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ON DOUBLE SIGNAL NUMBER OF A GRAPH

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Abstract: A set S of vertices in a connected graph $G=(V,E)$ is called a *signal set* if every vertex not in S lies on a signal path between two vertices from S. A set S is called a *double signal set* of G if S if for each pair of vertices $x,y\in G$ there exist $u,v\in S$ such that $x,y\in L[u,v]$. The double signal number $\mathrm{dsn}\,(G)$ of G is the minimum cardinality of a double signal set. Any double signal set of cardinality $\mathrm{dsn}\,(G)$ is called dsn-set of G. In this paper we introduce and initiate some properties on double signal number of a graph. We have also given relation between geodetic number, signal number and double signal number for some classes of graphs.

Keywords: Signal set, Geodetic set, Double signal set, Double signal number.

1. Introduction

By a graph $G=(V,E)$ we mean a finite, connected, undirected graph with neither loops nor multiple edges. The *order* $|V|$ and *size* $|E|$ of G are denoted by p and q respectively. For graph theoretic terminology we refer to [1]. The *open neighborhood* of any vertex v in G is $N(v)=\{x: xv\in E(G)\}$ and *closed neighborhood* of a vertex v in G is $N[v]=N(v)\cup\{v\}$. The *degree* of a vertex in the graph G is denoted by $\deg\,(v)$ and the maximum degree (minimum degree) in the graph G is denoted by $\triangle(G)(\delta(G))$. For a set $S\subseteq V(G)$ the open (closed) neighborhood $N(S)(N[S])$ in G is defined as

$$N(S)=\bigcup_{v\in S}N(v)(N[S]=\bigcup_{v\in S}N[v]).$$

A graph G is said to be *connected* if any two vertices in G are joined by a path. A maximal connected subgraph of G is called a *component* of G. A graph is said to be *disconnected* if it has at least two components. A *cut-vertex* of a connected graph is a vertex whose removal results a disconnected graph. A graph G is said to be *regular* if every vertex of G has equal degree.

If G is a connected graph the *distance* $d(x,y)$ is the length of a shortest x-y path in G. The *diameter* is defined by $\mathrm{diam}\,(G)=\max_{x,y\in V(G)}d(x,y)$. Two vertices u and v are said to be *antipodal vertices* if $d(u,v)=\mathrm{diam}\,(G)$. If $e=\{u,v\}$ is an edge of a graph G with $\deg\,(u)=1$ and $\deg\,(v)>1$, then we call e a pendant edge, u a pendant vertex and v a support vertex. A vertex v of G is said to be an *extreme vertex* if the subgraph induced by its neighborhood is complete. The set of all extreme vertices is denoted by $\mathrm{Ext}\,(G)$. An acyclic connected graph is called a *tree*. An x-y path of length $d(x,y)$ is called *geodesic*.

A set $S \subseteq V(G)$ is called a *geodetic set* of G, if every vertex in G lies on a geodesic joining a pair of vertices of S. The *geodetic number* of G, denoted by $g(G)$, is the minimum cardinality of a geodetic set of G. The geodetic number of a disconnected graph is the sum of the geodetic number of its components. Any geodetic set of cardinality $g(G)$ is called g-set of G.

A set S of vertices in G is called a *double geodetic set* of G if for each pair of vertices of G lie on any geodesic joining pair of vertices from S. The *double geodetic number* $\operatorname{dg}(G)$ is the minimum cardinality of a double geodetic set. Any geodetic set of cardinality $\operatorname{dg}(G)$ is called dg-set of G. The double geodetic number of a graph was introduced and studied in [7]. Various concepts inspired by geodetic sets are introduced in [1, 3, 4].

On a various study on the distance in graphs, we refer to [1]. In the meantime, Chartrand et al. introduced a new type of distance parameter called the detour distance in graphs. Once a new type of distance between two vertices was introduced by Chartrand et al., various new distance parameters such as Supreme distance, D-distance and many more, were introduced by different researchers. In continuation, Kathiresan et al. introduced a distance parameter, called the signal distance in graphs [5]. The signal distance $d_{SD}(u, v)$ between a pair of vertices u and v is defined by

$$d_{SD}(u,v) = \min\Big\{d(u,v) + \sum_{w \in V(G)} (\deg w - 2) + (\deg u - 1) + (\deg v - 1)\Big\},$$

where S is a path connecting u and v, $d(u, v)$ is the length of the path S and the sum $\sum_{w\in V(G)}$, where sum runs over all the internal vertices between u and v in the path S. The u-v signal path of length $d_{SD}(u, v)$ is also called *geosig*. A vertex v is said to lie on a geosig P if v is an internal vertex of P. The signal interval $L[x, y]$ consists of x, y and all vertices lying on some x-y geosig of G and for a non empty set $S \subseteq V(G)$, $L[S] = \bigcup_{x,y\in S} L[x, y]$.

