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A NEW CHARACTERIZATION OF SYMMETRIC DUNKL AND q-DUNKL-CLASSICAL ORTHOGONAL POLYNOMIALS

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Abstract: In this paper, we consider the following $\mathcal{L}$-difference equation

$$\Phi(x)\mathcal{L}P_{n+1}(x) = (\xi_n x + \vartheta_n)P_{n+1}(x) + \lambda_n P_n(x), \quad n \geq 0,$$

where Φ is a monic polynomial (even), $\deg \Phi \leq 2$, ξ_n, ϑ_n, λ_n, $n \geq 0$, are complex numbers and $\mathcal{L}$ is either the Dunkl operator T_μ or the the q-Dunkl operator $T_{(\theta,q)}$. We show that if $\mathcal{L} = T_\mu$, then the only symmetric orthogonal polynomials satisfying the previous equation are, up a dilation, the generalized Hermite polynomials and the generalized Gegenbauer polynomials and if $\mathcal{L} = T_{(\theta,q)}$, then the q^2-analogue of generalized Hermite and the q^2-analogue of generalized Gegenbauer polynomials are, up a dilation, the only orthogonal polynomials sequences satisfying the $\mathcal{L}$-difference equation.

Keywords: Orthogonal polynomials, Dunkl operator, q-Dunkl operator.

1. Introduction

The classical orthogonal polynomials (Hermite, Laguerre, Bessel, and Jacobi) have a lot of useful characterizations: they satisfy a Hahn's property, that the sequence of their monic derivatives is again orthogonal (see [1, 8, 14, 16]), they are characterized as the polynomial eigenfunctions of a second order homogeneous linear differential (or difference) hypergeometric operator with polynomial coefficients [4, 15, 16], their corresponding linear functionals satisfy a distribution equation of Pearson type (see [11, 13, 15]).

Another characterization was established by Al-Salam and Chihara in [1], in particular they showed that the sequences Hermite, Laguerre and Jacobi are the only monic orthogonal polynomial sequences $\{P_n\}_{n\geq 0}$ that satisfy an equation of the form:

$$\pi(x)P'_{n+1}(x) = (a_n x + b_n)P_{n+1} + c_n P_n(x), \quad n \geq 0, \tag{1.1}$$

where π is a monic polynomial, $\deg \pi \leq 2$.

Recently, Datta and J. Griffin [9] studied the q-analogue of (1.1). More precisely they studied a q-difference equation of the form:

$$\pi(x)D_q P_{n+1}(x) = (a_n x + b_n)P_{n+1} + c_n P_n(x), \quad n \geq 0, \tag{1.2}$$

where π is a monic polynomial, $\deg \pi \leq 2$ and D_q is the Hahn operator defined by

$$D_q f(x) = (f(qx) - f(x))/(q-1)x, \quad f \in \mathcal{P}.$$

In particular they showed that the only orthogonal polynomials satisfying (1.2) are the Al-Salam-Carlitz I, the little and big q-Laguerre, the little and big q-Jacobi and the q-Bessel polynomials. The aim of this paper is to study the equation of the form:

$$\Phi(x)\mathcal{L}P_{n+1}(x) = (\xi_n x + \vartheta_n)P_{n+1}(x) + \lambda_n P_n(x), \quad n \geq 0, \tag{1.3}$$

where Φ is a monic polynomial (even), $\deg\Phi \leq 2$ and $\mathcal{L} \in \{T_\mu, T_{(\theta,q)}\}$.

This paper is organized as follows. In Section 2, we introduce the basic background and some preliminary results that will be used in what follows. In Section 3, we show that the only symmetric orthogonal polynomials satisfying (1.3), are, up a dilation, the generalized Hermite polynomials and the generalized Gegenbauer polynomials if $\mathcal{L} = T_\mu$ and the q^2-analogue of generalized Hermite polynomials and the q^2-analogue of generalized Gegenbauer polynomials if $\mathcal{L} = T_{(\theta,q)}$.

2. Preliminaries and notations

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $\langle u, f\rangle$ the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(u)_n = \langle u, x^n\rangle$, $n \geq 0$, the moments of u. For any form u, any polynomial f and any $a \in \mathbb{C}\setminus\{0\}$, let fu and $h_a u$, be the forms defined by duality:

$$\langle fu, p\rangle = \langle u, fp\rangle, \quad \langle h_a u, p\rangle = \langle u, h_a p\rangle, \quad p \in \mathcal{P},$$

where $h_a p(x) = p(ax)$.

Let $\{P_n\}_{n\geq 0}$ be a sequence of monic polynomials (MPS, in short) with $\deg P_n = n$, $n \geq 0$. The dual sequence associated with $\{P_n\}_{n\geq 0}$ is the sequence $\{u_n\}_{n\geq 0}$, $u_n \in \mathcal{P}'$ such that $\langle u_n, P_m\rangle = \delta_{n,m}$, n, $m \geq 0$, where $\delta_{n,m}$ is the Kronecker symbol [14].

The linear functional u is called regular if there exists a MPS $\{P_n\}_{n\geq 0}$ such that (see [8, p. 7]):

$$\langle u, P_m P_n\rangle = r_n\delta_{n,m}, \quad n,\, m \geq 0, \quad r_n \neq 0, \quad n \geq 0.$$

Then the sequence $\{P_n\}_{n\geq 0}$ is said to be orthogonal with respect to u. In this case, we have

$$u_n = \left(\langle u_0, P_n^2\rangle\right)^{-1} P_n u_0, \quad n \geq 0.$$

Moreover, $u = \lambda u_0$, where $(u)_0 = \lambda \neq 0$ [17].

