7) for $z = \pm\lambda$ is extended by continuity.

P r o o f. We have

$$\left|\frac{2\lambda g(z)}{z^2-\lambda^2}\right| = \left|\frac{g(z)}{z-\lambda} - \frac{g(z)}{z+\lambda}\right| \leq \left|\frac{g(z)}{z-\lambda}\right| + \left|\frac{g(z)}{z+\lambda}\right|. \tag{3.8}$$

Let us estimate the first term on the right-hand side of (3.8).

If $|z-\lambda| > 1$, then

$$\left|\frac{g(z)}{z-\lambda}\right| \leq |g(z)| \leq \max_{|\zeta - z|\leq 2} |g(\zeta)|. \tag{3.9}$$

Assume that $|z-\lambda| \leq 1$. Then, applying the maximum-modulus principle to the entire function $g(\zeta)/(\zeta-\lambda)$, we obtain

$$\left|\frac{g(z)}{z-\lambda}\right| \leq \max_{|\zeta-\lambda|\leq 1} \left|\frac{g(\zeta)}{\zeta-\lambda}\right| = \max_{|\zeta-\lambda|=1} |g(\zeta)|.$$

Considering that the circle $|\zeta-\lambda| = 1$ is contained in the disc $|\zeta - z| \leq 2$, we arrive at the estimate

$$\left|\frac{g(z)}{z-\lambda}\right| \leq \max_{|\zeta - z|\leq 2} |g(\zeta)|, \tag{3.10}$$

which is valid for all $z \in \mathbb{C}$ (see (3.9)).

Similarly,

$$\left|\frac{g(z)}{z+\lambda}\right| \leq \max_{|\zeta - z|\leq 2} |g(\zeta)|, \quad z \in \mathbb{C}, \tag{3.11}$$

because $g(-\lambda) = 0$. From (3.10), (3.11), and (3.8) the required assertion follows. □

Lemma 3. *The function σ_r^λ satisfies the equation*

$$\Delta(\sigma_r^\lambda) + \lambda^2\sigma_r^\lambda = -\sigma_r, \quad \lambda \in \mathcal{Z}_+(\widetilde{\sigma}_r). \tag{3.12}$$

P r o o f. For every function $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we have

$$\langle \Delta(\sigma_r^\lambda) + \lambda^2\sigma_r^\lambda, \varphi\rangle = \langle \sigma_r^\lambda, (\Delta + \lambda^2)\varphi\rangle$$

$$= -\frac{1}{r\lambda^2}\int_{|x|\leq r} \frac{\mathbf{I}_{n/2-1}(\lambda|x|)}{\mathbf{I}_{n/2}(\lambda r)}\Delta\varphi(x)dx - \frac{1}{r}\int_{|x|\leq r} \frac{\mathbf{I}_{n/2-1}(\lambda|x|)}{\mathbf{I}_{n/2}(\lambda r)}\varphi(x)dx.$$

We apply Green's formula

$$\int_G (v\Delta u - u\Delta v)dx = \int_{\partial G}\left(v\frac{\partial u}{\partial \mathbf{n}} - u\frac{\partial v}{\partial \mathbf{n}}\right) d\sigma$$

to the former integral (see, for example, [22, Ch. 5, Sect. 21.2]). Since $\lambda \in \mathcal{Z}_+(\widetilde{\sigma}_r)$, we have

$$\langle \Delta(\sigma_r^\lambda) + \lambda^2\sigma_r^\lambda, \varphi\rangle = -\frac{1}{r\lambda^2}\int_{|x|\leq r} \Delta\left(\frac{\mathbf{I}_{n/2-1}(\lambda|x|)}{\mathbf{I}_{n/2}(\lambda r)}\right)\varphi(x)dx$$

$$+\frac{1}{r\lambda^2}\int_{S_r}\varphi(x)\frac{\partial}{\partial \mathbf{n}}\left(\frac{\mathbf{I}_{n/2-1}(\lambda|x|)}{\mathbf{I}_{n/2}(\lambda r)}\right)d\sigma(x) - \frac{1}{r}\int_{|x|\leq r} \frac{\mathbf{I}_{n/2-1}(\lambda|x|)}{\mathbf{I}_{n/2}(\lambda r)}\varphi(x)dx.$$

