REGULARIZATION OF PONTRYAGIN MAXIMUM PRINCIPLE IN OPTIMAL CONTROL OF DISTRIBUTED SYSTEMS[1]

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Abstract: This article is devoted to studying dual regularization method applied to parametric convex optimal control problem of controlled third boundary-value problem for parabolic equation with boundary control and with equality and inequality pointwise state constraints. This dual regularization method yields the corresponding necessary and sufficient conditions for minimizing sequences, namely, the stable, with respect to perturbation of input data, sequential or, in other words, regularized Lagrange principle in nondifferential form and Pontryagin maximum principle for the original problem. Regardless of the fact that the stability or instability of the original optimal control problem, they stably generate a minimizing approximate solutions in the sense of J. Warga for it. For this reason, we can interpret these regularized Lagrange principle and Pontryagin maximum principle as tools for direct solving unstable optimal control problems and reducing to them unstable inverse problems.

Key words: Optimal boundary control, Parabolic equation, Minimizing sequence, Dual regularization, Stability, Lagrange principle, Pontryagin maximum principle

Introduction

Pontryagin maximum principle is the central result of all optimal control theory, including optimal control for differential equations with partial derivatives. Its statement and proof assume, first of all, that the optimal control problem is considered in an ideal situation, when its input data are known exactly. However, in the vast number of important practical problems of optimal control, as well as numerous problems reducing to optimal control problems, the requirement of exact defining input data is very unnatural, and in many undoubtedly interest cases is simply impracticable. In similar problems, we can not, strictly speaking, take as an approximation to the solution of the initial (unperturbed) problem with the exact input data, a control formally satisfying the maximum principle in the perturbed problem. The reason of such situation lies in the natural instability of optimization problems with respect to perturbation of its input data. As a typical property of optimization problems in general, including constrained ones, instability fully manifests itself in optimal control problems (see., e.g., [10]). As a consequence, the above mentioned instability implies "instability" of the classical optimality conditions, including the conditions in the form of Pontryagin maximum principle. This instability manifests itself in selecting arbitrarily distant "perturbed" optimal elements from their unperturbed counterparts in the case of an arbitrarily small perturbations of the input data. The above applies, in full measure, both to discussed below optimal control problem with pointwise state constraints for linear parabolic equation in divergent form, and to the classical optimality conditions in the form of the Lagrange principle and the Pontryagin maximum principle for this problem.

In this paper we discuss how to overcome the problem of instability of the classical optimality conditions in optimal control problems applying dual regularization method (see., e.g., [11–13]) and simultaneous transition to the concept of minimizing sequence of admissible elements as the main concept of optimization theory. The latter role acts the concept of the minimizing approximate solution in the sense of J. Warga [23]. The main attention in the paper is given to the discussion of the so–called regularized or, in other words, stable, with respect to perturbation of input data, sequential Lagrange principle in the nondifferential form and Pontryagin maximum principle. Regardless of the stability or instability of the original optimal control problem, they stably generate minimizing approximate solutions for it. For this reason, we can interpret the regularized Lagrange principle and Pontryagin maximum principle that are obtained in the article as tools for direct solving unstable optimal control problems and reducing to them unstable inverse problems [10,14–16]. Thus, they contribute to a significant expansion of the range of applicability of the theory of optimal control in which a central role belongs to classic constructions of the Lagrange and Hamilton–Pontryagin functions. Finally, we note that discussed in this paper regularized Lagrange principle in the nondifferential form and Pontryagin maximum principle may have another kind, more convenient for applications [4,9,15]. Justification of these alternative forms of the regularized Lagrange principle and Pontryagin maximum principle is based on the so-called method of iterative dual regularization [11,12]. In this case, they take the form of iterative processes with the corresponding stopping rules when the error of input data is fixed and finite. Here these alternative forms are not considered.

1. Statement of optimal control problem

We consider the fixed-time parametric optimal control problem

$$g_0^\delta(\pi) \to \min, \quad \pi \equiv (u, w) \in \mathcal{D} \subset L_2(Q_T) \times L_2(S_T), \qquad (P_{p,r}^\delta)$$

$$g_1^\delta(\pi)(x,t) \equiv \varphi_1^\delta(x,t) z^\delta[\pi](x,t) = h^\delta(x,t) + p(x,t) \quad \text{for a.e. } (x,t) \in Q,$$

$$g_2^\delta(\pi)(x,t) \equiv \varphi_2^\delta(x,t, z^\delta[\pi](x,t)) \leq r(x,t) \quad \text{for a.e. } (x,t) \in Q$$

with equality and inequality pointwise state constraints understood as ones in the Hilbert space $\mathcal{H} \equiv L_2(Q)$;

$$\mathcal{D} \equiv \{u \in L_2(Q_T) : u(x,t) \in U \text{ for a.e. } (x,t) \in Q_T\} \times \{w \in L_2(S_T) : w(x,t) \in W \text{ for a.e. } (x,t) \in S_T\};$$

$U, W \subset \mathbb{R}^1$ are convex compact sets. In this problem, $p \in \mathcal{H}$ and $r \in \mathcal{H}$ are parameters; $g_0^\delta : L_2(Q_T) \times L_2(S_T)$ is a continuous convex functional, $Q \subset \overline{Q}_{\iota,T}$ is a compact set without isolated points with a nonempty interior, $\iota \in (0,T)$, $Q = \text{cl int} Q$; and $z^\delta[\pi] \in V_2^{1,0}(Q_T) \cap C(\overline{Q}_T)$ is a weak solution [6] to the third boundary-value problem[2]

$$z_t - \frac{\partial}{\partial x_i}(a_{i,j}(x,t) z_{x_j}) + a^\delta(x,t) z + u(x,t) = 0, \qquad (1.1)$$

$$z(x,0) = v_0(x), \quad x \in \Omega, \qquad \frac{\partial z}{\partial \mathcal{N}} + \sigma^\delta(x,t) z = w(x,t), \quad (x,t) \in S_T.$$

The superscript δ in the input data of Problem $(P_{p,r}^\delta)$ indicates that these data are exact ($\delta = 0$) or perturbed ($\delta > 0$), i.e., they are specified with an error, $\delta \in [0, \delta_0]$, where $\delta_0 > 0$ is a fixed number.

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[2]Here and below, we use the notations for the sets Q_T, S_T, $Q_{i,T}$ and also for functional spaces and norms of their elements adopted in monograph [6].

For definiteness, as a target functional we take the terminal one

$$g_0^\delta(\pi) \equiv \int_\Omega G^\delta(x, z^\delta[\pi](x,T))dx.$$

The input data for Problem $(P_{p,r}^0)$ are assumed to meet the following conditions:

a) It is true that $a_{i,j} \in L_\infty(Q_T)$, $i,j = 1,\ldots,n$, $a^\delta \in L_\infty(Q_T)$, $\sigma^\delta \in L_\infty(S_T)$, $v_0^\delta \in C(\overline{\Omega})$,

$$\nu|\xi|^2 \le a_{i,j}(x,t)\xi_i\xi_j \le \mu|\xi|^2 \quad \forall(x,t) \in Q_T, \quad \nu,\mu > 0,$$

$$a^\delta(x,t) \ge C_0 \text{ for a.e. } (x,t) \in Q_T, \quad \sigma^\delta(x,t) \ge C_0 \quad \text{for a.e. } (x,t) \in S_T;$$

b) It is true that ϕ_1^δ, $h^\delta \in L_\infty(Q)$; $\phi_2^\delta : Q \times \mathbb{R}^1 \to \mathbb{R}^1$ is Lebesgue measurable function that is continuous and convex with respect to z for a.e. $(x,t) \in Q$, $\varphi_2^\delta(\cdot,\cdot,z(\cdot,\cdot)) \in L_\infty(Q)$ $\forall z \in C(Q)$; $G^\delta : \Omega \times \mathbb{R}^1 \to \mathbb{R}^1$ is Lebesgue measurable function that is continuous and convex with respect to z for a.e. $x \in \Omega$, $G^\delta(\cdot,z(\cdot,T)) \in L_\infty(\Omega)$ $\forall z(\cdot,T) \in C(Q)$;

c) $\Omega \subset \mathbb{R}^n$ be a bounded domain with piece–wise smooth boundary S.

Assume that the following estimates hold:

$$|G^\delta(x,z) - G^0(x,z)| \le C_M\delta \quad \forall (x,z) \in \Omega \times S_M^1,$$

$$\|\varphi_1^\delta - \varphi_1^0\|_{\infty,Q} \le C\delta, \quad \|h^\delta - h^0\|_{\infty,Q} \le C\delta,$$

$$|\varphi_2^\delta(x,t,z) - \varphi_2^0(x,t,z)| \le C_M\delta \quad \forall (x,t,z) \in Q \times S_M^1, \tag{1.2}$$

$$\|a^\delta - a^0\|_{\infty,Q_T} \le C\delta, \quad |v_0^\delta - v_0^0|_{\overline{\Omega}}^{(0)} \le C\delta, \quad \|\sigma^\delta - \sigma^0\|_{\infty,S_T} \le C\delta,$$

where C, $C_M > 0$ are independent of δ; $S_M^n \equiv \{x \in \mathbb{R}^n : |x| < M\}$. Let's note, that the conditions on the input data of Problem $(P_{p,r}^\delta)$, and also the estimates of deviations of the perturbed input data from the exact ones can be weakened.