A set $S \subseteq V(G)$ in a connected graph is a *signal set* of G if $L[S] = V(G)$. The *signal number* $\operatorname{sn}(G)$ is the minimum cardinality of a signal set of G. A signal set of cardinality $\operatorname{sn}(G)$ is called a sn-set of G. The signal number of a graph was introduced in [8] and further studied in [2, 5]. The concept of signal number can be applied in the fields of electrical engineering and irrigation systems. It was shown that the determining the signal number of a graph is an NP-hard problem. Let 2^V denote the set of all the subsets of V. The mapping L: $V \times V \to 2^V$ defined by

$$L[x, y] = \{z \in V : z \text{ lies on a } x\text{-}y \text{ geosig in } G\}$$

is the signal function of G. One of the basic properties of L is that $x, y \in L[x, y]$ for any pair $x, y \in V$. Hence the signal function captures every pair of vertices and so the problem of double signal sets is trivially well-defined while it is clear that this fails in many graphs already for triplets (for example, complete graphs). This is the motivation for introducing and studying double signal sets.

The concepts of distance in graphs is a major component in graph theory with its convexity concepts having numerous applications in real life problems. There are several interesting applications of these concepts to facility location in real life situations, routing of transport problems and communication network designs. As the path involved in this discussion of this paper are geosig, no intervention by hackers or enemies is possible to the respective facilities provided. Further, as signal paths are secured and longer than geodesic paths, it is advantageous to more customers in getting the service with protection.

The following theorems will be used in the subsequent sections.

Theorem 1 [2]. *For any connected graph G, the set of all end vertices is a subset of every signal set of G.*

Theorem 2 [3]. *Each extreme vertex of a connected graph G belongs to every geodetic set of G.*

Theorem 3 [7]. *Each extreme vertex of a connected graph G belongs to every double geodetic set of G.*

The signal number of some standard classes of graph can be easily found and are given below:

- Path P_p of $p \geq 2$ vertices, $\operatorname{sn}(P_p) = 2$.
- Cycle C_p of $p \geq 3$ vertices, $\operatorname{sn}(C_p) = \begin{cases} 2, & \text{if } \; p \text{ is even}, \\ 3, & \text{if } \; p \text{ is odd}. \end{cases}$
- Complete graph K_p of $p \geq 2$ vertices, $\operatorname{sn}(K_p) = p$.
- Peterson graph G, $\operatorname{sn}(G) = 4$.
- Star graph $K_{1,p-1}$ of $p \geq 2$ vertices, $\operatorname{sn}(K_{1,p-1}) = p - 1$.
- Complete bipartite graph $K_{m,n}(2 \leq m \leq n)$, $\operatorname{sn}(C_p) = \begin{cases} m, & \text{if } \; m \leq 3, \\ 4, & \text{otherwise}. \end{cases}$

2. Double signal number of a graph

Definition 1. *Let G be a connected graph with at least two vertices. A set S of vertices of G is called a double signal set of G if for each pair of vertices $x, y \in G$ there exist $u, v \in S$ such that $x, y \in L[u, v]$. The double signal number* $\operatorname{dsn}(G)$ *of G is the minimum cardinality of a double signal set. Any double signal set of cardinality* $\operatorname{dsn}(G)$ *is called* dsn*-set of G.*

Example 1. For the graph G in Fig. 1, it is clear that no 2-element subset of G is a signal set of G. Now $S = \{v_1, v_4, v_5\}$ is a signal set of G and so $\operatorname{sn}(G) = 3$. Clearly the pair of vertices v_3, v_6 lies only the v_3-v_6 geosig. Similarly, the vertices v_6, v_8 lies only the v_6-v_8 geosig. Also the vertices v_2, v_6 and v_6, v_7 lies only the v_2-v_6 and v_6, v_7 geosig, respectively. Therefore that S is not a double signal set of G. Since v_2, v_3, v_7, v_8 be an internal vertices of v_1-v_4 geosig path, we need at least 6

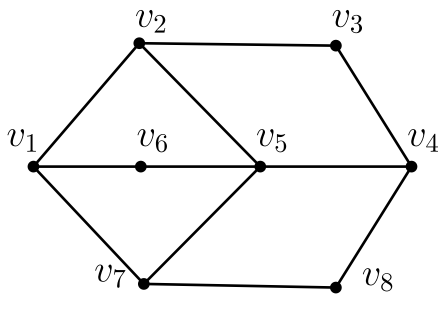

Figure 1. Graph G.

vertices to form a double signal set of G and so $\operatorname{dsn}(G) \geq 6$. Now, since $S_1 = \{v_1, v_3, v_4, v_5, v_6, v_8\}$ is a double signal set, it follows that $\operatorname{dsn}(G) = 6$.

Remark 1. For the graph G in Fig. 1, $S = \{v_1, v_4\}$ is the unique g-set and dg-set of G and so $g(G) = \operatorname{dg}(G) = 2$. Thus the double signal number is different from geodetic number and double geodetic number.

The following theorem directly follows by the definition of signal number and double signal number.

Theorem 4. *For any connected graph G of order p, $2 \leq \operatorname{sn}(G) \leq \operatorname{dsn}(G) \leq p$.*

Remark 2. The bounds in Theorem 4 are sharp. For the complete graph $K_p (p \geq 2)$, $\operatorname{dsn}(K_p) = p$. The set of the two end vertices of path graph $P_p (p \geq 2)$ forms a unique double signal set and so $\operatorname{dsn}(P_p) = 2$. Thus the nontrivial complete graph K_p has the largest possible double signal number and the nontrivial path graph P_p has the smallest double signal number. Also Example 2 shows that the bounds in Theorem 4 is sharp.

