ON THE STRUCTURE OF SINGULAR SET OF A PIECEWISE SMOOTH MINIMAX SOLUTION OF HAMILTON–JACOBI–BELLMAN EQUATION[1]

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Abstract: The properties of a minimax piecewise smooth solution of the Hamilton–Jacobi–Bellman equation are studied. We get a generalization of the nesessary and sufficient conditions for the points of nondifferentiability (singularity) of the minimax solution and the Rankine–Hugoniot condition. We describe the dimensions of smooth manifolds containing in the singular set of the piecewise smooth solution in terms of state characteristics crossing on this singular set. New structural properties of the singular set are obtained for the case of the Hamiltonian depending only on the impulse variable.

Key words: Hamilton–Jacobi–Bellman equation, Minimax solution, Singular set, Piecewise smooth solution, Tangent subspace, Rankine–Hugoniot condition.

Introduction

As is known [1–3], the first-order partial differential equations of the Hamilton–Jacobi–Bellman type are associated with problems of optimal control theory. In the present paper, we study the properties of the generalized (minimax) solution of the Hamilton–Jacobi–Bellman equation (HJBE) proposed by A.I. Subbotin. Necessary and sufficient conditions for a point belonging to the singular set of the minimax solution, i. e., to the set of points of nondifferentiability, were obtained by E.A. Kolpakova [4, 5]. These results are developed in the present paper. We study properties of the singular set of a minimax piecewise smooth solution and establish connections between the dimension of singular submanifolds and the state characteristics that come to these submanifolds. We also obtain the connection between the structure of the Hamiltonian and the structure of the singular set in the case when the Hamiltonian depends only on the impulse variable.

One of the close works in this area is the book by A.A. Melikyan [6], where partial differential equations of the first order. Where considered in the class of continuously differentiable and piecewise smooth input data. His work is mainly devoted to the issue of development of the method of characteristics and its further application to constructions of solutions in the following cases: a) the generalized viscosity solution is not smooth and then the Hamiltonian is smooth or non-smooth function; b) the solution is smooth, but the Hamiltonian is non smooth. The Poisson brackets are the main tools in the study [6].

Another work deserves special attention. It is the monograph P. Cannarsa and C. Sinestari [7]. The authors study solutions of the Hamilton–Jacobi–Bellman equation in the class of semiconvex or semiconcave functions. This solutions have the bounded second derivatives along any direction. Singular set has a simple structure in the case when a solution is in the class of semiconvex or semiconcave functions. The class of semiconvex or semiconcave functions is more narrow then the class of piecewise smooth functions in the presenting paper.

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1. Piecewise smooth solution of Hamilton–Jacobi–Bellman equation and its singular set

1.1. Problem statement

Consider the Cauchy boundary value problem for the Hamilton–Jacobi–Bellman equation

$$\frac{\partial\varphi(t,x)}{\partial t} + H(t,x,D_x\varphi(t,x)) = 0, \quad \varphi(T,x) = \sigma(x), \tag{1.1}$$

where $t \in [0,T]$, $x \in \mathbb{R}^n$, and $D_x\varphi(t,x) = \Big(\frac{\partial\varphi(t,x)}{\partial x_1}, \frac{\partial\varphi(t,x)}{\partial x_2}, \dots, \frac{\partial\varphi(t,x)}{\partial x_n}\Big) = s$.

Define $\Pi_T = \{(t,x) : t \in [0,T], x \in \mathbb{R}^n\}$, the symbol $int\,\Pi_T$ denote the interior of the set Π_T.

We investigate problem (1.1) under the following assumptions:

(A1) the function $H(t,x,s)$ is continuously differentiable with respect to the variables t,x,s and is concave with respect to the variable s;

(A2) the functions $D_sH(t,x,s)$, $D_xH(t,x,s)$ are Lipschitz continuous on the variables x and s, there exist constants $L_1 > 0$, $L_2 > 0$ such that:

$$||D_sH(t,x^{'},s^{'}) - D_sH(t,x^{''},s^{''})|| \le L_1(||x^{'} - x^{''}|| + ||s^{'} - s^{''}||),$$
$$||D_xH(t,x^{'},s^{'}) - D_xH(t,x^{''},s^{''})|| \le L_2(||x^{'} - x^{''}|| + ||s^{'} - s^{''}||)$$

for any $(t,x^{'})$, $(t,x^{''}) \in \Pi_T$ and for any $s^{'}$, $s^{''} \in \mathbb{R}^n$;

(A3) the function $\sigma(x)$ is continuously differentiable;

(A4) there exist $\alpha > 0$ and $\beta > 0$ such that

$$\|D_xH(t,x,s)\| \le \alpha(1 + \|x\| + \|s\|), \quad \|D_sH(t,x,s)\| \le \beta(1 + \|x\| + \|s\|)$$

for any point $(t,x,s) \in \Pi_T \times \mathbb{R}^n$. Here, the symbol $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$.

The aim of this paper is to study the structure of a solution $\varphi(\cdot)$ to problem (1.1).

1.2. Generalized solution to problem (1.1)

Under the above assumptions, a classical solution $\varphi(\cdot)$ to problem (1.1) may exist only locally in a neighborhood of the boundary manifold

$$C^T = \{(t,x,z)\colon t = T, x = \xi, z = \sigma(\xi);\ \xi \in \mathbb{R}^n\}.$$

This solution $\varphi(\cdot)$ can be constructed using the Cauchy method of characteristics [8]. Let us write the characteristic system with the boundary conditions at $t = T$ for problem (1.1):

$$\dot{\tilde{x}} = D_sH(t,\tilde{x},\tilde{s}), \quad \dot{\tilde{s}} = -D_xH(t,\tilde{x},\tilde{s}), \quad \dot{\tilde{z}} = \langle\tilde{s}, D_sH(t,\tilde{x},\tilde{s})\rangle - H(t,\tilde{x},\tilde{s}), \tag{1.2}$$

$$\tilde{x}(T,\xi) = \xi, \quad \tilde{s}(T,\xi) = D_x\sigma(\xi), \quad \tilde{z}(T,\xi) = \sigma(\xi) \ \ \forall\xi \in \mathbb{R}^n. \tag{1.3}$$

The symbol $\langle\cdot,\cdot\rangle$ denotes the inner product.