In what follows all regular linear functionals u will be taken normalized i.e., $(u)_0 = 1$. Therefore, $u = u_0$.

A polynomial set $\{P_n\}_{n\geq 0}$ is called symmetric if

$$P_n(-x) = (-1)^n P_n(x), \quad n \geq 0.$$

According to Favard's theorem [8], a sequence of monic orthogonal polynomials $\{P_n(x)\}_{n\geq 0}$ (MOPS, in short) satisfies a three-term recurrence relation

$$\begin{cases} P_0(x) = 1, \quad P_1(x) = x, \\ P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \quad n \geq 0, \quad \gamma_{n+1} \neq 0, \quad n \geq 0. \end{cases} \tag{2.1}$$

with

$$\beta_n = \frac{\langle u_0, xP_n^2\rangle}{\langle u_0, P_n^2\rangle}, \quad \gamma_{n+1} = \frac{\langle u_0, P_{n+1}^2\rangle}{\langle u_0, P_n^2\rangle}, \quad n \geq 0.$$

A dilatation preserves the property of orthogonality. Indeed, the sequence $\{\widetilde{P}_n(x)\}_{n\geq 0}$ defined by

$$\widetilde{P}_n(x) = a^{-n}P_n(ax), \quad n \geq 0, \quad a \in \mathbb{C}\setminus\{0\},$$

satisfies the recurrence relation [16]

$$\begin{cases} \widetilde{P}_0(x) = 1, \quad \widetilde{P}_1(x) = x - \widetilde{\beta}_0, \\ \widetilde{P}_{n+2}(x) = (x - \widetilde{\beta}_{n+1})\widetilde{P}_{n+1}(x) - \widetilde{\gamma}_{n+1}\widetilde{P}_n(x), \quad n \geq 0, \end{cases}$$

where

$$\widetilde{\beta}_n = \frac{\beta_n}{a}, \quad \widetilde{\gamma}_{n+1} = \frac{\gamma_{n+1}}{a^2}, \quad n \geq 0. \tag{2.2}$$

Moreover, if $\{P_n\}_{n\geq 0}$ is a MOPS with respect to the regular form u_0, then $\{\widetilde{P}_n\}_{n\geq 0}$ is a MOPS with respect to the regular form $\widetilde{u}_0 = h_{a^{-1}} u_0$.

Theorem 1 [8]. *Let $\{P_n\}_{n\geq 0}$ be a MOPS satisfying* (2.1) *and orthogonal with respect to a linear functional u. The following statements are equivalent*:

(*i*) *the sequence $\{P_n\}_{n\geq 0}$ is symmetric*;
(*ii*) $(u)_{2n+1} = 0$, $n \geq 0$;
(*iii*) $\beta_n = 0$, $n \geq 0$.

Next, we introduce the Dunkl operator T_μ defined on $\mathcal{P}$ by [10, 18]

$$(T_\mu f)(x) = f'(x) + \mu H_{-1} f(x), \quad \mu > -\frac{1}{2}, \quad f \in \mathcal{P},$$

where

$$(H_{-1} f)(x) = \frac{f(x) - f(-x)}{2x}.$$

For the Dunkl operator, we have the property [6]

$$T_\mu(fg)(x) = (T_\mu f)(x)g(x) + f(x)(T_\mu g)(x) - 4\mu x (H_{-1} f)(x)(H_{-1} g)(x), \quad f, g \in \mathcal{P}.$$

In particular,

$$T_\mu(x P_{n+1}) = \big(1 + 2\mu(-1)^{n+1}\big) P_{n+1}(x) + x(T_\mu P_{n+1})(x), \quad n \geq 0. \tag{2.3}$$

We define the operator T_μ from $\mathcal{P}'$ to $\mathcal{P}'$ as follows:

$$\langle T_\mu u, f\rangle = -\langle u, T_\mu f\rangle, \quad f \in \mathcal{P}, \quad u \in \mathcal{P}'.$$

In particular,

$$(T_\mu u)_n = -\mu_n (u)_{n-1}, \quad n \geq 0,$$

with the convention $(u)_{-1} = 0$, where

$$\mu_n = n + \mu(1 - (-1)^n), \quad n \geq 0.$$

We introduce also the q-Dunkl operator $T_{(\theta,q)}$ defined on $\mathcal{P}$ by [2, 5, 7]

$$(T_{(\theta,q)} f)(x) = \frac{f(qx) - f(x)}{(q-1)x} + \theta H_{-1} f(x), \quad f \in \mathcal{P}, \quad \theta \in \mathbb{C}.$$

Remark 1. Note that when $q \to 1$, we again meet the Dunkl operator.

