A GUARANTEED CONTROL PROBLEM FOR A LINEAR STOCHASTIC DIFFERENTIAL EQUATION[1]

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Abstract: A problem of guaranteed closed-loop control under incomplete information is considered for a linear stochastic differential equation (SDE) from the viewpoint of the method of open-loop control packages worked out earlier for the guidance of a linear control system of ordinary differential equations (ODEs) to a convex target set. The problem consists in designing a deterministic control providing (irrespective of a realized initial state from a given finite set) prescribed properties of the solution (being a random process) at a terminal point in time. It is assumed that a linear signal on some number of realizations is observed. By the equations of the method of moments, the problem for the SDE is reduced to an equivalent problem for systems of ODEs describing the mathematical expectation and covariance matrix of the original process. Solvability conditions for the problems in question are written.

Key words: Guidance problem, Guaranteed closed-loop control, Linear stochastic differential equation.

Introduction

The problem of constructing optimal strategies of guaranteed feedback control under conditions of uncertainty is one of the most important in mathematical control theory and its applications. In the present paper following the theory of closed-loop control developed by N.N. Krasovskii's school [1–3], the approach based on the so-called method of open-loop control packages originating from the technique of nonanticipating strategies from the theory of differential games [4] is applied to solving the guidance problem for a linear SDE. The method tested on the guidance problems under incomplete information for linear controlled systems of ODEs consists in reducing the problems of guaranteed control formulated in the class of closed-loop strategies to equivalent problems in the class of open-loop control packages. The latter class contains the families of open-loop controls parameterized by admissible initial states and possessing the property of nonanticipation with respect to the dynamics of observations [5–7].

This paper is devoted to the study of the problem of guiding (with a probability close to one) a trajectory of a linear SDE to some target set. The statement means that we should form a deterministic control providing (irrespective of the realized initial state from a specified finite set) prescribed properties of the solution (being a random process) at a terminal point in time. Here, we observe a linear signal on some number of realizations. Similar problems arise in practical situations, when it is possible to observe the behavior of a large number of identical objects described by a stochastic dynamics. By the equations of the method of moments [8, 9], the problem for the SDE is reduced to an equivalent problem for systems of ODEs describing the mathematical expectation and covariance matrix of the original process. The technique of the method of open-loop control packages developed in [5–7] is applied to the systems obtained.

The paper has the following structure. In Section 1, the main problem of closed-loop guaranteed control for a linear SDE under incomplete information is formulated. In Section 2, the procedure reducing the original problem to two auxiliary guidance problems for systems of ODEs is described and the equivalence of the problems for SDE and ODEs is established. In Section 3, a brief summary of results obtained earlier within the framework of the method of open-loop control packages for

[1]This work was supported by the Russian Science Foundation (project no. 14-11-00539).

ODEs is given. In Section 4, the necessary statistical estimates are analyzed. In Section 5, the main results of the paper are presented: a solvability criterion for the original problem and an assertion connecting the guidance accuracy and the number of trajectories of the original process that are available for measuring. In Section 6, we consider an example illustrating the application of the algorithm to solving the guidance problem for a first-order SDE. Note that the procedure reducing a problem for a linear SDE to the corresponding problem for the ODE, which is conceptually close to the one proposed in the paper, was used, in particular, in [10] for solving the problem of dynamic reconstruction of an unknown disturbance characterizing the level of random noise in a linear SDE on the base of measuring some realizations of the SDE's phase vector.

1. Problem statement

Consider a system of linear SDEs of the following form:

$$dx(t,\omega) = (A(t)x(t,\omega) + B_1(t)u_1(t) + f(t))\,dt + B_2(t)U_2(t)\,d\xi(t,\omega), \quad x(t_0,\omega) = x_0. \tag{1.1}$$

Here, $t \in T = [t_0, \vartheta]$, $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $\xi = (\xi_1, \xi_2, \dots, \xi_k) \in \mathbb{R}^k$ (all the vectors are treated as columns); $\omega \in \Omega$, (Ω, F, P) is a probability space; $\xi(t,\omega)$ is a standard Wiener process (i.e., a process starting from zero with zero mathematical expectation and covariance matrix equal to It (I is the unit matrix from $\mathbb{R}^{k\times k}$)); $f(t)$ is a continuous vector function with values in $\mathbb{R}^n$; $A(t) = \{a_{ij}(t)\}$, $B_1(t) = \{b_{1ij}(t)\}$, and $B_2(t) = \{b_{2ij}(t)\}$ are continuous matrix functions of dimensions $n \times n$, $n \times r$, and $n \times k$, respectively.

Two controls act in the system: a vector $u_1(t) = (u_{11}(t), u_{12}(t), \dots, u_{1r}(t)) \in \mathbb{R}^r$ and a diagonal matrix $U_2(t) = \{u_{21}(t), u_{22}(t), \dots, u_{2k}(t)\} \in \mathbb{R}^{k\times k}$, which are Lebesgue measurable on T and take values from specified instantaneous control resources S_{u1} and S_{u2} being convex compact sets in the corresponding spaces. The control u_1 enters the deterministic component and influences the mathematical expectation of the desired process. Since $U_2 d\xi = (u_{21}d\xi_1, u_{22}d\xi_2, \dots, u_{2k}d\xi_k)$, we can assume that the vector $u_2 = (u_{21}, u_{22}, \dots, u_{2k})$ characterizes the diffusion of the process (the amplitude of random noises).

The initial state x_0 belongs to a finite set of admissible initial states X_0, which consists of normally distributed random variables with numerical parameters (m_0, D_0), where $m_0 = Mx_0$ is the mathematical expectation, $m_0 \in \mathcal{M}_0 = \{m_0^1, m_0^2, \dots, m_0^{n_1}\}$, $D_0 = M(x_0 - m_0)(x_0 - m_0)^*$ is the covariance matrix (the asterisk means transposition), $D_0 \in \mathcal{D}_0 = \{D_0^1, D_0^2, \dots, D_0^{n_2}\}$. Thus, the set X_0 contains $n_1 n_2$ elements. Note that, if $0 \in \mathcal{D}_0$, then there are deterministic vectors in X_0. We assume that the system's initial state belongs to X_0 but is unknown.

