

APPROXIMATE CONTROLLABILITY OF IMPULSIVE STOCHASTIC SYSTEMS DRIVEN BY ROSENBLATT PROCESS AND BROWNIAN MOTION

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Abstract: In this paper we consider a class of impulsive stochastic functional differential equations driven simultaneously by a Rosenblatt process and standard Brownian motion in a Hilbert space. We prove an existence and uniqueness result and we establish some conditions ensuring the approximate controllability for the mild solution by means of the Banach fixed point principle. At the end we provide a practical example in order to illustrate the viability of our result.

Keywords: Approximate controllability, Fixed point theorem, Rosenblatt process, Mild solution stochastic impulsive systems.

1. Introduction

It is well known that approximate controllability is one of the fundamental concepts in mathematical control theory for infinite differential systems and plays a significant role in both deterministic and in stochastic dynamical systems. Approximate controllability means that the system can be moved to an arbitrary small neighborhood of the final state. Some recent researches on the existence results of approximate controllability are [8, 9, 14, 25].

Recently, there has been increasing interest in the analysis of control synthesis problems for impulsive systems due to their significance both in theory and applications, for example, in problems of sudden environmental changes, radiation of electromagnetic waves and changes in the interconnections of subsystems. For some recent researches on the existence results for impulsive stochastic differential equations, we refer the reader to monographs [3–5, 10, 23, 24, 29]. In these models, the processes are characterized by the fact that they undergo abrupt changes of state at certain moments of time between intervals of continuous evolution. For basic concepts about the impulsive systems see [12, 17].

In recent years, there has been a growing interest in stochastic functional differential equations driven by the Rosenblatt process [2, 19, 20, 22]. The theory of Rosenblatt process has been developed accordingly due to its nice properties see [13, 16, 27]. Tudor [28] investigated the Rosenblatt process which is Gaussian and the calculus for it is much easier than other processes. However, in concrete situations where the Gaussianity is not plausible for the model, one can employ the Rosenblatt process. There is corresponding literature devoted to various theoretical aspects of impulse systems controlled by Rosenblatt processes [7, 15, 18, 20].

Some dynamical systems of a special kind require a mixed process to model their dynamics [1, 26].

Inspired by the above studies, this article is devoted to demonstrating the approximate controllability of a soft solution for a class of neutral functional-stochastic differential equations controlled

by a Wiener process and a Rosenblatt process independent of the form

$$\begin{cases} dx(t) = Ax(t)dt + Bu(t)dt + f(t, x(t))dt + g(t, x(t))dW(t) + \sigma(t)dZ_H(t), \\ t \in [0, T], \quad t \neq t_k, \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad k = 1, 2, \dots, m, \\ x(0) = x_0 \in X, \end{cases} \quad (1.1)$$

where $x(\cdot)$ takes values in the separable Hilbert space X , $A : D(A) \subset X \rightarrow X$ is a closed, linear, and densely defined operator on X . Let B be a bounded linear operator from the Hilbert space U into X .

Let the control $u \in \mathcal{L}_2^{\mathcal{F}}([0, T], U)$ which is the Hilbert space of all square integrable and \mathcal{F}_t -adapted processes with values in U . Let Q_K be a positive, self adjoint and trace class operator on K and let $\mathcal{L}_2(K, X)$ be the space of all Q_K -Hilbert-Schmidt operators acting between K and X equipped with the Hilbert-Schmidt norm $\|\cdot\|_{\mathcal{L}_2}$. The W is a Q_K -Wiener process on Hilbert space K .

Let Q be a positive, self adjoint and trace class operator on Y and let $\mathcal{L}_2^0(Y, X)$ be the space of all Q -Hilbert-Schmidt operators acting between Y and X equipped with the Hilbert-Schmidt norm $\|\cdot\|_{\mathcal{L}_2^0}$. Let Z_H be a Q -Rosenblatt process on a Hilbert space Y . The process W and Z_H are independent. The functions f , g and σ will be specified later. Moreover, the fixed moments of times t_k satisfy $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$, $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$. Here $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ represents the jump in the state x at time t_k , where I_k determines the size of the jump.

Let $(\Omega, \mathcal{F}_T, P)$ be the complete probability space with the natural filtration $\{\mathcal{F}_t \mid t \in [0, T]\}$ generated by random variables $\{Z_H(s), W(s), s \in [0, T]\}$. Let x_0 be an \mathcal{F}_0 -measurable random variable independent of W and Z_H satisfying $\mathbf{E} \|x_0\|^2 < \infty$. We define the following classes of functions: let $\mathcal{L}_2(\Omega, \mathcal{F}_T, X)$ be the Hilbert space of all \mathcal{F}_T -measurable, square integrable variables with values in X , $\mathcal{L}_2^{\mathcal{F}}([0, T], X)$ is the Hilbert space of all square integrable and \mathcal{F}_t -adapted processes with values in X .

