EVOLUTION OF A MULTISCALE SINGULARITY OF THE SOLUTION OF THE BURGERS EQUATION IN THE 4-DIMENSIONAL SPACE-TIME

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Abstract: The solution of the Cauchy problem for the vector Burgers equation with a small parameter of dissipation ε in the 4-dimensional space-time is studied:

$$\mathbf{u}_t + (\mathbf{u}\nabla)\mathbf{u} = \varepsilon\triangle\mathbf{u}, \quad u_\nu(\mathbf{x}, -1, \varepsilon) = -x_\nu + 4^{-\nu}(\nu+1)x_\nu^{2\nu+1},$$

With the help of the Cole–Hopf transform $\mathbf{u} = -2\varepsilon\nabla\ln H$, the exact solution and its leading asymptotic approximation, depending on six space-time scales, near a singular point are found. A formula for the growth of partial derivatives of the components of the vector field $\mathbf{u}$ on the time interval from the initial moment to the singular point, called the formula of the gradient catastrophe, is established:

$$\frac{\partial u_\nu(0, t, \varepsilon)}{\partial x_\nu} = \frac{1}{t}\left[1 + O\left(\varepsilon|t|^{-1-1/\nu}\right)\right], \quad \frac{t}{\varepsilon^{\nu/(\nu+1)}} \to -\infty, \quad t \to -0.$$

The asymptotics of the solution far from the singular point, involving a multistep reconstruction of the space-time scales, is also obtained:

$$u_\nu(\mathbf{x}, t, \varepsilon) \approx -2\left(\frac{t}{\nu+1}\right)^{1/2\nu} \tanh\left[\frac{x_\nu}{\varepsilon}\left(\frac{t}{\nu+1}\right)^{1/2\nu}\right], \quad \frac{t}{\varepsilon^{\nu/(\nu+1)}} \to +\infty.$$

Keywords: Vector Burgers equation, Cauchy problem, Cole–Hopf transform, Singular point, Laplace's method, Multiscale asymptotics.

1. Statement of the problem

In the present work, we study the solution of the following Cauchy problem for the vector Burgers equation in the $(3+1)$-dimensional space-time:

$$\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\nabla)\mathbf{u} = \varepsilon\triangle\mathbf{u}, \quad t \geqslant -1, \tag{1.1}$$

$$u_\nu(\mathbf{x}, -1, \varepsilon) = -x_\nu + \frac{\nu+1}{4^\nu}x_\nu^{2\nu+1}, \tag{1.2}$$

where $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\mathbf{u} = (u_1, u_2, u_3)$ is a *potential* vector field, ε is a small positive parameter of dissipation frequently called viscosity, and the index ν changes from 1 to 3.

The evolutionary equation (1.1) is widely used in the mechanics of continuous media [5], in particular, for modeling the formation and the propagation of shock waves (in limit of vanishing viscosity ε), in addition, it successfully serves as a basic instrument of the theoretical investigation

of the large-scale structure of the Universe [6]; it is worth noting that the Burgers equation, in the case of small values of parameter ε, simulates good enough the observed mosaic distribution of the matter in the space at the distances measured by billions of light-years.

The aim of the present work is to study the arising microlocal singularity, i.e., the solution of problem (1.1), (1.2) as $\varepsilon \to +0$ near the singular point, which coincides with the origin because of the special choice of the initial data for $t = -1$. Our investigation has to find the scales of the localization of the singularity and explicit asymptotic formulas for the solution $\mathbf{u}(\mathbf{x}, t, \varepsilon)$.

In the context of the present paper, the terms "singularity" and "singular point" are understood in the sense of a large space gradient of the solution $\mathbf{u}(\mathbf{x}, t, \varepsilon)$ in some small neighborhood of the origin, while the solution itself is *smooth*; in this connexion, see [8, Sect. 2] or detailed explanations in the introductory part of survey [9].