Equation (1.1) is a symbolic notation for the integral identity

$$x(t,\omega) = x_0 + \int_{t_0}^{t} (A(s)x(s,\omega) + B_1(s)u_1(s) + f(s))\,ds + \int_{t_0}^{t} B_2(s)U_2(s)\,d\xi(s,\omega). \tag{1.2}$$

The latter integral on the right-hand side of equality (1.2) is stochastic and is understood in the sense of Ito. For any $\omega \in \Omega$, the formulated Cauchy problem has a unique solution and specifies the corresponding realization of the stochastic process $x(t,\omega)$, $t \in T$. A solution of equation (1.1) is defined as a stochastic process satisfying integral identity (1.2) for any t with probability 1. Under the above assumptions, there exists a unique solution, which is a normal Markov process with continuous realizations [11].

Note that equations similar to (1.1), (1.2) describe simplest linearized models, for example, of changing the size of a multi-species biological population in a stochastic medium, of price dynamics on goods markets under the influence of random factors, or of particle motion in some field.

The problem in question consists in the following. Let convex closed target sets $\mathcal{M} \in \mathbb{R}^n$ and $\mathcal{D} \in \mathbb{R}^{n\times n}$ and a continuous matrix observation function $Q(t)$ of dimension $q \times n$ be given.

At any time, it is possible to receive the information on some number N of realizations of the stochastic process $x(t)$ (we omit the symbol ω if we mean the process rather than its realization). The following signal is available:

$$y(t) = Q(t)x(t). \tag{1.3}$$

Assume that, for a finite set of some specified times $\tau_i \in T$, $i \in [1 : l]$, we can construct, using N realizations of the process $x(t)$, a statistical estimate m_i^N of the mathematical expectation $m(\tau_i)$ and a statistical estimate D_i^N of the covariance matrix $D(\tau_i)$ such that

$$P\Big(\max_{i\in[1:l]} \big\{ \|m_i^N - m(\tau_i)\|_{\mathbb{R}^n}, \|D_i^N - D(\tau_i)\|_{\mathbb{R}^{n\times n}} \big\} \le h(N)\Big) = 1 - g(N), \tag{1.4}$$

where $h(N)$ and $g(N) \to 0$ as $N \to \infty$. We show below that standard procedures of obtaining the estimates m_i^N and D_i^N admit modifications providing the validity of relation (1.4) and the specified convergences (a similar procedure was proposed in [10]).

Problem 1 of guaranteed closed-loop ε-guidance consists in forming a control $(u_1(\cdot), u_2(\cdot))$ guaranteeing, whatever the initial state x_0 from the set X_0, prescribed properties of the process x at the terminal time ϑ. Here, we mean that, for an arbitrary small (in advance specified) $\varepsilon > 0$, the mathematical expectation $m(\vartheta)$ and the covariance matrix $D(\vartheta)$ reach the ε-neighborhoods of the target sets $\mathcal{M}$ and $\mathcal{D}$, respectively. In the motion process, the sought control is formed using the information on N realizations of the signal $y(t)$. By virtue of estimate (1.4), it is reasonable to require that the probability of the desired event should be close to 1 for sufficiently large N and algorithm's parameters concordant with N in a special way.

2. Reduction of the original problem

Let us reduce the guidance problem for the SDE formulated above to two problems for systems of ODEs. By virtue of the linearity of the original system, the mathematical expectation $m(t)$ depends only on $u_1(t)$; its dynamics is described by the equation

$$\dot{m}(t) = A(t)m(t) + B_1(t)u_1(t) + f(t), \quad t \in T = [t_0, \vartheta], \quad m(t_0) = m_0 \in \mathcal{M}_0. \tag{2.1}$$

We assume that N ($N > 1$) trajectories $x^r(t)$, $r \in [1 : N]$, of the original SDE are measured; then, according to the statement of the guidance problem, we know values of signal (1.3) $y^r(t) = Q(t)x^r(t)$.

The signal on the trajectory of equation (2.1) is denoted by $y_m(t) = Q_m(t)m(t)$; its estimate formed by the information on y^r, $r \in [1 : N]$, by $y_m^N(t)$. The latter is constructed as follows:

$$y_m^N(t) = \frac{1}{N}\sum_{r=1}^{N} y^r(t) = Q(t)m^N(t), \quad m^N(t) = \frac{1}{N}\sum_{r=1}^{N} x^r(t). \tag{2.2}$$

Obviously, $Q_m(t) = Q(t)$ and, for the finite set of times $\tau_i \in T$, $i \in [1 : l]$, in view of relation (1.4), it holds that

$$P\big(\forall i \in [1 : l] \ \ \|y_m^N(\tau_i) - y_m(\tau_i)\|_{\mathbb{R}^q} \le C_1 h(N)\big) = 1 - g(N), \tag{2.3}$$

where the constant C_1 can be written explicitly. Recall that $\mathcal{M}$ is the target set for the trajectory of equation (2.1).

The covariance matrix $D(t)$ depends only on $U_2(t)$; its dynamics is described by the so-called equation of the method of moments [8, 9] in the following form:

$$\dot{D}(t) = A(t)D(t) + D(t)A^*(t) + B_2(t)U_2(t)U_2^*(t)B_2^*(t), \quad t \in T = [t_0, \vartheta], \quad D(t_0) = D_0 \in \mathcal{D}_0. \tag{2.4}$$

For our purposes, matrix equation (2.4) is conveniently rewritten in the form of a vector equation, which is more traditional for the problem under consideration. By virtue of the symmetry of the matrix $D(t)$, its dimension is defined as $n_d = (n^2+n)/2$. Let us introduce the vector $d(t) = \{d_s(t)\}$, $s \in [1 : n_d]$, whose coordinates are found by the elements of the matrix $D(t) = \{d_{ij}(t)\}$, $i, j \in [1 : n]$:

$$d_s(t) = d_{ij}(t), \quad i \le j, \quad s = (n - i/2)(i-1) + j. \tag{2.5}$$

Actually, the vector $d(t)$ consists of successively written and enumerated elements of the matrix $D(t)$, taken line by line starting with the element located at the main diagonal. Let us write the symmetric right-hand side of (2.4) in detail:

$$A(t)D(t) + D(t)A^*(t) = \begin{pmatrix} 2\langle \hat a_1, \hat d_1\rangle & \langle \hat a_1, \hat d_2\rangle + \langle \hat a_2, \hat d_1\rangle & \cdots & \langle \hat a_1, \hat d_n\rangle + \langle \hat a_n, \hat d_1\rangle \\ \langle \hat a_2, \hat d_1\rangle + \langle \hat a_1, \hat d_2\rangle & 2\langle \hat a_2, \hat d_2\rangle & \cdots & \langle \hat a_2, \hat d_n\rangle + \langle \hat a_n, \hat d_2\rangle \\ \cdots & \cdots & \cdots & \cdots \\ \langle \hat a_n, \hat d_1\rangle + \langle \hat a_1, \hat d_n\rangle & \langle \hat a_n, \hat d_2\rangle + \langle \hat a_2, \hat d_n\rangle & \cdots & 2\langle \hat a_n, \hat d_n\rangle \end{pmatrix},$$

$$B_2(t)U_2(t)U_2^*(t)B_2^*(t) = \begin{pmatrix} \sum\limits_{r=1}^{k} b_{21r}^2 u_{2r}^2 & \sum\limits_{r=1}^{k} b_{21r}b_{22r}u_{2r}^2 & \cdots & \sum\limits_{r=1}^{k} b_{21r}b_{2nr}u_{2r}^2 \\ \sum\limits_{r=1}^{k} b_{22r}b_{21r}u_{2r}^2 & \sum\limits_{r=1}^{k} b_{22r}^2 u_{2r}^2 & \cdots & \sum\limits_{r=1}^{k} b_{22r}b_{2nr}u_{2r}^2 \\ \cdots & \cdots & \cdots & \cdots \\ \sum\limits_{r=1}^{k} b_{2nr}b_{21r}u_{2r}^2 & \sum\limits_{r=1}^{k} b_{2nr}b_{22r}u_{2r}^2 & \cdots & \sum\limits_{r=1}^{k} b_{2nr}^2 u_{2r}^2 \end{pmatrix}.$$

Here, $\hat a_i$, $\hat d_i$, $i \in [1 : n]$, are the rows of the matrices A and D; the symbol $\langle \cdot, \cdot \rangle$ stands for the scalar product of vectors in the corresponding space. Note that the element (ij) of the matrix $A(t)D(t) + D(t)A^*(t)$, namely $\langle \hat a_i, \hat d_j\rangle + \langle \hat a_j, \hat d_i\rangle = \sum\limits_{r=1}^{n} a_{ir}d_{jr} + \sum\limits_{r=1}^{n} a_{jr}d_{ir}$, determines the coefficients for the coordinates of the vector d in the sth equation of the required system, where s is found from relation (2.5). Based on the representations above, by the matrices $A(t)$ and $B_2(t)$, we form the matrices $\bar A(t) : T \to R^{n_d \times n_d}$ and $\bar B(t) : T \to R^{n_d \times k}$ acting as follows.

1. To form the sth row ($s \in [1 : n_d]$) of the matrix $\bar A$, we should uniquely define by formula (2.5) the indices i and j ($i, j \in [1 : n]$, $i \le j$) corresponding to s, i.e., such that $(n-i/2)(i-1)+j = s$. Further, we find the elements $\bar a_{sr}$ ($r \in [1 : n_d]$). If s is such that $i = j$, then there can be at most n nonzero elements in the sth row: $\bar a_{sr} = 2a_{ih_1}$, $h_1 \in [1 : n]$, r corresponds to (in the sense of (2.5), and the first index is not greater than the second) the pair of indices $h_1 i$ (or ih_1). In the case when s is such that $i \ne j$, to fill in the row of the matrix $\bar A$, one should analyze the three variants: (i) r does not correspond to any pair of indices from the set $\{1i, 2i, \ldots, ni, i1, i2, \ldots, in, 1j, 2j, \ldots, nj, j1, j2, \ldots, jn\}$, then $\bar a_{sr} = 0$; (ii) r corresponds to some (unique) pair $h_1 i$ (ih_1), $h_1 \in [1 : n]$, then $\bar a_{sr} = a_{jh_1}$, or to a pair $h_2 j$ (jh_2), $h_2 \in [1 : n]$, then $\bar a_{sr} = a_{ih_2}$; (iii) r corresponds to two pairs of indices, in this case, $h_1 = j$ and $h_2 = i$, then $\bar a_{sr} = a_{ii} + a_{jj}$. Since such a coincidence is unique for $i \ne j$, obviously, we have at most $2n-1$ nonzero elements in the sth row of the matrix $\bar A$.

2. To form the sth row ($s \in [1 : n_d]$) of the matrix $\bar B$, we should uniquely define by formula (2.5) the indices i and j ($i, j \in [1 : n]$, $i \le j$) corresponding to s, i.e., such that $(n-i/2)(i-1)+j = s$. Further, we find the element $\bar b_{sr}$ ($r \in [1 : k]$) of the matrix $\bar B$ by the rule $\bar b_{sr} = b_{2ir}b_{2jr}$.

Obviously, the elements of the matrices obtained are continuous. Performing standard matrix operations, it is easy to verify that system (2.4) is rewritten in the form

$$\dot d(t) = \bar A(t)d(t) + \bar B(t)v(t), \quad t \in T = [t_0, \vartheta], \quad d(t_0) = d_0 \in \mathcal{D}_0. \tag{2.6}$$

The initial state d_0 is obtained by D_0; the notation for the set $\mathcal{D}_0$ is the same. The multiplication of the diagonal matrices $U_2(t)U_2^*(t)$ results in the appearance of the control vector $v(t) = (u_{21}^2(t), u_{22}^2(t), \dots, u_{2k}^2(t))$, whose elements take values from some convex compact set $S_v \in \mathbb{R}^k$ for all $t \in T$.