2. Basic concepts and auxiliary propositions

In this paper we use for discussing the main results, related to the stable sequential Lagrange principle and Pontryagin maximum principle in Problem $(P_{p,r}^0)$, a scheme of studying the similar optimization problems in the papers [17,19] for a system of controlled ordinary differential equations (see also [20,21] for the case of distributed systems). In these works, both spaces of admissible controls and spaces, containing lie images of the operators that define the pointwise state constraints, were presented as Hilbert spaces of square-integrable functions. For this reason, we put the set $\mathcal{D}$ of admissible controls π into a Hilbert space also, i.e., assume that

$$\mathcal{D} \subset Z \equiv L_2(Q_T) \times L_2(S_T), \quad \|\pi\| \equiv (\|u\|_{2,Q_T}^2 + \|w\|_{2,S_T}^2)^{1/2}.$$

At the same time, we note that the conditions on the input data of Problem $(P_{p,r}^\delta)$ allow formally to consider that the operators g_1^δ, g_2^δ, specifying the state constraints of the problem, act into space $L_p(Q)$ with any index $p \in [1,+\infty]$. However, in this paper, taking into account the above remark, we will put images of these functional operators in the Hilbert space $L_2(Q) \equiv \mathcal{H}$. We note here that the imbedding the images of the operators g_1^δ, g_2^δ, specifying the state constraints, into reflexive space $L_p(Q)$ with $1 < p < 2$, in general, permits significantly to weaken the conditions on the input data and to get, strictly speaking, a stronger result in Problem $(P_{p,r}^0)$.

If Problem $(P^0_{p,r})$ is solvable (it has a unique solution if g^0_0 is strictly (strongly) convex), then its solutions are denoted by $\pi^0_{p,r} \equiv (u^0_{p,r}, w^0_{p,r})$, and the set of all such solutions is designated as $U^0_{p,r}$. Define the Lagrange functional, a set of its minimizers and the concave dual problem

$$L^\delta_{p,r}(\pi, \lambda, \mu) \equiv g^\delta_0(\pi) + \langle \lambda, g^\delta_1(\pi) - h^\delta - p \rangle + \langle \mu, g^\delta_2(\pi) - r \rangle, \quad \pi \in \mathcal{D},$$

$$U^\delta[\lambda, \mu] \equiv \mathrm{Argmin}\, \{L^\delta_{p,r}(\pi, \lambda, \mu) : \pi \in \mathcal{D}\} \quad \forall (\lambda, \mu) \in \mathcal{H} \times \mathcal{H}_+,$$

$$\mathcal{H}_+ \equiv \{z \in \mathcal{H} : z(x,t) \geq 0 \text{ for a.e. } (x,t) \in Q\},$$

$$V^\delta_{p,r}(\lambda, \mu) \to \sup, \quad (\lambda, \mu) \in \mathcal{H} \times \mathcal{H}_+, \quad V^\delta_{p,r}(\lambda, \mu) \equiv \inf_{\pi \in \mathcal{D}} L^\delta_{p,r}(\pi, \lambda, \mu).$$

Since the Lagrange functional is continuous and convex for any pair $(\lambda, \mu) \in \mathcal{H} \times \mathcal{H}_+$, and the set $\mathcal{D}$ is bounded, the dual functional $V^\delta_{p,r}$, is obviously defined and finite for any $(\lambda, \mu) \in \mathcal{H} \times \mathcal{H}_+$.

The concept of a minimizing approximate solution in the sense of J. Warga [23] is of great importance for the design of a dual regularizing algorithm for Problem $(P^0_{p,r})$. Recall that a minimizing approximate solution is a sequence $\pi^i \equiv (u^i, w^i)$, $i = 1, 2, \ldots$ such that $g^0_0(\pi^i) \leq \beta(p,r) + \delta^i$, $\pi^i \in \mathcal{D}^{0,\epsilon^i}_{p,r}$ for some nonnegative number sequences δ^i and ϵ^i, $i = 1, 2, \ldots$, that converge to zero. Here, $\beta(p,r)$ is the generalized infimum, i.e., S-function:

$$\beta(p,r) \equiv \lim_{\epsilon \to +0} \beta_\epsilon(p,r), \quad \beta_\epsilon(p,r) \equiv \inf_{\pi \in \mathcal{D}^{0,\epsilon}_{p,r}} g^0_0(\pi), \quad \beta_\epsilon(p,r) \equiv +\infty \text{ if } \mathcal{D}^{0,\epsilon}_{p,r} = \emptyset,$$

$$\mathcal{D}^{\delta,\epsilon}_{p,r} \equiv \{\pi \in \mathcal{D} : \|g^\delta_1(\pi) - h^\delta - p\|_{2,Q} \leq \epsilon, \min_{z \in \mathcal{H}_-} \|g^\delta_2(\pi) - r - z\|_{2,Q} \leq \epsilon\}, \quad \epsilon \geq 0,$$

$$\mathcal{D}^{00}_{p,r} \equiv \mathcal{D}^0_{p,r}, \quad \mathcal{H}_- \equiv \{z \in \mathcal{H} : z(x,t) \leq 0 \text{ for a.e. } (x,t) \in Q\}, \quad \mathcal{H}_+ \equiv -\mathcal{H}_-.$$

Obviously, in the general situation, $\beta(p,r) \leq \beta_0(p,r)$, where $\beta_0(p,r)$ is the classical value of the problem. However, in the case of Problem $(P^0_{p,r})$, we have $\beta(p,r) = \beta_0(p,r)$. Simultaneously, we may asset that $\beta : \mathcal{H} \times \mathcal{H} \to \mathbb{R}^1 \cup \{+\infty\}$ is a convex and lower semicontinuous function. Note here that the existence of a minimizing approximate solution in Problem $(P^0_{p,r})$ obviously implies its solvability.

From the conditions a) – c) and from the theorem on the existence of a weak solution of the third boundary-value problem for a linear parabolic equation of the divergent type [6, ch. III, section 5] (see also [5, 7]), it follows that the direct boundary-value problem (1.1) and the corresponding adjoint problem are uniquely solvable in $V^{1,0}_2(Q_T)$.

Proposition 1. *For any pair $(u, w) \in L_2(Q_T) \times L_2(S_T)$ and for any $T > 0$ the direct boundary-value problem (1.1) is uniquely solvable in $V^{1,0}_2(Q_T)$ and we have the estimate*

$$|z^\delta[\pi]|_{Q_T} + \|z^\delta[\pi]\|_{2,S_T} \leq C_T(\|u\|_{2,Q_T} + \|v_0\|_{2,\Omega} + \|w\|_{2,S_T})$$

where the constant C_T is independent of $\delta \geq 0$ and pair $\pi \equiv (u, w) \in L_2(Q_T) \times L_2(S_T)$. Also the adjoint problem

$$-\eta_t - \frac{\partial}{\partial x_j} a_{i,j}(x,t)\eta_{x_i} + a^\delta(x,t)\eta = \chi(x,t),$$

$$\eta(x,T) = \psi(x), \quad x \in \Omega, \quad \frac{\partial \eta}{\partial \mathcal{N}} + \sigma^\delta(x,t)\eta = \omega(x,t), \quad (x,t) \in S_T$$

is uniquely solvable in $V^{1,0}_2(Q_T)$ for any $\chi \in L_2(Q_T)$, $\psi \in L_2(\Omega)$, $\omega \in L_2(S_T)$ and any $T > 0$. Its solution is denoted as $\eta[\chi, \psi, \omega]$. Simultaneously, the estimate

$$|\eta^\delta[\chi, \psi, \omega]|_{Q_T} + \|\eta^\delta[\chi, \psi, \omega]\|_{2,S_T} \leq C^1_T(\|\chi\|_{2,Q_T} + \|\psi\|_{2,\Omega} + \|\omega\|_{2,S_T}),$$

is true where the constant C^1_T is independent of $\delta \geq 0$ and a triple (χ, ψ, ω).

Simultaneously, from conditions a) – c) and the theorems on the existence of a weak (generalized) solution of the third boundary-value problem for a linear parabolic equation of the divergent type (see, e.g., [3, 8]), it follows that the direct boundary-value problem is uniquely solvable in $V_2^{1,0}(Q_T) \cap C(\overline{Q}_T)$.