The space $C([0, T], \mathcal{L}_2(\Omega, \mathcal{F}_T, X))$ is the Banach space of continuous maps except for a finite number of points t_k at which $x(t_k^-)$ and $x(t_k^+)$ exists and $x(t_k^-) = x(t_k)$ satisfying the condition

$$\sup_{t \in [0, T]} \mathbf{E} \|x(t)\|^2 < \infty$$

and $\mathbf{\Lambda}_2^T$ is the closed subspace of $C([0, T], \mathcal{L}_2(\Omega, \mathcal{F}_T, X))$ consisting of measurable and \mathcal{F}_t -adapted processes $x(t)$, then $\mathbf{\Lambda}_2^T$ is a Banach space with the norm defined by

$$\|x\|_{\mathbf{\Lambda}_2^T} = \left(\sup_{t \in [0, T]} \mathbf{E} \|x(t)\|^2 \right)^{1/2}.$$

Let $\{Z_H(t), t \in [0, T]\}$ be the one-dimensional Rosenblatt process with parameter $H \in (1/2, 1)$, Z_H has the following representation (see Tudor [28])

$$Z_H(t) = d(H) \int_0^t \int_0^t \left[\int_{y_1 \vee y_2}^t \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) du \right] dB(y_1) dB(y_2),$$

where

$$\begin{cases} B(t)_{t \in [0, T]} & \text{is the Wiener process,} \\ \mathcal{B}(\cdot, \cdot) & \text{is the Beta function,} \\ H' = \frac{H+1}{2}, \quad d(H) = \frac{1}{H+1} \sqrt{\frac{H}{2(2H-1)}}, \quad c_H = \sqrt{\frac{H(2H-1)}{\mathcal{B}(2-2H, H-1/2)}}, \\ K^H(t, s) = 1_{\{t>s\}} c_H s^{1/2-H} \int_s^t (u-s)^{H-3/2} u^{H-1/2} du. \end{cases}$$

Let X and Y be two real separable Hilbert spaces, $\mathcal{L}(Y; X)$ be the space of bounded linear operator from Y to X , $Q \in \mathcal{L}(Y; X)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace

$$\operatorname{tr} Q = \sum_{n=1}^{\infty} \lambda_n < \infty, \quad \lambda_n \geq 0$$

and $\{e_n\}$ is a complete orthonormal basis in Y .

We define the infinite dimensional Q -Rosenblatt process on Y as

$$Z_H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n z_n(t),$$

where $(z_n)_{n \geq 0}$ is a family of real independent Rosenblatt processes. Consider the following fundamental inequality.

Lemma 1 [21]. *If $\phi : [0, T] \rightarrow \mathcal{L}_2^0(Y; X)$ satisfies*

$$\int_0^T \|\phi(s)\|_{\mathcal{L}_2^0}^2 ds < \infty,$$

then we have

$$E \left\| \int_0^t \phi(s) dZ_H(s) \right\|^2 \leq 2Ht^{2H-1} \int_0^t \|\phi(s)\|_{\mathcal{L}_2^0}^2 ds.$$

Definition 1. *For each $u \in \mathcal{L}_2^{\mathcal{F}}([0, T], U)$, a stochastic process $x \in \mathbf{\Lambda}_2^T$ is a mild solution of (1.1) if we have*

$$\begin{aligned} x(t) = & S(t)x_0 + \int_0^t S(t-s) (Bu(s) + f(s, x(s))) ds \\ & + \int_0^t S(t-s)g(s, x(s))dW(s) + \int_0^t S(t-s)\sigma(s)dZ_H(s) + \sum_{0 < t_k < t} S(t-t_k)I_k(x(t_k^-)). \end{aligned}$$

Let $x(T; u)$ be the state value of system (1.1) at terminal time T corresponding to control u . The set

$$R(T) = \{x(T; u) : u \in \mathcal{L}_2^{\mathcal{F}}([0, T], U)\}$$

is called the reachable set of (1.1) at the terminal time T .

Definition 2. *The stochastic control system (1.1) is called approximately controllable on the interval $[0, T]$ if*

$$\overline{R(T)} = \mathcal{L}_2(\Omega, \mathcal{F}_T, X).$$

For the proof of the main result, we impose the following conditions on data of the problem.

(Hyp 1) A is the infinitesimal generator of a compact semigroup $\{S(t), t \geq 0\}$ on X such that $\|S(t)\| \leq M$, for some constant $M > 0$.

