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MATRIX RESOLVING FUNCTIONS IN THE LINEAR GROUP PURSUIT PROBLEM WITH FRACTIONAL DERIVATIVES[1]

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Abstract: In finite-dimensional Euclidean space, we analyze the problem of pursuit of a single evader by a group of pursuers, which is described by a system of differential equations with Caputo fractional derivatives of order α. The goal of the group of pursuers is the capture of the evader by at least m different pursuers (the instants of capture may or may not coincide). As a mathematical basis, we use matrix resolving functions that are generalizations of scalar resolving functions. We obtain sufficient conditions for multiple capture of a single evader in the class of quasi-strategies. We give examples illustrating the results obtained.

Keywords: Differential game, Group pursuit, Pursuer, Evader, Fractional derivatives.

1. Introduction

The theory of two-player differential games, originally considered by Isaacs [20], has grown to be a profound and substantial theory that develops various approaches to the analysis of conflict situations [3, 14, 15, 19, 21, 22, 24, 36, 40]. The following methods for solving game problems were developed: the Isaacs method based on the analysis of some partial differential equation and its characteristics, the method of stable bridges, Krasovskii's rule of extremal aiming, Pontryagin's method based on alternating integration of convex sets, etc.

In [6, 7], Chikrii proposed a method of scalar resolving functions using Pontryagin's condition and, based on it, measurable choice theorems.

The method of scalar resolving functions was developed further to investigate linear and quasi-linear group pursuit problems [2, 10, 18, 28–30, 38, 39]. In [8], Chikrii noted that scalar resolving functions attract the terminal set to the images of some multivalued maps. This attraction occurs in the conical hull of this set, which restricts the maneuverability of pursuers.

In [8, 11], for the analysis of two-player pursuit games, matrix resolving functions were proposed. In [26], matrix resolving functions were applied to studying the group pursuit problem described by a linear autonomous system of differential equations.

In the present paper, we consider matrix resolving functions in a linear group pursuit problem described by a system of differential equations with Caputo fractional derivatives. It should be noted that matrix resolving functions for solving group pursuit problems with fractional derivatives are used for the first time. Previously, scalar resolving functions were used in [23, 25, 27] devoted to this class of problems. We obtain sufficient conditions for multiple capture of a single evader.

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The multiple capture of a single evader in the simple pursuit problem was considered in [4, 17]; [4] investigated it in a discrete setting. In [31, 32], the problem of multiple capture of a single evader was presented in the example of L.S. Pontryagin, and in [1, 33] it was considered in linear differential games.

2. Statement of the problem

Definition 1 [5]. *Let $f\colon [0,\infty) \to \mathbb{R}^k$ be an absolutely continuous function and $\alpha \in (0,1)$. The Caputo derivative of order α of the function f is defined to be a function $D^{(\alpha)}f$ of the form*

$$\big(D^{(\alpha)}f\big)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^\alpha}\, ds, \quad \textit{where} \quad \Gamma(\beta) = \int_0^\infty e^{-s} s^{\beta-1}\, ds.$$

In the space $\mathbb{R}^k$ $(k \geq 2)$, we consider a differential game $G(n+1)$ involving $n+1$ players: n pursuers $P_1, \ldots, P_n$ and an evader E, which is described by a system of the form

$$D^{(\alpha)} z_i = A_i z_i + u_i - v, \quad z_i(0) = z_i^0, \quad u_i \in U_i, \quad v \in V. \tag{2.1}$$

Here $i \in I = \{1, \ldots, n\}$, $z_i, u_i, v \in \mathbb{R}^k$, U_i and V are compact sets from $\mathbb{R}^k$, $\alpha \in (0,1)$, $D^{(\alpha)}f$ is the Caputo derivative of order α of the function f, and A_i are constant square matrices of order $k \times k$. Assume that $z_i^0 \neq 0$ for all $i \in I$. Define $z^0 = \{z_i^0, i \in I\}$ to be the vector of initial positions.

Let $v : [0,\infty) \to V$ be a measurable function. Let us call the restriction of the function v to $[0,t]$ the prehistory $v_t(\cdot)$ of the function v at time t.

Definition 2. *We will say that a quasi-strategy $\mathcal{U}_i$ of a pursuer P_i is given if a map U_i^0 is defined that associates a measurable function $u_i(t)$ with values in U_i to the initial positions z^0, time t, and arbitrary prehistory of control $v_t(\cdot)$ of the evader E.*

Definition 3. *An m-fold capture (a capture for $m=1$) occurs in the game $G(n+1)$ if there exist a time $T>0$ and quasi-strategies $\mathcal{U}_1, \ldots, \mathcal{U}_n$ of pursuers $P_1, \ldots, P_n$ such that, for any measurable function $v(\cdot)$, $v(t) \in V$, $t \in [0,T]$, there exist times $\tau_1, \ldots, \tau_m \in [0,T]$ and pairwise different indices $i_1, \ldots, i_m \in I$ such that $z_{i_l}(\tau_l) = 0$ for all $l = 1, \ldots, m$.*

The aim of this paper is to obtain conditions for the solvability of the pursuit problem.