Hence, by (2.5), we obtain

$$\langle \Delta(\sigma_r^\lambda) + \lambda^2 \sigma_r^\lambda, \varphi\rangle = \frac{1}{r\lambda^2}\int_{S_r} \varphi(x) \frac{\partial}{\partial \mathbf{n}} \left(\frac{\mathbf{I}_{n/2-1}(\lambda|x|)}{\mathbf{I}_{n/2}(\lambda r)} \right) d\sigma(x).$$

Now, using the formula

$$\frac{\partial}{\partial \mathbf{n}} \big(f(|x|)\big) = f'(|x|), \quad \mathbf{n} = \frac{x}{|x|},$$

and relation (2.8), we find

$$\langle \Delta(\sigma_r^\lambda) + \lambda^2 \sigma_r^\lambda, \varphi\rangle = -\frac{1}{r}\int_{S_r} \varphi(x)\, |x|\, \frac{\mathbf{I}_{n/2}(\lambda|x|)}{\mathbf{I}_{n/2}(\lambda r)}\, d\sigma(x) = -\int_{S_r} \varphi(x) d\sigma(x) = -\langle \sigma_r, \varphi\rangle.$$

This proves equality (3.12). □

Remark 1. From (2.13) and the injectivity of the spherical transform, it follows that, for distributions $U, T \in \mathcal{E}'_\natural(\mathbb{R}^n)$ and $\lambda \in \mathcal{Z}_+(\widetilde{T})$,

$$\Delta U + \lambda^2 U = -T \quad \Leftrightarrow \quad \widetilde{U}(z) = \frac{\widetilde{T}(z)}{z^2 - \lambda^2}. \tag{3.13}$$

Therefore, relation (3.12) implies the equality

$$\widetilde{\sigma_r^\lambda}(z) = \frac{\widetilde{\sigma}_r(z)}{z^2 - \lambda^2}, \quad \lambda \in \mathcal{Z}_+(\widetilde{\sigma}_r). \tag{3.14}$$

Lemma 4. *Let* $\lambda \in \mathcal{Z}_+(\widetilde{\Omega}_r)$*. Then*

$$\widetilde{\Omega_r^\lambda}(z) = \frac{\widetilde{\Omega}_r(z)}{z^2 - \lambda^2}. \tag{3.15}$$

P r o o f. Formula (3.15) easily follows from (2.13) and Remark 1. Indeed, if $\lambda \in \mathcal{Z}_+(\widetilde{\sigma}_r)$, then, by (2.17), (2.13), (3.14), and (2.14), we have

$$\widetilde{\Omega_r^\lambda}(z) = P_r(-z^2)\widetilde{\sigma_r^\lambda}(z) = \frac{P_r(-z^2)\widetilde{\sigma}_r(z)}{z^2 - \lambda^2} = \frac{\widetilde{\Omega}_r(z)}{z^2 - \lambda^2}.$$

Similarly, if $P_r(-\lambda^2) = 0$, then

$$\widetilde{\Omega_r^\lambda}(z) = Q_{r,\lambda}(-z^2)\widetilde{\sigma}_r(z) = \frac{P_r(-z^2)\widetilde{\sigma}_r(z)}{z^2 - \lambda^2} = \frac{\widetilde{\Omega}_r(z)}{z^2 - \lambda^2}$$

(see (2.18), (2.19), (2.13), and (2.14)). □

Lemma 5. *Let*

$$\Psi_r^\lambda = \frac{2\lambda}{\widetilde{\Omega}_r{}'(\lambda)}\, \Omega_r^\lambda, \quad \lambda \in \mathcal{Z}_+\big(\widetilde{\Omega}_r\big). \tag{3.16}$$