Theorem 5. *Each extreme vertex of a connected graph G belongs to every signal set of G.*

P r o o f. Let u be an extreme vertex of G and let S be a signal set of G. If u is an end-vertex, then by Theorem 1 $u \in S$. Suppose $u \notin S$ be non end-vertex. Then u is an internal vertex of an x-y geosig path, say P, for some $x, y \in S$. Since $\deg(u) \geq 2$, u has at least two neighbours in P which are not adjacent and so that u is not an extreme vertex, which is a contradiction. Hence $u \in S$. □

The following result is an easy consequences of Theorem 5.

Result 1. For the complete graph $K_p (p \geq 2)$, $\operatorname{dsn}(K_p) = p$.

To aid in our discussion throughout this paper, we define a definition as follows.

Definition 2. *Let G be a connected graph of order $p \geq 2$. A vertex $v \in G$ is said to be a weak extreme vertex, if there exists a vertex u in G such that v is either an initial vertex or a terminal vertex of any signal interval containing both u and v.*

Theorem 6. *Every double signal set of a connected graph G contains all the weak extreme vertices of G. In particular, if the set S of all weak extreme vertices is a double signal set, then S is the unique* dsn*-set of G.*

P r o o f. Let S be a double signal set of G and let x be a weak extreme vertex of G. Suppose $x \notin S$. Let y be any vertex in G such that $x \neq y$. Since S is a double signal set of G, we have for some $u, v \in S$, that x, y lie on an u-v geosig path. Also, that x is a weak extreme vertex of G shows either $x = u$ or $x = v$. It follows that $x \in S$, which is a contradiction. □

Corollary 1. *Each extreme vertex of a connected graph G belongs to every double signal set of G.*

P r o o f. Since every extreme vertex of G is weak extreme, the result follows from Theorem 6. □

Example 2. For the graph G in Fig. 2, the set $S = \{v_1, v_5, v_7\}$ of extreme vertices form unique minimum signal set of G and so $\operatorname{sn}(G) = 3$. Since the pair of vertices v_3, v_6 does not lie on any geosig of any pair of vertices from S, that S is not a double signal set of G. Also the vertex v_6 is the only non-extreme vertex which became weak extreme. It is clear that the set $S_1 = S \cup \{v_6\}$ of all weak extreme vertices form a double signal set of G and so by Theorem 6 $\operatorname{dsn}(G) = 4$.

Result 2. For any cycle $C_p (p \geq 3)$,

$$\operatorname{dsn}(C_p) = \begin{cases} 2, & \text{if } p \text{ is even,} \\ p, & \text{if } p \text{ is odd.} \end{cases}$$

Result 3. For any wheel $W_p = K_1 + C_{p-1} (p \geq 3)$, $\text{dsn}\,(W_p) = p$.
Result 4. For the complete bipartite graph $K_{m,n} (m, n \geq 2)$, $\text{dsn}\,(K_{m,n}) = \min\{m, n\}$.
Result 5. For any fan $F_p = K_1 + P_{p-1} (p \geq 3)$,

$$\text{dsn}\,(F_p) = \begin{cases} p-2, & \text{if} \quad p \text{ is even,} \\ p, & \text{if} \quad p \text{ is odd.} \end{cases}$$

Result 6. For the star graph $K_{1,p-1}$, $\text{dsn}\,(K_{1,p-1}) = p - 1$.

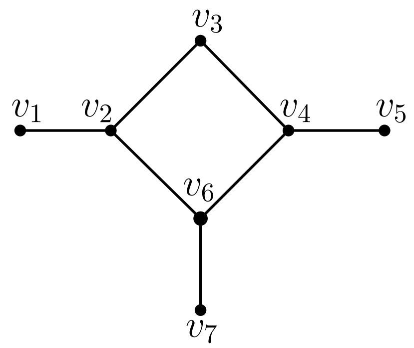

Figure 2. Graph G.

Theorem 7. *Let G be a connected graph with cut vertices and let S be a double signal set of G. If v is a cut vertex of G, then every component of $G - v$ contains at least one element of S.*

P r o o f. Let v be cut vertex of G and S be a double signal set of G. Suppose to the contrary, there exists a component, say H of $G - v$ such that H contains no vertex of S. By Theorem 6, S contains all the weak extreme vertices of G and hence, by assumption H does not contain any weak extreme vertex of G. Let $u \in V(H)$. Since, S is a double signal set of G, there exist vertices $x, y \in S$ such that $u, v \in L[x, y] \subseteq L[S]$. Let the x-y geosig path in G be $P : x = u_0, u_1, .., u, ..., u_l = y$ such that $u \neq x, y$. Since, v is a cut vertex, the x-u subpath of P and the u-y subpath of P both contain v, it implies that P is not a geosig path, which is a contradiction. Hence, every component of $G - v$ contains an element of S. □

Theorem 8. *No cut-vertex of a connected graph G belongs to any* dsn*-set of G.*

P r o o f. Suppose S be a dsn-set of a connected graph G that contains a cut-vertex v. Let $G_1, G_2, ..., G_n (n \geq 2)$ be the components of $G - v$. Let $S_1 = S - \{v\}$. We show that S_1 is a double signal set of G. Let $x, y \in V(G)$. Since S is a double signal set, then x, y lies on a geosig P joining a pair of vertices $a, b \in S$. If $v \notin \{a, b\}$, then $\{a, b\} \subseteq S_1$ and so that S_1 is a double signal set of G, which contradicts the minimality of S. Therefore, assume that $v \in \{a, b\}$ such that $v = b$ and $a \in G_1$. Since $S_1 \subseteq S$, that $a \in S_1$. By Theorem 7 we can fix a vertex $u \in G_k$ for $k \neq 1$ such that $u \in S$. Since $u \neq v$, that $u \in S$. Now, since v is a cut vertex of G, the signal interval of the path between a and v contained in the signal interval of the path between a and u. This shows that x, y lies on the geosig between $a, u \in S_1$. Therefore, that S_1 is a double signal set of G, which again contradicts the minimality of S. Hence no cut-vertex of G belongs to any dsn-set of G. □