The solutions $\tilde{x}$, $\tilde{s}$, and $\tilde{z}$ are called, respectively, the *state, impulse*, and *cost of characteristics* of the Hamilton–Jacobi–Bellman equation (1.1).

We note that, under conditions (A1)–(A4), for any $\xi \in \mathbb{R}^n$ a solution of the characteristic system exists, is unique, and can be extended to the interval $[0,T]$.

According to the Cauchy method [8], we have the formulas $x = \tilde{x}(t,\xi)$, $\varphi(t,x) = \tilde{z}(t,\xi)$ and $D_x\varphi(t,x) = \tilde{s}(t,\xi)$, $t \in [0,T]$, $\xi \in R^n$, if the Jacobian $\dfrac{\partial\tilde{x}(t,\xi)}{\partial(t,\xi)}$ is not equal to zero.

In what follows, we consider nonclassical, nonsmooth solutions to problem (1.1). We apply the following generalization of the notion of differentiability of a function [9]; this generalization is a useful tool of nonsmooth analysis.

Definition 1. The superdifferential $D^+\varphi(t_0, x_0)$ of a function $\varphi(\cdot)\colon \Pi_T \to \mathbb{R}$ at a point (t_0, x_0) is defined as the following set

$$D^+\varphi(t_0,x_0) = \mathrm{co}\Big\{(\alpha, s) \in \mathbb{R}^{n+1}:$$

$$\limsup_{\Delta t\to 0,\, \Delta x \to 0} \frac{\varphi(t_0+\Delta t, x_0+\Delta x) - \varphi(t_0,x_0) - \langle(\alpha,s),(\Delta t,\Delta x)\rangle}{|\Delta t| + \|\Delta x\|} \le 0\Big\}.$$

The superdifferential of a function $\varphi(\cdot)$ at the points of its differentiability consists of the unique element equal to the gradient of this function.

We recall (see [4, 5]) the definition of a generalized solution to problem (1.1).

Definition 2. A generalized solution to problem (1.1) is a locally Lipschitz superdifferentiable function $\Pi_T \ni (t, x) \mapsto \varphi(t, x) \in \mathbb{R}$ such that, for any point $(t_0, x_0) \in \Pi_T$, there exist $\xi_0 \in \mathbb{R}^n$ and solutions $\tilde{x}(\cdot, \xi_0)$, $\tilde{s}(\cdot, \xi_0)$, and $\tilde{z}(\cdot, \xi_0)$ of system (1.2), (1.3) satisfying the condition

$$\tilde{x}(t_0,\xi_0) = x_0, \quad \tilde{z}(t_0,\xi_0) = \varphi(t_0,x_0), \quad \tilde{z}(t,\xi_0) = \varphi(t, \tilde{x}(t,\xi_0)) \quad \forall t \in [t_0, T].$$

A superdifferential function is the function $\varphi(\cdot) : \Pi_T \to R$ such that $D^+\varphi(t, x) \neq \varnothing$ for any $(t, x) \in int\, \Pi_T$.

The following assertion on the connection of Definition 2 with the definitions of the minimax solution and the viscosity solution is a consequence of results in [4, 5, 10–12].

Proposition 1. *If conditions* (A1)–(A4) *are satisfied to problem* (1.1)*, then there exists a unique generalized solution to problem* (1.1) *in the sense of Definition* 2*. In addition, Definition* 2 *is equivalent to the definitions of minimax solution and viscosity solution to problem* (1.1).

1.3. Singular set

Let us recall the definition of the singular set of a generalized solution $\varphi(\cdot)$ to problem (1.1).

Definition 3. The singular set Q of a generalized solution $\varphi(\cdot)$ to problem (1.1) is the set of points $(t, x) \in \Pi_T$ where the function φ is not differentiable.

According to [4, 5], the following assertions hold.

Proposition 2. *Let conditions* (A1)–(A4) *be satisfied for problem* (1.1)*. Then* $(t, x) \in Q$ *if and only if there exist* $\xi_1, \xi_2 \in \mathbb{R}^n$, $\xi_1 \neq \xi_2$, *such that*

$$\tilde{x}(t,\xi_1) = \tilde{x}(t,\xi_2) = x, \quad \tilde{z}(t,\xi_1) = \tilde{z}(t,\xi_2) = \varphi(t,x), \quad \tilde{s}(t,\xi_1) \neq \tilde{s}(t,\xi_2),$$

where $\tilde{x}(\cdot, \xi_i)$, $\tilde{s}(\cdot, \xi_i)$, *and* $\tilde{z}(\cdot, \xi_i)$, $i = 1, 2$, *are solutions of the characteristic system* (1.2), (1.3).

Proposition 3. *If the singular set* Q *contains the curve given by a continuously differentiable function* $t \mapsto x(t)$, $0 < t_0 < t \le T$, *then*

$$\left\langle \tilde{s}(t,\xi_1) - \tilde{s}(t,\xi_2), \frac{dx(t)}{dt} \right\rangle = H(t, x(t), \tilde{s}(t,\xi_1)) - H(t, x(t), \tilde{s}(t,\xi_2)) \quad \forall t \in (t_0, T]\,.$$

This relation generalizes the known Rankine–Hugoniot condition to the case of the n-dimensional state variable x.