Here, it is appropriate to mention that Arnold's scientific school performed a detailed topological classification of singular points and reconstructions[1] of singular sets of the solution of equation (1.1) in limit of vanishing viscosity ε [1, Ch. 2, § 2.5], including the theorems forbidding some metamorphoses of singularities of solutions, for example, see [3, § 3; 4]; while, in the present investigation, we are mainly interested in analytical results of studying the asymptotic behavior of the solution for small, however, *not equal to zero*, values of parameter ε.

2. Exact solution and its asymptotics

By the standard Cole–Hopf transform

$$\mathbf{u} = -2\varepsilon \nabla \ln H \qquad (2.1)$$

equation (1.1) is reduced to the linear heat equation $\partial H / \partial t = \varepsilon \triangle H$, whose solution with the initial condition (1.2) is easily obtained in the explicit form:

$$H(\mathbf{x}, t, \varepsilon) = \frac{1}{8\pi^{3/2}(1+t)^{3/2}} \int_{\mathbb{R}^3} \exp\left\{ \frac{1}{\varepsilon}\left[-\frac{|\mathbf{x}-\mathbf{s}|^2}{4(1+t)} + \frac{|\mathbf{s}|^2}{4} - \sum_{\mu=1}^{3}\left(\frac{s_\mu}{2}\right)^{2\mu+2} \right] \right\} \prod_{\mu=1}^{3} ds_\mu. \qquad (2.2)$$

With the help of the scaling change of variables of integration

$$s_\mu = 2\varepsilon^{1/(2\mu+2)}\sigma_\mu,$$

from expression (2.2), we find the following formula:

$$H(\mathbf{x}, t, \varepsilon) = \frac{\varepsilon^{13/24}}{\pi^{3/2}(1+t)^{3/2}} \exp\left\{ -\frac{|\mathbf{x}|^2}{4\varepsilon(1+t)} \right\}$$

$$\times \int_{\mathbb{R}^3} \exp \sum_{\mu=1}^{3}\left[-\sigma_\mu^{2\mu+2} + \frac{t\sigma_\mu^2}{\varepsilon^{\mu/(\mu+1)}(1+t)} + \frac{x_\mu\sigma_\mu}{\varepsilon^{(2\mu+1)/(2\mu+2)}(1+t)} \right] \prod_{\mu=1}^{3} d\sigma_\mu. \qquad (2.3)$$

Whence, by elementary differentiation, we obtain

$$\frac{\partial H(\mathbf{x}, t, \varepsilon)}{\partial x_\nu} = -\frac{1}{2\varepsilon^{11/24}\pi^{3/2}(1+t)^{5/2}} \exp\left\{ -\frac{|\mathbf{x}|^2}{4\varepsilon(1+t)} \right\}$$

$$\times \int_{\mathbb{R}^3} \left(x_\nu - 2\varepsilon^{1/(2\nu+2)}\sigma_\nu \right) \exp \sum_{\mu=1}^{3}\left[-\sigma_\mu^{2\mu+2} + \frac{t\sigma_\mu^2}{\varepsilon^{\mu/(\mu+1)}(1+t)} + \frac{x_\mu\sigma_\mu}{\varepsilon^{(2\mu+1)/(2\mu+2)}(1+t)} \right] \prod_{\mu=1}^{3} d\sigma_\mu. \qquad (2.4)$$

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[1]The equivalent terms used in the relevant literature are as follows: bifurcations, metamorphoses, perestroikas, transitions.

Using formulas (2.3) and (2.4), from transform (2.1) we immediately get the exact solution of the Cauchy problem (1.1), (1.2) in the component-wise form:

$$u_\nu(\mathbf{x}, t, \varepsilon) = \frac{\displaystyle\int_{-\infty}^{+\infty} \left(x_\nu - 2\varepsilon^{1/(2\nu+2)}\sigma_\nu\right) \exp\left[-\sigma_\nu^{2\nu+2} + \frac{\Theta_\nu\sigma_\nu^2 + \Lambda_\nu\sigma_\nu}{(1+t)}\right] d\sigma_\nu}{(1+t)\displaystyle\int_{-\infty}^{+\infty} \exp\left[-\sigma_\nu^{2\nu+2} + \frac{\Theta_\nu\sigma_\nu^2 + \Lambda_\nu\sigma_\nu}{(1+t)}\right] d\sigma_\nu}, \tag{2.5}$$