(Hyp 2) **1.** The function $f : [0, T] \times X \rightarrow X$ is continuous and there exists a constant C_f such that for $x, y \in X$ and $t \in [0, T]$

$$\begin{aligned} \|f(t, x)\|^2 & \leq C_f(1 + \|x\|^2), \\ \|f(t, x) - f(t, y)\|^2 & \leq C_f \|x - y\|^2. \end{aligned}$$

2. The function $g : [0, T] \times X \rightarrow \mathcal{L}_2(K, X)$ is continuous and there exists a constant C_g such that for $x, y \in X$ and $t \in [0, T]$

$$\begin{aligned} \|g(t, x)\|_{\mathcal{L}_2}^2 &\leq C_g(1 + \|x\|^2), \\ \|g(t, x) - g(t, y)\|_{\mathcal{L}_2}^2 &\leq C_g \|x - y\|^2. \end{aligned}$$

(Hyp 3) The function $\sigma : [0, T] \rightarrow \mathcal{L}_2^0$ is bounded by a positive constant L for all $t \in [0, T]$.

(Hyp 4) $I_k : X \rightarrow X$ is continuous and there exist constants $d_k, q_k > 0$ such that, for $x, y \in X$

$$\begin{aligned} (i) \quad &\|I_k(x) - I_k(y)\|^2 \leq d_k \|x - y\|^2, \quad k \in \{1, \dots, m\}, \\ (ii) \quad &\|I_k(x)\|^2 \leq q_k (1 + \|x\|^2), \quad k \in \{1, \dots, m\}, \\ (iii) \quad &M^2 m \left(\sum_{k=1}^m d_k \right) < \frac{1}{4}. \end{aligned}$$

(Hyp 5) For each $0 \leq t < T$, the operator $\alpha(\alpha I + \Gamma_t^T)^{-1} \rightarrow 0$ in the strong operator topology as $\alpha \rightarrow 0^+$, with $\Gamma_s^T \in \mathcal{L}(X, X)$ and

$$\Gamma_s^T = \int_s^T S(T-t) B B^* S^*(T-t) dt.$$

(Hyp 6) 1. The function $f : [0, T] \times X \rightarrow X$ is continuous and there exists a constant C_f such that for $x, y \in X$ and $t \in [0, T]$

$$\|f(t, x) - f(t, y)\|^2 \leq C_f \|x - y\|^2.$$

2. The function $g : [0, T] \times X \rightarrow \mathcal{L}_2(K, X)$ is continuous and there exists a constant C_g such that for $x, y \in X$ and $t \in [0, T]$

$$\|g(t, x) - g(t, y)\|_{\mathcal{L}_2}^2 \leq C_g \|x - y\|^2.$$

3. The functions f and g are uniformly bounded, then there exists $C > 0$ such that

$$\|f(s, x(s))\|^2 + \|g(s, x(s))\|_{\mathcal{L}_2}^2 \leq C.$$

Lemma 2 [6]. For any $x_T \in \mathcal{L}_2(\Omega, \mathcal{F}_T, X)$ there exists a unique $\Psi \in \mathcal{L}_2^{\mathcal{F}}([0, T]; \mathcal{L}_2(K, X))$ such that

$$x_T = \mathbf{E}(x_T) + \int_0^T \Psi(s) dW(s).$$

For any $\alpha > 0$ and an arbitrary function $x(\cdot)$, we define the control function for system (1.1) in the following form

$$\begin{aligned} u^\alpha(t, x) &= B^* S^*(T-t) (\alpha I + \Gamma_0^T)^{-1} (\mathbf{E}(x_T) - S(T)x_0) \\ &+ B^* S^*(T-t) \int_0^t (\alpha I + \Gamma_s^T)^{-1} \Psi(s) dW(s) - B^* S^*(T-t) \int_0^t (\alpha I + \Gamma_s^T)^{-1} S(T-s) \sigma(s) dZ_H(s) \\ &\quad - B^* S^*(T-t) \int_0^t (\alpha I + \Gamma_s^T)^{-1} S(T-s) f(s, x(s)) ds \\ &\quad - B^* S^*(T-t) \int_0^t (\alpha I + \Gamma_s^T)^{-1} S(T-s) g(s, x(s)) dW(s) \\ &\quad - B^* S^*(T-t) (\alpha I + \Gamma_0^T)^{-1} \sum_{0 < t_k < t} S(t-t_k) I_k(x(t_k^-)), \end{aligned}$$

the function $u^\alpha(t, x)$ is defined so that the system driven by this command has a unique solution (see Theorem 1) and moreover the system is approximately controllable (see Theorem 2).

Lemma 3. *There exists positive real constant M_u such that, for all $x, y \in \Lambda_2^T$ we have*

$$\mathbf{E} \|u^\alpha(t, x) - u^\alpha(t, y)\|^2 \leq \frac{M_u}{\alpha^2} \|x - y\|_{\Lambda_2^T}^2, \quad (1.2)$$

$$\mathbf{E} \|u^\alpha(t, x)\|^2 \leq \frac{M_u}{\alpha^2} \left(1 + \|x\|_{\Lambda_2^T}^2\right). \quad (1.3)$$