1.4. Class of piecewise smooth functions

In the present paper, we consider generalized solutions $\varphi(\cdot)$ to problem (1.1) in the class of piecewise smooth functions (see, for instance, [10]).

Definition 4. A function $\varphi(\cdot) : \Pi_T \to \mathbb{R}$ is called piecewise smooth in Π_T if

(1) The domain of this function Π_T has the following structure:

$$int\, \Pi_T = \bigcup_{i \in I} M_i, \quad M_i \cap M_j = \varnothing \quad \text{for} \quad i, j \in I,\ i \neq j,$$

where $I = \{1, 2, \ldots, N\}$, M_i are differentiable submanifolds in Π_T.

(2) The restriction of a piecewise smooth function $\varphi(\cdot)$ to $\overline{M}_j$, $j \in J$, is a continuously differentiable function, where

$$J := \{i \in I : M_i \text{ is an } (n+1)\text{-dimensional manifold}\}, \qquad \overline{M}_j \text{ is the closure of the set } M_j.$$

(3) For any $i \in I$, $(t_1, x_1), (t_2, x_2) \in M_i$, the condition $J(t_1, x_1) = J(t_2, x_2)$ holds, where

$$J(t, x) := \{j \in J : x \in \overline{M}_j\}.$$

Let us explain Definition 4. The manifolds M_j, $j \in J \subset I$, of dimension $n+1$ are open, and $\bigcup_{j \in J} \overline{M}_j = \Pi_T$. All the remaining manifolds M_i, $i \in I \setminus J$, of dimension less then $n+1$ belong to the boundary of the closure of $(n+1)$-dimensional manifolds. In addition, the following property holds: $J(t_1, x_1) = J(t_2, x_2)$ for any points (t_1, x_1) and (t_2, x_2) belonging to the same manifold. Therefore, for any point $(t, x) \in M_i$, $i \in I$, we have $(t, x) \in \overline{M}_{j_1} \cap \ldots \cap \overline{M}_{j_k}$ for $j_1, \ldots, j_k \in J(t, x)$ and $(t, x) \notin \overline{M}_i$ for $i \in J \setminus J(t, x)$.

2. Characteristics and dimension of a singular manifold

2.1. Structure of a singular manifold

Let us consider a minimax solution $\varphi(\cdot)$ to problem (1.1) in the class of piecewise smooth functions.

We fix a manifold M_i, $i \in I$, of dimension $n + 1 - k$, where $k \in \overline{1, n}$, and denote it by $M_{[k]}$ to simplify the further presentation.

Let $L_{[k]}(t, x)$ be the tangent subspace to the manifold $M_{[k]}$ at the point (t, x). We call the set of vectors orthogonal to vectors from the tangent subspace at point (t, x) as the normal subspace at point (t, x). The projection of the superdifferential of the function $\varphi(\cdot)$ to the normal subspace at the point (t, x), is denoted as

$$S^+_{[k]}(t, x) := \left\{ q^+ \in \mathbb{R}^{n+1} : \exists p \in L_{[k]}(t, x), p + q^+ \in D^+\varphi(t, x), \left\langle q^+, (1, f) \right\rangle = 0\ \forall (1, f) \in L_{[k]}(t, x) \right\}.$$

The normal subspace at point (t, x) is the subspace of the minimal dimention containing the set $S^+_{[k]}(t, x)$.

Fix a point $(t, x) \in Q$. The symbol Index (t, x) denotes the set consisting two or more parameters $\xi \in R^n$, such that for any pairs ξ^*, $\xi^{**} \in$ Index (t, x) the following conditions are valid:

$$\tilde{x}(t, \xi^*) = \tilde{x}(t, \xi^{**}) = x, \quad \tilde{z}(t, \xi^*) = \tilde{z}(t, \xi^{**}) = \varphi(t, x), \quad \tilde{s}(t, \xi^*) \neq \tilde{s}(t, \xi^{**}), \quad \xi^* \neq \xi^{**}. \tag{2.1}$$

According to Assertion 2, this set is nonempty for all $(t, x) \in Q$.

Let us fix a point $(t, x) \in M_{[k]} \subset Q$.

Lemma 1. *If the superdifferential $D^+\varphi(t,x)$ of a piecewise smooth minimax solution $\varphi(\cdot)$ to problem* (1.1) *at a point $(t,x) \in M_{[k]} \subset \Pi_T$ consists of more then one element, then the difference of two elements d^* and d_* of this superdifferential belongs to the normal subspace to manifold at the point (t,x). If the dimension of the normal subspace is k, where $1 \le k \le n$, then there exist k linearly independent vectors of the form $d^* - d_*$, where $d^*, d_* \in D^+\varphi(t,x)$.*

P r o o f. The proof follows from the properties of a piecewise smooth minimax solution $\varphi(\cdot)$ to problem (1.1) [10]. Any element d of the superdifferential $D^+\varphi(t,x)$ can be represented as the sum $p + q^+$, where p belongs to the tangent subspace to the singular set at the point (t,x) and q^+ belongs to the normal subspace at the same point. It was shown in [10] that the projection of the superdifferential $D^+\varphi(t,x)$ to the tangent subspace is a singleton.