From the last definition, it is easy to prove that

$$T_{(\theta,q)}(fg) = (T_{(\theta,q)} f) g + (h_q f)(T_{(\theta,q)} g) + \theta(h_{-1} f - h_q f) H_{-1} g, \quad f, g \in \mathcal{P}.$$

In particular,

$$T_\mu(x P_{n+1}) = \left(1 + \theta - \theta(q+1)\frac{1 - (-1)^{n+1}}{2}\right) P_{n+1}(x) + qx(T_{(\theta,q)} P_{n+1})(x), \quad n \geq 0. \tag{2.4}$$

We define the operator $T_{(\theta,q)}$ from $\mathcal{P}'$ to $\mathcal{P}'$ as follows:

$$\langle T_{(\theta,q)}u, f\rangle = -\langle u, T_{(\theta,q)}f\rangle, \quad f \in \mathcal{P}, \quad u \in \mathcal{P}'.$$

In particular,

$$(T_{(\theta,q)})_n = -\theta_{n,q}(u)_{n-1}, \quad n \geq 0,$$

where $(u)_{-1} = 0$ and

$$\theta_{n,q} = [n]_q + \theta\frac{1-(-1)^n}{2}, \quad n \geq 0, \tag{2.5}$$

here $[n]_q$, $n \geq 0$, denotes the basic q-number defined by

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + ... + q^{n-1}, \quad n \geq 1, \quad [0]_q = 0.$$

According to the definitions of T_μ and $T_{(\theta,q)}$, we have

$$T_\mu(x^n) = \mu_n x^{n-1}, \quad T_{(\theta,q)}(x^n) = \theta_{n,q}x^{n-1}.$$

3. The main results

In this section, we will look for all symmetric MOPS satisfying (1.3). We distinguish two cases. The first case is when $\mathcal{L} = T_\mu$ and the second one is when $\mathcal{L} = T_{(\theta,q)}$.

3.1. First case: when $\mathcal{L} = T_\mu$

Theorem 2. *The only symmetric MOPS satisfying a T_μ-difference equation of the form*

$$\Phi(x)T_\mu P_{n+1}(x) = (\xi_n x + \vartheta_n)P_{n+1}(x) + \lambda_n P_n(x), \quad n \geq 0, \tag{3.1}$$

where Φ is a monic polynomial (even), $\deg\Phi \leq 2$, are, up a dilation, the generalized Hermite polynomials and the generalized Gegenbauer polynomials.

P r o o f. Let $\{P_n\}_{n\geq 0}$ be a symmetric MOPS satisfying (3.1). Since Φ is a monic, even and $\deg\Phi \leq 2$, then we distinguish two cases: $\Phi(x) = 1$ and $\Phi(x) = x^2 + c$.

Case 1. $\Phi(x) = 1$, then (3.1) becomes

$$T_\mu P_{n+1}(x) = (\xi_n x + \vartheta_n)P_{n+1}(x) + \lambda_n P_n(x), \quad n \geq 0.$$

By comparing the degrees in the last equation (in x^{n+2} and x^{n+1}), we obtain $\xi_n = \vartheta_n = 0$, $n \geq 0$ and then

$$T_\mu P_{n+1}(x) = \lambda_n P_n(x), \quad n \geq 0. \tag{3.2}$$

Identifying coefficients in the monomials of degree n in the last equation, we obtain

$$\lambda_n = \mu_{n+1}, \quad n \geq 0. \tag{3.3}$$

On the other hand, applying the operator T_μ to (2.1) with $\beta_{n+1} = 0$ and using (2.3), we get

$$T_\mu P_{n+2}(x) = \big(1 + 2\mu(-1)^{n+1}\big)P_{n+1}(x) + x(T_\mu P_{n+1})(x) - \gamma_{n+1}(T_\mu P_n)(x), \quad n \geq 0.$$

Substituting (3.2) and (3.3) in the last equation and taking into account the fact that

$$1 + 2\mu(-1)^{n+1} = \mu_{n+2} - \mu_{n+1},$$

we get

$$\mu_{n+1}P_{n+1}(x) = \mu_{n+1}xP_n(x) - \mu_n\gamma_{n+1}P_{n-1}(x), \quad n \geq 0.$$

From (2.1), the last equation is equivalent to

$$\mu_{n+1}\gamma_n P_{n-1}(x) = \mu_n\gamma_{n+1}P_{n-1}(x), \quad n \geq 0,$$

hence,

$$\mu_{n+1}\gamma_n = \mu_n\gamma_{n+1}, \quad n \geq 1.$$

Therefore,

$$\gamma_{n+1} = \frac{\gamma_1}{\mu_1}\mu_{n+1}, \quad n \geq 1.$$

Since the last relation remains valid for $n = 0$, then we have

$$\gamma_{n+1} = \frac{\gamma_1}{\mu_1}\mu_{n+1}, \quad n \geq 0.$$

Using (2.2), where $a^2 = 2\gamma_1/\mu_1$, we obtain

$$\widetilde{\beta}_n = 0, \quad \widetilde{\gamma}_{n+1} = \frac{\mu_{n+1}}{2}, \quad n \geq 0.$$

So, we meet the recurrence coefficients for the generalized Hermite polynomial sequence (see [8]).