Proposition 2. *Let us $l > n + 1$. For any pair $(u, w) \in L_l(Q_T) \times L_l(S_T)$ and any $T > 0$, $\delta \in [0, \delta_0]$ the direct boundary-value problem (1.1) is uniquely solvable in $V_2^{1,0}(Q_T) \cap C(\overline{Q}_T)$ and the estimate*

$$|z^\delta[\pi]|_{\overline{Q}_T}^{(0)} \leq C_T(\|u\|_{l,Q_T} + |v_0|_{\Omega}^{(0)} + \|w\|_{l,S_T}),$$

is true where the constant C_T is independent of pair $\pi \equiv (u, w)$ and δ.

Further, the minimization problem for Lagrange functional

$$L_{p,r}^\delta(\pi, \lambda, \mu) \to \min, \quad \pi \in \mathcal{D}, \quad \text{when} \quad (\lambda, \mu) \in \mathcal{H} \times \mathcal{H}_+ \tag{2.1}$$

plays the central role in all subsequent constructions. It is usual problem without equality and inequality constraints. It is solvable as a minimization problem for weakly semicontinuous functional on the weak compact set $\mathcal{D} \subset L_2(Q_T) \times L_2(S_T)$. Here, the weak semicontinuity is a consequence of the convexity and continuity with respect to π of the Lagrange functional. Minimizers $\pi^\delta[\lambda, \mu] \in U^\delta[\lambda, \mu]$ for this optimal control problem satisfy the Pontryagin maximum principle under supplementary assumption of the existence of Lebesgue measurable with respect to $(x, t) \in Q$ for all $z \in \mathbb{R}^1$ and continuous with respect to z for a.e. x, t gradients $\nabla_z \varphi_2^\delta(x, t, z)$, $\nabla_z G^\delta(x, z)$ with the estimates

$$|\nabla_z \varphi_2^\delta(x, t, z)| \leq C_M, \quad |\nabla_z G^\delta(x, z)| \leq C_M, \quad \forall z \in S_M^1,$$

where $C_M > 0$ is independent of δ. The following lemma is true due to the estimates of the propositions 1, 2 and to the so called two-parameter variation [22] of the pair $\pi^\delta[\lambda, \mu]$ that is needle-shaped with respect to control u and classical with respect to control w.

Lemma 1. *Let $H(y, \eta) \equiv -\eta y$ and the additional condition that specified above is fulfilled. Any pair $\pi^\delta[\lambda, \mu] = (u^\delta[\lambda, \mu], w^\delta[\lambda, \mu]) \in U^\delta[\lambda, \mu]$, $(\lambda, \mu) \in \mathcal{H} \times \mathcal{H}_+$ satisfies the (usual) Pontryagin maximum principle in the problem* (2.1): *for $\pi = \pi^\delta[\lambda, \mu]$ the following maximum relations*

$$H(u(x, t), \eta^\delta(x, t)) = \max_{u \in U} H(u, \eta^\delta(x, t)) \text{ for a.e. } Q_T, \tag{2.2}$$

$$H(w(s, t), \eta^\delta(s, t)) = \max_{w \in W} H(w, \eta^\delta(s, t)) \text{ for a.e. } S_T$$

hold, where $\eta^\delta(x, t)$, $(x, t) \in Q_T$ is a solution for $\pi = \pi^\delta[\lambda, \mu]$ of the adjoint problem

$$-\eta_t - \frac{\partial}{\partial x_j}(a_{i,j}(x, t)\eta_{x_i}) + a^\delta(x, t)\eta = \varphi_1^\delta(x, t)\lambda(x, t) + \nabla_z \varphi_2^\delta(x, t, z^\delta[\pi](x, t))\mu(x, t), \quad (x, t) \in Q_T,$$

$$\eta(x, T) = \nabla_z G^\delta(x, z^\delta[\pi](x, T)), \quad x \in \Omega,$$

$$\frac{\partial \eta(x, t)}{\partial \mathcal{N}} + \sigma^\delta(x, t)\eta = 0, \quad (x, t) \in S_T.$$

Remark 1. Note that here and below, if the functions φ_1^δ, $\nabla_z \varphi_2^\delta(\cdot, \cdot, z(\cdot, \cdot))$, $\lambda, \mu \in \mathcal{H}$ are considered on the entire cylinder Q_T, we set that the equalities $\varphi_1^\delta(x, t) = \nabla_z \varphi_2^\delta(x, t, z(x, t)) = \lambda(x, t) = \mu(x, t) = 0$ take place for $(x, t) \in Q_T \setminus Q$; the same notation is preserved if these functions are taken on the entire cylinder.

An important result for the subsequent presentation is the following lemma, which is a consequence of the classical asymmetric minimax theorem [2, Chapter 6, Section 2, Theorem 7].

Lemma 2. *The minimax equality*

$$\inf_{\pi \in \mathcal{D}} \sup_{(\lambda,\mu) \in \mathcal{H} \times \mathcal{H}_+} L^0_{p,r}(\pi, \lambda, \mu) = \sup_{(\lambda,\mu) \in \mathcal{H} \times \mathcal{H}_+} \inf_{\pi \in \mathcal{D}} L^0_{p,r}(\pi, \lambda, \mu),$$

is true. It can be rewritten as the duality relation

$$g^0_0(\pi^0_{p,r}) = \sup_{(\lambda,\mu) \in \mathcal{H} \times \mathcal{H}_+} V^0_{p,r}(\lambda, \mu). \tag{2.3}$$

In the next section we construct minimizing approximate solutions for Problem $(P^0_{p,r})$ from the elements $\pi^\delta[\lambda,\mu]$, $(\lambda,\mu) \in (\mathcal{H}) \times \mathcal{H}_+$. As consequence, this construction leads us to various versions of the stable sequential Lagrange principle and Pontragin maximum principle. In the case of strong convexity and subdifferentiability of the target functional g^0_0, these versions are statements about stable approximations of the solutions of Problem $(P^0_{p,r})$ in the metric of $Z \equiv L_2(Q_T) \times L_2(S_T)$ by the points $\pi^\delta[\lambda,\mu]$. Due to the estimates (1.2) and the propositions 1, 2 we may assert that the estimates

$$|g^\delta_0(\pi) - g^0_0(\pi)| \le C_1 \delta \quad \forall \pi \in \mathcal{D}, \quad \|g^\delta_1(\pi) - g^0_1(\pi)\|_{2,Q} \le C_2 \delta(1 + \|\pi\|) \quad \forall \pi \in Z, \tag{2.4}$$

$$\|h^\delta - h^0\|_{2,Q} \le C\delta, \quad \|g^\delta_2(\pi) - g^0_2(\pi)\|_{2,Q} \le C_3 \delta \quad \forall \pi \in \mathcal{D},$$

hold, in which the constants C_1, C_2, $C_3 > 0$ are independent of $\delta \in (0, \delta_0]$, π.

Since the set $\mathcal{D}$ is bounded, the dual functional is obviously defined and finite for any element $(\lambda,\mu) \in \mathcal{H} \times \mathcal{H}$. Moreover, it is also obvious that the value $V^\delta_{p,r}(\lambda,\mu)$ is reached at elements $\pi^\delta[\lambda,\mu]$ of the set $U^\delta[\lambda,\mu] \equiv \text{Argmin}\,\{L^\delta_{p,r}(\pi, \lambda, \mu), \pi \in \mathcal{D}\}$ for $(\lambda,\mu) \in \mathcal{H} \times \mathcal{H}_+$,

$$\mathcal{H}_+ \equiv \{z \in \mathcal{H} : z(x,t) \ge 0 \text{ for a.e. } (x,t) \in Q\}.$$

Note also that, by virtue of estimates (2.4) and since $\mathcal{D}$ is bounded, we have the estimate

$$|V^\delta_{p,r}(\lambda,\mu) - V^0_{p,r}(\lambda,\mu)| \le C\delta(1 + \|\lambda\| + \|\mu\|), \tag{2.5}$$

where $C > 0$ is a constant independent of λ, μ, δ.

3. Stable sequential Pontryagin maximum principle

In this section we discuss the so-called regularized or, in other words, stable, with respect to errors of input data, sequential Pontryagin maximum principle for Problem $(P^0_{p,r})$ as necessary and sufficient condition for elements of minimizing approximate solutions. Simultaneously, we may treat this condition as one for existence of a minimizing approximate solutions in Problem $(P^0_{p,r})$ with perturbed input data or as condition of stable construction of a minimizing sequence in this problem. The proof of the necessity of this condition is based on the dual regularization method [11–13] that is a stable algorithm of constructing a minimizing approximate solutions in Problem $(P^0_{p,r})$.