As is known (see [4]), the superdifferential of a locally Lipschitz minimax solution $\varphi(\cdot)$ to problem (1.1) is a closed bounded set and has the form

$$D^+\varphi(t,x) := \mathrm{co}\{d(\xi^*) \in \mathbb{R}^{n+1} : \xi^* \in \mathrm{Index}\,(t,x)\},\ d(\xi^*) = (-H(t,\tilde{x}(t,\xi^*),\tilde{s}(t,\xi^*)),\tilde{s}(t,\xi^*)). \quad (2.2)$$

Let the tangent subspace to the singular set at a point (t,x) have dimension $n+1-k$. We consider the vectors $d(\xi^*) - d(\xi_1^*)$, where $\xi^*, \xi_1^* \in \mathrm{Index}\,(t,x)$ and $d(\xi^*) - d(\xi_1^*) = q^+(\xi^*) - q^+(\xi_1^*)$. The symbol $q^+(\xi^*)$ denotes the projection of $d(\xi^*)$ to the normal subspace at the point (t,x). Since $q^+(\xi^*), q^+(\xi_1^*) \in S^+_{[k]}(t,x)$, $\xi^*, \xi_1^* \in \mathrm{Index}\,(t,x)$, we find that the vectors $q^+(\xi^*) - q^+(\xi_1^*)$, $\xi^*, \xi_1^* \in \mathrm{Index}\,(t,x)$, also lie in the normal subspace of dimension k. We will show that there exist no less than k linearly independent differences following form

$$q^+(\xi_i^*) - q^+(\xi_1^*), \xi_i^*, \xi_1^* \in \mathrm{Index}\,(t,x), i \in \overline{2,k+1}.$$

We show that there is no element $\|q*\| \ne 0$ satisfing the following condition: $q*$ is ortogonal to the set $S^+_{[k]}(t,x)$ and $q*$ belongs to the normal subspace.

We prove the fact by reductio ad absurdum. Let the convex set $S^+_{[k]}(t,x)$ has dimention $k-l$, $0 < l \le k$ [9]. Then there is an element $q*$ such that $q*$ is ortogonal to the set $S^+_{[k]}(t,x)$ and $q*$ belongs to the normal subspace. The equality $\langle p, q*\rangle = 0$ follows from $p \in L_{[k]}(t,x)$, $q*$ belongs to the normal subspace, $\langle q^+(\xi^*), q*\rangle = 0$, $\xi^* \in \mathrm{Index}\,(t,x)$, $q^+(\xi^*) \in S^+_{[k]}(t,x)$, $q*$ is ortogonal to the set $S^+_{[k]}(t,x)$. This implies that

$$\big\langle p + q^+(\xi^*), q*\big\rangle = 0,$$

for any $p \in L_{[k]}(t,x)$, $\xi^* \in \mathrm{Index}\,(t,x)$, $q^+(\xi^*) \in S^+_{[k]}(t,x)$, as any supergradient may be represented as the sum $p + q^+(\xi^*)$ [10], than $q*$ is ortogonal to any supergradient. Therefore $q*$ belongs in intersection of hyperplanes whose normals are the supergradients. Consecuently, $q*$ belongs to tangent subspace, but $q*$ belongs to normal subspace and $\|q*\| \ne 0$. This is a contradiction. □

Theorem 1. *Let conditions* (A1)–(A4) *be satisfied for problem* (1.1)*, and let $(t,x) \in Q$. Then $(t,x) \in M_{[k]}$, where $\dim M_{[k]} = n+1-k$, $k \in \overline{1,n}$, if and only if there exist solutions $\tilde{x}(\cdot,\xi_i^*)$, $\tilde{s}(\cdot,\xi_i^*)$, and $\tilde{z}(\cdot,\xi_i^*)$, of system* (1.2), (1.3)*, $\xi_i^* \in \mathrm{Index}\,(t,x)$, $i \in \overline{1,k+1}$, such that properties* (2.1) *hold and the $k \times (n+1)$-matrix*

$$D = \begin{pmatrix} -(H_2 - H_1) & s_2^1 - s_1^1 & s_2^2 - s_1^2 & \dots & s_2^n - s_1^n \\ -(H_3 - H_1) & s_3^1 - s_1^1 & s_3^2 - s_1^2 & \dots & s_3^n - s_1^n \\ \dots & \dots & \dots & \dots & \dots \\ -(H_{k+1} - H_1) & s_{k+1}^1 - s_1^1 & s_{k+1}^2 - s_1^2 & \dots & s_{k+1}^n - s_1^n \end{pmatrix} \quad (2.3)$$

has the rank equal to k. Here $(s_i^1, s_i^2, \dots, s_i^n) = \tilde{s}(t,\xi_i)$ and $H_i = H(t,\tilde{x}(t,\xi_i),\tilde{s}(t,\xi_i))$. If one adds any row of the form

$$(-(H_{k+2} - H_1) \quad s_{k+2}^1 - s_1^1 \quad s_{k+2}^2 - s_1^2 \quad \dots \quad s_{k+2}^n - s_1^n)$$

where $(s^1_{k+1}, s^2_{k+1}, \dots, s^n_{k+1}) = \tilde{s}(t, \xi_{k+1})$, $H_{k+1} = H(t, \tilde{x}(t, \xi_{k+1}), \tilde{s}(t, \xi_{k+1}))$, $\xi_{k+1} \in \mathrm{Index}\,(t, x)$, *to the matrix of* D, *then the rank of the received* $(k+1) \times (n+1)$*-matrix equal to* k.

P r o o f. *Necessity.* Let $(t, x) \in M_{[k]} \subset Q$. We note that the dimension of the tangent subspace $L_{[k]}(t, x)$ coincides with the dimension of the manifold $M_{[k]}$.

Since $\dim L_{[k]}(t, x) = n+1-k$, we conclude that the dimension of the normal subspace $S^+_{[k]}(t, x)$ is $n + 1 - (n + 1 - k) = k$.