Definition 3. *Let u be a vertex in G. A vertex v in G is said to be an $u -$ signal vertex if for any vertex $w \neq u, v$ with $d_{SD}(u, v) < d_{SD}(u, w)$, w lies on an u-v signal path.*

Theorem 9. *For any connected graph G,* $\mathrm{sn}\,(G) = 2$ *if and only if there exist vertices u, v such that v is an u-signal vertex of G.*

P r o o f. Let $\mathrm{sn}\,(G) = 2$ and $S = \{u, v\}$ be a sn-set of G. Then every vertex w in G lies on this u-v signal path and so that $d_{SD}(u, v)$ is minimum. Thus, $d_{SD}(u, v) < d_{SD}(u, w)$ for every $w \neq u, v$. Hence v is an u-signal vertex of G. The converse part is obvious. □

Theorem 10. *For a nontrivial connected graph G,* $\mathrm{dsn}\,(G) = 2$ *if and only if* $\mathrm{sn}\,(G) = 2$.

P r o o f. Let $S = \{u, v\}$ be a sn-set of G such that $\mathrm{sn}\,(G) = 2$. Then every pair of vertices of G lies on a u-v geosig and so that S itself forms a double signal set. Hence, $\mathrm{dsn}\,(G) = 2$. Converse part follows from Theorem 4. □

The following result follows from Theorem 6 and Theorem 8.

Result 7. If T is a tree with l end vertices, then $\mathrm{dsn}\,(T) = l$. In fact, the set of all end vertices of T is the unique dsn-set of T.

Lemma 1. *Let G be a connected graph of order $p \geq 2$. If there exists a vertex $v \in G$ such that*

$$\deg\,(v) > \sum_{w \in G} \deg\,(w) + l(P),$$

where $l(P)$ is the length of a geosig path P between any two antipodal vertices and the sum $\sum_{w \in G} \deg\,(w)$ runs over all the internal vertices between the antipodal vertices in P, then $\mathrm{dsn}\,(G) = p$.

For every connected graph G, it is clear that $\mathrm{rad}\,(G) \leq \mathrm{diam}\,(G) \leq 2\ \mathrm{rad}\,(G)$ [6]. Ostrand showed that any two positive integers a and b with $a \leq b \leq 2a$ are realizable as the radius and diameter, respectively. This theorem can be extended so that the double geodetic number can be prescribed as well.

Theorem 11. *For positive integers r, d and $a \geq 2$ with $r \leq d \leq 2r$, there exists a connected graph G with* $\mathrm{rad}\,(G) = r$, $\mathrm{diam}\,(G) = d$ *and* $\mathrm{dsn}\,(G) = a$.

P r o o f. If $r = 1$, then consider $G = K_a$ or $G = K_{1,a}$ according to whether $d = 1$ or $d = 2$, respectively. If $r = d \geq 2$ and $a = 2$, then we take $G = C_{2r}$.

Now assume that $r = d \geq 2$ and $a \geq 3$. Let $C_{2r} : u_1, u_2, ..., u_r, u_{r+1}, ..., u_{2r}, u_1$ be a cycle of order $2r$. Add $p - 1$ pendant edges $v_1u_1, v_2u_1, ..., v_{a-1}u_1$ to obtained the graph G. Clearly $\mathrm{rad}\,(G) = \mathrm{diam}\,(G) = r$. The graph G has $a - 1$ extreme vertices, that is, $S = \{v_1, v_2, ..., v_{a-1}\}$. By Corollary 1, each double signal set of G must contain S and that $L[S] \neq V(G)$. Hence, $\mathrm{dsn}\,(G) \geq a - 1$. On the other hand, we have $L[S \cup \{u_{r+1}\}] = V(G)$ and every pair of vertices of G lies on a geosig of some pair of vertices from $S \cup \{u_{r+1}\}$, implying that $\mathrm{dsn}\,(G) = a$.

Finally assume $2 \leq r < d$. First assume $a \geq 3$. Let G be the graph obtained from the disjoint union of a cycle $C_{2r} : u_1, u_2, ..., u_r, u_{r+1}, ..., u_{2r}, u_1$ of order $2r$ and a path $P_{d-r+1} : v_0, v_1, ..., v_{d-r}$ of order $d - r + 1$ by identifying u_1 and v_0. Add new pendant edges $u_rw_1, u_rw_2, u_rw_3, ..., u_rw_{a-3}$. Then G has radius r and diameter d. This graph G is shown in Fig. 3.