The signal on the trajectory of equation (2.6) is denoted by $y_d(t) = Q_d(t)d(t)$; its estimate formed by the information on y^r, $r \in [1:N]$, by $y_d^N(t)$. The latter is constructed as follows:

$$\frac{1}{N-1}\sum_{r=1}^{N}(y^r(t) - y_m^N(t))(y^r(t) - y_m^N(t))^*$$

$$= Q(t)\frac{1}{N-1}\sum_{r=1}^{N}(x^r(t) - m^N(t))(x^r(t) - m^N(t))^* Q^*(t) = Q(t)D^N(t)Q^*(t), \tag{2.7}$$

where $D^N(t) = \{d_{ij}^N(t)\}$, $i,j \in [1:n]$ is the standard estimate of the covariance matrix $D(t)$ for an unknown (estimated by $m^N(t)$) mathematical expectation $m(t)$. Symmetric $q \times q$-matrix (2.7) can be represented in the form

$$Q(t)D^N(t)Q^*(t) = \begin{pmatrix} \sum\limits_{r=1}^{n}\sum\limits_{p=1}^{n} q_{1r}q_{1p}d_{pr}^N & \sum\limits_{r=1}^{n}\sum\limits_{p=1}^{n} q_{2r}q_{1p}d_{pr}^N & \cdots & \sum\limits_{r=1}^{n}\sum\limits_{p=1}^{n} q_{qr}q_{1p}d_{pr}^N \\ \sum\limits_{r=1}^{n}\sum\limits_{p=1}^{n} q_{1r}q_{2p}d_{pr}^N & \sum\limits_{r=1}^{n}\sum\limits_{p=1}^{n} q_{2r}q_{2p}d_{pr}^N & \cdots & \sum\limits_{r=1}^{n}\sum\limits_{p=1}^{n} q_{qr}q_{2p}d_{pr}^N \\ \cdots & \cdots & \cdots & \cdots \\ \sum\limits_{r=1}^{n}\sum\limits_{p=1}^{n} q_{1r}q_{qp}d_{pr}^N & \sum\limits_{r=1}^{n}\sum\limits_{p=1}^{n} q_{2r}q_{qp}d_{pr}^N & \cdots & \sum\limits_{r=1}^{n}\sum\limits_{p=1}^{n} q_{qr}q_{qp}d_{pr}^N \end{pmatrix}.$$

Obviously, it is sufficient to consider $n_q = (q^2 + q)/2$ elements of this matrix. The element $(i_1 j_1)$, $i_1 \le j_1$, $i_1, j_1 \in [1:q]$, namely $\sum\limits_{r=1}^{n}\sum\limits_{p=1}^{n} q_{i_1 r}q_{j_1 p}d_{pr}^N$, defines the s_1th row ($s_1 = (q - i_1/2)(i_1 - 1) + j_1$ by analogy with (2.5)) in the formed relation $y_d^N(t) = Q_d(t)d^N(t)$, where $Q_d(t)$ is the continuous $n_q \times n_d$-matrix, $d^N(t)$ is the n_d-vector extracted from $D^N(t)$ by rule (2.5). Then, to write the element $Q_d[s_1, s_2]$, it is necessary to find, using s_1 and s_2, the indices i_1, j_1 ($i_1, j_1 \in [1:q]$, $i_1 \le j_1$, $(q - i_1/2)(i_1 - 1) + j_1 = s_1$) and i_2, j_2 ($i_2, j_2 \in [1:n]$, $i_2 \le j_2$, $(n - i_2/2)(i_2 - 1) + j_2 = s_2$), respectively. It is easy to verify that

$$Q_d[s_1, s_2] = \begin{cases} q_{i_1 i_2}q_{j_1 j_2}, & i_2 = j_2 \\ q_{i_1 i_2}q_{j_1 j_2} + q_{i_1 j_2}q_{j_1 i_2}, & i_2 \ne j_2 \end{cases}.$$

Thus, by matrix (2.7), we construct the estimate $y_d^N(t) = Q_d(t)d^N(t)$ of the signal $y_d(t) = Q_d(t)d(t)$. Obviously, for the finite set of times $\tau_i \in T$, $i \in [1:l]$, we have the relation of type (1.4):

$$P\big(\forall i \in [1:l] \ \ \|y_d^N(\tau_i) - y_d(\tau_i)\|_{\mathbb{R}^{n_q}} \le C_2 h(N)\big) = 1 - g(N), \tag{2.8}$$

where the constant C_2 can be written explicitly. We denote the target set for the trajectory of equation (2.6) by the prior symbol $\mathcal{D}$.

Thus, original problem 1 of guaranteed closed-loop ε-guidance for the SDE can be reformulated as follows.

Problem 2. For an arbitrary small (in advance specified) $\varepsilon > 0$, it is required to choose controls $u_1(\cdot)$ in equation (2.1) and $v(\cdot)$ in equation (2.6) such that the trajectories of (2.1) and (2.6) reach the ε-neighborhoods of the target sets $\mathcal{M}$ and $\mathcal{D}$ at the terminal time ϑ, whatever the initial states m_0 from the set $\mathcal{M}_0$ and d_0 from the set $\mathcal{D}_0$. It is important that the probability of the desired event should be close to 1.

The required controls are formed through the estimates of the signals y_m and y_d satisfying relations (2.3) and (2.8); actually, these controls define the control in SDE (1.1). The dependence of the number N of measurable trajectories (necessary for the estimation) on the value ε is given below. The next theorem follows from the procedure of constructing problem 2.

Theorem 1. *Problem* 1 *and problem* 2 *are equivalent.*

Thus, to solve problem 1, one should establish some conditions of consistent solvability of the problems of ε-guidance for ODEs (2.1) and (2.6), i.e., solvability conditions for problem 2, and should find the form of concordance of parameters N and ε as well.

3. The method of open-loop control packages: a brief review of results for ODE

Let us present briefly the approach by A.V. Kryazhimskii and Yu.S. Osipov to solving the problem of closed-loop guidance under incomplete information for a linear ODE [5–7].

Consider a dynamical control system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + f(t), \quad t \in T = [t_0, \vartheta], \quad x(t_0) = x_0 \in X_0, \tag{3.1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in P \subset \mathbb{R}^m$ (P is a convex compact set); $A(\cdot)$, $B(\cdot)$, $f(\cdot)$ are continuous matrix function of dimension $n \times n$, $n \times m$, and $n \times 1$, respectively; X_0 is a finite set of possible initial states. The real initial state of the system is assumed to be unknown. A convex closed target set $\mathcal{M} \subset \mathbb{R}^n$ and a continuous matrix observation function $Q(t)$ of dimension $q \times n$ are given.

The problem of guaranteed closed-loop ε-guidance consists in forming by the signal $y(t) = Q(t)x(t)$ a control guaranteeing that the system's state $x(\vartheta)$ reaches an arbitrarily small ε-neighborhood of the target set $\mathcal{M}$ ($\varepsilon > 0$ is in advance specified) at the terminal time ϑ. The solution of the problem is sought in the class of closed-loop control strategies with memory. The correction of the values of a control $u(\cdot)$ is possible at in advance specified times. In [6], the equivalence of the formulated problem of closed-loop control to the so-called problem of package guidance is established; the terminology of the latter problem and conditions of its solvability are discussed in [7].