It follows from Lemma 1 that there exist elements $q^+(\xi^*_i) \in S^+_{[k]}(t, x)$, $\xi^*_i \in \mathrm{Index}\,(t, x)$ $i \in \overline{1, k+1}$, such that the vectors $q^+(\xi^*_i) - q^+(\xi^*_1)$, $\xi^*_i, \xi^*_1 \in \mathrm{Index}\,(t, x)$ $i \in \overline{2, k+1}$, are linearly independent.

We denote by $\mathrm{Basic}_{[k]}(t, x)$ the following set

$$\{\{\xi^*_1, \xi^*_2, \dots, \xi^*_{k+1}\} :\ \xi^*_i \in \mathrm{Index}\,(t, x),\ i \in \overline{1, k+1},\ q^+(\xi^*_i) - q^+(t, \xi^*_1) - \text{linearly independent}\}.$$

Consequently, the matrix D consisting of rows of the form $q^+(\xi^*_i) - q^+(\xi^*_1)$, $\xi^*_i, \xi^*_1 \in \mathrm{Index}\,(t, x)$, $i \in \overline{2, k+1}$, has the rank k.

If we add a row of the form $q^+(\xi^*) - q^+(\xi^*_1)$, $\xi^*, \xi^*_1 \in \mathrm{Index}\,(t, x)$, to the matrix consisting of rows of the form $q^+(\xi^*_i) - q^+(\xi_1)$, $\{\xi^*_1, \xi^*_2, \dots, \xi^*_{k+1}\} \in \mathrm{Basic}_{[k]}(t, x)$, then the rank of the matrix remains the same.

Sufficiency. Assume that the rank of the matrix

$$D = \begin{pmatrix} -(H_2 - H_1) & s^1_2 - s^1_1 & s^2_2 - s^2_1 & \dots & s^n_2 - s^n_1 \\ -(H_3 - H_1) & s^1_3 - s^1_1 & s^2_3 - s^2_1 & \dots & s^n_3 - s^n_1 \\ \dots & \dots & \dots & \dots & \dots \\ -(H_{k+2} - H_1) & s^1_{k+2} - s^1_1 & s^2_{k+2} - s^2_1 & \dots & s^n_{i+1} - s^n_1 \end{pmatrix}$$

is k, for any parameter $\xi^*_{k+2} \in \mathrm{Index}\,(t, x)$. The rows of this matrix are elements $d(\xi^*_i) - d(\xi^*_1)$, where $d(\xi^*_i), d(\xi^*_1) \in D^+\varphi(t, x)$, $i \in \overline{2, k+1}$. In addition, they can be considered as normals to hyperplanes of dimension n. Moreover, k of these normals are linearly independent.

From Lemma 1, the vectors $d(\xi^*_i) - d(\xi^*_1) = q^+(\xi^*_i) - q^+(\xi^*_1)$, $\{\xi^*_1, \xi^*_2, \dots, \xi^*_{k+1}\} \in \mathrm{Basic}_{[k]}(t, x)$, belong to the normal subspace at point (t, x). The vectors $d(\xi^*_i) - d(\xi^*_1)$, $\{\xi^*_1, \xi^*_2, \dots, \xi^*_{k+1}\} \in \mathrm{Basic}_{[k]}(t, x)$ form a basic of the normal subspace. It implies that the dimension of the normal subspace at the point (t, x) is k and the dimension of the tangent subspace is $n + 1 - k$. Hence, $(t, x) \in M_{[k]}$.

Remark 1. According to [5], the inclusions $(t, x) \in Q$ and $d(\xi^*_i) \in D^+\varphi(t, x)$, $\{\xi^*_1, \xi^*_2, \dots, \xi^*_{k+1}\} \in \mathrm{Basic}_{[k]}(t, x)$, imply that the following Rankine–Hugoniot condition holds for a curve $x(\cdot)$ lying on $M_{[k]}$:

$$\Big\langle \tilde{s}(t, \xi^*_i) - \tilde{s}(t, \xi^*_1), \frac{dx(t)}{dt} \Big\rangle = H(t, x(t), \tilde{s}(t, \xi^*_i)) - H(t, x(t), \tilde{s}(t, \xi^*_1)), \tag{2.4}$$

$\{\xi^*_1, \xi^*_2, \dots, \xi^*_{k+1}\} \in \mathrm{Basic}_{[k]}(t, x)$.

We rewrite this condition in the form

$$\Big\langle \Big(- \big(H(t, x(t), \tilde{s}(t, \xi^*_i)) - H(t, x(t), \tilde{s}(t, \xi^*_1))\big), \tilde{s}(t, \xi^*_i) - \tilde{s}(t, \xi^*_1)\Big), (1, \dot{x}(t)) \Big\rangle$$
$$= \big\langle q^+(\xi^*_i) - q^+(\xi^*_1), (1, \dot{x}(t))\big\rangle = 0, \tag{2.5}$$

$\{\xi^*_1, \xi^*_2, \dots, \xi^*_{k+1}\} \in \mathrm{Basic}_{[k]}(t, x)$. We see from condition (2.5) that the vector $(1, \dot{x}(t))$ is orthogonal to all the vectors $q^+(\xi^*_i) - q^+(\xi^*_1)$, $\xi^*_i, \xi^*_1 \in \mathrm{Basic}_{[k]}(t, x)$, $i \in \overline{2, k+1}$, that form a basis in the normal subspace. Hence, we conclude that the vector $(1, \dot{x}(t))$ belongs to the tangent subspace $L_{[k]}(t, x)$. The Theorem 1 is proven. $\square$