3.1. Dual regularization for optimal control problem with pointwise state constraints

The estimates (2.4) give a possibility to organize the procedure of the dual regularization in accordance with a scheme of the paper [19] for constructing a minimizing approximate solution in Problem $(P^0_{p,r})$. In accordance with this scheme the dual regularization consists in the direct solving dual of Problem $(P^0_{p,r})$ and Tikhonov stabilized problem

$$R^{\delta,\alpha(\delta)}_{p,r}(\lambda,\mu) \equiv V^\delta_{p,r}(\lambda,\mu) - \alpha(\delta)\|(\lambda,\mu)\|^2 \to \max, \quad (\lambda,\mu) \in \mathcal{H} \times \mathcal{H}_+$$

under consistency condition

$$\frac{\delta}{\alpha(\delta)} \to 0, \quad \alpha(\delta) \to 0, \quad \delta \to 0. \tag{3.1}$$

Let us denote $(\lambda_{p,r}^{\delta,\alpha}, \mu_{p,r}^{\delta,\alpha}) \equiv \operatorname{argmax}\{R_{p,r}^{\delta,\alpha}(\lambda,\mu) : (\lambda,\mu) \in \mathcal{H} \times \mathcal{H}_+\}$. The above dual regularization leads to constructing minimizing approximate solution in Problem $(P_{p,r}^0)$ from the elements $\pi^\delta[\lambda_{p,r}^{\delta,\alpha(\delta)}, \mu_{p,r}^{\delta,\alpha(\delta)}] \in \operatorname{Argmin}\{L_{p,r}^\delta(\pi,\lambda,\mu) : \pi \in \mathcal{D}\}$, when $\delta \to 0$.

In this section, we extend the algorithm of the dual regularization [12, 18] to the case of Problem $(P_{p,r}^0)$ in which the objective functional is only convex. Below we prove convergence theorem for dual regularization method in exact accordance with a scheme of proving the similar theorem in [19]. We note only that, as in [19], this proving uses a weak continuity of the operators g_1^δ, g_2^δ that is consequence of the conditions on the input data of Problem $(P_{p,r}^0)$ and a regularity of the bounded solutions of the boundary-value problem (1.1) (see Proposition 2) inside of the cylinder Q_T [6, ch.III, theorem 10.1].

Let Problem $(P_{p,r}^0)$ be solvable. To prove the convergence theorem for dual regularization method, first of all, we give a formula for the superdifferential (in the sense of a convex analysis) of the concave value functional $V_{p,r}^\delta$. The proof of this formula can be found in [12].

Lemma 3. *The superdifferential of the concave value functional* $V_{p,r}^\delta(\lambda,\mu)$ *at the point* $(\lambda,\mu) \in \mathcal{H} \times \mathcal{H}$ *is equal*

$$\partial V_{p,r}^\delta(\lambda,\mu) = \partial_C V_{p,r}^\delta(\lambda,\mu) = \operatorname{cl}\operatorname{conv}\{w - \lim_{i\to\infty}(g_1^\delta(u^i) - h^\delta - p, g_2^\delta(u^i) - r) : \pi^i \in \mathcal{D},$$

$$L_{p,r}^\delta(\pi^i,\lambda,\mu) \to \inf_{\pi\in\mathcal{D}} L_{p,r}^\delta(\pi,\lambda,\mu), \ i \to \infty\},$$

where $\partial_C V_{p,r}^\delta(\lambda,\mu)$ *is Clarke's generalized gradient of the functional* $V_{p,r}^\delta(\lambda,\mu)$ *at the point* (λ,μ) *and the limit* $w - \lim$ *is understood in the sense of weak convergence in the space* $\mathcal{H} \times \mathcal{H}$.

Further, to substantiate the dual regularization method in the case under consideration, we write the inequality $\forall(\lambda',\mu') \in \mathcal{H} \times \mathcal{H}_+$

$$\langle(I_1, I_2) - 2\alpha(\delta)(\lambda_{p,r}^{\delta,\alpha(\delta)}, \mu_{p,r}^{\delta,\alpha(\delta)}), (\lambda',\mu') - (\lambda_{p,r}^{\delta,\alpha(\delta)}, \mu_{p,r}^{\delta,\alpha(\delta)})\rangle \le 0$$

for some element $(I_1, I_2) \in \partial V_{p,r}^\delta(\lambda_{p,r}^{\delta,\alpha(\delta)}, \mu_{p,r}^{\delta,\alpha(\delta)})$.

By Lemma 3 and the classical properties of closed convex hulls (see [1, p. 210, 217]), we obtain

$$\langle\lim_{s\to\infty}\sum_{i=1}^{l(s,\delta)}\gamma_i(s,\delta)(w - \lim_{j\to\infty}(g_1^\delta(\pi_{s,i}^j) - h^\delta - p, g_2^\delta(\pi_{s,i}^j)) - r) - 2\alpha(\delta)(\lambda_{p,r}^{\delta,\alpha(\delta)}, \mu_{p,r}^{\delta,\alpha(\delta)}), \tag{3.2}$$

$$(\lambda',\mu') - (\lambda_{p,r}^{\delta,\alpha(\delta)}, \mu_{p,r}^{\delta,\alpha(\delta)})\rangle \le 0 \quad \forall(\lambda',\mu') \in \mathcal{H} \times \mathcal{H}_+,$$

where $\sum_{i=1}^{l(s,\delta)}\gamma_i(s,\delta) = 1$, $\gamma_i(s,\delta) \ge 0$, $i = 1, \ldots, l(s,\delta)$, and $\pi_{s,i}^j \in \mathcal{D}$, $j = 1, 2, \ldots$ is a sequence such that

$$L_{p,r}^\delta(\pi_{s,i}^j, \lambda_{p,r}^{\delta,\alpha(\delta)}, \mu_{p,r}^{\delta,\alpha(\delta)}) \to \min_{\pi\in\mathcal{D}} L_{p,r}^\delta(\pi, \lambda_{p,r}^{\delta,\alpha(\delta)}, \mu_{p,r}^{\delta,\alpha(\delta)}), \ j \to \infty.$$

Assume without loss of generality that the sequence $\pi_{s,i}^j \in \mathcal{D}$, $j = 1, 2, \ldots$, converges weakly as $j \to \infty$ to an element $\pi_{s,i} \in \mathcal{D}$, which obviously belongs to the set $U^\delta[\lambda_{p,r}^{\delta,\alpha(\delta)}, \mu_{p,r}^{\delta,\alpha(\delta)}]$. Due to weak continuity of the operators g_i^δ, $i = 1, 2$, and boundedness of $\mathcal{D}$, from (3.2) the inequality follows

$$\langle\lim_{s\to\infty}\sum_{i=1}^{l(s,\delta)}\gamma_i(s,\delta)(g_1^\delta(\pi_{s,i}) - h^\delta - p, g_2^\delta(\pi_{s,i}) - r) - 2\alpha(\delta)(\lambda_{p,r}^{\delta,\alpha(\delta)}, \mu_{p,r}^{\delta,\alpha(\delta)}),$$

$$(\lambda', \mu') - (\lambda_{p,r}^{\delta,\alpha(\delta)}, \mu_{p,r}^{\delta,\alpha(\delta)}))\rangle \leq 0 \quad \forall (\lambda', \mu') \in \mathcal{H} \times \mathcal{H}_+.$$

The above inequality implies the limit relations

$$\lim_{s\to\infty} \sum_{i=1}^{l(s,\delta)} \gamma_i(s,\delta)(g_1^\delta(\pi_{s,i}) - h^\delta - p) = 2\alpha(\delta)\lambda_{p,r}^{\delta,\alpha(\delta)}, \tag{3.3}$$

$$\lim_{s\to\infty} \sum_{i=1}^{l(s,\delta)} \gamma_i(s,\delta)(g_2^\delta(\pi_{s,i})(x,t) - r(x,t)) = 2\alpha(\delta)\mu_{p,r}^{\delta,\alpha(\delta)}(x,t) \tag{3.4}$$

$$\text{for a.e. } (x,t) \in \{(x,t) \in Q : \mu_{p,r}^{\delta,\alpha(\delta)}(x,t) > 0\},$$

$$\lim_{s\to\infty} \sum_{i=1}^{l(s,\delta)} \gamma_i(s,\delta)(g_2^\delta(\pi_{s,i})(x,t) - r(x,t)) \leq 0 \quad \text{for a.e. } (x,t) \in \{(x,t) \in Q : \mu_{p,r}^{\delta,\alpha(\delta)}(x,t) = 0\}. \tag{3.5}$$

In turn, the limit relations (3.3)–(3.5) imply the limit equalities

$$\lim_{s\to\infty} \langle \sum_{i=1}^{l(s,\delta)} \gamma_i(s,\delta)(g_1^\delta(\pi_{s,i}) - h^\delta - p), \lambda_{p,r}^{\delta,\alpha(\delta)}\rangle = 2\alpha(\delta)\|\lambda_{p,r}^{\delta,\alpha(\delta)}\|^2 \geq 0, \tag{3.6}$$

$$\lim_{s\to\infty} \langle \sum_{i=1}^{l(s,\delta)} \gamma_i(s,\delta)g_2^\delta(\pi_{s,i}) - r, \mu_{p,r}^{\delta,\alpha(\delta)}\rangle = 2\alpha(\delta)\|\mu_{p,r}^{\delta,\alpha(\delta)}\|^2 \geq 0.$$