where, for convenience, the inner variables

$$\Theta_\nu = \frac{t}{\varepsilon^{\nu/(\nu+1)}}, \quad \Lambda_\nu = \frac{x_\nu}{\varepsilon^{(2\nu+1)/(2\nu+2)}} \tag{2.6}$$

are introduced.

First of all, we must find the leading approximation of the exact solution obtained above, since the explicit expression (2.5) itself tells us few about the asymptotic structure of the solution.

Statement 1. *As* $|\mathbf{x}| + |t| \to 0$ *and* $\varepsilon \to +0$, *for the solution of problem* (1.1), (1.2), *there holds the asymptotic formula*

$$u_\nu(\mathbf{x}, t, \varepsilon) = -2\varepsilon^{1/(2\nu+2)}\frac{\partial}{\partial\Lambda_\nu}\ln\int_{-\infty}^{+\infty}\exp\left(-\sigma_\nu^{2\nu+2} + \Theta_\nu\sigma_\nu^2 + \Lambda_\nu\sigma_\nu\right) d\sigma_\nu + O(|\mathbf{x}| + |t|). \tag{2.7}$$

P r o o f. Near the origin, by the elementary passage to the limit $|\mathbf{x}| + |t| \to 0$ in formula (2.5), we obtain the expression for the leading approximation:

$$U_\nu(\mathbf{x}, t, \varepsilon) = -2\varepsilon^{1/(2\nu+2)}\frac{\displaystyle\int_{-\infty}^{+\infty}\sigma_\nu\exp\left(-\sigma_\nu^{2\nu+2} + \Theta_\nu\sigma_\nu^2 + \Lambda_\nu\sigma_\nu\right) d\sigma_\nu}{\displaystyle\int_{-\infty}^{+\infty}\exp\left(-\sigma_\nu^{2\nu+2} + \Theta_\nu\sigma_\nu^2 + \Lambda_\nu\sigma_\nu\right) d\sigma_\nu}. \tag{2.8}$$

In the argument of the integrand exponent of this expression, we recognize the truncated generating family (in other words, the truncated versal deformation of the germ) of the Lagrange singularities $A_{2\nu+1}$; the first of them A_3 is usually called the Whitney fold; see [1, Ch. 2; 2, Ch. II, §11, §17].

With the help of some obvious transforms of formula (2.8), by formula (2.5) for small values of the independent variables $(\mathbf{x}, t)$, we arrive at the desired result. $\qquad\square$

Now, we are ready to move to the very center of the singularity of the solution $u_\nu(\mathbf{x}, t, \varepsilon)$.

Statement 2. *As* $\Theta_\nu = \varepsilon^{-\nu/(\nu+1)}t \to -\infty$, *there holds the asymptotic formula*

$$\frac{\partial U_\nu(0, t, \varepsilon)}{\partial x_\nu} = \frac{1}{t}\left[1 + O\left(\varepsilon|t|^{-1-1/\nu}\right)\right]. \tag{2.9}$$

P r o o f. Using formula (2.8), let us show that the point $(\mathbf{x}, t) = (0, 0)$ is singular by computing the asymptotics of the derivative

$$\frac{\partial U_\nu(0, t, \varepsilon)}{\partial x_\nu} = -2\varepsilon^{-\nu/(\nu+1)}\frac{\displaystyle\int_{-\infty}^{+\infty}\sigma^2\exp\left(-\sigma^{2\nu+2} - |\Theta_\nu|\sigma^2\right) d\sigma}{\displaystyle\int_{-\infty}^{+\infty}\exp\left(-\sigma^{2\nu+2} - |\Theta_\nu|\sigma^2\right) d\sigma} \tag{2.10}$$