Assumption 1. *For all $i \in I$, it is true that $0 \in \bigcap\limits_{v \in V} \big(U_i - v\big)$.*

In what follows, we assume that Assumption 1 holds. We introduce the following notation:

$$E_\rho(B,\mu) = \sum_{l=0}^{\infty} \frac{B^l}{\Gamma(l\rho^{-1}+\mu)},$$

which is a generalized Mittag-Leffler function [16], where B is a square matrix of order $k \times k$, $\rho > 0$, and $\mu \in \mathbb{R}^1$; $\Delta = \big\{(t,\tau) \ : \ t \geq 0, \ 0 \leq \tau \leq t\big\}$, $J = \{1, \ldots, k\}$,

$$g_i(t,\tau) = (t-\tau)^{\alpha-1} E_{\frac{1}{\alpha}}(A_i(t-\tau)^\alpha, \alpha), \quad \tau \neq t, \quad g(t,t) = 0,$$
$$f_i(t) = E_{\frac{1}{\alpha}}(A_i t^\alpha, 1) z_i^0, \quad W_i(t,\tau,v) = g_i(t,\tau)\big(U_i - v\big),$$
$$W_i(t,\tau) = \bigcap_{v \in V} W_i(t,\tau,v), \quad i \in I, \quad 0 \leq \tau \leq t,$$

where $(t,\tau) \in \Delta$ and $v \in V$.

Consider an arbitrary diagonal square matrix L_i of order $k \times k$ of the form

$$L_i = \begin{pmatrix} \lambda_{i1} & 0 & \dots & 0 \\ 0 & \lambda_{i2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_{ik} \end{pmatrix} = \mathrm{diag}\,(\lambda_{i1}, \lambda_{i2}, \dots, \lambda_{ik}).$$

We identify the matrix L_i with the vector $(\lambda_{i1}, \dots, \lambda_{ik})$, understand the inequality $L_i \geq 0$ coordinatewise, and introduce the multivalued maps

$$\mathcal{M}_i(t,\tau,v) = \bigl\{L_i \ : \ L_i \geq 0, -L_i f_i(t) \in W_i(t,\tau,v)\bigr\}, \quad (t,\tau) \in \Delta, \quad v \in V.$$

By Assumption 1, for all $i \in I$, $v \in V$, and t,τ such that $0 \leq \tau \leq t$, the sets $W_i(t,\tau,v)$ are not empty and $0 \in \mathcal{M}_i(t,\tau,v)$. By the properties of the parameters of the conflict-controlled process (2.1), the maps $\mathcal{M}_i(t,\tau,v)$ are measurable in τ [12]. Then the maps $W_i(t,\tau)$ are measurable in τ [12].

Define the scalar functions

$$\lambda_i^0(t,\tau,v) = \sup_{L_i \in \mathcal{M}_i(t,\tau,v)} \min_{j \in J} \lambda_{ij}(t,\tau,v), \quad (t,\tau) \in \Delta, \quad v \in V. \tag{2.2}$$

Assumption 2. *For all $(t,\tau) \in \Delta$ and $v \in V$, the supremum in (2.2) is attained.*

Assuming that the supremum in (2.2) is attained, we define the sets

$$\mathcal{M}_i^*(t,\tau,v) = \bigl\{L_i(t,\tau,v) \in \mathcal{M}_i(t,\tau,v) \ : \ \lambda_i^0(t,\tau,v) = \min_j \lambda_{ij}(t,\tau,v)\bigr\}.$$

It follows from [12] that, under the above assumptions, $\mathcal{M}_i(t,\tau,v)$ and $\mathcal{M}_i^*(t,\tau,v)$ are measurable in (τ,v) and closed-valued for any $t \geq 0$. By the measurable choice theorem [35, Theorem 20.6], for each $i \in I$ in $\mathcal{M}_i^*(t,\tau,v)$, there exists at least one selector measurable in (τ,v) for any $t \geq 0$. We fix these selectors $L_i^*(t,\tau,v)$ and define $\lambda_i^*(t,\tau,v) = \min_j \lambda_{ij}^*(t,\tau,v)$. Next, define

$$\Omega(m) = \{(i_1,\dots,i_m) \ : \ i_1,\dots,i_m \in I \text{ and are pairwise different}\},$$
$$\delta(t,\tau) = \inf_{v \in V} \max_{\Lambda \in \Omega(m)} \min_{l \in \Lambda} \lambda_l^*(t,\tau,v).$$