From (3.4) we obtain also: if $\mu_{p,r}^{\delta,\alpha(\delta)}(x,t) > 0$ for some (x,t) belonging to a set of full measure in $\{(x,t) \in Q : \mu_{p,r}^{\delta,\alpha(\delta)}(x,t) > 0\}$, then

$$\lim_{s\to\infty} \sum_{i=1}^{l(s,\delta)} \gamma_i(s,\delta)(g_2^\delta(\pi_{s,i})(x,t) - r(x,t)) - 2\alpha(\delta)\mu_{p,r}^{\delta,\alpha(\delta)}(x,t) = 0, \tag{3.7}$$

$$\lim_{s\to\infty} \sum_{i=1}^{l(s,\delta)} \gamma_i(s,\delta)(g_2^\delta(\pi_{s,i})(x,t) - r(x,t))\mu_{p,r}^{\delta,\alpha(\delta)}(t) > 0.$$

This implies that for a.e. $(x,t) \in Q$ such that $\lim_{s\to\infty} \sum_{i=1}^{l(s,\delta)} \gamma_i(s,\delta)(g_2^\delta(\pi_{s,i})(x,t) - r(x,t)) < 0$, the equality $\mu_{p,r}^{\delta,\alpha(\delta)}(x,t) = 0$ holds. From (3.4) and (3.7) we obtain simultaneously that

$$\mu_{p,r}^{\delta,\alpha(\delta)}(x,t) \lim_{s\to\infty} \sum_{i=1}^{l(s,\delta)} \gamma_i(s,\delta)(g_2^\delta(\pi_{s,i})(x,t) - r(x,t)) \geq 0 \quad \text{for a.e. } (x,t) \in Q.$$

Besides, from (3.6) we get the inequality

$$\lim_{s\to\infty} \langle \sum_{i=1}^{l(s,\delta)} \gamma_i(s,\delta)(g_1^\delta(\pi_{s,i}) - h^\delta - p, g_2^\delta(\pi_{s,i}) - r), (\lambda_{p,r}^{\delta,\alpha(\delta)}, \mu_{p,r}^{\delta,\alpha(\delta)})\rangle =$$
$$2\alpha(\delta)(\|\lambda_{p,r}^{\delta,\alpha(\delta)}\|^2 + \|\mu_{p,r}^{\delta,\alpha(\delta)}\|^2) \geq 0. \tag{3.8}$$

Further, since for any $\pi_{p,r}^0 \in U_{p,r}^0$

$$L_{p,r}^\delta(\pi_{s,i}, \lambda_{p,r}^{\delta,\alpha(\delta)}, \mu_{p,r}^{\delta,\alpha(\delta)}) \equiv g_0^\delta(\pi_{s,i}) + \langle \lambda_{p,r}^{\delta,\alpha(\delta)}, g_1^\delta(\pi_{s,i}) - h^\delta - p\rangle + \langle \mu_{p,r}^{\delta,\alpha(\delta)}, g_2^\delta(\pi_{s,i}) - r\rangle \leq$$

$$L_{p,r}^{\delta}(\pi_{p,r}^0, \lambda_{p,r}^{\delta,\alpha(\delta)}, \mu_{p,r}^{\delta,\alpha(\delta)}) \equiv g_0^{\delta}(\pi_{p,r}^0) + \langle \lambda_{p,r}^{\delta,\alpha(\delta)}, g_1^{\delta}(\pi_{p,r}^0) - h^{\delta} - p \rangle + \langle \mu_{p,r}^{\delta,\alpha(\delta)}, g_2^{\delta}(\pi_{p,r}^0) - r \rangle \leq$$

$$g_0^0(\pi_{p,r}^0) + [g_0^{\delta}(\pi_{p,r}^0) - g_0^0(\pi_{p,r}^0)] + \|\lambda_{p,r}^{\delta,\alpha(\delta)}\| \|g_1^{\delta}(\pi_{p,r}^0) - h^{\delta} - p\| + \|\mu_{p,r}^{\delta,\alpha(\delta)}\| \|g_2^{\delta}(\pi_{p,r}^0) - g_2^0(\pi_{p,r}^0)\|,$$

due to the estimates (2.4) and the limit equality (3.8) and doing some elementary transformation, we obtain the estimate

$$2\alpha(\delta)(\|\lambda_{p,r}^{\delta,\alpha(\delta)}\|^2 + \|\mu_{p,r}^{\delta,\alpha(\delta)}\|^2) \leq$$

$$C_1 \delta \|\lambda_{p,r}^{\delta,\alpha(\delta)}\| + C_1 \delta \|\mu_{p,r}^{\delta,\alpha(\delta)}\| + g_0^0(\pi_{p,r}^0) + C_1 \delta - \min_{\pi \in \mathcal{D}} g_0^{\delta}(\pi) \leq$$

$$\sqrt{2} C_1 \delta \sqrt{\|\lambda_{p,r}^{\delta,\alpha(\delta)}\|^2 + \|\mu_{p,r}^{\delta,\alpha(\delta)}\|^2} + g_0^0(\pi_{p,r}^0) + C_1 \delta - \min_{\pi \in \mathcal{D}} g_0^{\delta}(\pi)$$

or

$$\alpha(\delta)(\|\lambda_{p,r}^{\delta,\alpha(\delta)}\|^2 + \|\mu_{p,r}^{\delta,\alpha(\delta)}\|^2) - C_2 \delta \sqrt{\|\lambda_{p,r}^{\delta,\alpha(\delta)}\|^2 + \|\mu_{p,r}^{\delta,\alpha(\delta)}\|^2} - g_0^0(\pi_{p,r}^0) - C_1 \delta + \min_{\pi \in \mathcal{D}} g_0^{\delta}(\pi) \leq 0,$$

where $C_1, C_2 > 0$ are independent of constant δ. From here, the estimate follows

$$\sqrt{\|\lambda_{p,r}^{\delta,\alpha(\delta)}\|^2 + \|\mu_{p,r}^{\delta,\alpha(\delta)}\|^2} \leq \frac{C_2 \delta + \sqrt{(C_2 \delta)^2 - 4\alpha(\delta)K(\delta)}}{2\alpha(\delta)},$$

where $K(\delta) \equiv \min_{\pi \in \mathcal{D}} g_0^{\delta}(\pi) - g_0^0(\pi_{p,r}^0) - C\delta$. In turn, this estimate implies the limit realtions

$$\alpha(\delta)\|\lambda_{p,r}^{\delta,\alpha(\delta)}\| \to 0, \quad \alpha(\delta)\|\mu_{p,r}^{\delta,\alpha(\delta)}\| \to 0, \quad \delta \to 0. \tag{3.9}$$

Further, the limit relations (3.3)–(3.5), (3.9) imply

$$\lim_{s \to \infty} \sum_{i=1}^{l(s,\delta)} \gamma_i(s,\delta)(g_1^{\delta}(\pi_{s,i}) - h^{\delta} - p) \to 0, \quad \delta \to 0,$$

$$\lim_{s \to \infty} \sum_{i=1}^{l(s,\delta)} \gamma_i(s,\delta)(g_2^{\delta}(\pi_{s,i}) - r) \leq \phi(\delta), \quad \|\phi(\delta)\| \to 0, \quad \delta \to 0,$$

where the inequality $\lim_{s \to \infty} \sum_{i=1}^{l(s,\delta)} \gamma_i(s,\delta)(g_2^{\delta}(\pi_{s,i}) - r) \leq \phi(\delta)$ is understood in the sense of ordering on the cone of nonpositive functions $\mathcal{H}_-$.

Denoting by $\pi_{\delta} \in U^{\delta}[\lambda_{p,r}^{\delta,\alpha(\delta)}, \mu_{p,r}^{\delta,\alpha(\delta)}]$ any weak limit point of the sequence $\sum_{i=1}^{l(s,\delta)} \gamma_i(s,\delta)\pi_{s,i}$, $s = 1, 2, \ldots$ and taking into account the inequality

$$g_2^{\delta}\left(\sum_{i=1}^{l(s,\delta)} \gamma_i(s,\delta)\pi_{s,i}\right) \leq \sum_{i=1}^{l(s,\delta)} \gamma_i(s,\delta)g_2^{\delta}(\pi_{s,i}),$$

which is understood also in the sense of ordering on the cone of nonpositive functions, we obtain the limit relations

$$g_1^{\delta}(\pi_{\delta}) - h^{\delta} - p \to 0, \quad g_2^{\delta}(\pi_{\delta}) - r \leq \lim_{s \to \infty} \sum_{i=1}^{l(s,\delta)} \gamma_i(s,\delta)(g_2^{\delta}(\pi_{s,i}) - r) \leq \phi(\delta), \quad \delta \to 0,$$

and, as a consequence, due to the boundedness of $\mathcal{D}$, the limit relations

$$g_1^0(\pi_{\delta}) - h^0 - p \to 0, \quad g_2^0(\pi_{\delta}) - r \leq \bar{\phi}(\delta), \quad \|\bar{\phi}(\delta)\| \to 0, \quad \delta \to 0. \tag{3.10}$$