as $\Theta_\nu \to -\infty$. After the change of the variable of integration $\sigma = |\Theta_\nu|^{1/2\nu}\eta$, we have:

$$\frac{\partial U_\nu(0,t,\varepsilon)}{\partial x_\nu} = -2\varepsilon^{-\nu/(\nu+1)}|\Theta_\nu|^{1/\nu} \frac{\int_{-\infty}^{+\infty} \eta^2 \exp\left(-|\Theta_\nu|^{1+1/\nu}S(\eta)\right) d\eta}{\int_{-\infty}^{+\infty} \exp\left(-|\Theta_\nu|^{1+1/\nu}S(\eta)\right) d\eta},$$

where the phase function $S(\eta) = \eta^{2\nu+2} + \eta^2$ has clearly only a unique point of minimum: $\eta = 0$.

In this special case, it is convenient to make use of the asymptotic formula for integrals of the Laplacian type:

$$\int_{-\infty}^{+\infty} A(\eta)\exp\left(-\omega S(\eta)\right) d\eta = \exp\left(-\omega S(0)\right)\sqrt{\frac{2\pi}{\omega S''(0)}}$$

$$\times \left\{ A(0) + \frac{1}{2\omega}\left[\frac{A''(0)}{S''(0)} + \frac{A'(0)S'''(0)}{(S''(0))^2} + A(0)\left(\frac{5(S'''(0))^2}{12(S''(0))^3} + \frac{S''''(0)}{4(S''(0))^2}\right)\right] + O\left(\frac{1}{\omega^2}\right)\right\},$$

where $\omega \to +\infty$. Taking into account that $S''(0) = 2$, for our phase function, after elementary calculations, we find a very simple approximation:

$$\frac{\partial U_\nu(0,t,\varepsilon)}{\partial x_\nu} = -\varepsilon^{-\nu/(\nu+1)}|\Theta_\nu|^{-1}\left[1 + O\left(|\Theta_\nu|^{-1-1/\nu}\right)\right], \quad \Theta_\nu \to -\infty;$$

whence we obtain the necessary result. $\qquad\square$

Remark 1. Relation (2.9) as $t \to -0$ can be called the *formula of the gradient catastrophe*, because the variable t enters the denominator. Strictly speaking, exactly in the sense of this statement, we use the term "singular point" with reference to the point $(\mathbf{x},t) = (0,0)$. Let us emphasize that for $t = 0$ the gradient $\partial U_\nu/\partial x_\nu$ still does not become infinite, although it has, according to formula (2.10), the order of the value $\varepsilon^{-\nu/(\nu+1)} \to +\infty$ as $\varepsilon \to +0$.

Now, let us look into the future: in other words, let us calculate the asymptotics of the function of the leading approximation $U_\nu(\mathbf{x},t,\varepsilon)$ as $\Theta_\nu \to +\infty$.

Statement 3. *As $\Theta_\nu \to +\infty$ there holds the asymptotic formula*

$$U_\nu(\mathbf{x},t,\varepsilon) = -2\varepsilon^{1/(2\nu+2)}\left(\frac{\Theta_\nu}{\nu+1}\right)^{1/2\nu}\tanh\left[\Lambda_\nu\left(\frac{\Theta_\nu}{\nu+1}\right)^{1/2\nu}\right] + O\left(\Theta_\nu^{-(1+1/4\nu)}\right).$$

P r o o f. Using the change of the variable $\sigma_\nu = \Theta_\nu^{1/2\nu}z_\nu$, for the integral in the denominator of expression (2.8), we obtain

$$\int_{-\infty}^{+\infty} \exp\left(-\sigma_\nu^{2\nu+2} + \Theta_\nu\sigma_\nu^2 + \Lambda_\nu\sigma_\nu\right) d\sigma_\nu = \Theta_\nu^{1/2\nu}\int_{-\infty}^{+\infty} \exp\left[\Theta_\nu^{1+1/\nu}(z_\nu^2 - z_\nu^{2\nu+2}) + \Lambda_\nu\Theta_\nu^{1/2\nu}z_\nu\right] dz_\nu.$$