Now, we prove the set $\{w_1, w_2, ...w_{a-3}, u_{r+1}, u_{2r}, v_{d-r}\}$ forms a double signal set of G. By Corollary 1, $w_1, w_2, ..., w_{a-3}, v_{d-r} \in S$, where S is a double signal set of G. Further, as vertices $u_{r+1}, u_{r+2}, ..., u_{2r}$ in $V(G) - \{w_1, w_2, ..., w_{a-3}, v_{d-r}\}$ cannot covered by using the vertices $w_1, w_2, ..., w_{a-2}, v_{d-r}$, $|S| \geq a - 3 + 1 = a - 2$. Now it is clear that u_{r+1} is either an internal

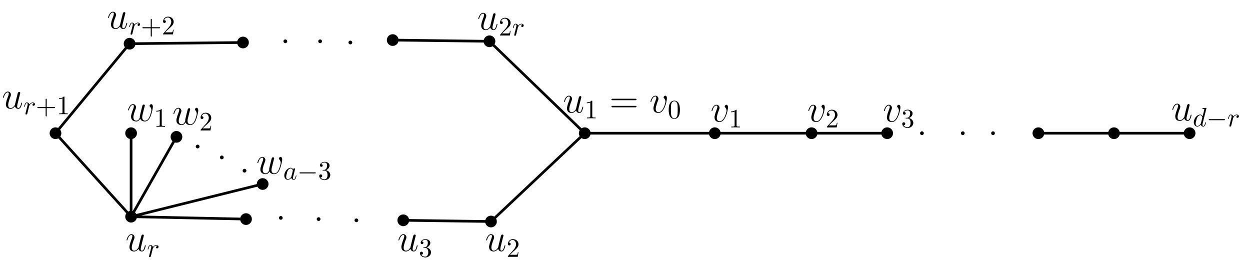

Figure 3. Graph G.

vertex or a terminal vertex of any signal path containing the pair of vertices u_{r+1}, v_i. Similarly, u_{2r} is either an internal vertex or a terminal vertex of any signal path containing the pair of vertices u_{2r}, w_i. Thus u_{r+1}, u_2r are weak extreme vertices. Therefore by Theorem 6, $u_{r+1}, u_{2r} \in S$ and so $\operatorname{dsn}(G) \geq a$. Since $\{w_1, w_2, ...w_{a-3}, u_{r+1}, u_{2r}, v_{d-r}\}$ forms a double signal set of G, that $\operatorname{dsn}(G) = a$. For the case $a = 2$, we remove the pendant edges $u_r w_1, u_r w_2, u_r w_3, ..., u_r w_{a-3}$ of G in Fig. 3. Clearly G has radius r and diameter d. Also $\{u_{r+1}, v_{d-r}\}$ is the unique double signal set of G and so by Theorem 4 we conclude that $\operatorname{dsn}(G) = 2$. This complete the proof. □

Theorem 12. *For every pair a, p of integers with $2 \leq a \leq p$, there exists a connected graph G of order p such that* $\operatorname{dsn}(G) = a$.

P r o o f. If $2 \leq a = p$, we take $G = K_p$. For $2 \leq a < p$, we consider a tree graph G of order p with a end-vertices. □

3. The double signal number and double geodetic number of a graph

In this section, we consider the realization result connecting the double signal number and double geodetic number of connected graphs. For this, first we focus the signal number and geodetic number. Because, for the graph G in Fig. 2, $g(G) = 3$ and $\operatorname{sn}(G) = 3$ so that $\operatorname{sn}(G) = g(G)$. Similarly, for the graph G in Fig. 1, $g(G) = 2$ and $\operatorname{sn}(G) = 3$ so that $\operatorname{sn}(G) > g(G)$ and for the graph G in Fig. 4, $g(G) = 3$ and $\operatorname{sn}(G) = 2$ so that $\operatorname{sn}(G) < g(G)$.

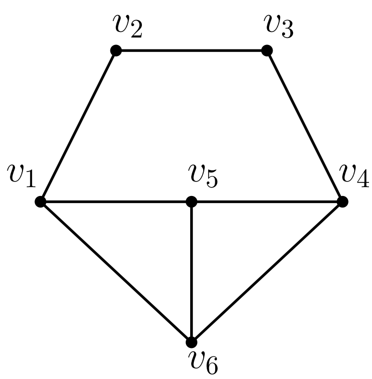

Figure 4. Graph G.

It is easily seen that a signal set is not in general a geodetic set in a graph G. Also the converse ir true. We verify that if S is a signal set and D is a geodetic set of G, either $S \subseteq D$ or $D \subseteq S$. Hence the signal set and the geodetic set depend one to another. Therefore we can't find out which one is the bigger set.

Result 8. If G is a tree, then $\operatorname{sn}(G) = g(G)$.

Theorem 13. *If G is a regular graph, then* $\operatorname{sn}(G) = g(G)$.

P r o o f. Since G is regular, the degree of every vertices of G is unique. So the signal distance between any pair of vertices depends only the geodesic distance between this pair of vertices. Hence, $\operatorname{sn}(G) = g(G)$. □

In view of Theorem 4, we have the following realization theorems.

Theorem 14. *For any integers a, b and c with $3 \le a \le b \le c$, there exists a connected graph G with $g(G) = a$, $\operatorname{sn}(G) = b$ and $\operatorname{dsn}(G) = c$.*

P r o o f. This theorem is proved by considering four cases.

Case 1. $a = b = c$. Then for the complete graph K_a, $g(G) = \operatorname{sn}(G) = \operatorname{dsn}(G) = a$.