Consider the homogeneous system

$$\dot{x}(t) = A(t)x(t), \quad t \in T = [t_0, \vartheta], \quad x(t_0) = x_0 \in X_0;$$

its fundamental matrix is denoted by $F(\cdot,\cdot)$. For any admissible initial state $x_0 \in X_0$, the *homogeneous signal* corresponding to x_0 is the function $g_{x_0}(t) = Q(t)F(t,t_0)x_0$, $t \in [t_0, \vartheta]$. The set of all admissible initial states x_0 corresponding to a homogeneous signal $g(\cdot)$ till a time τ is denoted by $X_0(\tau|g(\cdot)) = \{x_0 \in X_0 \colon g_{x_0}(\cdot)|_{[t_0,\tau]} = g(\cdot)|_{[t_0,\tau]}\}$, where $g(\cdot)|_{[t_0,\tau]}$ is the restriction of the homogeneous signal $g(\cdot)$ onto the interval $[t_0, \tau]$.

A family $(u_{x_0}(\cdot))_{x_0 \in X_0}$ of open-loop controls is called an *open-loop control package* if it satisfies the condition of nonanticipation: for any homogeneous signal $g(\cdot)$, time $\tau \in (t_0, \vartheta]$, and admissible initial states x_0', $x_0'' \in X_0(\tau|g(\cdot))$, the equality $u_{x_0'}(t) = u_{x_0''}(t)$ holds for all $t \in [t_0, \tau]$. An open-loop control package $(u_{x_0}(\cdot))_{x_0 \in X_0}$ is guiding if, for any $x_0 \in X_0$, the motion from x_0 corresponding to $u_{x_0}(\cdot)$ takes a value exactly in the target set $\mathcal{M}$ at the time ϑ. If there exists a guiding open-loop package, we say that the idealized problem of package guidance [5,6] corresponding to the original problem of guaranteed closed-loop is solvable. Note that it is assumed in these problems that the signal is known exactly.

Let G be the set of all homogeneous signals (their number does not exceed the number of elements of the set X_0). We introduce the set $G_0(g(\cdot))$ of all homogeneous signals coinciding with $g(\cdot)$ in a right-sided neighborhood of the initial time t_0. The first splitting moment of the homogeneous signal $g(\cdot)$ is the time

$$\tau_1(g(\cdot)) = \max\Big\{\tau \in [t_0, \vartheta] \colon \max_{g'(\cdot) \in G_0(g(\cdot))} \max_{t \in [t_0,\tau]} \big\|g'(t) - g(t)\big\|_{\mathbb{R}^q} = 0\Big\}.$$

If $\tau_1(g(\cdot)) < \vartheta$, then, by analogy with $G_0(g(\cdot))$, we introduce the set $G_1(g(\cdot))$ of all homogeneous signals from $G_0(g(\cdot))$ coinciding with $g(\cdot)$ in a right-sided neighborhood of the splitting moment $\tau_1(g(\cdot))$. Actually, the difference $G_0(g(\cdot))\backslash G_1(g(\cdot))$ informs on the number of the signals from $G_0(g(\cdot))$ noncoincident with $g(\cdot)$ after $\tau_1(g(\cdot))$. By analogy with $\tau_1(g(\cdot))$, we define the second splitting moment of the homogeneous signal $g(\cdot)$ and so on. Finally, we introduce the set of all the splitting moments of the homogeneous signal $g(\cdot)$: $T(g(\cdot)) = \{\tau_j(g(\cdot))\colon j = 1, \dots, k_g\}$, $k_g \geq 1$, $\tau_{k_g}(g(\cdot)) = \vartheta$. Then, we consider the set (ascending ordering) of all the splitting moments of all the homogeneous signals (in [7], it is shown that these moments are possible switching moments for the "ideal" guiding open-loop control): $T = \bigcup_{g(\cdot)\in G} T(g(\cdot))$, $T = \{\tau_1, \dots, \tau_K\}$, $K \leq \sum_{g(\cdot)\in G} k_{g(\cdot)}$ is the number of elements of the set T. Obviously, the sets $T(g(\cdot))$ and T are finite due to the finiteness of the sets X_0 and G. For any $k = 1, \dots, K$, the set $\mathcal{X}_0(\tau_k) = \{X_0(\tau_k|g(\cdot))\colon g(\cdot) \in G\}$ is called the cluster position at the time τ_k, whereas each of its elements X_{0k} is called the cluster of initial states at this moment.

The constructions described are used for the characterization of open-loop control packages. Further, in [7], by the introduction of an auxiliary extended open-loop guidance problem for the system consisting of copies of system (3.1) parameterized by the admissible initial states, a criterion for the solvability of the original problem based on solving a finite-dimensional optimization problem is obtained. Let us formulate the main result of [7], as far as possible without using the terminology of the extended problem.

Theorem 2 [7, Theorem 2]. *The problem of package guidance for system* (3.1) *is solvable if and only if*

$$\sup_{(l_{x_0})_{x_0\in X_0}\in\mathcal{S}} \gamma((l_{x_0})_{x_0\in X_0}) \leq 0,$$

$$\gamma((l_{x_0})_{x_0\in X_0}) = \sum_{x_0\in X_0} \langle l_{x_0}, F(\vartheta, t_0)x_0\rangle + \sum_{k=1}^{K} \int\limits_{\tau_{k-1}}^{\tau_k} \sum_{X_{0k}\in\mathcal{X}_0(\tau_k)} \rho^-\Big(\sum_{x_0\in X_{0k}} B^*(s)F^*(\vartheta, s)l_{x_0}\big|P\Big)\, ds$$

$$+ \int\limits_{t_0}^{\vartheta} \Big\langle \sum_{x_0\in X_0} l_{x_0}, F(\vartheta, s)f(s)\Big\rangle\, ds - \sum_{x_0\in X_0} \rho^+(l_{x_0}|\mathcal{M}). \tag{3.2}$$

Here, $(l_{x_0})_{x_0\in X_0}$ is a family of vectors from $\mathbb{R}^n$ parameterized by the admissible initial states (the number of vectors coincides with the number of elements in X_0); $\mathcal{S}$ is the set of families $(l_{x_0})_{x_0\in X_0}$ such that $\sum_{x_0\in X_0} \|l_{x_0}\|^2_{\mathbb{R}^n} = 1$; $\langle\cdot,\cdot\rangle$ is the scalar product in the corresponding finite-dimensional Euclidean space; $\rho^-(\cdot|P)$ is the lower support function of the set P, and $\rho^+(\cdot|\mathcal{M})$ is the upper support function of the set $\mathcal{M}$.