Following now Laplace's method, for the phase function

$$F(z) = z^2 - z^{2\nu+2},$$

we find two stationary points

$$z^\pm = \pm\frac{1}{(\nu+1)^{1/2\nu}}, \quad F'(z^\pm) = 0, \quad F''(z^\pm) = -4\nu,$$

and the necessary formula of the leading approximation for the integral

$$\int\limits_{-\infty}^{+\infty} \exp\left(-\sigma_\nu^{2\nu+2} + \Theta_\nu\sigma_\nu^2 + \Lambda_\nu\sigma_\nu\right) d\sigma_\nu$$

$$= \Theta_\nu^{1/4\nu} \left(\frac{2\pi}{\nu}\right)^{1/2} \exp\left[\nu\left(\frac{\Theta_\nu}{\nu+1}\right)^{1+1/\nu}\right] \cosh\left[\Lambda_\nu\left(\frac{\Theta_\nu}{\nu+1}\right)^{1/2\nu}\right] + O\left(\Theta_\nu^{-(1+3/4\nu)}\right);$$

in addition, by the same method, the asymptotic formula

$$\int\limits_{-\infty}^{+\infty} \sigma_\nu \exp\left(-\sigma_\nu^{2\nu+2} + \Theta_\nu\sigma_\nu^2 + \Lambda_\nu\sigma_\nu\right) d\sigma_\nu$$

$$= \Theta_\nu^{3/4\nu} \left(\frac{2\pi}{\nu(\nu+1)^{1/\nu}}\right)^{1/2} \exp\left[\nu\left(\frac{\Theta_\nu}{\nu+1}\right)^{1+1/\nu}\right] \sinh\left[\Lambda_\nu\left(\frac{\Theta_\nu}{\nu+1}\right)^{1/2\nu}\right] + O\left(\Theta_\nu^{-(1+1/4\nu)}\right)$$

is established. Substituting these formulas into expression (2.8), we easily arrive at the desired result. □

Remark 2. Using the change (2.6), from Statement 3 in the leading approximation, we obtain the relation

$$U_\nu(\mathbf{x}, t, \varepsilon) \approx -2\left(\frac{t}{\nu+1}\right)^{1/2\nu} \tanh\left[\frac{x_\nu}{\varepsilon}\left(\frac{t}{\nu+1}\right)^{1/2\nu}\right], \quad \Theta_\nu \to +\infty,$$

which gives a mathematical formulation of happening reconstructions of the scales of time and space in the solution of the problem under consideration:

$$\Theta_\nu = \frac{t}{\varepsilon^{\nu/(\nu+1)}} \longmapsto t, \quad \Lambda_\nu = \frac{x_\nu}{\varepsilon^{(2\nu+1)/(2\nu+2)}} \longmapsto \frac{x_\nu}{\varepsilon}.$$

3. Survey of the asymptotic structure

Using the form of the inner variables (2.6) and Statements 1–3, we can establish the boundaries of domains, where the obtained asymptotic approximations of the solution of problem (1.1), (1.2) remain valid.

Since in the formula for the solution (2.5) we have specific space-time scales, which are determined by changes (2.6), it is natural to define correspondingly the following sets of the independent variables:

$$\Omega_\nu = \{(\mathbf{x}, t) : |x_\nu| < \varepsilon^{(2\nu+1)/(2\nu+2)}, \ |t| < \varepsilon^{\nu/(\nu+1)}\}.$$

In the smallest domain Ω_3 (in Figures 1–3, it is conventionally shown with lilac color), bounded in time by the value of order $\varepsilon^{3/4}$, for the solution $\mathbf{u}(\mathbf{x}, t, \varepsilon)$, a fortiori, there holds the asymptotic formula (2.7) of the leading approximation.