Case 2. $a = b < c$. let G be the graph in Fig. 5 obtained from the path $P_3 : u_1, u_2, u_3$ of order 3, by adding c vertices $v_1, v_2, ..., v_{a-2}, w_1, w_2, ..., w_{c-a+2}$ to P_3 and joining each vertex v_i $(1 \le i \le a-2)$ to u_2; and joining each vertex $w_i (1 \le j \le c - a + 2)$ to u_1 and u_3. By Theorem 2, Theorem 5 and Corollary 1, every geodetic set, every signal set and every double signal set of G contains the set $S = \{v_1, v_2, ..., v_{a-2}\}$ of all extreme vertices of G. Clearly, S is not a geodetic set of G. Also, for any $x \in V(G) - S$, $S \cup \{x\}$ is not a geodetic set or a signal set of G and so $g(G) \ge a$. Now it is easy to check that $S_1 = S \cup \{u_1, u_2\}$ is a geodetic set of G. Since every vertex in $V(G) - S_1$ lies on the signal path between some vertices from S_1, so S_1 is the minimum geodetic set as well as signal set of G. Thus $g(G) = \operatorname{sn}(G) = a$. It is clear that the pair of vertices v_i, w_j for $(1 \le i \le a - 2)$ and $(1 \le j \le c - a + 2)$ do not lie on any u-v geosig path, for any $u, v \in S_1$ and so that S_1 is not a double signal set of G. It is easy to verify that $S_2 = S \cup \{w_1, w_2, ..., w_{c-a+2}\}$ is a minimum double signal set of G and so $\operatorname{dsn}(G) = c$.

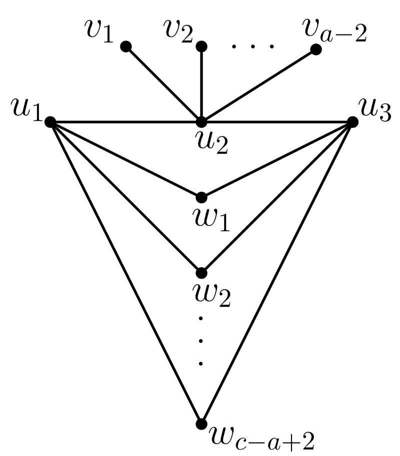

Figure 5. Graph G.

Case 3. $a < b = c$. Let G be the graph in Fig. 6 got from the complete graph K_{b-a+2} and the path $P_3 : x, y, z$ of order 3 by joining all the vertices of K_{b-a+2} to x and y and adding $a-2$ new pendant edges $v_1, v_2, ..., v_{a-2}$. By Theorem 2, Theorem 5 and Corollary 1, every geodetic set, every signal set and every double signal set of G contain the set $S = \{v_1, v_2, ..., v_{a-2}\}$ of all extreme vertices of G. Clearly, S is not a geodetic set of G. Also, for any $u \in V(G) - S$, $S \cup \{u\}$ is not a geodetic set or a signal set of G and so $g(G) \geq a$. Since $S_1 = S \cup \{x, y\}$ is a geodetic set of G, it follows that $g(G) = a$. It is clear that the vertices of K_{b-a+2} do not lie on any signal path between vertices from S_1, that S_1 is not a signal set of G. Clearly, every signal set and every double signal set contains every vertices of K_{b-a+2}. Now it is easily to verify that $S \cup V(K_{b-a+2})$ is a minimum signal set and minimum double signal of G. Hence, $\mathrm{sn}(G) = \mathrm{dsn}(G) = a - 2 + b - a + 2 = b$.

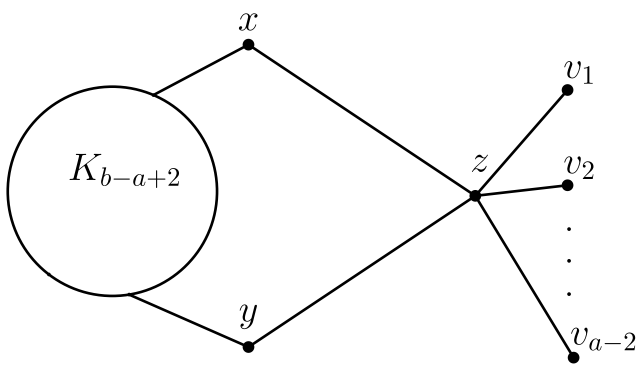

Figure 6. Graph G.

Case 4. $a < b < c$. Let G be the graph in Fig. 7 obtained from the path $P_3 : x, y, z$ of order 3 by adding c new vertices $u_1, u_2, ..., u_{a-2}, w_1, w_2, ..., w_{b-a}, v_1, v_2, ..., v_{c-b+2}$ to P_3 and joining each vertex $w_i (1 \leq i \leq b-a)$ to the vertices x, y and z ; joining each vertex $v_j (1 \leq j \leq c-b+2)$ to the vertices x and z; joining each vertex $u_k (1 \leq k \leq a-2)$ to the vertex y. By Theorems 2, Theorem 5 and Corollary 1, every geodetic set, every signal set and every double signal set of G contain the set $S = \{v_1, v_2, ..., v_{a-2}\}$ of all extreme vertices of G. Clearly, S is not a geodetic set of G. Also, for any $v \in V(G) - S$, $S \cup \{v\}$ is not a geodetic set of G and so $g(G) \geq a$. Since $S_1 = S \cup \{x, z\}$ is a geodetic set of G, it follows that $g(G) = a$. Since each vertex w_j does not lie on any geosig of vertices of S_1, that S_1 is not a signal set of G. It is clear that every signal set of G contains $\{w_1, w_2, ..., w_{b-a}\}$. Then $S_2 = S_1 \cup \{w_1, w_2, ..., w_{b-a}\}$ is a minimum signal set of G and so $\mathrm{sn}(G) = b$. Now, each pair v_j is either an initial vertex or terminal vertex of any signal path containing the vertices v_j and w_i. Hence $v_1, v_2, ..., v_{c-b+2}$ are weak extreme vertices. It is easily verified that the set $S_3 = S \cup \{w_1, w_2, ..., w_{b-a}, v_1, v_2, ..., v_{c-b+2}\}$ is the unique minimum double signal set of G and so $\mathrm{dsn}(G) = c$.