P r o o f. Let $x, y \in \Lambda_2^T$, we have

$$\begin{aligned} \mathbf{E} \|u^\alpha(t, x) - u^\alpha(t, y)\|^2 &\leq 3\mathbf{E} \left\| B^* S^*(T-t) \int_0^t (\alpha I + \Gamma_s^T)^{-1} S(T-s) [f(s, x(s)) - f(s, y(s))] ds \right\|^2 \\ &\quad + 3\mathbf{E} \left\| B^* S^*(T-t) \int_0^t (\alpha I + \Gamma_s^T)^{-1} S(T-s) [g(s, x(s)) - g(s, y(s))] dW(s) \right\|^2 \\ &\quad + 3\mathbf{E} \left\| B^* S^*(T-t) (\alpha I + \Gamma_0^T)^{-1} \sum_{k=1}^m S(T-t_k) [I_k(x(t_k^-)) - I_k(y(t_k^-))] \right\|^2. \end{aligned}$$

Using the Holder inequality, Ito isometric theorem and the assumptions on the data, we obtain

$$\begin{aligned} \mathbf{E} \|u^\alpha(t, x) - u^\alpha(t, y)\|^2 &\leq \frac{3}{\alpha^2} \|B\|^2 M^4 T C_f \int_0^t \mathbf{E} \|x(s) - y(s)\|^2 ds \\ &\quad + \frac{3}{\alpha^2} \|B\|^2 M^4 C_g \int_0^t \mathbf{E} \|x(s) - y(s)\|^2 ds + \frac{3}{\alpha^2} \|B\|^2 M^4 m \left(\sum_{k=1}^m d_k \right) \mathbf{E} \left\| [I_k(x(t_k^-)) - I_k(y(t_k^-))] \right\|^2 \\ &\leq \frac{3}{\alpha^2} \|B\|^2 M^4 T C_f T \sup_{s \in [0, T]} \mathbf{E} \|x(s) - y(s)\|^2 \\ &\quad + \frac{3}{\alpha^2} \|B\|^2 M^4 C_g T \sup_{s \in [0, T]} \mathbf{E} \|x(s) - y(s)\|^2 + m \left(\sum_{k=1}^m d_k \right) \sup_{s \in [0, T]} \mathbf{E} \|x(s) - y(s)\|^2 \\ &\leq \frac{3}{\alpha^2} \|B\|^2 M^4 \left[T^2 C_f + T C_g + m \left(\sum_{k=1}^m d_k \right) \right] \|x - y\|_{\Lambda_2^T}^2 \\ &= \frac{M_u}{\alpha^2} \|x - y\|_{\Lambda_2^T}^2, \end{aligned}$$

where

$$M_u = 3 \|B\|^2 M^4 \left[T^2 C_f + T C_g + m \left(\sum_{k=1}^m d_k \right) \right].$$

The proof of the second (1.3) is similar. \square

2. Approximate controllability

For any $\alpha > 0$, define the operator $F_\alpha : \Lambda_2^T \rightarrow \Lambda_2^T$ by

$$\begin{aligned} (F_\alpha x)(t) &= S(t)x_0 + \int_0^t S(t-s) (Bu^\alpha(s, x) + f(s, x(s))) ds \\ &\quad + \int_0^t S(t-s) g(s, x(s)) dW(s) + \int_0^t S(t-s) \sigma(s) dZ_H(s) + \sum_{0 < t_k < t} S(t-t_k) I_k(x(t_k^-)). \end{aligned}$$

The first main result is the following theorem.

Theorem 1. *Under assumptions (Hyp 1)–(Hyp 5), the system (1.1) has a mild solution on $[0, T]$.*

P r o o f. Step 1. Let $0 \leq t_1 \leq t_2 \leq T$. Then for any fixed $x \in \mathbf{\Lambda}_2^T$

$$\begin{aligned} & \mathbf{E} \|(F_\alpha x)(t_2) - (F_\alpha x)(t_1)\|^2 \leq 6\mathbf{E} \|(S(t_2) - S(t_1))x_0\|^2 \\ & + 6\mathbf{E} \left\| \int_0^{t_2} S(t_2 - s)f(s, x(s))ds - \int_0^{t_1} S(t_1 - s)f(s, x(s))ds \right\|^2 \\ & + 6\mathbf{E} \left\| \int_0^{t_2} S(t_2 - s)g(s, x(s))dW(s) - \int_0^{t_1} S(t_1 - s)g(s, x(s))dW(s) \right\|^2 \\ & + 6\mathbf{E} \left\| \int_0^{t_2} S(t_2 - s)\sigma(s)dZ_H(s) - \int_0^{t_1} S(t_1 - s)\sigma(s)dZ_H(s) \right\|^2 \\ & + 6\mathbf{E} \left\| \sum_{0 < t_k < t_2} S(t_2 - t_k)I_k(x(t_k^-)) - \sum_{0 < t_k < t_1} S(t_1 - t_k)I_k(x(t_k^-)) \right\|^2 \\ & + 6\mathbf{E} \left\| \int_0^{t_2} S(t_2 - s)Bu^\alpha(s, x(s))ds - \int_0^{t_1} S(t_1 - s)Bu^\alpha(s, x(s))ds \right\|^2 \\ & = 6(J_1 + J_2 + J_3 + J_4 + J_5 + J_6). \end{aligned}$$