Simultaneously, due to the inclusion $\pi_{s,i} \in U^\delta[\lambda_{p,r}^{\delta,\alpha(\delta)}, \mu_{p,r}^{\delta,\alpha(\delta)}]$ we have the inequality

$$g_0^\delta(\pi_{s,i}) + \langle \lambda_{p,r}^{\delta,\alpha(\delta)}, g_1^\delta(\pi_{s,i}) - h^\delta - p \rangle + \langle \mu_{p,r}^{\delta,\alpha(\delta)}, g_2^\delta(\pi_{s,i}) - r \rangle \leq$$

$$g_0^\delta(\pi) + \langle \lambda_{p,r}^{\delta,\alpha(\delta)}, g_1^\delta(\pi) - h^\delta - p \rangle + \langle \mu_{p,r}^{\delta,\alpha(\delta)}, g_2^\delta(\pi) - r \rangle \quad \forall \, \pi \in \mathcal{D}.$$

Hence, due to the limit relation (3.8), we can write for any $u_{p,r}^0 \in U_{p,r}^0$

$$\liminf_{s \to \infty} \sum_{i=1}^{l(s,\delta)} \gamma_i(s,\delta) g_0^\delta(\pi_{s,i}) \leq g_0^\delta(\pi_{p,r}^0) + \langle \lambda_{p,r}^{\delta,\alpha(\delta)}, g_1^\delta(\pi_{p,r}^0) - h^\delta - p \rangle + \langle \mu_{p,r}^{\delta,\alpha(\delta)}, g_2^\delta(\pi_{p,r}^0) - r \rangle.$$

In turn, from here, due to the consistency condition (3.1), the estimates (1.2) and the boundedness of $\mathcal{D}$ we derive

$$\liminf_{s \to \infty} \sum_{i=1}^{l(s,\delta)} \gamma_i(s,\delta) g_0^0(\pi_{s,i}) \leq g_0^0(\pi_{p,r}^0) + \tilde{\phi}(\delta), \quad \tilde{\phi}(\delta) \to 0, \quad \delta \to 0$$

or

$$g_0^0(\pi_\delta) \leq \liminf_{s \to \infty} g_0^0\left(\sum_{i=1}^{l(s,\delta)} \gamma_i(s,\delta) \pi_{s,i} \right) \leq \liminf_{s \to \infty} \sum_{i=1}^{l(s,\delta)} \gamma_i(s,\delta) g_0^0(\pi_{s,i}) \leq$$

$$g_0^0(\pi_{p,r}^0) + \tilde{\phi}(\delta), \quad \tilde{\phi}(\delta) \to 0, \quad \delta \to 0.$$

Thus, by virtue of the boundedness of $\mathcal{D}$, weak lower semicontinuity of g_0^0 and weak continuity of g_i^0, $i = 1, 2$, we constructed the family of elements $\pi_\delta \in U^\delta[\lambda_{p,r}^{\delta,\alpha(\delta)}, \mu_{p,r}^{\delta,\alpha(\delta)}]$, depending on δ, such that the limit relations (3.10) hold and simultaneously

$$g_0^0(\pi_\delta) \to \min_{\pi \in \mathcal{D}_{p,r}^0} g_0^0(\pi), \quad \delta \to 0.$$

Moreover, weak limit point $\bar{\pi}$ of any weakly converging sequence π_{δ^k}, $k = 1, 2, \ldots, \delta^k \to 0$, $k \to \infty$, is obviously a solution of Problem $(P_{p,r}^0)$.

We can assert that simultaneously the family of elements $(\lambda_{p,r}^{\delta,\alpha(\delta)}, \mu_{p,r}^{\delta,\alpha(\delta)})$, in view of the estimates (1.2), (2.5) and the consistency condition (3.1), satisfies the limit relation (see [12, 13, 15, 18])

$$\lim_{\delta \to +0} V_{p,r}^0(\lambda_{p,r}^{\delta,\alpha(\delta)}, \mu_{p,r}^{\delta,\alpha(\delta)}) = \sup_{(\lambda,\mu) \in \mathcal{H} \times \mathcal{H}_+} V_{p,r}^0(\lambda, \mu), \tag{3.11}$$

which, combined with the estimate (2.5), the consistency condition (3.1), and the duality relation (2.3) yields the limit relation (see [12, 13, 15, 18])

$$\langle (\lambda_{p,r}^{\delta,\alpha(\delta)}, \mu_{p,r}^{\delta,\alpha(\delta)}), (g_1^\delta(\pi_\delta) - h^\delta - p, g_2^\delta(\pi_\delta)) - r \rangle \to 0, \quad \delta \to 0.$$

Let us prove the limit relation (3.11). Since

$$V_{p,r}^\delta(\lambda_{p,r}^{\delta,\alpha(\delta)}, \mu_{p,r}^{\delta,\alpha(\delta)}) - \alpha(\delta)\|\lambda_{p,r}^{\delta,\alpha(\delta)}\|^2 - \alpha(\delta)\|\mu_{p,r}^{\delta,\alpha(\delta)}\|^2 \geq$$

$$V_{p,r}^\delta(\lambda, \mu) - \alpha(\delta)\|\lambda\|^2 - \alpha(\delta)\|\mu\|^2 \quad \forall \, (\lambda, \mu) \in \mathcal{H} \times \mathcal{H}_+,$$

we can write, thanks to (2.5), the estimates

$$V_{p,r}^\delta(\lambda_{p,r}^{\delta,\alpha(\delta)}, \mu_{p,r}^{\delta,\alpha(\delta)}) \geq V_{p,r}^0(\lambda, \mu) + \alpha(\delta)\|\lambda_{p,r}^{\delta,\alpha(\delta)}\|^2 + \alpha(\delta)\|\mu_{p,r}^{\delta,\alpha(\delta)}\|^2 -$$

$$C\delta(1 + \|\lambda\| + \|\mu\|) - \alpha(\delta)\|\lambda\|^2 - \alpha(\delta)\|\mu\|^2,$$

$$V_{p,r}^0(\lambda_{p,r}^{\delta,\alpha(\delta)}, \mu_{p,r}^{\delta,\alpha(\delta)}) = V_{p,r}^\delta(\lambda_{p,r}^{\delta,\alpha(\delta)}, \mu_{p,r}^{\delta,\alpha(\delta)}) + [V_{p,r}^0(\lambda_{p,r}^{\delta,\alpha(\delta)}, \mu_{p,r}^{\delta,\alpha(\delta)}) - V_{p,r}^\delta(\lambda_{p,r}^{\delta,\alpha(\delta)}, \mu_{p,r}^{\delta,\alpha(\delta)})] \geq$$

$$V_{p,r}^\delta(\lambda_{p,r}^{\delta,\alpha(\delta)}, \mu_{p,r}^{\delta,\alpha(\delta)}) - C\delta(1 + \|\lambda_{p,r}^{\delta,\alpha(\delta)}\| + \|\mu_{p,r}^{\delta,\alpha(\delta)}\|),$$

whence we obtain

$$V_{p,r}^0(\lambda_{p,r}^{\delta,\alpha(\delta)}, \mu_{p,r}^{\delta,\alpha(\delta)}) \geq V_{p,r}^0(\lambda, \mu) + \alpha(\delta)\|\lambda_{p,r}^{\delta,\alpha(\delta)}\|^2 + \alpha(\delta)|\mu_{p,r}^{\delta,\alpha(\delta)}|^2 - C\delta(1 + \|\lambda_{p,r}^{\delta,\alpha(\delta)}\| + \|\mu_{p,r}^{\delta,\alpha(\delta)}\|)-$$

$$C\delta(1 + \|\lambda\| + \|\mu\|) \quad \forall(\lambda,\mu) \in \mathcal{H} \times \mathcal{H}_+.$$

From here, we deduce, due to the consistency condition (3.1) and limit relations (3.9), that for any fixed $M > 0$ and for any fixed $\epsilon > 0$ there exists such $\delta(\epsilon) > 0$ for which the estimate

$$V_{p,r}^0(\lambda_{p,r}^{\delta,\alpha(\delta)}, \mu_{p,r}^{\delta,\alpha(\delta)}) \geq \sup_{(\lambda,\mu)\in\mathcal{H}\times\mathcal{H}_+:\|\lambda\|\leq M, \|\mu\|\leq M} V_{p,r}^0(\lambda, \mu) - \epsilon \qquad (3.12)$$

$$\forall\delta \leq \delta(\epsilon) \ \forall(\lambda,\mu) \in \{(\lambda,\mu) \in \mathcal{H} \times \mathcal{H}_+ : \|\lambda\| \leq M, \|\mu\| \leq M\}$$

holds.

Suppose now that the limit relation (3.11) is not true. Then there exists such a sequence δ_s, $s = 1, 2, \ldots$ convergent to zero that the inequality

$$V_{p,r}^0(\lambda_{p,r}^{\delta_s,\alpha(\delta_s)}, \mu_{p,r}^{\delta_s,\alpha(\delta_s)}) \leq \sup_{(\lambda,\mu)\in\mathcal{H}\times\mathcal{H}_+} V_{p,r}^0(\lambda, \mu) - l, \quad s = 1, 2, \ldots$$

is fulfilled for some $l > 0$.