Case 2. $\Phi(x) = x^2 + c$, then (3.1) becomes

$$(x^2 + c)T_\mu P_{n+1}(x) = (\xi_n x + \vartheta_n)P_{n+1}(x) + \lambda_n P_n(x), \quad n \geq 0. \tag{3.4}$$

Identifying the coefficients of higher degree in both sides of (3.4), we obtain $\xi_n = \mu_{n+1}$, $n \geq 0$. Therefore, (3.4) becomes

$$(x^2 + c)T_\mu P_{n+1}(x) = (\mu_{n+1}x + \vartheta_n)P_{n+1}(x) + \lambda_n P_n(x), \quad n \geq 0. \tag{3.5}$$

Applying the operator T_μ to (2.1) with $\beta_{n+1} = 0$ and using (2.3) and the fact that

$$1 + 2\mu(-1)^{n+1} = \mu_{n+2} - \mu_{n+1},$$

we get

$$T_\mu P_{n+2}(x) = (\mu_{n+2} - \mu_{n+1})P_{n+1}(x) + x(T_\mu P_{n+1})(x) - \gamma_{n+1}(T_\mu P_n)(x), \quad n \geq 0.$$

Multiplying the previous equation by $x^2 + c$ and using (3.5), we get

$$(\mu_{n+2}x + \vartheta_{n+1})P_{n+2}(x) + \lambda_{n+1}P_{n+1}(x) = (\mu_{n+2} - \mu_{n+1})(x^2 + c)P_{n+1}(x) + (\mu_{n+1}x^2 + \vartheta_n x)P_{n+1}(x) + \lambda_n xP_n(x) - \gamma_{n+1}\big((\mu_n x + \vartheta_{n-1})P_n(x) + \lambda_{n-1}P_{n-1}(x)\big), \quad n \geq 1,$$

or, equivalently,

$$\begin{aligned}(\vartheta_{n+1} - \vartheta_n)xP_{n+1}(x) - c(\mu_{n+2} - \mu_{n+1})P_{n+1} + \lambda_{n+1}P_{n+1}(x)\\ = \lambda_n xP_n(x) + \gamma_{n+1}\big((\mu_{n+2}x + \vartheta_{n+1})P_n(x) - (\mu_n x + \vartheta_{n-1})P_n(x) - \lambda_{n-1}P_{n-1}(x)\big), \quad n \geq 1.\end{aligned} \tag{3.6}$$

Comparing the degrees in the last equation, we obtain $\vartheta_{n+1} = \vartheta_n$, $n \geq 1$. But, from (3.5) and the fact that $\{P_n\}_{n\geq 0}$ is symmetric, where $n = 0$ and $n = 1$, we get, respectively,

$$\begin{aligned} v_0 &= 0, \quad \lambda_0 = c(1 + 2\mu),\\ v_1 &= 0, \quad \lambda_1 = 2(\gamma_1 + c).\end{aligned} \tag{3.7}$$

Thus,

$$\vartheta_n = 0, \quad n \geq 0.$$

Therefore, (3.6) becomes

$$\begin{aligned} & c(\mu_{n+1} - \mu_{n+2})P_{n+1}(x) + \lambda_{n+1}P_{n+1}(x) \\ & = \lambda_n x P_n(x) + \gamma_{n+1}\big((\mu_{n+2} - \mu_n)xP_n(x) - \lambda_{n-1}P_{n-1}(x)\big), \quad n \geq 1. \end{aligned}$$

Taking into account (2.1), we get

$$\begin{aligned} & \big(\lambda_{n+1} + c(\mu_{n+1} - \mu_{n+2})\big)xP_n(x) - \gamma_n\big(\lambda_{n+1} + c(\mu_{n+1} - \mu_{n+2})\big)P_{n-1}(x) \\ & = \big(\lambda_n + (\mu_{n+2} - \mu_n)\gamma_{n+1}\big)xP_n(x) - \lambda_{n-1}\gamma_{n+1}P_{n-1}(x), \quad n \geq 1. \end{aligned}$$

Then,

$$\lambda_{n+1} + c(\mu_{n+1} - \mu_{n+2}) = \lambda_n + (\mu_{n+2} - \mu_n)\gamma_{n+1}, \quad n \geq 1, \tag{3.8}$$

$$\big(\lambda_{n+1} + c(\mu_{n+1} - \mu_{n+2})\big)\gamma_n = \lambda_{n-1}\gamma_{n+1}, \quad n \geq 1. \tag{3.9}$$

Since $\mu_{n+2} - \mu_n = 2$, then, substitution of (3.8) in (3.9) gives

$$(\lambda_n + 2\gamma_{n+1})\gamma_n = \lambda_{n-1}\gamma_{n+1}, \quad n \geq 1.$$

Therefore,

$$\frac{\lambda_n}{\gamma_{n+1}} = \frac{\lambda_{n-1}}{\gamma_n} - 2, \quad n \geq 1.$$

So,

$$\lambda_n = \frac{\lambda_0 - 2n\gamma_1}{\gamma_1}\gamma_{n+1}, \quad n \geq 1. \tag{3.10}$$

It is clear that (3.10) remains valid for $n = 0$. Then, we have

$$\lambda_n = \frac{\lambda_0 - 2n\gamma_1}{\gamma_1}\gamma_{n+1}, \quad n \geq 0. \tag{3.11}$$

Substitution of (3.11) in (3.8) gives

$$\lambda_{n+1} = \frac{\lambda_0 - 2(n-1)\gamma_1}{\lambda_0 - 2n\gamma_1}\lambda_n + c(\mu_{n+2} - \mu_{n+1}), \quad n \geq 1.$$

By virtue of fourth equality in (3.7), we obtain that the previous equation remains valid for $n = 0$. Hence,

$$\lambda_{n+1} = \frac{\lambda_0 - 2(n-1)\gamma_1}{\lambda_0 - 2n\gamma_1}\lambda_n + c(\mu_{n+2} - \mu_{n+1}), \quad n \geq 0. \tag{3.12}$$

We will distinguish two situations: $c = 0$ and $c \neq 0$.