Then

$$\sum_{\lambda \in \mathcal{Z}_+(\widetilde{\Omega}_r)} \Psi_r^\lambda = \delta, \tag{3.17}$$

where the series in (3.17) converges unconditionally in the space $\mathcal{D}'(\mathbb{R}^n)$*.*

P r o o f. For an arbitrary function $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we define a function $\psi \in \mathcal{S}(\mathbb{R}^n)$ as follows:

$$\psi(y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \varphi(x) e^{i(x,y)} dx, \quad y \in \mathbb{R}^n.$$

Then (see (2.1), (2.6), and (3.15))

$$\langle \Psi_r^\lambda, \varphi \rangle = \langle \Psi_r^\lambda, \widehat{\psi} \rangle = \langle \widehat{\Psi_r^\lambda}, \psi \rangle = \int_{\mathbb{R}^n} \psi(x) \widetilde{\Psi_r^\lambda}(|x|) dx = \frac{2}{\widetilde{\Omega}_r{}'(\lambda)} \int_{\mathbb{R}^n} \psi(x) \frac{\lambda \widetilde{\Omega}_r(|x|)}{|x|^2 - \lambda^2} dx.$$

Using this representation and Lemma 2, we get

$$\left|\langle \Psi_r^\lambda, \varphi \rangle\right| \leq \frac{2}{\left|\widetilde{\Omega}_r{}'(\lambda)\right|} \int_{\mathbb{R}^n} |\psi(x)| \max_{|\zeta - |x|| \leq 2} \left|\widetilde{\Omega}_r(\zeta)\right| dx.$$

From (2.14), (2.7), and (3.1), we obtain

$$\max_{|\zeta - |x|| \leq 2} \left|\widetilde{\Omega}_r(\zeta)\right| = (2\pi)^{n/2} r^{n-1} \max_{|\zeta - |x|| \leq 2} \left|P_r(-\zeta^2)\right| \left|\mathbf{I}_{n/2-1}(r\zeta)\right|$$
$$\leq \frac{2\pi^{n/2} r^{n-1}}{\Gamma(n/2)} \max_{|\zeta - |x|| \leq 2} \left|P_r(-\zeta^2)\right| \cdot e^{r|\mathrm{Im}\zeta|} \leq \frac{2\pi^{n/2} r^{n-1} e^{2r}}{\Gamma(n/2)} \max_{|\zeta - |x|| \leq 2} \left|P_r(-\zeta^2)\right|.$$

Therefore,

$$\left|\langle \Psi_r^\lambda, \varphi \rangle\right| \leq \frac{4\pi^{n/2} r^{n-1} e^{2r}}{\Gamma(n/2) \left|\widetilde{\Omega}_r{}'(\lambda)\right|} \int_{\mathbb{R}^n} |\psi(x)| \max_{|\zeta - |x|| \leq 2} \left|P_r(-\zeta^2)\right| dx. \tag{3.18}$$

This inequality and Corollary 1 show that the series in (3.17) converges unconditionally in the space $\mathcal{D}'(\mathbb{R}^n)$ to some distribution f supported in $\overline{B}_r$. By Lemma 4, the spherical transform of this distribution satisfies the equality

$$\widetilde{f}(z) = \sum_{\lambda \in \mathcal{Z}_+(\widetilde{\Omega}_r)} \widetilde{\Psi_r^\lambda}(z) = \sum_{\lambda \in \mathcal{Z}_+(\widetilde{\Omega}_r)} \frac{2\lambda}{\widetilde{\Omega}_r{}'(\lambda)} \frac{\widetilde{\Omega}_r(z)}{z^2 - \lambda^2}. \tag{3.19}$$