Theorem 15. *For integers a, b and c with $3 \leq a \leq b \leq c$, there exists a connected graph G with $\mathrm{sn}(G) = a$, $g(G) = b$ and $\mathrm{dsn}(G) = c$.*

P r o o f. This theorem is proved by considering three cases.

Case 1. $a = b = c$. Then for the complete graph K_a, $g(G) = \mathrm{sn}(G) = \mathrm{dsn}(G) = a$.

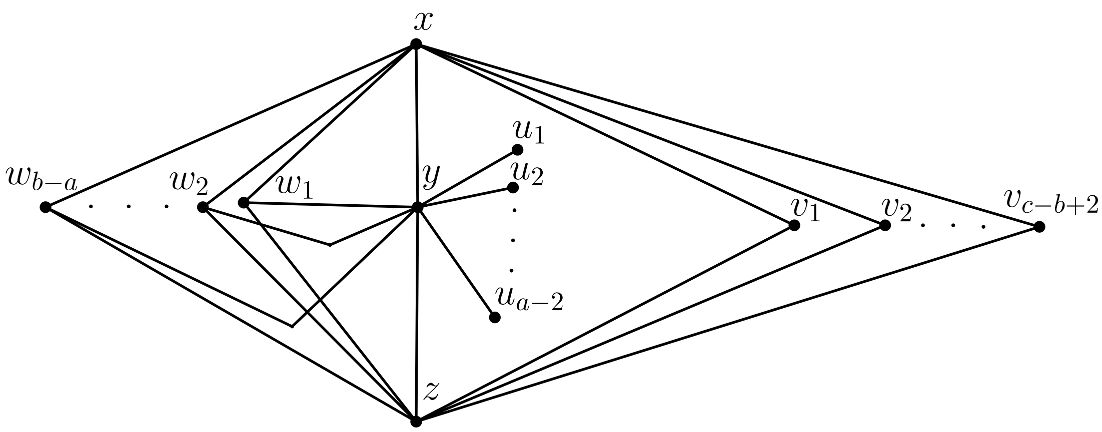

Figure 7. Graph G.

Case 2. $a = b < c$. The proof is similar to the proof of case 2 in Theorem 14.

Case 3. $a < b \leq c$. Let H be the graph obtained from the path $P_6 : u, v, w, x, y, z$ of order 6, $b - a$ copies of path $P_i : x_i, y_i, z_i$ $(1 \leq i \leq b - a)$ of order 3 and 2 copies of path $P_j : x'_j, y'_j$ $(1 \leq j \leq 2)$ of order 3 by joining each vertex $x_i (1 \leq i \leq b - a)$ to the vertex u of P_6, each vertex $z_i (1 \leq i \leq b - a)$ to the vertex v of P_6, each vertex x'_j $(1 \leq j \leq 2)$ to the vertex u of P_6 and each vertex y'_j $(1 \leq j \leq 2)$ to the vertex v of P_6 and add an edge $x'_1\ y'_2$. Let G be the graph in Fig. 8 obtained from H by adding the following new vertices to H.

(i) Add $a - 1$ new vertices $u_1, u_2, ..., u_{a-1}$ to H and join each $u_i (1 \leq i \leq a - 1)$ to w.

(ii) Add $b - a - 1$ new vertices $v_1, v_2, ..., v_{b-a-1}$ to H and join each $v_i (1 \leq i \leq b - a - 1)$ to both w and y.

(iii) Add $c - b$ new vertices $w_1, w_2, ..., w_{c-b}$ to H and join each $w_i (1 \leq i \leq c - b)$ to both v and x.

Let $S = \{u_1, u_2, ..., u_{a-1}, z\}$ be the set of extreme vertices of G. By Theorem 5, Theorem 2 and Corollary 1, every signal set, every geodetic set and every double signal set contains S. Clearly S itself is not a signal set of G and so $\text{sn}(G) \geq a$. It is clear that $S_1 = S \cup \{u\}$ is a signal set of G and hence $\text{sn}(G) = a$. Since the vertices $x_i, y_i, z_i (1 \leq i \leq b - a)$ do not lie on any geodesic joining a pair of vertices from S_1, S_1 is not a geodetic set of G. Let $S_2 = S \cup \{y_1, y_2, ..., y_{b-a}\}$. It is easy to verify that S_2 is a minimum geodetic set of G and so $g(G) = a + b - a = b$. Since the pair of vertices $w_i (1 \leq i \leq c - b), v_j (1 \leq j \leq b - a - 1)$ do not lie on any signal path between a pair of vertices from S_1, S_1 is not a double signal set of G. Also, x is either an initial vertex or terminal vertex of any geosig containing the vertices x and v_1 and so x is a weak extreme vertex. Hence $w_1, w_2, ..., w_{c-b}, v_1, v_2, ..., v_{b-a-1}, x$ are weak extreme vertices.