- If $c = 0$, then from (3.7) we have $\lambda_0 = 0$. Therefore, $\lambda_n = 0$, $n \geq 0$. Consequently, according to (3.11) and the fourth equality in (3.7), $\gamma_{n+1} = 0$, $n \geq 0$. This contradicts the orthogonality of $\{P_n\}_{n\geq 0}$.
- If $c \neq 0$, using a dilatation, we can take $c = -1$. Putting

$$\gamma_1 = \frac{1 + 2\mu}{3 + 2\mu + 2\alpha},$$

then (3.12) becomes

$$\lambda_{n+1} = \frac{2n+2\alpha+2\mu+1}{2n+2\alpha+2\mu+3}\lambda_n + \mu_{n+1} - \mu_{n+2}, \quad n \geq 0. \tag{3.13}$$

From (3.13), we can easily prove by induction that

$$\lambda_n = -\frac{\mu_{n+1}(\mu_{n+1}+2\alpha)}{2n+2\alpha+2\mu+1}, \quad n \geq 0.$$

Thus, (3.11) gives

$$\gamma_{n+1} = \frac{\mu_{n+1}(\mu_{n+1}+2\alpha)}{(2n+2\alpha+2\mu+1)(2n+2\alpha+2\mu+3)}, \quad n \geq 0.$$

So, we meet the recurrence coefficients for the generalized Gegenbauer polynomial (see [3, 8]). □

Remark 2. Notice that when $\mu = 0$ in (3.1), we again meet (1.1) for the symmetric case.

3.2. Second case: when $\mathcal{L} = T_{(\theta,q)}$

Theorem 3. *The only symmetric MOPS satisfying a $T_{(\theta,q)}$-difference equation of the form:*

$$\Phi(x)T_{(\theta,q)}P_{n+1}(x) = (\xi_n x + \vartheta_n)P_{n+1}(x) + \lambda_n P_n(x), \quad n \geq 0, \tag{3.14}$$

where Φ is a monic polynomial (even), $\deg \Phi \leq 2$, are, up a dilation, the q^2-analogue of generalized Hermite polynomials and the q^2-analogue of generalized Gegenbauer polynomials.

P r o o f. Let $\{P_n\}_{n\geq 0}$ be a symmetric MOPS satisfying (3.1). As in proof of Theorem 2, we distinguish two cases: $\Phi(x) = 1$ and $\Phi(x) = x^2 + c$.

Case 1. $\Phi(x) = 1$, then (3.14) becomes

$$T_{(\theta,q)}P_{n+1}(x) = (\xi_n x + \vartheta_n)P_{n+1}(x) + \lambda_n P_n(x), \quad n \geq 0. \tag{3.15}$$

By comparing the degrees in (3.15), we obtain $\xi_n = \vartheta_n = 0,\ n \geq 0$. Then,

$$T_{(\theta,q)}P_{n+1}(x) = \lambda_n P_n(x), \quad n \geq 0.$$

The comparison of the coefficients of x^n in the previous equation leads to $\lambda_n = \theta_{n+1,q},\ n \geq 0$. Therefore,

$$T_{(\theta,q)}P_{n+1}(x) = \theta_{n+1,q}P_n(x), \quad n \geq 0. \tag{3.16}$$

Now, applying $T_{(\theta,q)}$ to (2.1) with $\beta_{n+1} = 0$ and using (2.4), we get

$$\begin{aligned} T_{(\theta,q)}P_{n+2}(x) = &\left(1+\theta-\theta(q+1)\frac{1-(-1)^{n+1}}{2}\right)P_{n+1}(x) \\ &+q\,x(T_{(\theta,q)}P_{n+1})(x) - \gamma_{n+1}(T_{(\theta,q)}P_n)(x), \quad n \geq 0. \end{aligned}$$

Substituting (3.16) in the last equation, we get

$$\begin{aligned} \theta_{n+2,q}P_{n+1}(x) = &\left(1+\theta-\theta(q+1)\frac{1-(-1)^{n+1}}{2}\right)P_{n+1}(x) \\ &+q\theta_{n+1,q}\,xP_n(x) - \gamma_{n+1}\theta_{n,q}P_{n-1}(x), \quad n \geq 0. \end{aligned}$$

Using the fact that

$$xP_n = P_{n+1} + \gamma_n P_{n-1},$$

we obtain

$$\Big(\theta_{n+2,q} - 1 - \theta + \theta(q+1)\frac{1-(-1)^{n+1}}{2} - q\theta_{n+1,q}\Big)P_{n+1}(x) = q\theta_{n+1,q}\gamma_n P_{n-1}(x) - \theta_{n,q}\gamma_{n+1}P_{n-1}(x), \quad n \geq 0.$$

After easy calculations from (2.5), we have

$$\theta_{n+2,q} - 1 - \theta + \theta(q+1)\frac{1-(-1)^{n+1}}{2} - q\theta_{n+1,q} = 0, \quad n \geq 0. \tag{3.17}$$

Therefore,

$$(q\theta_{n+1,q}\gamma_n - \theta_{n,q}\gamma_{n+1})P_{n-1}(x) = 0, \quad n \geq 0.$$

Hence,

$$q\theta_{n+1,q}\gamma_n = \theta_{n,q}\gamma_{n+1}, \quad n \geq 1.$$

Then, we can deduce by induction that

$$\gamma_{n+1} = \frac{\gamma_1}{1+\theta}\, q^n\, \theta_{n+1,q}, \quad n \geq 1.$$