Thus we obtain by Holder inequality, Ito isometric theorem and the assumptions (Hyp 1)–(Hyp 5)

$$\begin{aligned} J_1 & \leq \|S(t_2) - S(t_1)\|^2 \mathbf{E} \|x_0\|^2, \\ J_2 & \leq 2\mathbf{E} \left\| \int_0^{t_1} (S(t_2 - s) - S(t_1 - s))f(s, x(s))ds \right\|^2 + 2\mathbf{E} \left\| \int_{t_1}^{t_2} S(t_2 - s)f(s, x(s))ds \right\|^2 \\ & \leq 2t_1 \int_0^{t_1} \mathbf{E} \|(S(t_2 - s) - S(t_1 - s))f(s, x(s))\|^2 ds + 2M^2(t_2 - t_1) \int_{t_1}^{t_2} \mathbf{E} \|f(s, x(s))\|^2 ds, \\ J_3 & \leq 2\mathbf{E} \left\| \int_0^{t_1} (S(t_2 - s) - S(t_1 - s))g(s, x(s))dW(s) \right\|^2 + 2\mathbf{E} \left\| \int_{t_1}^{t_2} S(t_2 - s)g(s, x(s))dW(s) \right\|^2 \\ & \leq 2 \int_0^{t_1} \mathbf{E} \|(S(t_2 - s) - S(t_1 - s))g(s, x(s))\|_{\mathcal{L}_2}^2 ds + 2M^2 \int_{t_1}^{t_2} \mathbf{E} \|g(s, x(s))\|_{\mathcal{L}_2}^2 ds, \\ J_4 & \leq 2\mathbf{E} \left\| \int_0^{t_1} (S(t_2 - s) - S(t_1 - s))\sigma(s)dZ_H(s) \right\|^2 + 2\mathbf{E} \left\| \int_{t_1}^{t_2} S(t_2 - s)\sigma(s)dZ_H(s) \right\|^2 \\ & \leq 4Ht_1^{2H-1} \int_0^{t_1} \mathbf{E} \|(S(t_2 - s) - S(t_1 - s))\sigma(s)\|_{\mathcal{L}_2^0}^2 ds + 4M^2H(t_2^{2H-1} - t_1^{2H-1}) \int_{t_1}^{t_2} \|\sigma(s)\|_{\mathcal{L}_2^0}^2 ds, \\ J_5 & \leq 2m \sum_{t_1 < t_k < t_2} \mathbf{E} \|S(t_2 - s)I_k(x(t_k^-))\|^2 + 2m \sum_{0 < t_k < t_1} \mathbf{E} \|(S(t_2 - s) - S(t_1 - s))I_k(x(t_k^-))\|^2 \\ & \leq 2mM^2 \sum_{t_1 < t_k < t_2} \mathbf{E} \|I_k(x(t_k^-))\|^2 + 2m \sum_{0 < t_k < t_1} \mathbf{E} \|(S(t_2 - s) - S(t_1 - s))I_k(x(t_k^-))\|^2, \\ J_6 & \leq 2\mathbf{E} \left\| \int_0^{t_1} (S(t_2 - s) - S(t_1 - s))Bu^\alpha(s, x)ds \right\|^2 + 2\mathbf{E} \left\| \int_{t_1}^{t_2} S(t_2 - s)Bu^\alpha(s, x)ds \right\|^2 \\ & \leq 2t_1 \int_0^{t_1} \mathbf{E} \|(S(t_2 - s) - S(t_1 - s))Bu^\alpha(s, x)\|^2 ds + 2M^2 \|B\|^2 (t_2 - t_1) \int_{t_1}^{t_2} \mathbf{E} \|u^\alpha(s, x)\|^2 ds. \end{aligned}$$

Consequently, using the strong continuity of $S(t)$, as well as the Lebesgue's dominated convergence theorem, we conclude that the right side of the above inequality tends to zero when $t_2 - t_1 \rightarrow 0$. Thus we conclude that $(F_\alpha x)(t)$ is continuous in $[0, T]$.