2.2. Properties of the superdifferential

Theorem 2. *Let conditions* (A1)–(A4) *be satisfied for problem* (1.1). *Let* $(t,x) \in Q$ *and* $(t,x) \in M_{[k]}$, *where* $\dim M_{[k]} = n+1-k$ *and* $1 \le k \le n$. *Assume that the Hamiltonian* $H = H(s)$ *is concave in variable* s. *For any characteristics* $\tilde{x}(\cdot,\xi_i), \tilde{s}(\cdot,\xi_i), \tilde{z}(\cdot,\xi_i)$, $\{\xi_1^*, \xi_2^*, \dots, \xi_{k+1}^*\} \in \mathrm{Basic}_{[k]}(t,x)$ *such that the* $k \times (n+1)$*-matrix* D *of the form* (2.3) *has the rank* k, *there is no characteristic* $\tilde{x}(\cdot,\xi_{k+2}), \tilde{s}(\cdot,\xi_{k+2}), \tilde{z}(\cdot,\xi_{k+2})$, $\xi_{k+2} \in \mathrm{Index}\,(t,x)$ *satysfing the condition*

$$\tilde{s}(t,\xi_{k+2}) = \sum_{i=1}^{k+1} \alpha_i \tilde{s}(t,\xi_i), \quad \alpha_i \ge 0, \quad \sum_{i=1}^{k+1} \alpha_i = 1.$$

P r o o f. Let us introduce the convenient notation

$$q_{i,j}(t,x) = q^+(\xi_i) - q^+(\xi_j) = d(\xi_i) - d(\xi_j) = \Big(-\big(H(\tilde{s}(t,\xi_i)) - H(\tilde{s}(t,\xi_j))\big),\ \tilde{s}(t,\xi_i) - \tilde{s}(t,\xi_j) \Big), \tag{2.6}$$

$\xi_i, \xi_j \in \mathrm{Index}\,(t,x)$.

Assume that the statement of Theorem 2 does not hold and there exists a characteristic $\tilde{x}(t,\xi_{k+2})$, $\tilde{z}(t,\xi_{k+2})$, $\tilde{s}(t,\xi_{k+2})$, $\xi_{k+2} \in \mathrm{Index}\,(t,x)$, satisfying condition (2.1) and such that

$$\tilde{s}(t,\xi_{k+2}) = \sum_{i=1}^{k+1} \alpha_i \tilde{s}(t,\xi_i), \quad \alpha_i \ge 0, \quad \sum_{i=1}^{k+1} \alpha_i = 1. \tag{2.7}$$

In view of Theorem 1 and the inclusion $(t,x) \in M_{[k]}$, the rank of the $(k+1) \times (n+1)$-matrix $\tilde{D}$ obtained by adding to the matrix D the row $q_{k+2,1}(t,x)$ is equal to k. The added row $q_{k+2,1}(t,x)$ is a linear combination of rows of the matrix D, i.e. exist $b_i \in \mathbb{R}$, $i \in \overline{1,k}$, such that

$$q_{k+2,1}(t,x) = \sum_{i=1}^{k} b_i q_{i+1,1}(t,x). \tag{2.8}$$

Relations (2.6) and (2.8) imply that

$$d(\xi_{k+2}) = \Big(1 - \sum_{i=1}^{k} b_i\Big) d(\xi_1) + \sum_{i=1}^{k} b_i d(\xi_{i+1}). \tag{2.9}$$

Since equality (2.9) is applicable to all the components of the vector $d(\xi_{k+2})$, we rewrite equality (2.9) in the form of the following two equalities:

$$H(\tilde{s}(t,\xi_{k+2})) = \Big(1 - \sum_{i=1}^{k} b_i\Big) H(\tilde{s}(t,\xi_1)) + \sum_{i=1}^{k} b_i H(\tilde{s}(t,\xi_{i+1})), \tag{2.10}$$

$$\tilde{s}(t,\xi_{k+2}) = \Big(1 - \sum_{i=1}^{k} b_i\Big) \tilde{s}(t,\xi_1) + \sum_{i=1}^{k} b_i \tilde{s}(t,\xi_{i+1}). \tag{2.11}$$

It follows from (2.10) and (2.11) that

$$H\Big(\Big(1 - \sum_{i=1}^{k} b_i\Big)\tilde{s}(t,\xi_1) + \sum_{i=1}^{k} b_i \tilde{s}(t,\xi_{i+1})\Big) = \Big(1 - \sum_{i=1}^{k} b_i\Big) H(\tilde{s}(t,\xi_1)) + \sum_{i=1}^{k} b_i H(\tilde{s}(t,\xi_{i+1})).$$

We also note that the sum of the coefficients at $H(\tilde{s}(t,\xi_i))$ and $\tilde{s}(t,\xi_i)$ is equal to 1, $i \in \overline{1,k+1}$.

There exist two representations for $\tilde{s}(t,\xi_{k+2})$: formula (2.7) from the above assumption condition of the problem that $\tilde{s}(t,\xi_{k+2})$ is a convex combination of $\tilde{s}(t,\xi_i)$, $i \in \overline{1,k+1}$, and formula (2.11) obtained from the linear of the row $q_{k+2,1}(t,x)$ reletive rows of the matrix D.