Moreover, the previous identity remains valid for $n = 0$, thus

$$\gamma_{n+1} = \frac{\gamma_1}{1+\theta}\, q^n\, \theta_{n+1,q}, \quad n \geq 0.$$

Then, according to (2.2), with the choice

$$a^2 = q(q+1)\frac{\gamma_1}{1+\theta}$$

and putting

$$\mu = \frac{1+\theta}{q(q+1)} - \frac{1}{2},$$

we obtain

$$\widetilde{\beta}_n = 0, \quad \widetilde{\gamma}_{n+1} = q^n\, \frac{\theta_{n+1,q}}{q(q+1)}, \quad n \geq 0,$$

which are the recurrence coefficients for the q^2-analogue of generalized Hermite polynomial $H_n^{(\mu,q^2)}$ [12], with

$$\mu = \frac{1+\theta}{q(q+1)} - \frac{1}{2}.$$

Case 2: $\Phi(x) = x^2 + c$, then in this case (3.14) becomes

$$(x^2+c)T_{(\theta,q)}P_{n+1}(x) = (\xi_n x + \vartheta_n)P_{n+1}(x) + \lambda_n P_n(x), \quad n \geq 0. \tag{3.18}$$

By comparing terms of higher degree in the previous equation, we obtain

$$\xi_n = \theta_{n+1,q}, \quad n \geq 0.$$

Then, equation (3.18) becomes

$$(x^2+c)T_{(\theta,q)}P_{n+1}(x) = (\theta_{n+1,q}x + \vartheta_n)P_{n+1}(x) + \lambda_n P_n(x), \quad n \geq 0. \tag{3.19}$$

Applying the operator $T_{(\theta,q)}$ to (2.1) with $\beta_{n+1}=0$ and using (2.4), we get

$$\begin{aligned}T_{(\theta,q)}P_{n+2}(x) &= \left(1+\theta-\theta(q+1)\frac{1-(-1)^{n+1}}{2}\right)P_{n+1}(x)\\ &+qx(T_{(\theta,q)}P_{n+1})(x)-\gamma_{n+1}(T_{(\theta,q)}P_n)(x), \quad n\geq 0.\end{aligned}$$

By (3.17), the last equation becomes

$$\begin{aligned}T_{(\theta,q)}P_{n+2}(x) &= (\theta_{n+2,q}-q\theta_{n+1,q})P_{n+1}(x)\\ &+qx(T_{(\theta,q)}P_{n+1})(x)-\gamma_{n+1}(T_{(\theta,q)}P_n)(x), \quad n\geq 0.\end{aligned}$$

Multiplying the above equation by x^2+c and substituting (3.19) into the result, we get

$$\begin{aligned}(\theta_{n+2,q}x+\vartheta_{n+1})P_{n+2}(x)+\lambda_{n+1}P_{n+1}(x) &= (\theta_{n+2,q}-q\theta_{n+1,q})(x^2+c)P_{n+1}(x)\\ +q(\theta_{n+1,q}x^2+\vartheta_n x)P_{n+1}(x)+q\lambda_n xP_n(x) &- \gamma_{n+1}\big((\theta_{n,q}x+\vartheta_{n-1})P_n(x)+\lambda_{n-1}P_{n-1}(x)\big), \quad n\geq 1.\end{aligned}$$

Substituting of (2.1) in the previous equation, we get

$$\begin{aligned}(\vartheta_{n+1}-q\vartheta_n)xP_{n+1}(x)+\big(\lambda_{n+1}-c(\theta_{n+2,q}-q\theta_{n+1,q})\big)P_{n+1}(x) &=\\ q\lambda_n xP_n(x)+\gamma_{n+1}\big(((\theta_{n+2,q}-\theta_{n,q})x+\vartheta_{n+1}-\vartheta_{n-1})P_n(x)-\lambda_{n-1}P_{n-1}(x)\big), &\quad n\geq 1.\end{aligned}$$

The comparison of the coefficients of x^{n+2} in the previous equation gives $\vartheta_{n+1}=q\vartheta_n$, $n\geq 1$ and putting $n=0$ and $n=1$ in (3.19), we get respectively

$$\begin{aligned}v_0=0, &\quad \lambda_0=c(1+\theta),\\ v_1=0, &\quad \lambda_1=(1+q)(\gamma_1+c).\end{aligned} \tag{3.20}$$

Hence, $\vartheta_n=0,\ n\geq 0$.

Therefore, the last equation becomes

$$\begin{aligned}&\big(\lambda_{n+1}-c(\theta_{n+2,q}-q\theta_{n+1,q})\big)P_{n+1}(x)\\ &=q\lambda_n xP_n(x)+\gamma_{n+1}\big((\theta_{n+2,q}-\theta_{n,q})xP_n(x)-\lambda_{n-1}P_{n-1}(x)\big), \quad n\geq 1.\end{aligned}$$

Using the fact that $P_{n+1}=xP_n(x)-\gamma_nP_{n-1}$, the above equation is equivalent to

$$\begin{aligned}\big(\lambda_{n+1}-c(\theta_{n+2,q}-q\theta_{n+1,q})\big)xP_n(x)-\gamma_n\big(\lambda_{n+1}-c(\theta_{n+2,q}-q\theta_{n+1,q})\big)P_{n-1}(x)\\ =\big(q\lambda_n+(\theta_{n+2,q}-\theta_{n,q})\gamma_{n+1}\big)xP_n(x)-\lambda_{n-1}\gamma_{n+1}P_{n-1}(x), \quad n\geq 1.\end{aligned}$$