4. Properties of the statistical estimates

Lemma 1. *For a finite set of some specified times $\tau_i \in T$, $i \in [1:l]$, the standard estimates m_i^N of the mathematical expectation $m(\tau_i)$ and D_i^N of the covariance matrix $D(\tau_i)$ constructed through N ($N > 1$) realizations $x^1(\tau_i), x^2(\tau_i), \dots, x^N(\tau_i)$ of the random variables $x(\tau_i)$ by the following rules* [12]:

$$m_i^N = \frac{1}{N}\sum_{r=1}^{N} x^r(\tau_i), \tag{4.1}$$

$$D_i^N = \frac{1}{N-1}\sum_{r=1}^{N} (x^r(\tau_i) - m_i^N)(x^r(\tau_i) - m_i^N)^*, \tag{4.2}$$

provide the validness of relation (1.4) (*consequently,* (2.3) *and* (2.8)).

P r o o f. We present the proof of the lemma for case $n = 1$ (the argument for $n > 1$ is similar; however, the formulas are too cumbersome). In this case, the scalar variable $x(\tau_i)$ is normally distributed with the mathematical expectation $m(\tau_i)$ and dispersion $D(\tau_i)$. Consider estimate (4.1). Let us show that it has the following property:

$$P\left(\forall i \in [1:l]\ \left|m_i^N - m(\tau_i)\right| \le h_m(N)\right) = 1 - g_m(N); \quad h_m(N),\, g_m(N) \to 0 \text{ as } N \to \infty. \tag{4.3}$$

First, we prove that

$$\forall i \in [1:l]\ P\left(\left|m_i^N - m(\tau_i)\right| \le h_m(N)\right) = 1 - f_m(N); \quad f_m(N) \to 0 \text{ as } N \to \infty. \tag{4.4}$$

As is known [12], the random variable $\xi = (m_i^N - m(\tau_i))\sqrt{N/D_i^N}$ has the t-distribution (Student's distribution) with $N-1$ degrees of freedom, which is close to the standard normal law $N(0,\,1)$ for $N > 30$. Our aim is to modify the procedure of constructing the estimate of the mathematical expectation in case of unknown dispersion in such a way that relation (4.4) is fulfilled.

Consider the following expression for $0 < \gamma < 1/2$:

$$P\Big(-N^\gamma \le (m_i^N - m(\tau_i))\sqrt{N/D_i^N} \le N^\gamma\Big) = P\Big(-N^\gamma\sqrt{D_i^N/N} \le m_i^N - m(\tau_i) \le N^\gamma\sqrt{D_i^N/N}\Big)$$
$$= P\Big(|m_i^N - m(\tau_i)| \le N^\gamma\sqrt{D_i^N/N}\Big) = 1 - f_m(N).$$

We can assume that $h_m(N) = C_1 N^{\gamma-1/2}$ in (4.4) (here and below, by C_i we denote auxiliary constants, which can be written explicitly). On the other hand,

$$P\Big(-N^\gamma \le (m_i^N - m(\tau_i))\sqrt{N/D_i^N} \le N^\gamma\Big) = 2(F_{t,N-1}(N^\gamma) - F_{t,N-1}(0)) = 2F_{t,N-1}(N^\gamma) - 1$$
$$= 2\big(F_{t,N-1}(N^\gamma) - \Phi(N^\gamma\sqrt{1-2/N}\,)\big) + 2\Phi(N^\gamma\sqrt{1-2/N}\,) - 1$$
$$\ge 2\Phi(N^\gamma\sqrt{1-2/N}\,) - 1 - 2\left|F_{t,N-1}(N^\gamma) - \Phi(N^\gamma\sqrt{1-2/N}\,)\right|$$
$$= 1 - 2\big(1 - \Phi(N^\gamma\sqrt{1-2/N}\,)\big) - 2\left|F_{t,N-1}(N^\gamma) - \Phi(N^\gamma\sqrt{1-2/N}\,)\right|.$$

Here, $F_{t,N-1}(x)$ is the probability function of the t-distribution with $N-1$ degrees of freedom, $F_{t,N-1}(0) = 1/2$; $\Phi(x)$ is the function of the normal distribution, $\Phi(x) = \displaystyle\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}\, e^{-y^2/2}\, dy$. Using the inequality describing the closedness of the functions $F_{t,N-1}(x)$ and $\Phi(x)$ for large N in the form [13]

$$\left|F_{t,N-1}(x) - \Phi(x\sqrt{1-2/N}\,)\right| \le C_2/N, \tag{4.5}$$

as well as the asymptotics of the normal distribution as $x \to \infty$ in the form [12]

$$1 - \Phi(x) = \frac{1}{x\sqrt{2\pi}}\, e^{-x^2/2} + o\Big(\frac{e^{-x^2/2}}{x}\Big), \tag{4.6}$$

we obtain

$$P\big(-N^\gamma \le (m_i^N - m(\tau_i))\sqrt{N/D_i^N} \le N^\gamma\big) \ge 1 - 2C_2/N - C_3 e^{-N^{2\gamma}/2}/N^\gamma + o\big(e^{-N^{2\gamma}/2}/N^\gamma\big).$$

Since the asymptotics of the normal distribution (see (4.6)) is suppressed by the error of the normal approximation (see (4.5)), we can assume (see (4.4)) that $f_m(N) \le C_4/N$. Relation (4.4) holds. Denote

$$A_{mi} = \left\{\left|m_i^N - m(\tau_i)\right| \le h_m(N)\right\}, \quad P(A_{mi}) = 1 - f_m(N) \quad \forall i \in [1:l].$$

Then,

$$P\left(\forall i \in [1:l]\ \ \left|m_i^N - m(\tau_i)\right| \le h_m(N)\right) = P\left(A_{m1}A_{m2}\ldots A_{ml}\right)$$

$$= P(A_{m1})P(A_{m2}|A_{m1})P(A_{m3}|A_{m1}A_{m2})\dots P\Big(A_{ml}\,|\prod_{k=1}^{l-1}A_{mk}\Big) \ge (1-f_m(N))^l \ge (1-l f_m(N))\,.$$