Subtracting (2.7) from (2.11), we get

$$0 = \Big(\Big(1 - \sum_{i=1}^{k} b_i\Big) - \alpha_1\Big)\tilde{s}(t,\xi_1) + \sum_{i=1}^{k}(b_i - \alpha_{i+1})\tilde{s}(t,\xi_{i+1}). \tag{2.12}$$

We subtract and add the term

$$\sum_{i=1}^{k}(b_i - \alpha_{i+1})\tilde{s}(t,\xi_1)$$

to the right-hand side of (2.12). Then, grouping the terms, we get

$$0 = \Big(1 - \alpha_1 - \sum_{i=1}^{k}(b_i - b_i + \alpha_{i+1})\Big)\tilde{s}(t,\xi_1) + \sum_{i=1}^{k}(b_i - \alpha_{i+1})\big(\tilde{s}(t,\xi_{i+1}) - \tilde{s}(t,\xi_1)\big).$$

Hence, taking into account that the coefficient at $\tilde{s}(t,\xi_1)$ is equal to zero, we obtain

$$0 = \sum_{i=1}^{k}(b_i - \alpha_{i+1})\big(\tilde{s}(t,\xi_{i+1}) - \tilde{s}(t,\xi_1)\big). \tag{2.13}$$

If the differences $\tilde{s}(t,\xi_{i+1}) - \tilde{s}(t,\xi_1)$, $i \in \overline{1,k}$, are linearly independent, then equality (2.13) is equivalent to the equalities $b_i = \alpha_{i+1}$, $i \in \overline{1,k}$. In this case, equality (2.13) and the condition $\sum_{i=1}^{k+1}\alpha_i = 1$ imply $\alpha_1 = (1 - \sum_{i=1}^{k} b_i)$.

Let $b_1 \neq \alpha_2$ and vector $\tilde{s}(t,\xi_2) - \tilde{s}(t,\xi_1)$ be a linear combination of the differences $\tilde{s}(t,\xi_{i+1}) - \tilde{s}(t,\xi_1)$, $i \in \overline{2,k}$. Let multiply scalarly both sides of identity (2.13) by $\dot{x} \neq 0$, where $(1,\dot{x}) \in L_{[k]}(t,x)$. Taking into account the Rankine–Hugoniot condition (2.4), we get

$$0 = \sum_{i=1}^{k}(b_i - \alpha_{i+1})\big(H(\tilde{s}(t,\xi_1)) - H(\tilde{s}(t,\xi_{i+1}))\big).$$

This implies that the difference $H(t,\xi_1) - H(t,\xi_2)$ is a linear combination of the differences $H(\tilde{s}(t,\xi_1)) - H(\tilde{s}(t,\xi_{i+1}))$, $i \in \overline{2,k}$, and the rank of the matrix D is equal $k-1$.

However, this is not possible, because of the assumption of the Theorem 2, that the rank of the $k \times (n+1)$-matrix D is equal k.

The Rankine–Hugoniot condition (2.4) for $\dot{x} = 0$ imply, that $H(t,\xi_1) - H(t,\xi_{i+1}) = 0$, $i \in \overline{1,k}$.

Therefore, the differences $\tilde{s}(t,\xi_{i+1}) - \tilde{s}(t,\xi_1)$, $i \in \overline{1,k}$, are linearly independent. Then, as mentioned above, $b_i = \alpha_{i+1}$, $i \in \overline{1,k}$, and $\alpha_1 = 1 - \sum_{i=1}^{k} b_i$.

It follows from (2.10) and (2.11), that

$$H(\tilde{s}(t,\xi_{k+2})) = \sum_{i=1}^{k+1}\alpha_i H(\tilde{s}(t,\xi_i)), \quad \tilde{s}(t,\xi_{k+2}) = \sum_{i=1}^{k+1}\alpha_i \tilde{s}(t,\xi_i), \tag{2.14}$$

$$\sum_{i=1}^{k+1}\alpha_i = 1, \quad \alpha_i \geq 0, \quad i \in \overline{1,k+1}.$$

Let's consider the simplex S of dimension of k spanned by the points

$$(-H(\tilde{s}(t,\xi_i)), \tilde{s}(t,\xi_i)), \quad i \in \overline{1,k+1}.$$

Convexity of function $s \to -H(s)$ and (2.14) imply that the simplex S lies on the graph of function $s \to -H(s)$. Smoothness of function of $H(s)$ in variable s (condition A1), implies that the supporting hyperplane to the hypograph of $-H(\cdot)$ at any point $(-H(\tilde{s}(t,\xi_i)), \tilde{s}(t,\xi_i)) \in S$, $i \in \overline{1,k+1}$, is unique and contains the simplex S. We will denote the normal to the supporting hyperplane containing simplex S by $(-1,-N) \in \mathbb{R}^{n+1}$, where

$$D_s H(\tilde{s}(t,\xi_i)) = N, \quad i \in \overline{1,k+2}. \tag{2.15}$$

Since $H = H(s)$, we can obtained from (1.2) and (1.3) the relations

$$\dot{\tilde{s}}(t,\xi_i) = -D_x H(\tilde{s}(t,\xi_i)) = 0, \quad \xi_i \in Index(t,x), \quad i \in \overline{1,k+2}, \quad t \le T.$$

Therefore, $\tilde{s}(t,\xi_i) \equiv D_x\sigma(\xi_i)$.

It follows from (2.1) and (2.15), that

$$x = x(t) = \xi_i - \int_t^T \frac{\partial H}{\partial s}(D_x\sigma(\xi_i))d\tau = \xi_i - \int_t^T N d\tau = \xi_j - \int_t^T N d\tau, \tag{2.16}$$

$\xi_i, x_j \in \mathrm{Index}\,(t,x), \quad i,j \in \overline{1,k+2}$.

Relation (2.16) implies that $\xi_i = \xi_j$ where $i,j \in \overline{1,k+2}$ what the contradicts condition (2.1). The Theorem 2 is proven. □

Remark 2. It follows from the statement of Theorem 2 in case H=H(s) that the points of the form

$$(-H(\tilde{s}(\xi^*)), \tilde{s}(\xi^*)), \quad \xi^* \in \mathrm{Index}\,(t,x), \quad \forall (t,x) \in Q, \tag{2.17}$$

are corner points of the convex set $D^+\varphi(t,x)$.