Then, we deduce

$$\lambda_{n+1}-c(\theta_{n+2,q}-q\theta_{n+1,q})=q\lambda_n+(\theta_{n+2,q}-\theta_{n,q})\gamma_{n+1}, \quad n\geq 1, \tag{3.21}$$

$$\Big(\lambda_{n+1}-c(\theta_{n+2,q}-q\theta_{n+1,q})\Big)\gamma_n=\lambda_{n-1}\gamma_{n+1}, \quad n\geq 1. \tag{3.22}$$

Since

$$\theta_{n+2,q}-\theta_{n,q}=(1+q)q^n,$$

then the substitution of (3.21) in (3.22) gives

$$(q\lambda_n+(1+q)q^n\gamma_{n+1})\gamma_n=\lambda_{n-1}\gamma_{n+1}, \quad n\geq 1,$$

therefore,

$$q\lambda_n=\left(\frac{\lambda_{n-1}}{\gamma_n}-(1+q)q^n\right)\gamma_{n+1}, \quad n\geq 1.$$

We can easily deduce by induction that

$$q^n\lambda_n = \left(\frac{\lambda_0}{\gamma_1} - q(q+1)[n]_{q^2}\right)\gamma_{n+1}, \quad n \geq 1.$$

It is clear that the previous identity remains valid for $n = 0$. Then, we have

$$q^n\lambda_n = \left(\frac{\lambda_0}{\gamma_1} - q(q+1)[n]_{q^2}\right)\gamma_{n+1}, \quad n \geq 0. \tag{3.23}$$

Now, we will determine λ_n. By (3.23), we have

$$\gamma_{n+1} = q^n \frac{\gamma_1}{\lambda_0 - q(q+1)\gamma_1[n]_{q^2}}\lambda_n, \quad n \geq 0. \tag{3.24}$$

Therefore, (3.21) becomes

$$\lambda_{n+1} = \frac{q\lambda_0 - (q+1)\gamma_1([n]_{q^2} - 1)}{\lambda_0 - q(q+1)\gamma_1[n]_{q^2}}\lambda_n + c(\theta_{n+2,q} - q\theta_{n+1,q}), \quad n \geq 1.$$

By virtue of (3.20), we obtain that the previous equation remains valid for $n = 0$.

Then,

$$\lambda_{n+1} = \frac{q\lambda_0 - (q+1)\gamma_1([n]_{q^2} - 1)}{\lambda_0 - q(q+1)\gamma_1[n]_{q^2}}\lambda_n + c(\theta_{n+2,q} - q\theta_{n+1,q}), \quad n \geq 0. \tag{3.25}$$

We will distinguish two situations: $c = 0$ and $c \neq 0$.

- If $c = 0$, then from (3.20) $\lambda_0 = 0$. Therefore, $\lambda_n = 0$, $n \geq 0$. Consequently, according to (3.24) and the fourth equality in (3.20), $\gamma_{n+1} = 0$, $n \geq 0$. This contradicts the orthogonality of $\{P_n\}_{n\geq 0}$.
- If $c \neq 0$, using a suitable dilatation, we can suppose that $c = -1$. Putting

$$\gamma_1 = \frac{1+\theta}{1+\theta+q(q+1)(\alpha+1)}. \tag{3.26}$$

Equation (3.25) becomes

$$\lambda_{n+1} = q\frac{q(q+1)(\alpha+1) + \theta_{2n-1,q}}{q(q+1)(\alpha+1) + \theta_{2n+1,q}}\lambda_n - (\theta_{n+2,q} - q\theta_{n+1,q}), \quad n \geq 0. \tag{3.27}$$

Therefore, from (3.27), we can prove by induction that

$$\lambda_n = -\frac{\theta_{n+1,q}\Big(q(q+1)(\alpha+1) + \theta_{n-1,q}(1+\theta(1-q)(1-(-1)^n)/2)\Big)}{q(q+1)(\alpha+1) + \theta_{2n-1,q}}, \quad n \geq 0. \tag{3.28}$$

By virtue of (3.24), (3.26) and (3.28), we get

$$\gamma_{n+1} = q^n\frac{\theta_{n+1,q}\Big(q(q+1)(\alpha+1) + \theta_{n-1,q}(1+\theta(1-q)(1-(-1)^n)/2)\Big)}{\Big(q(q+1)(\alpha+1) + \theta_{2n-1,q}\Big)\Big(q(q+1)(\alpha+1) + \theta_{2n+1,q}\Big)}, \quad n \geq 0.$$

So, we meet the recurrence coefficients for the q^2-anlogue of generalized Gegenbauer polynomial $S_n^{(\alpha,\beta,q^2)}$, with

$$\beta = \frac{1+\theta}{q(q+1)} - 1$$

(see [12]). □

Remark 3. Notice that when $q \to 1$, we recover the result in Theorem 2 and when $\theta = 0$ in (3.14), we again meet (1.2) for symmetric case.

4. Conclusion

To conclude this paper, we will present two tables in which we give the only symmetric MOPS verifying the $\mathcal{L}$-difference (1.3).