Step 2. Let $x \in \Lambda_2^T$, then we have

$$\begin{aligned} \mathbf{E} \|(F_\alpha x)(t)\|^2 &\leq 6\mathbf{E} \|S(t)x_0\|^2 + 6\mathbf{E} \left\| \int_0^t S(t-s)Bu^\alpha(s,x)ds \right\|^2 \\ +6\mathbf{E} \left\| \int_0^t S(t-s)f(s,x(s))ds \right\|^2 &+ 6\mathbf{E} \left\| \int_0^t S(t-s)g(s,x(s))dW(s) \right\|^2 \\ +6\mathbf{E} \left\| \int_0^t S(t-s)\sigma(s)dZ_H(s) \right\|^2 &+ 6\mathbf{E} \left\| \sum_{0 < t_k < t} S(t-t_k)I_k(x(t_k^-)) \right\|^2. \end{aligned}$$

By Holder inequality, Lemma 3, Ito isometric theorem and the assumptions (Hyp 1)–(Hyp 5), we have

$$\begin{aligned} \mathbf{E} \|(F_\alpha x)(t)\|^2 &\leq 6\mathbf{E} \|S(t)x_0\|^2 + 6M^2 \|B\|^2 T \mathbf{E} \int_0^t \|u^\alpha(s,x)\|^2 ds \\ +6M^2 T \mathbf{E} \int_0^t \|f(s,x(s))\|^2 ds &+ 6M^2 \mathbf{E} \int_0^t \|g(s,x(s))\|_{\mathcal{L}_2}^2 ds \\ +12M^2 HT^{2H-1} \mathbf{E} \int_0^t \|\sigma(s)\|_{\mathcal{L}_2^0}^2 ds &+ 6mM^2 \sum_{k=1}^m \mathbf{E} \|I_k(x(t_k^-))\|^2. \end{aligned}$$

Hence

$$\begin{aligned} \mathbf{E} \|(F_\alpha x)(t)\|^2 &\leq 6M^2 \mathbf{E} \|x_0\|^2 + 6M^2 \|B\|^2 T^2 \frac{M_u}{\alpha^2} \left(1 + \|x\|_{\Lambda_2^T}^2\right) \\ +6M^2 T^2 C_f \left(1 + \|x\|_{\Lambda_2^T}^2\right) &+ 6M^2 TC_g \left(1 + \|x\|_{\Lambda_2^T}^2\right) \\ +12M^2 HT^{2H-1} TL &+ 6mM^2 \left(\sum_{k=1}^m q_k\right) \left(1 + \|x\|_{\Lambda_2^T}^2\right) \\ &\leq 6M^2 \left(\mathbf{E} \|x_0\|^2 + 2HT^{2H-1} TL\right) \\ +6M^2 \left(\|B\|^2 T^2 \left[\frac{M_u}{\alpha^2} + C_f\right] &+ TC_g + m \left(\sum_{k=1}^m q_k\right)\right) \left(1 + \|x\|_{\Lambda_2^T}^2\right), \end{aligned}$$

we thus obtain that $\|(F_\alpha x)\|_{\Lambda_2^T}^2 < \infty$. Since $(F_\alpha x)(t)$ is continuous on $[0, T]$, therefore F_α maps Λ_2^T , in itself.

Step 3. Let $x, y \in \Lambda_2^T$, then for any fixed $t \in [0, T]$ we have

$$\begin{aligned} \|(F_\alpha x)(t) - (F_\alpha y)(t)\|^2 &\leq 4\mathbf{E} \left\| \int_0^t S(t-s)B(u^\alpha(s,x) - u^\alpha(s,y)) ds \right\|^2 \\ +4\mathbf{E} \left\| \int_0^t S(t-s)(f(s,x(s)) - f(s,y(s))) ds \right\|^2 \\ +4\mathbf{E} \left\| \int_0^t S(t-s)(g(s,x(s)) - g(s,y(s))) dW(s) \right\|^2 \\ +4\mathbf{E} \left\| \sum_{0 < t_k < t} S(t-t_k)(I_k(x(t_k^-)) - I_k(y(t_k^-))) \right\|^2. \end{aligned}$$

By assumptions (Hyp 1)–(Hyp 5) combined with Hölder's inequality, Lemma 3 and Ito isometric

theorem, we get that

$$\begin{aligned} & \|(F_\alpha x)(t) - (F_\alpha y)(t)\|^2 \\ & \leq 4M^2 \|B\|^2 t \int_0^t \|u^\alpha(s, x) - u^\alpha(s, y)\|^2 ds + 4M^2 t \int_0^t \|f(s, x(s)) - f(s, y(s))\|^2 ds \\ & + 4M^2 \int_0^t \|g(s, x(s)) - g(s, y(s))\|_{\mathcal{L}_2}^2 ds + 4M^2 m \left(\sum_{k=1}^m d_k \right) \|I_k(x(t_k^-)) - I_k(y(t_k^-))\|^2. \end{aligned}$$

Therefore,

$$\begin{aligned} & \|(F_\alpha x)(t) - (F_\alpha y)(t)\|^2 \\ & \leq 4M^2 \|B\|^2 t \frac{M_\mu}{\alpha^2} \int_0^t \|x(s) - y(s)\|^2 ds + 4M^2 t C_f \int_0^t \|x(s) - y(s)\|^2 ds \\ & + 4M^2 C_g \int_0^t \|x(s) - y(s)\|^2 ds + 4M^2 m \left(\sum_{k=1}^m d_k \right) \|x(t_k^-) - y(t_k^-)\|^2. \end{aligned}$$