As $\Theta_3 \to +\infty$, according to Statement 3 for $\nu = 3$, the natural scale of the space localization in x_3 is constricted to the value of order ε and for the component u_3 one should use the following approximate relation:

$$u_3(\mathbf{x}, t, \varepsilon) \approx -2\left(\frac{t}{4}\right)^{1/6} \tanh\left[\frac{x_3}{\varepsilon}\left(\frac{t}{4}\right)^{1/6}\right], \quad \Theta_3 \to +\infty, \tag{3.1}$$
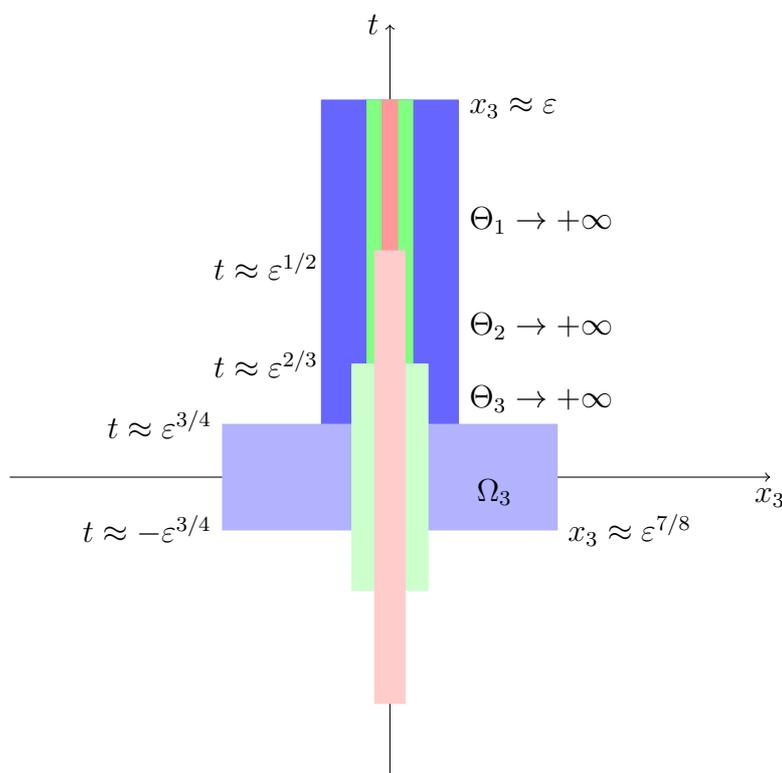
Figure 1. A schematic projection of the domain of the localization of the singularity in the plane (x_3, t).

where $t = \varepsilon^{3/4}\Theta_3$.

In the intermediate domain Ω_2, (in our figures, it is shown with light-green color) bounded in time by the value of order $\varepsilon^{2/3}$, the "remote future" from the point of view of domain Ω_3, i.e., the times such that $\Theta_3 \to +\infty$, turns out to be a relatively short interval, because we have the relation $\Theta_2 = \varepsilon^{1/12}\Theta_3$.

Remark 3.   Elegantly confirming Newton's principle *relativus de relativo in relativum*[2] from his famous "*Philosophiæ Naturalis Principia Mathematica*", a similar phenomenon is also observed in another Cauchy problem for equation (1.1) with an additional small parameter in the initial condition [8].

As $\Theta_2 \to +\infty$, according to Statement 3 for $\nu = 2$, the natural scale of the space localization in x_2 is constricted to the value of order ε and for the component u_2 one should use the following approximate relation:

$$u_2(\mathbf{x}, t, \varepsilon) \approx -2\left(\frac{t}{3}\right)^{1/4}\tanh\left[\frac{x_2}{\varepsilon}\left(\frac{t}{3}\right)^{1/4}\right], \quad \Theta_2 \to +\infty,$$

where $t = \varepsilon^{2/3}\Theta_2$.

At last, in the largest domain Ω_1 (in figures, it is shown with pink color), bounded in time by the value of order $\varepsilon^{1/2}$, the "remote future" already from the point of view of domain Ω_2, i.e. the times such that $\Theta_2 \to +\infty$, again turns out to be only a short interval, because we have the relation $\Theta_1 = \varepsilon^{1/6}\Theta_2$.