Polynomial	Φ	ξ_n	ϑ_n	λ_n
Generalized Hermite $H_n^{(\mu,q^2)}$	1	0	0	$\mu_{n+1}, \quad n \geq 0$
Generalized Gegenbauer $S_n^{(\alpha,\beta,q^2)}$	x^2-1	μ_{n+1}	0	$-\dfrac{\mu_{n+1}(\mu_{n+1}+2\alpha)}{2n+2\alpha+2\mu+1}, \quad n \geq 0$

Table 1: Case when $\mathcal{L} = T_\mu$

Polynomial	Φ	ξ_n	ϑ_n	λ_n
q^2-analogue of generalized Hermite $H_n^{(\mu,q^2)}$	1	0	0	$\theta_{n+1,q}, \quad n \geq 0$
q^2-analogue of generalized Gegenbauer $S_n^{(\alpha,\beta,q^2)}$	x^2-1	$\theta_{n+1,q}$	0	$-\dfrac{\theta_{n+1,q}\Big(q(q+1)(\alpha+1)+\theta_{n-1,q}(1+\theta(1-q)(1-(-1)^n)/2)\Big)}{q(q+1)(\alpha+1)+\theta_{2n-1,q}},$ $n \geq 0.$

Table 2: Case when $\mathcal{L} = T_{(\theta,q)}$

Remark 4. In this paper, we have studied only the symmetric case. The question for non-symmetric case remains open.

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REFERENCES

1. Al-Salam W. A., Chihara T. S. Another characterization of the classical orthogonal polynomials. *SIAM J. Math. Anal.*, 1972. Vol. 3, No. 1. P. 65–70. DOI: 10.1137/0503007
2. Aloui B., Souissi J. Characterization of q-Dunkl-classical symmetric orthogonal q-polynomials. *Ramanujan J.*, 2022. Vol. 57, No. 2. P. 1355–1365. DOI: 10.1007/s11139-021-00425-8
3. Belmehdi S. Generalized Gegenbauer orthogonal polynomials. *J. Comput. Appl. Math.*, 2001. Vol. 133, No. 1–2. P. 195–205. DOI: 10.1016/S0377-0427(00)00643-9
4. Bochner S. Über Sturm–Liouvillesche polynomsysteme. *Math. Z.*, 1929. Vol. 29. P. 730–736. DOI: 10.1007/BF01180560 (in German)
5. Bouanani A., Khériji L., Ihsen Tounsi M. Characterization of q-Dunkl Appell symmetric orthogonal q-polynomials. *Expo. Math.*, 2010. Vol. 28, No. 4. P. 325–336. DOI: 10.1016/j.exmath.2010.03.003
6. Bouras B. Some characterizations of Dunkl-classical orthogonal polynomials. *J. Difference Equ. Appl.*, 2014. Vol. 20, No. 8. P. 1240–1257. DOI: 10.1080/10236198.2014.906590
7. Bouras B., Habbachi Y., Marcellán F. Characterizations of the symmetric $T_{(\theta,q)}$-classical orthogonal q-polynomials. *Mediterr. J. Math.*, 2022. Vol. 19, No. 2. Art. no. 66. 18 p. DOI: 10.1007/s00009-022-01986-8
8. Chihara T. S. *An Introduction to Orthogonal Polynomials.* New York: Gordon and Breach, Sci. Publ., Inc., 1978. 249 p.
9. Datta S., Griffin J. A characterization of some q-orthogonal polynomials. *Ramanujan J.*, 2006. Vol. 12. P. 425–437. DOI: 10.1007/s11139-006-0152-5

10. Dunkl C. F. Integral kernels reflection group invariance. *Canad. J. Math.*, 1991. Vol. 43, No. 6. P. 1213–1227. DOI: 10.4153/CJM-1991-069-8
11. Geronimus J. L. On polynomials orthogonal with respect to numerical sequences and on Hahn's theorem. *Izv. Akad. Nauk.*, 1940. Vol. 250. P. 215–228. (in Russian)
12. Ghressi A., Khériji L. The symmetrical H_q-semiclassical orthogonal polynomials of class one. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 2009. Vol. 5. Art. no. 076. 22 p. DOI: 10.3842/SIGMA.2009.076
13. Hildebrandt E. H. Systems of polynomials connected with the Charlier expansions and the Pearson differential and difference equation. *Ann. Math. Statist.*, 1931. Vol. 2, No. 4. P. 379–439. DOI: 10.1214/aoms/1177732950
14. Marcellán F., Branquinho A., Petronilho J. Classical orthogonal polynomials: a functional approach. *Acta. Appl. Math.*, 1994. Vol. 34. P. 283–303. DOI: 10.1007/BF00998681
15. Maroni P. Fonctions eulériennes. Polynômes orthogonaux classiques. *Techniques de L'ingénieur*, 1994. Vol. 154. P. 1–30. DOI: 10.51257/a-v1-a154 (in French)
16. Maroni P. Une théorie algébrique des polynômes orthogonaux. Application aux polynômes orthogonaux semi-classiques. *IMACS Ann. Comput. Appl. Math.*, 1991. Vol. 9. P. 95–130. (in French)
17. Maroni P. Variations around classical orthogonal polynomials. Connected problems. *J. Comput. Appl. Math.*, 1993. Vol. 48, No. 1–2. P. 133–155. DOI: 10.1016/0377-0427(93)90319-7
18. Sghaier M. A note on Dunkl-classical orthogonal polynomials. *Integral Transforms Spec. Funct.*, 2012. Vol. 23, No. 10. P. 753–760. DOI: 10.1080/10652469.2011.631186