Since

$$\sup_{(\lambda,\mu)\in\mathcal{H}\times\mathcal{H}_+} V_{p,r}^0(\lambda, \mu) - \sup_{(\lambda,\mu)\in\mathcal{H}\times\mathcal{H}_+:\|\lambda\|\leq M, \|\mu\|\leq M} V_{p,r}^0(\lambda, \mu) \to 0,$$

for $M \to +\infty$, we deduce from the last estimate that for all sufficiently large positive M the inequality

$$V_{p,r}^0(\lambda_{p,r}^{\delta_s,\alpha(\delta_s)}, \mu_{p,r}^{\delta_s,\alpha(\delta_s)}) \leq \sup_{(\lambda,\mu)\in\mathcal{H}\times\mathcal{H}_+:\|\lambda\|\leq M, \|\mu\|\leq M} V_{p,r}^0(\lambda, \mu) - l/2$$

is true. This estimate contradicts to the estimate obtained above (3.12). The last contradiction proves correctness of the limit relation (3.11).

Summarizing the above arguments, we assert that the following "convergence" theorem for the dual regularization method in Problem $(P_{p,r}^0)$ is valid.

Theorem 1. *Let Problem $(P_{p,r}^0)$ be solvable. Regardless of the properties of the solvability of the dual problem to Problem $(P_{p,r}^0)$ or, in other words, regardless of the properties of the subdifferential $\partial\beta(p,r)$ (it is empty or not empty), it is true that exist elements $\pi^\delta \in U^\delta[\lambda_{p,r}^{\delta,\alpha(\delta)}, \mu_{p,r}^{\delta,\alpha(\delta)}]$ such that the relations*

$$g_0^0(\pi^\delta) \to g_0^0(\pi_{p,r}^0), \quad g_1^0(\pi^\delta) - h^0 - p \to 0, \quad g_2^0(\pi^\delta) - r \leq \kappa(\delta), \quad \|\kappa(\delta)\| \to 0, \quad \delta \to 0, \quad (3.13)$$

$$\langle(\lambda_{p,r}^{\delta,\alpha(\delta)}, \mu_{p,r}^{\delta,\alpha(\delta)}), (g_1^\delta(\pi^\delta) - h^\delta - p, g_2^\delta(\pi^\delta) - r)\rangle \to 0, \quad \delta \to 0$$

hold, in which the inequality $g_2^0(\pi^\delta) - r \leq \kappa(\delta)$ is understood in the sense of ordering on a cone of nonpositive functions in $\mathcal{H}$. Simultaneously, the equality

$$\lim_{\delta\to+0} V_{p,r}^0(\lambda_{p,r}^{\delta,\alpha(\delta)}, \mu_{p,r}^{\delta,\alpha(\delta)}) = \sup_{(\lambda,\mu)\in\mathcal{H}\times\mathcal{H}_+} V_{p,r}^0(\lambda, \mu)$$

is valid. In addition, the duality relation (2.3) holds. If the dual of Problem $(P_{p,r}^0)$ is solvable, then the limit relation $(\lambda_{p,r}^{\delta,\alpha(\delta)}, \mu_{p,r}^{\delta,\alpha(\delta)}) \to (\lambda_{p,r}^0, \mu_{p,r}^0)$, $\delta \to 0$ is valid also, where $(\lambda_{p,r}^0, \mu_{p,r}^0)$ denotes the minimum-norm solution of the dual problem.

3.2. Stable sequential Lagrange principle for optimal control problem with pointwise state constraints

We formulate in this subsection the necessary and sufficient condition for existence of a minimizing approximate solution in Problem $(P_{p,r}^0)$. Also, for this problem it can be called by stable sequential Lagrange principle in nondifferential form. Simultaneously, as we deal only with regular Lagrange function, the formulated theorem may be called by Kuhn–Tucker theorem in nondifferential form. Note that the necessity of the conditions of the theorem formulated below follows from the theorem 1. At the same time, their sufficiency is a simple consequence of the convexity of Problem $(P_{p,r}^0)$ and the conditions on its input data. A verification of these propositions for similar situation of the convex programming problem in a Hilbert space may be found in [10, 15, 16].

Theorem 2. *Regardless of the properties of the subdifferential $\partial\beta(p,r)$ (it is empty or not empty) or, in other words, regardless of the properties of the solvability of the dual problem to Problem $(P_{p,r}^0)$, necessary and sufficient conditions for Problem $(P_{p,r}^0)$ to have a minimizing approximate solution is that there is a sequence of dual variables $(\lambda^k, \mu^k) \in \mathcal{H} \times \mathcal{H}_+$, $k = 1, 2, \ldots$, such that $\delta^k \|(\lambda^k, \mu^k)\| \to 0$, $k \to \infty$, and relations*

$$\pi^{\delta^k}[\lambda^k, \mu^k] \in \mathcal{D}_{p,r}^{\delta^k, \epsilon^k}, \quad \epsilon^k \to 0, \quad k \to \infty, \tag{3.14}$$

$$\left\langle (\lambda^k, \mu^k), (g_1^{\delta^k}(\pi^{\delta^k}[\lambda^k, \mu^k]) - h^{\delta^k} - p, g_2^{\delta^k}(\pi^{\delta^k}[\lambda^k, \mu^k]) - r) \right\rangle \to 0, \quad k \to \infty \tag{3.15}$$

hold for some elements $\pi^{\delta^k}[\lambda^k, \mu^k] \in U^{\delta^k}[\lambda^k, \mu^k]$. The sequence $\pi^{\delta^k}[\lambda^k, \mu^k]$, $k = 1, 2, \ldots$, is the desired minimizing approximate solution and each of its weak limit points is a solution of Problem $(P_{p,r}^0)$. As $(\lambda^k, \mu^k) \in \mathcal{H} \times \mathcal{H}_+$, $k = 1, 2, \ldots$, we can use the sequence of the points $(\lambda_{p,r}^{\delta^k, \alpha(\delta^k)}, \mu_{p,r}^{\delta^k, \alpha(\delta^k)})$, $k = 1, 2, \ldots$, generated by the dual regularization method of the theorem 1. If the dual of Problem $(P_{p,r}^0)$ is solvable, the sequence $(\lambda^k, \mu^k) \in \mathcal{H} \times \mathcal{H}_+$, $k = 1, 2, \ldots$, should be assumed to be bounded. The limit relation

$$V_{p,r}^0(\lambda^k, \mu^k) \to \sup_{(\lambda, \mu) \in \mathcal{H} \times \mathcal{H}_+} V_{p,r}^0(\lambda, \mu) \tag{3.16}$$

holds as a consequence of the relations (3.14), (3.15). Furthermore, each weak limit point (if such points exist) of the sequence $(\lambda^k, \mu^k) \in \mathcal{H} \times \mathcal{H}_+$, $k = 1, 2, \ldots$ is a solution of the dual problem $V_{p,r}^0(\lambda, \mu) \to \max$, $(\lambda, \mu) \in \mathcal{H} \times \mathcal{H}_+$.

P r o o f. To prove the necessity, we first note that problem $(P_{p,r}^0)$ is solvable (i.e., $U_{p,r}^0 \neq \emptyset$) due to the conditions on the initial data and to the existence of a minimizing approximate solution. Now the existence of the indicated sequence $(\lambda^k, \mu^k) \in \mathcal{H} \times \mathcal{H}_+$, $k = 1, 2, \ldots$ and the limit relations (3.14) and (3.15) follow from Theorem 1 if the points (λ^k, μ^k) and $\pi^{\delta^k}[\lambda^k, \mu^k]$ are defined as $(\lambda_{p,r}^{\delta^k, \alpha(\delta^k)}, \mu_{p,r}^{\delta^k, \alpha(\delta^k)})$, and π_{δ^k}, $k = 1, 2, \ldots$, respectively. These limit relations imply that (3.16) holds as well. Really, combining estimates (2.4) with the limit relation $\delta^k \|(\lambda^k, \mu^k)\| \to 0$, $k \to \infty$, we conclude (see the estimate (2.5)) that $V_{p,r}^{\delta^k}(\lambda^k, \mu^k) - V_{p,r}^0(\lambda^k, \mu^k) \to 0$, $k \to \infty$. Then, in view of (2.3), (3.15), and the limit relation $f^0(z^{\delta^k}[\lambda^k, \mu^k]) \to f^0(z_{p,r}^0)$, $k \to \infty$ (see (3.13)), we have

$$V_{p,r}^{\delta^k}(\lambda^k, \mu^k) = f^{\delta^k}(z^{\delta^k}[\lambda^k, \mu^k]) + \left\langle (\lambda^k, \mu^k), (A^{\delta^k} z^{\delta^k}[\lambda^k, \mu^k] - h^{\delta^k} - p, g^{\delta^k}(z^{\delta^k}[\lambda^k, \mu^k]) - r) \right\rangle \to f^0(z_{p,r}^0),$$

therefore, the limit relation (3.16) holds true.