The last but one inequality is based on the assumption that all the conditional probabilities are not less than the corresponding unconditional ones due to the continuity of trajectories of the process under consideration; the last inequality follows from the smallness of $f_m(N)$. Obviously, we can assume that $g_m(N) = l f_m(N) = lC_4/N$ in (4.3). Thus, it holds that

$$P\left(\forall i \in [1:l] \quad \left|m_i^N - m(\tau_i)\right| \le C_1 N^{\gamma-1/2}\right) = 1 - lC_4/N. \tag{4.7}$$

Now, pass to the proof of the following property of estimate (4.2):

$$P\left(\forall i \in [1:l] \quad \left|D_i^N - D(\tau_i)\right| \le h_d(N)\right) = 1 - g_d(N); \quad h_d(N), g_d(N) \to 0 \quad \text{as } N \to \infty. \tag{4.8}$$

First, we show that the relation

$$\forall i \in [1:l] \quad P\left(\left|D_i^N - D(\tau_i)\right| \le h_d(N)\right) \;=\; 1 - f_d(N), \quad f_d(N) \to 0 \text{ as } N \to \infty \tag{4.9}$$

is valid.

As is known [12], the random variable $\xi = (N-1)D_i^N/D(\tau_i)$ has the distribution χ^2 with $N-1$ degrees of freedom and the distribution of the variable $\sqrt{2\xi} - \sqrt{2N-3}$ is close to the standard normal law $N(0,\,1)$ for $N > 30$. Our aim is to modify the procedure of constructing the estimate of the dispersion in case of unknown mathematical expectation in such a way that relation (4.9) is fulfilled. Let us choose the numerical parameters:

$$\chi_1^2 = \left(\sqrt{2N-3} - N^\gamma\right)^2/2 = N^{2\gamma}/2 - N^\gamma\sqrt{2N-3} + N - 3/2,$$

$$\chi_2^2 = \left(\sqrt{2N-3} + N^\gamma\right)^2/2 = N^{2\gamma}/2 + N^\gamma\sqrt{2N-3} + N - 3/2, \quad 0 < \gamma < 1/2.$$

For such a choice, we obtain $\chi_1^2/N \to 1$ and $\chi_2^2/N \to 1$ as $N \to \infty$. We have

$$P\left(\chi_1^2 \le (N-1)D_i^N/D(\tau_i) \le \chi_2^2\right)$$

$$= P\left(\left(\chi_1^2/(N-1) - 1\right) D(\tau_i) \le D_i^N - D(\tau_i) \le \left(\chi_2^2/(N-1) - 1\right) D(\tau_i)\right) = 1 - f_d(N).$$

Then, by virtue of the definition of χ_1^2 and χ_2^2, we get

$$P\left(\left|D_i^N - D(\tau_i)\right| \le h_d(N)\right) = 1 - f_d(N),$$

where $h_d(N) = \max\limits_{i\in[1:l]} \left\{\left|\chi_1^2/(N-1) - 1\right| D(\tau_i),\, \left|\chi_2^2/(N-1) - 1\right| D(\tau_i)\right\} = C_5 N^{\gamma-1/2}$. On the other hand,

$$P\left(\chi_1^2 \le (N-1)D_i^N/D(\tau_i) \le \chi_2^2\right)$$

$$= P\left(\left(\sqrt{2N-3} - N^\gamma\right)^2/2 \le (N-1)D_i^N/D(\tau_i) \le \left(\sqrt{2N-3} + N^\gamma\right)^2/2\right)$$

$$= P\left(-N^\gamma \le \sqrt{2(N-1)D_i^N/D(\tau_i)} - \sqrt{2N-3} \le N^\gamma\right)$$

$$= P\left(\sqrt{2(N-1)D_i^N/D(\tau_i)} - \sqrt{2N-3} \le N^\gamma\right) - P\left(\sqrt{2(N-1)D_i^N/D(\tau_i)} - \sqrt{2N-3} \le -N^\gamma\right)$$

$$= F_{i,N-1}(N^\gamma) - F_{i,N-1}(-N^\gamma),$$

where $F_{i,N-1}(x)$ is the distribution function of the variable $\sqrt{2(N-1)D_i^N/D(\tau_i)}-\sqrt{2N-3}$. Using the improved Berry–Esseen inequality [12] describing the closedness of the function $F_{i,N-1}(x)$ and the function of the normal distribution in the form

$$|F_{i,N-1}(x)-\Phi(x)| < C_6/(2\sqrt{N-1}(1+|x|^3)), \tag{4.10}$$

as well as the specified above asymptotics of the normal distribution (see (4.6)), we derive

$$F_{i,N-1}(N^\gamma)-F_{i,N-1}(-N^\gamma) = F_{i,N-1}(N^\gamma)-\Phi(N^\gamma)+\Phi(-N^\gamma)-F_{i,N-1}(-N^\gamma)+\Phi(N^\gamma)-\Phi(-N^\gamma)$$
$$\geq 2\Phi(N^\gamma)-1-|F_{i,N-1}(N^\gamma)-\Phi(N^\gamma)|-|F_{i,N-1}(-N^\gamma)-\Phi(-N^\gamma)| > 2\Phi(N^\gamma)-1-C_6/(\sqrt{N-1}(1+N^{3\gamma}))$$
$$= 1-2e^{-N^{2\gamma}/2}/(N^\gamma\sqrt{2\pi})-C_6/(\sqrt{N-1}(1+N^{3\gamma}))+o\left(e^{-N^{2\gamma}/2}/N^\gamma\right).$$

Since the asymptotics of the normal distribution (see (4.6)) is suppressed by the error of the normal approximation (see (4.10)), we can assume (see (4.9)) that $f_d(N)\leq C_7N^{-1/2-3\gamma}$. Relation (4.9) holds. Introducing the notation

$$A_{di}=\left\{\left|D_i^N-D(\tau_i)\right|\leq h_d(N)\right\},\ P(A_{di})\ =\ 1-f_d(N),\ \forall i\in[1:l],$$

by analogy with the events A_{mi}, we conclude that

$$P\left(\forall i\in[1:l]\quad \left|D_i^N-D(\tau_i)\right|\leq h_d(N)\right) =\ P\left(A_{d1}A_{d2}\dots A_{dl}\right)\geq\left(1-lf_d(N)\right).$$