In this case, if $\mu \in \mathcal{Z}_+(\widetilde{\Omega}_r)$, then

$$\widetilde{f}(\mu) = \frac{2\mu}{\widetilde{\Omega}_r{}'(\mu)} \lim_{z \to \mu} \frac{\widetilde{\Omega}_r(z)}{z^2 - \mu^2} = 1. \tag{3.20}$$

Further, since $\widetilde{f}(z) - 1$ and $\widetilde{\Omega}_r(z)$ are even entire functions of exponential type, by (3.20) and the simplicity of the zeros of $\widetilde{\Omega}_r$, their ratio

$$h(z) = \frac{\widetilde{f}(z) - 1}{\widetilde{\Omega}_r(z)}$$

is an entire function of at most first order (see [15, Ch. 1, Sect. 9, Corollary of Theorem 12]). For

$\operatorname{Im} z = \pm \operatorname{Re} z$, $z \neq 0$, it is estimated as follows:

$$|h(z)| \leq \frac{|\widetilde{f}(z)|}{|\widetilde{\Omega}_r(z)|} + \frac{1}{|\widetilde{\Omega}_r(z)|}$$

$$= \left| \sum_{\lambda \in \mathcal{Z}_+(\widetilde{\Omega}_r)} \frac{1}{\widetilde{\Omega}_r{}'(\lambda)} \left(\frac{1}{z-\lambda} - \frac{1}{z+\lambda} \right) \right| + \frac{1}{(2\pi)^{n/2} r^{n-1} |P_r(-z^2) \mathbf{I}_{n/2-1}(rz)|}$$

$$\leq \sum_{\lambda \in \mathcal{Z}_+(\widetilde{\Omega}_r)} \frac{1}{|\widetilde{\Omega}_r{}'(\lambda)|} \left(\frac{1}{|z-\lambda|} + \frac{1}{|z+\lambda|} \right) + \frac{1}{(2\pi)^{n/2} r^{n-1} |P_r(-z^2) \mathbf{I}_{n/2-1}(rz)|}$$

$$\leq \frac{2\sqrt{2}}{|z|} \sum_{\lambda \in \mathcal{Z}_+(\widetilde{\Omega}_r)} \frac{1}{|\widetilde{\Omega}_r{}'(\lambda)|} + \frac{1}{(2\pi)^{n/2} r^{n-1} |P_r(-z^2) \mathbf{I}_{n/2-1}(rz)|}.$$

It can be seen from this estimate and relations (3.6) and (3.2) that

$$\lim_{\substack{z \to \infty \\ \operatorname{Im} z = \pm \operatorname{Re} z}} h(z) = 0. \tag{3.21}$$

Then, according to the Phragmén–Lindelöf principle, h is bounded on $\mathbb{C}$. Now it follows from (3.21) and Liouville's theorem that $h = 0$. Hence, $\widetilde{f} = 1$, i.e., $f = \delta$. Thus, Lemma 5 is proved. □

Lemma 6. *Let $\lambda \in \mathcal{Z}_+(\widetilde{\Omega}_{r_1})$, $\mu \in \mathcal{Z}_+(\widetilde{\Omega}_{r_2})$. Then*

$$(\lambda^2 - \mu^2) \Psi_{r_1}^{\lambda} * \Psi_{r_2}^{\mu} = \frac{4\lambda\mu}{\widetilde{\Omega}_{r_1}{}'(\lambda) \widetilde{\Omega}_{r_2}{}'(\mu)} \left(\Omega_{r_2} * \Omega_{r_1}^{\lambda} - \Omega_{r_1} * \Omega_{r_2}^{\mu} \right). \tag{3.22}$$

P r o o f. By (3.15), (3.13), and (3.16), we have

$$(\Delta + \lambda^2) \left(\Psi_{r_1}^{\lambda} \right) = -\frac{2\lambda}{\widetilde{\Omega}_{r_1}{}'(\lambda)} \Omega_{r_1}, \tag{3.23}$$

$$(\Delta + \mu^2) \left(\Psi_{r_2}^{\mu} \right) = -\frac{2\mu}{\widetilde{\Omega}_{r_2}{}'(\mu)} \Omega_{r_2}. \tag{3.24}$$