To prove the sufficiency, we first note also that the set $U_{p,r}^0 \subset \mathcal{D}_{p,r}^{0\epsilon^k}$ is not empty. This follows from the inclusion $\pi^{\delta^k}[\lambda^k, \mu^k] \in \mathcal{D}_{p,r}^{\delta^k, \epsilon^k}$, from the fact that the sequence $\pi^{\delta^k}[\lambda^k, \mu^k]$, $k = 1, 2, \ldots$

is bounded, and from the conditions on the initial data in Problem $(P_{p,r}^0)$. Furthermore, since the point $\pi^{\delta^k}[\lambda^k, \mu^k]$ minimizes on $\mathcal{D}$ the functional $L_{p,r}^{\delta^k}(\cdot, \lambda^k, \mu^k)$, we have

$$g_0^{\delta^k}(\pi^{\delta^k}[\lambda^k, \mu^k]) + \langle(\lambda^k, \mu^k), (g_1^{\delta^k}(\pi^{\delta^k}[\lambda^k, \mu^k]) - h^{\delta^k} - p, g_2^{\delta^k}(\pi^{\delta^k}[\lambda^k, \mu^k]) - r)\rangle \leq$$

$$g_0^{\delta^k}(\pi) + \langle(\lambda^k, \mu^k), (g_1^{\delta^k}(\pi) - h^{\delta^k} - p, g_2^{\delta^k}(z) - r)\rangle \quad \forall\, \pi \in \mathcal{D}.$$

By the assumptions of the theorem, it follows that

$$g_0^{\delta^k}(\pi^{\delta^k}[\lambda^k, \mu^k]) \leq g_0^{\delta^k}(\pi) + \langle(\lambda^k, \mu^k), (g_1^{\delta^k}(\pi) - h^{\delta^k} - p, g_2^{\delta^k}(\pi) - r)\rangle + \psi^k \quad \forall \pi \in \mathcal{D}, \quad \psi^k \to 0, \quad k \to \infty.$$

Setting $\pi = \pi_{p,r}^0 \in U_{p,r}^0$ and using the consistency condition $\delta^k \|(\lambda^k, \mu^k)\| \to 0$, $k \to \infty$, we obtain $g_0^0(\pi^{\delta^k}[\lambda^k, \mu^k]) \leq g_0^0(\pi_{p,r}^0) + \widetilde{\psi}^k$, $\widetilde{\psi}^k \to 0$, $k \to \infty$. Since we also have the inclusion $\pi^{\delta^k}[\lambda^k, \mu^k] \in \mathcal{D}_{p,r}^{\delta^k, \epsilon^k}$, using the classical weak compactness properties of a bounded convex closed set and the weak lower semicontinuity of a continuous convex functional in a Hilbert space, we easily derive $g_0^0(\pi^{\delta^k}[\lambda^k, \mu^k]) \to g_0^0(\pi_{p,r}^0)$, $k \to \infty$; i.e., the sequence $\pi^{\delta^k}[\lambda^k, \mu^k]$, $k = 1, 2, \ldots$ is a minimizing approximate solution in Problem $(P_{p,r}^0)$. In view of (3.15) and the obtained limit relation $g_0^0(\pi^{\delta^k}[\lambda^k, \mu^k]) \to g_0^0(\pi_{p,r}^0)$, $k \to \infty$, we can write

$$V_{p,r}^{\delta^k}(\lambda^k, \mu^k) = g_0^{\delta^k}(\pi^{\delta^k}[\lambda^k, \mu^k]) + \langle(\lambda^k, \mu^k), (g_1^{\delta^k}(\pi^{\delta^k}[\lambda^k, \mu^k]) - h^{\delta^k} - p, g_2^{\delta^k}(\pi^{\delta^k}[\lambda^k, \mu^k]) - r)\rangle \to g_0^0(\pi_{p,r}^0),$$

therefore, limit relation (3.16) holds by virtue of estimate (2.5), equality (2.3), and the limit relation $\delta^k \|(\lambda^k, \mu^k)\| \to 0$, $k \to \infty$. To conclude, we note that, it is easy to show that each weak limit point of the sequence $(\lambda^k, \mu^k) \in \mathcal{H} \times \mathcal{H}_+$, $k = 1, 2, \ldots$ (if such points exist) is a solution of the dual problem $V_{p,r}^0(\lambda, \mu) \to \max$, $(\lambda, \mu) \in \mathcal{H} \times \mathcal{H}_+$.

Remark 1. *If the functional g_0^0 is strongly convex and subdifferentiable on $\mathcal{D}$ then from the weak convergence of the unique in this case elements $\pi^{\delta^k}[\lambda^k, \mu^k]$ to unique element $\pi_{p,r}^0$ as $k \to \infty$, and numerical convergence $g_0^0(\pi^{\delta^k}[\lambda^k, \mu^k]) \to g_0^0(\pi_{p,r}^0)$, $k \to \infty$ follows the strong convergence $\pi^{\delta^k}[\lambda^k, \mu^k] \to \pi_{p,r}^0$, $k \to \infty$. Problem $(P_{p,r}^0)$ with the strongly convex g_0^0 for linear system of ordinary differential equations but with exact input data is studied in [17].*

3.3. Stable sequential Pontryagin maximum principle for optimal control problem with pointwise state constraints

Denote by $U_{max}^\delta[\lambda, \mu]$ a set of elements $\pi_{\max}^\delta[\lambda, \mu] \in \mathcal{D}$ that satisfy all relations of the maximum principle (2.2) of the lemma 1. Under the supplementary condition of existence of continuous with respect to z gradients $\nabla_z \varphi_2^\delta(x, t, z)$, $\nabla_z G^\delta(x, z)$ with corresponding estimates, it follows that the proposition of the Theorem 2 may be rewritten in the form of the stable sequential Pontryagin maximum principle. It is obviously that the equality $U_{max}^\delta[\lambda, \mu] = U^\delta[\lambda, \mu]$ takes place under mentioned supplementary condition.

Theorem 3. *Regardless of the properties of the subdifferential $\partial\beta(p, r)$ (it is empty or not empty) or, in other words, regardless of the properties of the solvability of the dual problem to Problem $(P_{p,r}^0)$, necessary and sufficient conditions for Problem $(P_{p,r}^0)$ to have a minimizing approximate solution is that there is a sequence of dual variables $(\lambda^k, \mu^k) \in \mathcal{H} \times \mathcal{H}_+$, $k = 1, 2, \ldots$, such that $\delta^k \|(\lambda^k, \mu^k)\| \to 0$, $k \to \infty$, and relations (3.14), (3.15) hold for some elements $\pi^{\delta^k}[\lambda^k, \mu^k] \in U_{max}^{\delta^k}[\lambda^k, \mu^k]$. Moreover, the sequence $\pi^{\delta^k}[\lambda^k, \mu^k]$, $k = 1, 2, \ldots$, is the desired minimizing approximate solution and each of its weak limit points is a solution of Problem $(P_{p,r}^0)$. As $(\lambda^k, \mu^k) \in \mathcal{H} \times \mathcal{H}_+$,*

$k = 1, 2, \ldots$, we can use the sequence of the points $(\lambda_{p,r}^{\delta^k, \alpha(\delta^k)}, \mu_{p,r}^{\delta^k, \alpha(\delta^k)})$, $k = 1, 2, \ldots$, generated by the dual regularization method of the theorem 1. If the dual of Problem $(P_{p,r}^0)$ is solvable, the sequence $(\lambda^k, \mu^k) \in \mathcal{H} \times \mathcal{H}_+$, $k = 1, 2, \ldots$, should be assumed to be bounded. The limit relation (3.16) holds as a consequence of the relations (3.14), (3.15).

Remark 2. When the inequality constraint in Problem $(P_{p,r}^0)$ is absent, i.e., $(P_{p,r}^0) = (P_p^0)$, and $\phi_1(x, t) \equiv 1$, the target functional g_0^0 is taken, for example, in the form $g_0^0(\pi) \equiv \|\pi\|^2 \equiv \|u\|^2 + \|w\|^2$ then Problem (P_p^0) acquires the typical form of unstable inverse problem. In this case the stable sequential Pontryagin maximum principle of the Theorem 3 becomes a tool for the direct solving such unstable inverse problem.

Remark 3. In important partial case of Problem $(P_{p,r}^0) = (P_r^0)$, when it has only the inequality constraint $(\varphi_1^\delta(x, t) = h^\delta(x, t) = p(x, t) = 0, \ (x, t) \in Q)$, "weak" passage to the limit in the relations of the Theorem 3 leads to usual for similar optimal control problems Pontryagin maximum principle (see, e.g., [3,8]) with nonnegative Radon measures in the input data of the adjoint equation.

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