Then we have

$$\begin{aligned} & \sup_{s \in [0, t]} \mathbf{E} \|(F_\alpha x)(t) - (F_\alpha y)(t)\|^2 \\ & \leq 4M^2 \left(\|B\|^2 t^2 \frac{M_\mu}{\alpha^2} + t(tC_f + C_g) + m \left(\sum_{k=1}^m d_k \right) \right) \sup_{s \in [0, t]} \mathbf{E} \|x(s) - y(s)\|^2 \\ & = \varphi(t) \sup_{s \in [0, t]} \mathbf{E} \|x(s) - y(s)\|^2, \end{aligned}$$

where

$$\varphi(t) = 4M^2 \|B\|^2 t^2 \frac{M_\mu}{\alpha^2} + 4M^2 t(tC_f + C_g) + 4M^2 m \left(\sum_{k=1}^m d_k \right).$$

We have (see (Hyp 4)–(iii))

$$\varphi(0) = 4M^2 m \left(\sum_{k=1}^m d_k \right) < 1.$$

So there is T_1 with $0 < T_1 \leq T$ such that $0 < \varphi(T_1) < 1$ and F_α is a contraction mapping on $\mathbf{\Lambda}_2^{T_1}$ and consequently has a unique fixed point. So by repeating the procedure, we extend the solution to the interval $[0, T]$ in several finite steps. \square

The second main result is the following theorem.

Theorem 2. *Under assumptions (Hyp 1), (Hyp 3), (Hyp 4), (Hyp 5) and (Hyp 6), the system (1.1) is approximately controllable on $[0, T]$.*

P r o o f. Let x_α the solution of system (1.1) corresponding to $\mu(t, x) = \mu^\alpha(t, x)$. We obtain by the stochastic Fubini theorem

$$\begin{aligned} x_\alpha(T) & = x_T - \alpha(\alpha I + \Gamma_0^T)^{-1} (\mathbf{E} \bar{x}_T - S(T)x_0) \\ & + \alpha \int_0^T (\alpha I + \Gamma_s^T)^{-1} S(T-s) f(s, x(s)) ds + \alpha \int_0^T (\alpha I + \Gamma_s^T)^{-1} [S(T-s)g(s, x(s)) - \Psi(s)] dW(s) \\ & + \alpha \int_0^T (\alpha I + \Gamma_s^T)^{-1} S(T-s) \sigma(s) dZ_H(s) + \alpha(\alpha I + \Gamma_0^T)^{-1} \sum_{k=1}^m S(T-t_k) I_k(x^\alpha(t_k^-)). \end{aligned}$$

By the hypotheses (Hyp 6-2), there is a subsequence still designated by $\{f(s, x_\alpha(s)), g(s, x_\alpha(s))\}$ which converges weakly to some $\{f(s), g(s)\}$ in $X \times \mathcal{L}_2$ and $\{I_k(x^\alpha(t_k^-))\}$ weakly converging to $\{I_k(w)\}$ in X . By the compactness of $\{S(t) : t \geq 0\}$, we have

$$\begin{aligned} S(T-s)f(s, x_\alpha(s)) &\rightarrow S(T-s)f(s), \\ S(T-s)g(s, x_\alpha(s)) &\rightarrow S(T-s)g(s), \\ S(T-t_k)I_k(x^\alpha(t_k^-)) &\rightarrow S(T-t_k)I_k(w). \end{aligned}$$

By hypothesis (Hyp 5), we have

$$\begin{cases} \alpha(\alpha I + \Gamma_s^T)^{-1} \rightarrow 0 \text{ strongly as } \alpha \rightarrow 0^+, & \text{for all } 0 \leq s \leq T, \\ \|\alpha(\alpha I + \Gamma_s^T)^{-1}\| \leq 1. \end{cases}$$

So, by the Lebesgue dominated convergence theorem we obtain

$$\begin{aligned} \mathbf{E} \|x_\alpha(T) - x_T\|^2 &\leq 9\mathbf{E} \|\alpha(\alpha I + \Gamma_0^T)^{-1} (\mathbf{E}x_T - S(T)x_0)\|^2 + 9\mathbf{E} \int_0^T \|\alpha(\alpha I + \Gamma_s^T)^{-1} \Psi(s)\|_{\mathcal{L}_2}^2 ds \\ &+ 18HT^{2H-1} \int_0^T \|\alpha(\alpha I + \Gamma_s^T)^{-1} S(T-s)\sigma(s)\|_{\mathcal{L}_2^2}^2 ds + 9\mathbf{E} \left(\int_0^T \|\alpha(\alpha I + \Gamma_s^T)^{-1} S(T-s)f(s)\| ds \right)^2 \\ &+ 9\mathbf{E} \left(\int_0^T \|\alpha(\alpha I + \Gamma_s^T)^{-1}\| \|S(T-s)(f(s, x_\alpha(s)) - f(s))\| ds \right)^2 \\ &+ 9\mathbf{E} \int_0^T \|\alpha(\alpha I + \Gamma_s^T)^{-1} S(T-s)g(s)\|_{\mathcal{L}_2}^2 ds \\ &+ 9\mathbf{E} \int_0^T \|\alpha(\alpha I + \Gamma_s^T)^{-1}\|^2 \|S(T-s)(g(s, x_\alpha(s)) - g(s))\|_{\mathcal{L}_2}^2 ds \\ &+ 9\mathbf{E} \left\| \sum_{k=1}^m \alpha(\alpha I + \Gamma_s^T)^{-1} S(T-t_k)I_k(w) \right\|^2 \\ &+ 9\mathbf{E} \|\alpha(\alpha I + \Gamma_s^T)^{-1}\|^2 \left\| \sum_{k=1}^m S(T-t_k)I_k(x^\alpha(t_k^-)) - \sum_{k=1}^m S(T-t_k)I_k(w) \right\|^2 \rightarrow 0 \text{ as } \alpha \rightarrow 0^+. \end{aligned}$$