Let $S' = S_1 \cup \{w_1, w_2, ..., w_{c-b}, v_1, v_2, ..., v_{b-a-1}, x\}$. It is easily verified that S' is the set of all weak extreme vertices of G. Since S' is a double signal set of G, by Theorem 6 it follows that $\text{dsn}(G) = c$. □

Theorem 16. *For every pair a, b of integers with $4 \leq a \leq b$ and $b \neq a + 1$, there exists a connected graph G with $dg(G) = a$ and* $\text{dsn}(G) = b$.

P r o o f. For $4 \leq a = b$, then the complete graph K_a has the desired properties. So, assume that $4 \leq a < b$ and $b \neq a + 1$. Let G be the graph in Fig. 9 formed from the path $P_4 : u, v, w, y$ of

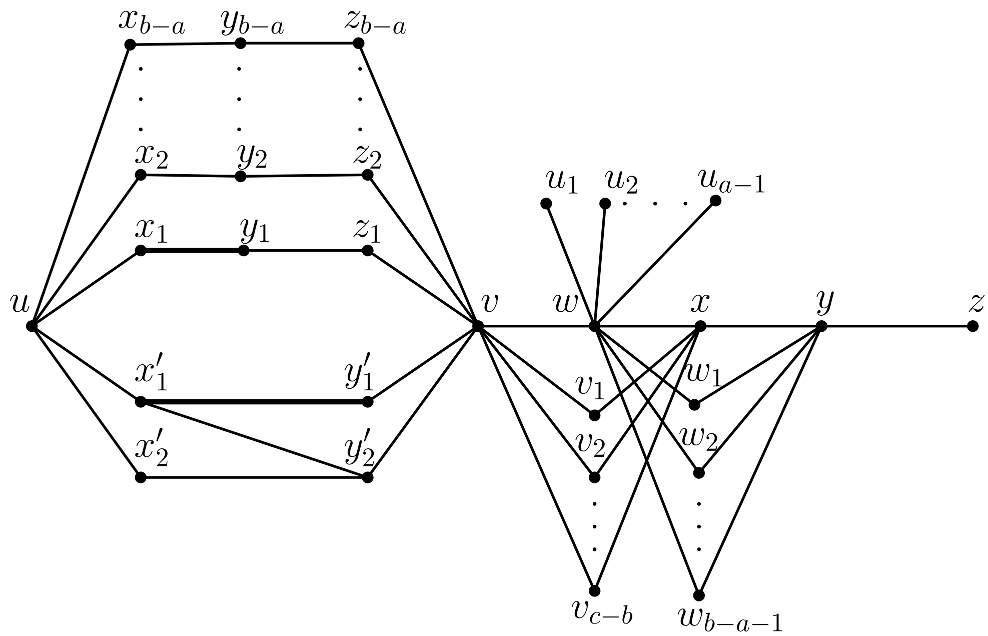

Figure 8. Graph G.

order 4, by adding b new vertices $u_1, u_2, ..., u_{a-3}, v_1, v_2, ..., v_{b-a-1}, x$ to P_4 and joining each vertex $u_i (1 \leq i \leq a-3)$ to v; and joining each vertex $v_j (1 \leq j \leq b-a-1)$ to both v and y; and join the vertex x to u and w. Let $S = \{u_1, u_2, ..., u_{a-3}\}$ be the set of extreme vertices of G. By Theorem 3 and Corollary 1, every double geodetic and every double signal set contains S. Now it is clear that $S_1 = S \cup \{u, x, y\}$ is a minimum double geodetic set of G and so $\mathrm{dg}\,(G) = a$. Since $L[u, y]$ contains u, x, w, y, the pair of vertices $x, v_i (1 \leq i \leq b-a-1)$ do not lie on any geosig of a pair of vertices from S. So that S_1 is not a double signal set of G. It is easy to verify that $S_2 = S \cup \{v_1, v_2, ..., v_{b-a-1}, w\}$ be the unique minimum double signal set of G. Hence, $\mathrm{dsn}\,(G) = b$. $\square$

4. Closing open problems

We close with the following list of open problems that we have yet to settle.

Problem 1. Determine the class of graphs G for which $g(G) = \mathrm{sn}\,(G)$.

Problem 2. Determine the class of graphs G for which $\mathrm{sn}\,(G) = \mathrm{dsn}\,(G)$.

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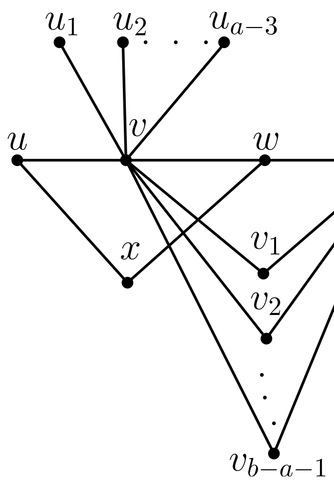

Figure 9. Graph G.

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