From (3.23), (3.16) and the permutation of the differentiation operator with convolution, we obtain

$$(\Delta + \lambda^2) \left(\Psi_{r_1}^{\lambda} * \Psi_{r_2}^{\mu} \right) = \frac{-4\lambda\mu}{\widetilde{\Omega}_{r_1}{}'(\lambda) \widetilde{\Omega}_{r_2}{}'(\mu)} \Omega_{r_1} * \Omega_{r_2}^{\mu}.$$

Similarly, it follows from (3.24) that

$$-(\Delta + \mu^2) \left(\Psi_{r_1}^{\lambda} * \Psi_{r_2}^{\mu} \right) = \frac{4\lambda\mu}{\widetilde{\Omega}_{r_1}{}'(\lambda) \widetilde{\Omega}_{r_2}{}'(\mu)} \Omega_{r_2} * \Omega_{r_1}^{\lambda}.$$

Adding the last two equalities, we arrive at relation (3.22). □

4. Proof of Theorem 3

By Lemma 5, we obtain

$$\sum_{\lambda\in\mathcal{Z}_+(\widetilde{\Omega}_{r_1})}\Psi_{r_1}^{\lambda}=\delta,\qquad \sum_{\mu\in\mathcal{Z}_+(\widetilde{\Omega}_{r_2})}\Psi_{r_2}^{\mu}=\delta. \tag{4.1}$$

We claim that

$$\sum_{\lambda\in\mathcal{Z}_+(\widetilde{\Omega}_{r_1})}\sum_{\mu\in\mathcal{Z}_+(\widetilde{\Omega}_{r_2})}\Psi_{r_1}^{\lambda}*\Psi_{r_2}^{\mu}=\delta, \tag{4.2}$$

where the series in (4.2) converges unconditionally in the space $\mathcal{D}'(\mathbb{R}^n)$. Let $\varphi\in\mathcal{D}(\mathbb{R}^n)$, $\psi\in\mathcal{S}(\mathbb{R}^n)$, and let $\varphi=\widehat{\psi}$. For $\lambda\in\mathcal{Z}_+(\widetilde{\Omega}_{r_1})$ and $\mu\in\mathcal{Z}_+(\widetilde{\Omega}_{r_2})$, we have (see (2.3) and the proof of estimate (3.18))

$$\begin{aligned}\big|\big\langle\Psi_{r_1}^{\lambda}*\Psi_{r_2}^{\mu},\varphi\big\rangle\big|&=\big|\big\langle\Psi_{r_1}^{\lambda}*\Psi_{r_2}^{\mu},\widehat{\psi}\big\rangle\big|=\big|\big\langle\widehat{\Psi_{r_1}^{\lambda}}\,\widehat{\Psi_{r_2}^{\mu}},\psi\big\rangle\big|=\left|\int_{\mathbb{R}^n}\psi(x)\widetilde{\Psi_{r_1}^{\lambda}}(|x|)\widetilde{\Psi_{r_2}^{\mu}}(|x|)dx\right|\\&=\frac{4}{\big|\widetilde{\Omega}_{r_1}'(\lambda)\widetilde{\Omega}_{r_2}'(\mu)\big|}\left|\int_{\mathbb{R}^n}\psi(x)\frac{\lambda\widetilde{\Omega}_{r_1}(|x|)}{|x|^2-\lambda^2}\frac{\mu\widetilde{\Omega}_{r_2}(|x|)}{|x|^2-\mu^2}dx\right|\\&\leq\frac{16\pi^n(r_1r_2)^{n-1}e^{2(r_1+r_2)}}{\big|\widetilde{\Omega}_{r_1}'(\lambda)\widetilde{\Omega}_{r_2}'(\mu)\big|\Gamma^2(n/2)}\int_{\mathbb{R}^n}|\psi(x)|\max_{|\zeta-|x||\leq 2}\big|P_{r_1}(-\zeta^2)\big|\max_{|\zeta-|x||\leq 2}\big|P_{r_2}(-\zeta^2)\big|dx.\end{aligned}$$