3. Sufficient conditions for capture

Lemma 1. *Suppose that Assumptions 1 and 2 hold and*

$$\lim_{t \to +\infty} \int_0^t \delta(t,s)ds = +\infty.$$

Then there exists a time $T > 0$ such that, for every measurable function $v(\cdot)$, $v(t) \in V$, $t \in [0,T]$, there is a set $\Lambda \in \Omega(m)$ such that the following inequalities hold for all $l \in \Lambda$, $j \in J$:

$$\int_0^T \lambda_{lj}^*(T,s,v(s))ds \geq 1.$$

P r o o f. Let $v(\cdot)$ be an arbitrary measurable function, $v\colon [0,\infty) \to V$. Then the inequalities

$$\lambda^*_{lj}(t,s,v(s)) \geq \lambda^*_l(t,s,v(s))$$

hold for all $t > 0$, $s \in [0,t]$, $l \in I$, and $j \in J$. Therefore, the inequalities

$$\int_0^t \lambda^*_{lj}(t,s,v(s))ds \geq \int_0^t \lambda^*_l(t,s,v(s))ds \tag{3.1}$$

hold for all $t \geq 0$, $l \in I$, and $j \in J$. In addition,

$$\max_{\Lambda\in\Omega(m)} \min_{l\in\Lambda} \int_0^t \lambda^*_l(t,s,v(s))ds \geq \max_{\Lambda\in\Omega(m)} \int_0^t \min_{l\in\Lambda} \lambda^*_l(t,s,v(s))ds. \tag{3.2}$$

Since, for any nonnegative numbers $a_\Lambda(\Lambda \in \Omega(m))$, one has

$$\max_{\Lambda\in\Omega(m)} a_\Lambda \geq \frac{1}{C_n^m} \sum_{\Lambda\in\Omega(m)} a_\Lambda, \quad \text{where} \quad C_n^m = \frac{n!}{(n-m)!\,m!},$$

it follows from (3.2) that

$$\max_{\Lambda\in\Omega(m)} \min_{l\in\Lambda} \int_0^t \lambda^*_l(t,s,v(s))ds \geq \frac{1}{C_n^m} \int_0^t \sum_{\Lambda\in\Omega(m)} \min_{l\in\Lambda} \lambda^*_l(t,s,v(s))ds \geq$$

$$\geq \frac{1}{C_n^m} \int_0^t \max_{\Lambda\in\Omega(m)} \min_{l\in\Lambda} \lambda^*_l(t,s,v(s))ds \geq \frac{1}{C_n^m} \int_0^t \delta(t,s)ds.$$

Since

$$\int_0^t \delta(t,s)ds = +\infty,$$

there exists $T > 0$ such that

$$\frac{1}{C_n^m} \int_0^T \delta(T,s)ds \geq 1.$$

Hence,

$$\max_{\Lambda\in\Omega(m)} \min_{l\in\Lambda} \int_0^T \lambda^*_l(T,s,v(s))ds \geq 1.$$

Therefore, there exists $\Lambda \in \Omega(m)$ such that the following inequalities hold for all $l \in \Lambda$:

$$\int_0^T \lambda^*_l(T,s,v(s))ds \geq 1.$$

This inequality and inequality (3.1) imply the validity of the lemma. $\square$

Let $\mathcal{V}$ be the set of all measurable functions $v\colon [0,\infty) \to V$. Let us define the number

$$\hat{T} = \inf\{t \geq 0 \;:\; \inf_{v(\cdot)\in\mathcal{V}} \max_{\Lambda\in\Omega(m)} \min_{l\in\Lambda} \min_{j\in J} \int_0^t \lambda^*_{lj}(t,s,v(s))ds \geq 1\}.$$

Consider the sets $(i \in I, j \in J, v(\cdot) \in \mathcal{V})$

$$T_{ij}(v(\cdot)) = \Big\{t \geq 0 \;:\; \int_0^t \lambda^*_{ij}(\hat{T},s,v(s))ds \geq 1\Big\}.$$

Define the quantities $(i \in I, j \in J, v(\cdot) \in \mathcal{V})$

$$t^*_{ij}(v(\cdot)) = \begin{cases} \inf\{t \;:\; t \in T_{ij}(v(\cdot))\} & \text{if} \quad T_{ij}(v(\cdot)) \neq \emptyset, \\ +\infty & \text{if} \quad T_{ij}(v(\cdot)) = \emptyset. \end{cases} \tag{3.3}$$

Assumption 3. *For any $\tau \in [0,\hat{T}]$, $v \in V$, $l \in I$, and $J_0 \subset J$, the selector*

$$B_l(\hat{T},\tau,v) = \operatorname{diag}\big(\beta_{l1}(\hat{T},\tau,v),\dots,\beta_{lk}(\hat{T},\tau,v)\big),$$

where

$$\beta_{lj}(\hat{T},\tau,v) = \begin{cases} \lambda^*_{lj}(\hat{T},\tau,v), & j \in J_0, \\ 0, & j \notin J_0, \end{cases}$$

satisfies the condition $B_l(\hat{T},\tau,v) \in \mathcal{M}_l(\hat{T},\tau,v)$.