Then the system (1.1) is approximately controllable. \square

3. Example

In this section we present an example. Let $X = L_2[0, \pi]$, $U = L_2[0, \pi]$ and $x_0 \in L_2[0, \pi]$. Let $A \subset D(A) : X \rightarrow X$ be the linear operator given by $Ay = y''$, where

$$D(A) = \{y \in X / y, y' \text{ are absolutely continuous } y'' \in X, y(0) = y(\pi) = 0\}.$$

Let $B \in L(\mathbb{R}, X)$ be defined as

$$(Bu)(z) = b(x)u, \quad 0 \leq z \leq \pi, \quad u \in \mathbb{R}, \quad b(x) \in L_2[0, \pi].$$

Here $W(t)$ denotes a one dimensional standard Brownian motion and Z_H is a Rosenblatt process, the processes W and Z_H are independent.

Consider the control system driven by the process W and Z_H to illustrate the obtained theory

$$\begin{cases} dx(t, z) = \left(\frac{\partial^2}{\partial z^2} x(t, z) + b(z)u(t) + f_1(t, x(t, z)) \right) dt \\ \quad + g_1(t, x(t, z)) dw(t) + \sigma(t) dZ_H, \quad t \in [0, T], \quad z \in [0, \pi], \\ \Delta x(t_k, z) = x(t_k^+, z) - x(t_k^-, z) = \frac{1}{2k} x(t_k, z), \quad t = t_k, \quad k = 1, \dots, m, \\ x(t, 0) = x(t, \pi) = 0, \quad t \in [0, T], \\ x(0, z) = x_0(z), \quad z \in [0, \pi]. \end{cases} \quad (3.1)$$

Suppose $f_1, g_1: \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous, satisfy the Lipschitz condition and the linear growth condition and are uniformly bounded.

First of all, note that there exists a complete orthonormal set $\{e_n\}_{n \geq 1}$ of eigenvectors of A with

$$e_n(z) = \sqrt{(2/\pi)} \sin nz, \quad 0 \leq z \leq \pi, \quad n = 1, 2, \dots$$

and the compact semigroup $S(t)$, $t \geq 0$, that is generated by A such that

$$\begin{aligned} Ay &= - \sum_{n=1}^{\infty} n^2 \langle y, e_n \rangle e_n(y), \quad y \in D(A), \\ S(t)y &= \sum_{n=1}^{\infty} e^{-n^2 t} \langle x, e_n \rangle e_n(y), \quad y \in X. \end{aligned}$$

Now define the functions: $f: [0, T] \times X \rightarrow X$, $g: [0, T] \times X \rightarrow L(K; X)$ as follows

$$\begin{aligned} f(t, x)(z) &= f_1(t, x(z)), \\ g(t, x)(z) &= g_1(t, x(z)) \end{aligned}$$

for $t \in [0, T]$, $x \in X$ and $0 < z < \pi$. Consequently, by [11, Theorem 4.1.7], we have that the deterministic linear system (3.1) is approximately controllable on every $[0, t]$, $t > 0$, provided that

$$\int_0^{\pi} b(z) e_n(z) dz \neq 0, \quad \text{for } n = 1, 2, 3, \dots$$

Hence, all conditions of Theorem 2 are satisfied, and consequently system (3.1) is approximately controllable on $[0, T]$.

4. Conclusion

Approximate controllability of a class of impulsive stochastic functional differential equations driven simultaneously by a Rosenblatt process and standard Brownian motion in a Hilbert space are obtained. The controllability problem is transformed into a fixed point problem for an appropriate nonlinear operator in a function space. By using some famous fixed point theorems and the approximating technique some new existence and controllability results are obtained.

We also remark that the same idea can be used to study the controllability and the exponential stability of impulsive stochastic functional differential equations driven simultaneously by a Rosenblatt process and standard Brownian motion under non-Lipschitz condition and with non local conditions.

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