Really, let $(t,x) \in M_{[k]} \subset Q$. It follows from (2.2) and Theorem 1, that the superdifferential is convex, closed set and it lies in the subspace of dimension k. According to the Caratheodory theorem [9] any element of $D^+u(t,x)$ can be presented as a convex combination of no more then $(k+1)$ supergradients of the form (2.17) and such that the differences

$$d(\xi_i) - d(\xi_1) = (-H(\tilde{s}(\xi_i)) + H(\tilde{s}(\xi_1)), \tilde{s}(\xi_i) - \tilde{s}(\xi_1)),$$

where $\xi_i, \xi_1 \in \mathrm{Index}\,(t,x)$, $i \in \overline{2,k+1}$, are linearly independent.
If a supergradient $(-H(\tilde{s}(\xi^*)), \tilde{s}(\xi^*)) \in D^+u(t,x)$ would be presented in the form

$$(-H(\tilde{s}(\xi^*)), \tilde{s}(\xi^*)) = \sum_{i=1}^{k+1} \alpha_i(-H(\tilde{s}(\xi_i)), \tilde{s}(\xi_i)) \quad \xi_i \in \mathrm{Index}\,(t,x), \quad i \in \overline{1,k+1},$$

where $\alpha_i \ge 0$, $\sum_{i=1}^{k+1} \alpha_i = 1$, then his component $\tilde{s}(\xi^*)$ should be also presented as the convex combination $\tilde{s}(\xi_i)$ $\xi_i \in \mathrm{Index}\,(t,x)$, $i \in \overline{1,k+1}$. That contradicts Theorem 2. Therefore, the statement of remark 2 is true.

Corollary 1. *If conditions* (A1)–(A4) *are satisfied,* $(t,x) \in Q$ *and* $H = H(s)$, *then the relation*

$$\langle \tilde{s}(\xi^*) - \tilde{s}(\xi^{**}), D_s H(\tilde{s}(\xi^{**})) \rangle \ne H(\tilde{s}(\xi^*)) - H(\tilde{s}(\xi^{**})),$$

is valid for any ξ^*, $\xi^{**} \in \mathrm{Index}\,(t,x)$ *and* $\xi^* \ne \xi^{**}$.

We will prove the corollary by reductio ad absurdum. Let there exist $\xi^*, \xi^{**} \in \mathrm{Index}\,(t,x)$, $\xi^* \ne \xi^{**}$, satisfied the relations

$$\langle \tilde{s}(\xi^*) - \tilde{s}(\xi^{**}), D_s H(\tilde{s}(\xi^{**})) \rangle = H(\tilde{s}(\xi^*)) - H(\tilde{s}(\xi^{**})).$$

The equality can be rewritten in the form

$$\langle (\tilde{s}(\xi^*) - \tilde{s}(\xi^{**}), H(\tilde{s}(\xi^*)) - H(\tilde{s}(\xi^{**}))), (D_s H(\tilde{s}(\xi^{**})), -1) \rangle = 0. \tag{2.18}$$

Note that the vector $(D_sH(\tilde{s}(\xi^{**})), -1)$ is the normal to the supporting hyperplane Γ to the hypograph of function $s \to H(s)$ at point $(\tilde{s}(\xi^{**}), H(\tilde{s}(\xi^{**})))$. The difference $(\tilde{s}(\xi^{*})-\tilde{s}(\xi^{**}), H(\tilde{s}(\xi^{*}))-H(\tilde{s}(\xi^{**})))$ lies in the supporting hyperplane Γ following condition (2.18). The point $(\tilde{s}(\xi^{*}), H(\tilde{s}(\xi^{*})))$ belongs to the graph of the concave function $s \to H(s)$ and lies in the hyperplane Γ. It follows that Γ is the supporting hyperplane to the hypograph of $H(\cdot)$ at this point. As $H(\cdot)$ is continuously differentiable, then Γ is the tangent hyperplane to the hypograph of $H(\cdot)$ at point $(\tilde{s}(\xi^{*}), H(\tilde{s}(\xi^{*})))$. Consequently,

$$(D_sH(\tilde{s}(\xi^{**})), -1) = (D_sH(\tilde{s}(\xi^{*})), -1).$$

We remain that

$$D_sH(\tilde{s}(\xi^{**})) = D_sH(\tilde{s}(\xi^{*})) = N.$$

As the Hamiltonian has the form $H = H(s)$, then $\dot{\tilde{s}} = -D_xH(\tilde{s}) = 0$ (1.2), (1.3). It implies that $\tilde{s}(t,\xi^{*}) \equiv D_x\sigma(\xi^{*})$, $\tilde{s}(t,\xi^{**}) \equiv D_x\sigma(\xi^{**})$.

Using condition (2.1), we get that states of characteristics $\tilde{x}(\cdot,\xi)$ with parameters ξ^{*} and ξ^{**} satisfy the relation

$$x = x(t) = \xi^{*} - \int_t^T \frac{\partial H}{\partial s}(D_x\sigma(\xi^{*}))d\tau = \xi^{**} - \int_t^T \frac{\partial H}{\partial s}(D_x\sigma(\xi^{**}))d\tau.$$

This equality can be rewritten in the following form

$$x = x(t) = \xi^{*} - \int_t^T N d\tau = \xi^{**} - \int_t^T N d\tau.$$

This equality imply that $\xi^{*} = \xi^{**}$. It contradicts the assumption $\xi^{*} \neq \xi^{**}$.

3. Conclusion

In this paper results presented in [13] are modified and developed. New properties of superdifferentials of a piece-smooth minimax solution of the HJBE and characteristics of the HJBE on the singular set are obtained and discussed.

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