Theorem 1. *Suppose that Assumptions 1, 2, and 3 hold and*

$$\lim_{t\to+\infty} \int_0^t \delta(t,s)ds = +\infty.$$

Then an m-fold capture occurs in the game $G(n+1)$.

P r o o f. By Lemma 1, $\hat{T} < +\infty$. Let $v : [0,\hat{T}] \to V$ be an arbitrary measurable function and $\tau \in [0,\hat{T}]$. Let us introduce functions $\big(\beta_{i1}(\hat{T},\tau,v),\dots,\beta_{ik}(\hat{T},\tau,v)\big)$ of the form

$$\beta_{ij}(\hat{T},\tau,v) = \begin{cases} \lambda^*_{ij}(\hat{T},\tau,v), & \tau \in [0,t^*_{ij}(v(\cdot))], \\ 0, & \tau \in (t^*_{ij}(v(\cdot)),\hat{T}], \end{cases}$$

where $t^*_{ij}(v(\cdot))$ are defined by formula (3.3). Let $B^*_i(\hat{T},s,v)$ be a matrix of the form

$$B^*_i(\hat{T},s,v) = \begin{pmatrix} \beta^*_{i1}(\hat{T},s,v) & 0 & \dots & 0 \\ 0 & \beta^*_{i2}(\hat{T},s,v) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \beta^*_{ik}(\hat{T},s,v) \end{pmatrix}.$$

Consider the multivalued maps $(s \in [0,\hat{T}],\ v \in V)$

$$\tilde{U}_i(\hat{T},s,v) = \big\{u_i \in U_i \;:\; g_i(\hat{T},s)(u_i - v) = -B^*_i(\hat{T},s,v)f_i(\hat{T})\big\}.$$

By Assumption 3, $B_i^*(\hat{T}, s, v)$ is a measurable selector of $\mathcal{M}_i(\hat{T}, s, v)$. Therefore, the sets $\tilde{U}_i(\hat{T}, s, v)$ are nonempty for all $i \in I$, $s \in [0, \hat{T}]$, and $v \in V$. Hence, by the measurable choice theorem [35, Theorem 20.6], there exists at least one measurable selector $u_i^*(\hat{T}, s, v)$. We define the controls of pursuers P_i, $i \in I$, assuming

$$u_i(\tau) = u_i^*(\hat{T}, \tau, v(\tau)).$$

By [12], the functions $u_i(\cdot)$ are measurable. We show that these controls of the pursuers guarantee the m-fold capture of the evader. The solution of the Cauchy problem for system (2.1) has the form [9]:

$$z_i(t) = f_i(t) + \int_0^t g_i(t, s)(u_i(s) - v(s))ds.$$

By the choice of controls of the pursuers, we obtain

$$z_i(\hat{T}) = f_i(\hat{T}) - \int_0^{\hat{T}} B_i(\hat{T}, s, v(s)) f_i(\hat{T}) ds = \Big(E - \int_0^{\hat{T}} B_i(\hat{T}, s, v(s)) ds\Big) f_i(\hat{T}),$$

where E is an identity matrix. It follows from the definition of $B_i(\hat{T}, s, v(s))$ that there exists $\Lambda \in \Omega(m)$ such that $z_l(\hat{T}) = 0$ for all $l \in \Lambda$. This proves the theorem. □

Assumption 4. *The matrices A_i are diagonal matrices of the form*

$$A_i = \begin{pmatrix} a_{i1} & 0 & \dots & 0 \\ 0 & a_{i2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{ik} \end{pmatrix} \quad \text{with} \quad a_{ij} \le 0 \quad \text{for all} \quad i \in I, \quad j \in J.$$

Let us introduce multivalued maps ($v \in V$)

$$\mathcal{M}_i^0(v) = \big\{L_i \ : \ L_i \ge 0, -L_i z_i^0 \in \big(U_i - v\big)\big\}.$$

By Assumption 1, the sets $\mathcal{M}_i^0(v)$ for all $i \in I$ and $v \in V$ are nonempty and $0 \in \mathcal{M}_i^0(v)$. Next, we define functions $\overline{\lambda}_i(v)$ of the form

$$\overline{\lambda}_i(v) = \sup_{L_i \in \mathcal{M}_i^0(v)} \min_j \lambda_{ij}(v). \tag{3.4}$$

Assumption 5. *For all $v \in V$, the supremum in (3.4) is attained.*

Assuming that the supremum in (3.4) is attained, we define the sets ($v \in V$)

$$\overline{\mathcal{M}}_i(v) = \big\{L_i(v) \in \mathcal{M}_i^0(v) \ : \ \overline{\lambda}_i(v) = \min_j \lambda_{ij}(v)\big\}.$$