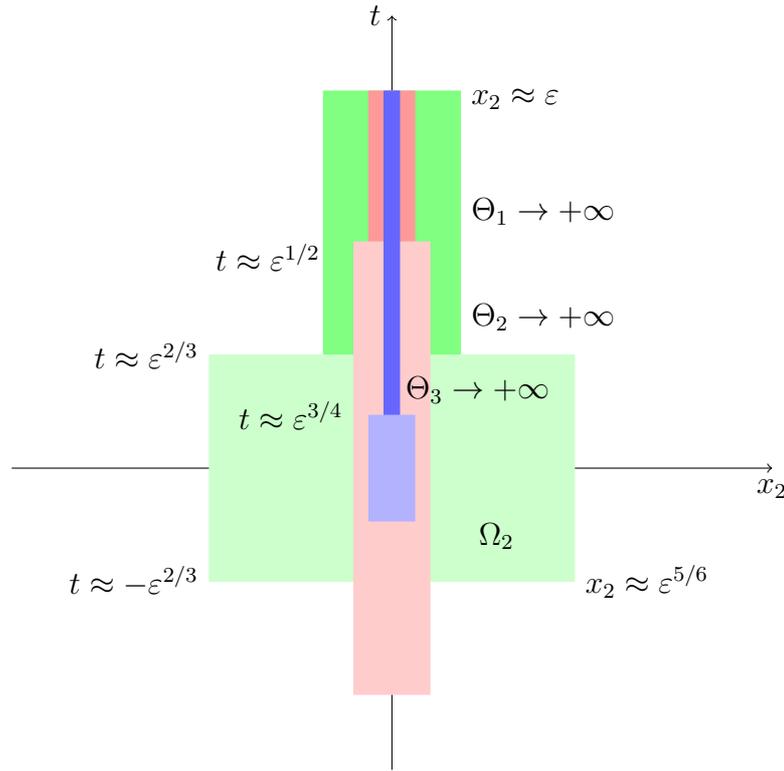[2]In author's free translation from Latin: anything relative [passes] from relative to relative.

Figure 2. A schematic projection of the domain of the localization of the singularity in the plane (x_2, t).

As $\Theta_1 \to +\infty$, according to Statement 3 for $\nu = 2$, the natural scale of the space localization in x_1 is constricted to the value of order ε and for the component u_1 one should use the following approximate relation:

$$u_1(\mathbf{x}, t, \varepsilon) \approx -\sqrt{2t} \, \tanh\left[\frac{x_1}{\varepsilon}\sqrt{\frac{t}{2}}\right], \quad \Theta_1 \to +\infty, \tag{3.2}$$

where $t = \varepsilon^{1/2}\Theta_1$.

Remark 4.   For correct understanding of the whole picture presented above, one important explanation should be given, although it is a quite trivial moment when one uses the standard matching method [7]. The indicated boundaries of the fragments of the asymptotic structure of the singularity of the solution $\mathbf{u}(\mathbf{x}, t, \varepsilon)$, that is the domains of the space-time scales of its localization, are not perfectly defined, since they can be displaced to some inessential distances, for example, with the help of multiplication by the value ε^δ, where $0 < \delta \ll 1$, even without prejudice to the strictness of mathematical statements if only the overlapping of the transition regions, i.e. the domains of the reconstructions of the scales, is not upset.[3]