This and (3.6) imply that

$$\sum_{\lambda\in\mathcal{Z}_+(\widetilde{\Omega}_{r_1})}\Bigg(\sum_{\mu\in\mathcal{Z}_+(\widetilde{\Omega}_{r_2})}\big|\big\langle\Psi_{r_1}^{\lambda}*\Psi_{r_2}^{\mu},\varphi\big\rangle\big|\Bigg)<\infty.$$

Therefore (see, for example, [14, Ch. 1, Theorem 1.24]), the series in (4.2) converges unconditionally in the space $\mathcal{D}'(\mathbb{R}^n)$. In addition (see (2.2) and (4.1)),

$$\begin{aligned}\sum_{\lambda\in\mathcal{Z}_+(\widetilde{\Omega}_{r_1})}\sum_{\mu\in\mathcal{Z}_+(\widetilde{\Omega}_{r_2})}\langle\Psi_{r_1}^{\lambda}*\Psi_{r_2}^{\mu},\varphi\rangle&=\sum_{\lambda\in\mathcal{Z}_+(\widetilde{\Omega}_{r_1})}\Bigg(\sum_{\mu\in\mathcal{Z}_+(\widetilde{\Omega}_{r_2})}\big\langle\Psi_{r_2}^{\mu}(y),\langle\Psi_{r_1}^{\lambda}(x),\varphi(x+y)\rangle\big\rangle\Bigg)\\&=\sum_{\lambda\in\mathcal{Z}_+(\widetilde{\Omega}_{r_1})}\langle\Psi_{r_1}^{\lambda}(x),\varphi(x)\rangle=\varphi(0),\end{aligned}$$

which proves (4.2).

Convolving both parts of (4.2) with f and taking into account the separate continuity of the convolution of $f\in\mathcal{D}'(\mathbb{R}^n)$ with $g\in\mathcal{E}'(\mathbb{R}^n)$, (3.22) and (2.16), we find

$$f=\sum_{\lambda\in\mathcal{Z}_+(\widetilde{\Omega}_{r_1})}\sum_{\mu\in\mathcal{Z}_+(\widetilde{\Omega}_{r_2})}\frac{4\lambda\mu}{(\lambda^2-\mu^2)\widetilde{\Omega}_{r_1}'(\lambda)\widetilde{\Omega}_{r_2}'(\mu)}\big(f*(\Omega_{r_2}*\Omega_{r_1}^{\lambda})-f*(\Omega_{r_1}*\Omega_{r_2}^{\mu})\big). \tag{4.3}$$

Finally, using (4.3), (2.12), and the commutativity of the convolution operator with the differentiation operator, we arrive at formula (2.20). Thus, Theorem 3 is proved. □

5. Conclusion

The proof of Theorem 3 shows that the key role in formula (2.20) is played by the expansion of the delta function into a series of distributions Ψ_r^λ, $\lambda \in \mathcal{Z}_+(\widetilde{\Omega}_r)$ (see Lemma 5). This system of distributions is biorthogonal to the system of spherical functions φ_μ, $\mu \in \mathcal{Z}_+(\widetilde{\Omega}_r)$, i.e.,

$$\langle \Psi_r^\lambda, \varphi_\mu \rangle = \begin{cases} 0 & \text{if } \ \mu \neq \lambda, \\ 1 & \text{if } \ \mu = \lambda \end{cases}$$

(see (2.4), (3.15) and (3.16)). Using similar expansions, it is possible to obtain inversion formulas for other convolution operators with radial distributions.

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