Next, suppose that $\overline{\lambda}_i^*(v)$ is a measurable selector of $\overline{\mathcal{M}}_i(v)$ and

$$\delta = \inf_{v \in V} \max_{\Lambda \in \Omega(m)} \min_{l \in \Lambda} \overline{\lambda}_l^*(v).$$

Define $((t,\tau)\in\Delta)$

$$
\begin{gathered}
a=\max_{i,j}(-a_{ij}),\\
g_{ij}(t,s)=(t-s)^{\alpha-1}E_{\frac{1}{\alpha}}\big(a_{ij}(t-s)^{\alpha},\alpha\big),\quad t\neq s,\\
g(t,s)=(t-s)^{\alpha-1}E_{\frac{1}{\alpha}}\big(-a(t-s)^{\alpha},\alpha\big),\quad t\neq s,\\
g_{ij}(t,t)=g(t,t)=0.
\end{gathered}
$$

Lemma 2. *Suppose that Assumptions 1, 4, and 5 hold, and $\delta>0$ and $a_{ij}<0$ for all $i\in I$ and $j\in J$. Then there exists $T>0$ such that, for every admissible function $v(\cdot)$, there is a set $\Lambda\in\Omega(m)$ such that the following inequalities hold for all $l\in\Lambda$ and $j\in J$:*

$$
E_{\frac{1}{\alpha}}(a_{lj}T^{\alpha},1)-\int_0^T g_{lj}(T,s)\overline{\lambda}^*_{lj}(v(s))\,ds\leq 0.
$$

P r o o f. Let $v(\cdot)$ be an admissible function. Then $0<-a_{ij}\leq a$ for all i and j. Therefore, the following inequalities hold [34] for all $t\geq 0$, $s\in[0,t]$, $i\in I$, and $i\in J$:

$$
E_{\frac{1}{\alpha}}(a_{ij}(t-s)^{\alpha},\alpha)\geq E_{\frac{1}{\alpha}}(-a(t-s)^{\alpha},\alpha).
$$

It follows from [37, Theorem 4.1.1] that $E_{\frac{1}{\alpha}}(z,\mu)\geq 0$ for all $z\in\mathbb{R}^1$ and $\mu\in[\alpha,+\infty)$. Hence, the inequalities

$$
\int_0^t g_{ij}(t,s)\overline{\lambda}^*_{ij}(v(s))ds\geq\int_0^t g(t,s)\overline{\lambda}^*_{i}(v(s))ds
$$

hold for all $t\geq 0$, $i\in I$, and $j\in J$. Next, we have

$$
\max_{\Lambda\in\Omega(m)}\min_{l\in\Lambda}\int_0^t g(t,s)\overline{\lambda}^*_{l}(v(s))ds\geq\max_{\Lambda\in\Omega(m)}\int_0^t g(t,s)\min_{l\in\Lambda}\overline{\lambda}^*_{l}(v(s))ds. \tag{3.5}
$$

Using inequality (3.5), we obtain

$$
\begin{gathered}
\max_{\Lambda\in\Omega(m)}\int_0^t g(t,s)\min_{l\in\Lambda}\overline{\lambda}^*_{l}(v(s))ds\geq\frac{1}{C_n^m}\int_0^t g(t,s)\sum_{\Lambda\in\Omega(m)}\min_{l\in\Lambda}\overline{\lambda}^*_{l}(v(s))ds\geq\\
\geq\frac{1}{C_n^m}\int_0^t g(t,s)\max_{\Lambda\in\Omega(m)}\min_{l\in\Lambda}\overline{\lambda}^*_{l}(v(s))ds\geq\frac{\delta}{C_n^m}\int_0^t g(t,s)ds.
\end{gathered}
$$

By [13, Ch. 3, formula (1.15)],

$$
\int_0^t g(t,s)ds=t^{\alpha}E_{\frac{1}{\alpha}}(-at^{\alpha},\alpha+1).
$$

Consider the functions ($t \in [0, \infty)$)

$$h_{ij}(t) = E_{\frac{1}{\alpha}}(a_{ij} t^{\alpha}, 1) - \frac{\delta}{C_n^m} t^{\alpha} E_{\frac{1}{\alpha}}(-a t^{\alpha}, \alpha + 1).$$

Since $a_{ij} < 0, -a < 0$, it follows from [37, Theorem 1.2.1] that the following asymptotic representation holds as $t \to +\infty$:

$$E_{1/\alpha}(a_{ij} t^{\alpha}, 1) = -\frac{1}{a_{ij} t^{\alpha} \Gamma(1-\alpha)} + O\left(\frac{1}{t^{2\alpha}}\right), \quad E_{1/\alpha}(-a t^{\alpha}, \alpha + 1) = \frac{1}{a t^{\alpha}} + O\left(\frac{1}{t^{2\alpha}}\right).$$

Therefore,

$$h_{ij}(t) = \frac{c_{ij}}{t^{\alpha}} - \frac{\delta}{a C_n^m} + O\left(\frac{1}{t^{2\alpha}}\right).$$