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[3]The dialectic image of these metamorphoses and the place of particular ones in the whole structure of the singularity may be excellently reproduced by the sharp-witted phrase from the first book of Hegel's "*Wissenschaft der Logik*": "So ist das Endliche in dem Vergehen nicht vergangen; es ist zunächst nur ein *anderes* Endliches geworden... ." (In author's translation: "Thus the finite had not passed in the passage; first of all, it became only some *other* finite.")
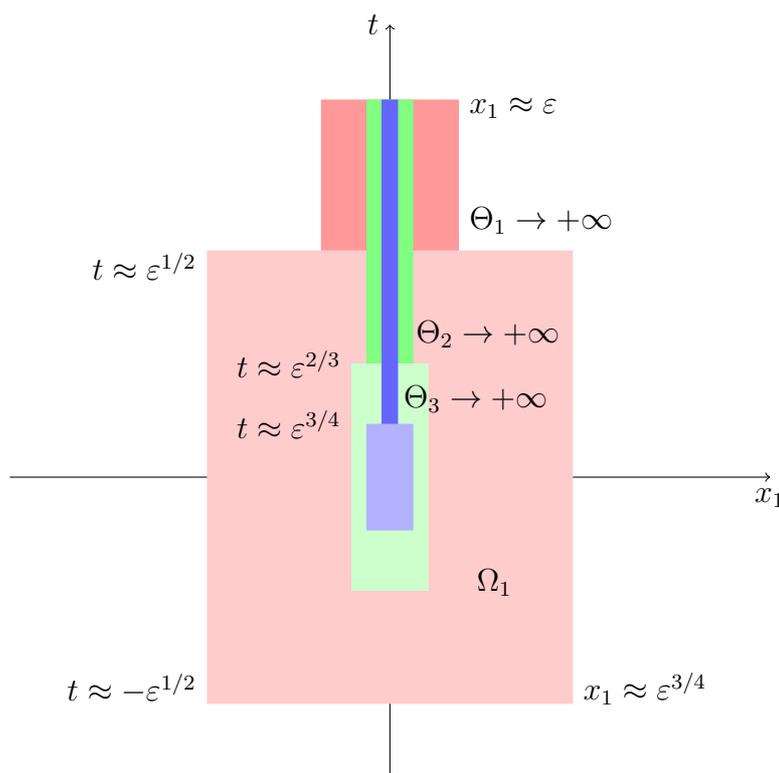
Figure 3. A schematic projection of the domain of the localization of the singularity in the plane (x_1, t).

4. Summary

1. The explicit formula (2.5) for the exact solution of the investigated Cauchy problem and expression (2.8) for its leading asymptotic approximation clearly show that the specific form the initial condition (1.2) in a finite time generates a peculiar multiscale microlocal singularity, whose evolution is determined by the joint effect of the Lagrange singularities A_3, A_5, and A_7; as we have seen, their truncated versal deformations appear in the arguments of the corresponding integrand exponents.

2. As shown by Statement 3 and further detailed explanations in Section 3, in particular, see relations (3.1)–(3.2), moving away from the singular point $(\mathbf{x}, t) = (0, 0)$ is accompanied by the multistep reconstruction of the natural space-time scales of the asymptotics of the solution, in other words, by a successive "switching" of the orders of their values with respect to the small parameter of dissipation.

    In view of this interesting property, the case of the origin and the evolution of the multiscale singularity of the solution under consideration is conceptually close to the nontrivial hierarchy of the space-time reconstructions corresponding to the multiscale evolution of the initial singularity obtained in [8]. Taking into account the picture of asymptotic relations clarified above, we may say that the case considered in the present paper has the advantage of the statement of the problem itself, since the vector field (1.2) at the initial moment of time is smooth and does not depend on additional small parameters.

3. The summarizing thesis of the present paper, that confirms Hilbert's thought about the

importance of studying specific problems[4], can be expressed as follows: the results of our study give an obvious example when a rather simple experiment in mathematical physics — the exactly solvable Cauchy problem for an evolutionary differential equation with only one small parameter — was able to generate the *multiscale* structure of metamorphoses of the "life" of the solution in the 4-dimensional space-time.

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[4]In his well-known talk, in 1900, David Hilbert literally said: "Eine noch wichtigere Rolle als das Verallgemeinern spielt — wie ich glaube — bei der Beschäftigung mit mathematischen Problemen das Specialisiren." (In author's translation: "An even more important role is played, as I believe, by studying rather special mathematical problems than general ones".)