Consequently, $\lim\limits_{t \to +\infty} h_{ij}(t) < 0$ for all $i \in I$ and $j \in J$. Hence, there exists $T > 0$ such that $h_{ij}(T) \leq 0$ for all $i \in I$ and $j \in J$. Next, let $\Lambda \in \Omega(m)$ be such that

$$\max_{\Lambda \in \Omega(m)} \min_{l \in \Lambda} \int_0^T g(T, s) \overline{\lambda}_l^*(v(s)) ds = \min_{l \in \Lambda} \int_0^T g(T, s) \overline{\lambda}_l^*(v(s)) ds.$$

Then, for all $l \in \Lambda$, one has

$$\int_0^T g(T, s) \overline{\lambda}_l^*(v(s)) ds \geq \frac{\delta}{C_n^m} T^{\alpha} E_{\frac{1}{\alpha}}(-a T^{\alpha}, \alpha + 1).$$

Therefore,

$$-\int_0^T g_{lj}(T, s) \overline{\lambda}_{lj}^*(v(s)) ds \leq -\int_0^T g(T, s) \overline{\lambda}_l^*(v(s)) ds \leq -\frac{\delta}{C_n^m} T^{\alpha} E_{\frac{1}{\alpha}}(-a T^{\alpha}, \alpha + 1).$$

Hence, the inequalities

$$E_{\frac{1}{\alpha}}(a_{lj} T^{\alpha}, 1) - \int_0^T g_{lj}(T, s) \overline{\lambda}_{lj}^*(v(s))\, ds \leq E_{\frac{1}{\alpha}}(a_{lj} T^{\alpha}, 1) - \frac{\delta}{C_n^m} T^{\alpha} E_{\frac{1}{\alpha}}(-a T^{\alpha}, \alpha + 1) \leq 0$$

hold for all $l \in \Lambda$ and $j \in J$. This proves the lemma. □

Lemma 3. *Suppose that Assumptions 1, 4, and 5 hold, $a_{ij} \leq 0$, and $\delta > 0$. Then there exists $T > 0$ such that, for every admissible function $v(\cdot)$, there is a set $\Lambda \in \Omega(m)$ such that the following inequalities hold for all $l \in \Lambda$ and $j \in J$:*

$$E_{\frac{1}{\alpha}}(a_{lj} T^{\alpha}, 1) - \int_0^T g_{lj}(T, s) \overline{\lambda}_{lj}^*(v(s))\, ds \leq 0.$$

P r o o f. The proof is similar to the proof of Lemma 2.

□

Assumption 6. *For all $v \in V$, $l \in I$, and $J_0 \subset J$, the selector*

$$B_l(v) = \operatorname{diag}(\beta_{l1}(v), \ldots, \beta_{lk}(v)),$$

where

$$\beta_{lj}(v) = \begin{cases} \lambda^*_{lj}(v), & j \in J_0, \\ 0, & j \notin J_0, \end{cases}$$

satisfies the condition $B_l(v) \in \mathcal{M}^0_l(v)$.

Remark 1. Note that Assumption 6 does not always hold. Suppose that, in system (2.1), $k = 2$, $n = 1$, $m = 1$, $z^0_1 = (1, 2)$, A_1 is a zero matrix, and

$$U_1 = V = \{(u_1, u_2) : \ u_1 = u_2, \ u_2 \in [-1, 1]\}.$$

Let $v = 0$. Then

$$\mathcal{M}^0_1(0) = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda/2 \end{pmatrix}, \quad \lambda \in [0, 1] \right\}.$$

Therefore,

$$\sup_{L \in \mathcal{M}^0_1(0)} \min_j \lambda_{1j} = \frac{1}{2}.$$

Hence,

$$\overline{\mathcal{M}}_1(0) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \right\}$$

and the extremal selector is $\lambda^*_1(0) = \operatorname{diag}(1, 1/2)$. However, the selector $B_1(0) = \operatorname{diag}(1, 0) \notin \mathcal{M}^0_1(0)$. Similarly, the selector $B_2(0) = \operatorname{diag}(0, 1/2) \notin \mathcal{M}^0_1(0)$.

Remark 2. If Assumption 1 holds, in particular, if the sets U_i have the form $U_i = [a_{i1}, b_{i1}] \times [a_{i2}, b_{i2}] \times \ldots \times [a_{ik}, b_{ik}]$ for all i, then Assumption 6 also holds.

Theorem 2. *Suppose that Assumptions 1, 4, 5, and 6 hold and $\delta > 0$. Then an m-fold capture occurs in the game $G(n + 1)$.*

P r o o f. Define the number

$$\hat{T} = \inf\left\{ t \geq 0 \ : \ \sup_{v(\cdot) \in \mathcal{V}} \min_{\Lambda \in \Omega(m)} \max_{l \in \Lambda} \max_{j \in J} \left(E_{\frac{1}{\alpha}}(a_{lj} t^\alpha, 1) - \int_0^t g_{lj}(t, s) \overline{\lambda}^*_{lj}(v(s)) ds \right) \leq 0 \right\}.$$

Then, by Lemma 3, $\hat{T} < +\infty$. Let $v(\cdot)$ be the admissible control of the evader. Consider the sets $(i \in I, j \in J, v(\cdot) \in \mathcal{V})$

$$T_{ij}(v(\cdot)) = \{t \ : \ E_{\frac{1}{\alpha}}(a_{lj} \hat{T}^\alpha, 1) - \int_0^t g_{lj}(\hat{T}, s) \overline{\lambda}^*_{lj}(v(s)) ds \leq 0\}.$$

Next, let

$$t^*_{ij}(v(\cdot)) = \begin{cases} \inf\{t : t \in T_{ij}(v(\cdot))\} & \text{if } \ T_{ij}(v(\cdot)) \neq \emptyset, \\ +\infty & \text{if } \ T_{ij}(v(\cdot)) = \emptyset, \end{cases} \quad \beta_{lj}(t) = \begin{cases} \lambda^*_{lj}(v(t)), & t \in [0, t^*_{ij}(v(\cdot))], \\ 0, & t \in (t^*_{ij}(v(\cdot)), \hat{T}], \end{cases}$$

$$B_i(t) = \operatorname{diag}(\beta_{i1}(t), \ldots, \beta_{ik}(t)).$$

Define the controls of pursuers P_i, $i \in I$, assuming

$$u_i(t) = v(t) - B_i(t) z_i^0.$$

The solution of the Cauchy problem for system (2.1) has the form [9]

$$z_i(t) = E_{\frac{1}{\alpha}}(A_i t^\alpha, 1) z_i^0 + \int_0^t (t-s)^{\alpha-1} E_{\frac{1}{\alpha}}(A_i(t-s)^{\alpha-1}, \alpha)(u_i(s) - v(s)) ds.$$

Therefore,

$$z_{lj}(\hat{T}) = \Big(E_{\frac{1}{\alpha}}(a_{ij}\hat{T}^\alpha, 1) - \int_0^{\hat{T}} g_{ij}(\hat{T}, s) B_{ij}(s) ds\Big) z^0_{ij} =$$

$$= \Big(E_{\frac{1}{\alpha}}(a_{ij}\hat{T}^\alpha, 1) - \int_0^{t^*_{ij}(v(\cdot))} g_{ij}(\hat{T}, s) \overline{\lambda}^*_{ij}(v(s)) ds\Big) z^0_{ij}.$$

It follows from the assumptions of the theorem and the definition of $B_i(t)$ $(i \in I,\ t \in [0, \infty))$ that there exists $\Lambda \in \Omega(m)$ such that $z_{lj}(\hat{T}) = 0$ for all $l \in \Lambda$ and $j \in J$, which implies that an m-fold capture occurs in the game $G(n+1)$. This proves the theorem. □

Example 1. Suppose that, in system (2.1), $k = 2$, $n = 1$, $m = 1$, $z_1^0 = (1, 2)$, A_1 is a zero matrix, $V = \{0\}$, and

$$U_1 = \{(u_1, u_2) : u_1{=}0, u_2{\in}[-1, 1]\} \cup \{(u_1, u_2) : u_2{=}0, u_1{\in}[-1, 1]\} \cup \{(u_1, u_2) : u_1{=}u_2{\in}[-1, 1]\}.$$

Then

$$\mathcal{M}_1^0(0) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}, \lambda \in [0, 1/2] \right\} \bigcup \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}, \lambda \in [0, 1] \right\} \bigcup \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda/2 \end{pmatrix}, \lambda \in [0, 1] \right\}.$$

Hence,

$$\sup_{L \in \mathcal{M}_1^0(0)} \min_j \lambda_{1j} = 1/2.$$

Consequently,

$$\overline{\mathcal{M}}_1(0) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \right\}$$

and the extremal selector is $\overline{\lambda}^*_1(0) = \operatorname{diag}(1, 1/2)$. Therefore, $\hat{T} = \big(2\alpha\Gamma(\alpha)\big)^{1/\alpha}$, and the control of the pursuer P_1 has the form

$$u_1(t) = \begin{cases} (-1, -1), & t \in [0, T_1], \\ (0, -1), & t \in (T_1, \hat{T}], \end{cases}$$

where $T_1 = \hat{T} - \big(\alpha\Gamma(\alpha)\big)^{1/\alpha}$. Then [9]

$$z_1(\hat{T}) = z_1^0 + \frac{1}{\Gamma(\alpha)} \int\limits_0^{\hat{T}} (\hat{T} - s)^{\alpha-1} u_1(s) ds.$$

Therefore,

$$z_{11}(\hat{T}) = z_{11}^0 - \frac{1}{\Gamma(\alpha)} \int\limits_0^{T_1} (\hat{T} - s)^{\alpha-1} ds = 0, \quad z_{12}(\hat{T}) = z_{12}^0 - \frac{1}{\Gamma(\alpha)} \int\limits_0^{\hat{T}} (\hat{T} - s)^{\alpha-1} ds = 0.$$

Note that the use of scalar resolving functions, i.e., functions of the form

$$L = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix},$$

does not allow one to get the capture since, in this case, the condition $-Lz_0 \in U_1 - v$ is satisfied only for the zero matrix L.

We now present conditions on the game parameters under which the capture is guaranteed when scalar resolving functions are used.

Assumption 7. *In system (2.1), the matrices A_i have the form $A_i = a_i E$, $a_i \leq 0$, $i \in I$, E is an identity matrix, and*

$$\delta_0 = \inf_{v \in V} \max_{\Lambda \in \Omega(m)} \min_{l \in \Lambda} \mu_l(v) > 0,$$

where $\mu_l(v) = \sup\{\mu \geq 0 \ : -\mu z_l^0 \in U_l - v\}$.

Theorem 3. *Suppose that Assumptions 1 and 7 hold. Then an m-fold capture occurs in the game $G(n+1)$.*

P r o o f. It follows from the conditions of the theorem that the following equations hold for all $i \in I$, $j \in J$:

$$g_{ij}(t,s) = (t-s)^{\alpha-1} E_{\frac{1}{\alpha}}\big(a_i (t-s)^\alpha, \alpha\big) = g_i(t,s), \quad t \neq s, \quad g_{ij}(t,t) = 0,$$
$$E_{\frac{1}{\alpha}}(a_{ij} t^\alpha, 1) = E_{\frac{1}{\alpha}}(a_i t^\alpha, 1).$$

Therefore, it follows from Lemma 3 that there exists a time $T > 0$ such that, for every admissible function $v(\cdot) \in \mathcal{V}$, there is a set $\Lambda \in \Omega(m)$ such that the inequalities

$$E_{\frac{1}{\alpha}}(a_l T^\alpha, 1) - \int\limits_0^T g_l(T,s) \mu_l(v(s)) ds \leq 0$$

hold for all $l \in \Lambda$. Define the number

$$T_0 = \inf\Big\{ t > 0 \ : \ \sup_{v(\cdot)} \min_{\Lambda \in \Omega(m)} \max_{l \in \Lambda} \Big(E_{\frac{1}{\alpha}}(a_l t^\alpha, 1) - \int\limits_0^t g_l(t,s) \mu_l(v(s)) ds \Big) \leq 0 \Big\}.$$

Next, let $v(\cdot)$ be the admissible control of the evader:

$$\tau_l = \inf\Big\{t > 0 \ : \ E_{\frac{1}{\alpha}}(a_i T_0^{\alpha}, 1) - \int\limits_0^t g_l(T_0, s)\mu_l(v(s))ds \le 0\Big\}.$$

It follows from the above proof that there exists a set $\Lambda_0 \in \Omega(m)$ such that the inequalities $\tau_l \le T_0$ hold for all $l \in \Lambda_0$. Define the controls of pursuers P_i, $i \in I$, assuming

$$u_i(t) = \begin{cases} v(t) - \mu_i(v(t))z_i^0, & t \in [0, \tau_i], \\ v(t), & t \in [\tau_i, T_0]. \end{cases}$$

The solution of the Cauchy problem for system (2.1) has the form [9]

$$z_l(T_0) = \Big(E_{\frac{1}{\alpha}}(a_l T_0^{\alpha}, 1) - \int\limits_0^{T_0} g_l(T_0, s)\mu_l(v(s))ds\Big) z_l^0.$$

This equation and the definition of Λ_0 imply that $z_l(T_0) = 0$ for all $l \in \Lambda_0$. This proves the theorem. □

Corollary 1. *Suppose that, in system (2.1), the matrices A_i have the form $A_i = a_i E$, $a_i \le 0$, $i \in I$, E is an identity matrix, $U_i = V$ for all $i \in I$, V is a strictly convex compact set with a smooth boundary, and*

$$0 \in \bigcap_{\Lambda \in \Omega(n-m+1)} \operatorname{Int}\operatorname{co}\{z_l^0, l \in \Lambda\}, \tag{3.6}$$

where $\operatorname{Int} A$ *and* $\operatorname{co} A$ *denote the interior and the convex hull of the set A, respectively. Then an m-fold capture occurs in the game $G(n+1)$.*

Indeed, in this case, condition (3.6) implies that $\delta_0 > 0$ [30].

4. Conclusion

We obtained new sufficient conditions for multiple capture of the evader in the group pursuit problem with fractional derivatives. To solve the problem, we introduced matrix resolving functions.

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