Then, obviously, we can assume that $g_d(N)=lf_d(N)=lC_7N^{-1/2-3\gamma}$ in (4.8). Thus,

$$P\left(\forall i\in[1:l]\quad \left|D_i^N-D(\tau_i)\right|\leq C_5N^{\gamma-1/2}\right)=1-lC_7N^{-1/2-3\gamma}. \tag{4.11}$$

Comparing (4.3) and (4.8), we conclude that, in relations (1.4), (2.3), and (2.8), we can consider the same parameters, namely,

$$h(N)=C_hN^{\gamma-1/2},\quad g(N)=C_gN^{\max\{-1,-1/2-3\gamma\}}, \tag{4.12}$$

where C_h and C_g are constants, which can be written explicitly. For example, choosing the value $\gamma=1/6$, we obtain $h(N)=C_hN^{-1/3}$, $g(N)=C_g/N$. For $\gamma\to+0$, $h(N)$ and $g(N)$ have the power exponents of the value $1/N$, which are asymptotically equal to 1/2. □

5. Criterion for the solvability of problems 1 and 2. Concordance conditions for parameters

In addition, we define several notions for ODEs (2.1) and (2.6) based on the information from Section 3. Let $G^1=\{g^1(\cdot)\}$ and $G^2=\{g^2(\cdot)\}$ be the sets of all homogeneous signals for (2.1) and (2.6), respectively. The sets of all splitting moments of all homogeneous signals for (2.1) and (2.6) are denoted by $T^1=\{\tau_1^1,\dots,\tau_{K_1}^1\}$ and $T^2=\{\tau_1^2,\dots,\tau_{K_2}^2\}$; the cluster positions and clusters of initial states at the times τ_k^1 and τ_k^2, by $\mathcal{M}_0(\tau_k^1)$ and M_{0k}, $\mathcal{D}_0(\tau_k^2)$ and D_{0k}, respectively. Recall that the last splitting moment always coincides, by definition, with ϑ: $\tau_{K_1}^1=\tau_{K_2}^2=\vartheta$. To simplify the presentation, we assume $\tau_0^1=\tau_0^2=t_0$ (t_0 is not a splitting moment). Let us introduce the sets of pairs of homogeneous signals splitted (in the sense of definitions from Section 3) at the moments τ_k^1, $k\in[0:K_1-1]$ and τ_k^2, $k\in[0:K_2-1]$: $G_k^{1*}=\{(g_i^1(\tau_k^1),g_j^1(\tau_k^1))\}$, $G_k^{2*}=\{(g_i^2(\tau_k^2),g_j^2(\tau_k^2))\}$, $i\neq j$. A moment from the interval $(\tau_k^1,\tau_k^1+C\varepsilon]$ ($\tau_k^1+C\varepsilon<\tau_{k+1}^1$, the sense of the constant C is clarified below), at which all the pairs from G_k^{1*} are distinguishable, is denoted by τ_k^{1*} and is called a distinguishing moment for all the signals splitted at the time τ_k^1. Similarly, we define a distinguishing moment τ_k^{2*}. Obviously, we can believe that, for all $\tau_k^1\in T^1$, $k\in[0:K_1-1]$ and $\tau_k^2\in T^2$, $k\in[0:K_2-1]$, the corresponding moments τ_k^{1*} and τ_k^{2*} are defined uniquely; at these moments, the signal's values in all the pairs from G_k^{1*} and G_k^{2*} are different. Indeed, if we assume that, in some time after splitting, the signals coincide once again (that is stipulated by the form of the observation matrix Q), then the finiteness of the number of the signals, their continuity, and the definition of splitting moments imply the existence of a minimum moment from $(\tau_k^1, \tau_k^1 + C\varepsilon]$ $((\tau_k^2, \tau_k^2 + C\varepsilon])$ such that the signals in any pair from G_k^{1*} (G_k^{2*}) do not coincide till this moment. The set of all such distinguishing moments for (2.1) and (2.6)

$$T^* = T^{1*} \cup T^{2*}, \quad T^{1*} = \{\tau_0^{1*}, \dots, \tau_{K_1-1}^{1*}\}, \quad T^{2*} = \{\tau_0^{2*}, \dots, \tau_{K_2-1}^{2*}\}, \tag{5.1}$$

determines both the aforesaid set of l $(l < K_1 + K_2)$ times, at which the N trajectories of the original process are measured, and the set of times, which are possible for switching the closed-loop control. Note that the case $T^* = \varnothing$ takes place only if all the signals coincide on the whole interval $[t_0, \vartheta]$.

Let us formulate according to (3.2) the solvability condition for the guidance problem for (2.1):

$$\sup_{(l_{m_0})_{m_0\in\mathcal{M}_0}\in\mathcal{S}_1} \gamma_1((l_{m_0})_{m_0\in\mathcal{M}_0}) \le 0,$$

$$\gamma_1((l_{m_0})_{m_0\in\mathcal{M}_0}) = \sum_{m_0\in\mathcal{M}_0} \langle l_{m_0}, F_1(\vartheta, t_0)m_0\rangle + \sum_{k=1}^{K_1} \int_{\tau_{k-1}^1}^{\tau_k^1} \sum_{M_{0k}\in\mathcal{M}_0(\tau_k^1)} \rho^-\Big(\sum_{m_0\in M_{0k}} B_1^*(s)F_1^*(\vartheta, s)l_{m_0}\big|S_{u1}\Big)ds$$

$$+ \int_{t_0}^{\vartheta} \Big\langle \sum_{m_0\in\mathcal{M}_0} l_{m_0},\, F_1(\vartheta, s)f(s)\Big\rangle\, ds - \sum_{m_0\in\mathcal{M}_0} \rho^+(l_{m_0}|\mathcal{M}), \tag{5.2}$$

and the solvability condition for the guidance problem for (2.6):

$$\sup_{(l_{d_0})_{d_0\in\mathcal{D}_0}\in\mathcal{S}_2} \gamma_2((l_{d_0})_{d_0\in\mathcal{D}_0}) \le 0, \quad \gamma_2((l_{d_0})_{d_0\in\mathcal{D}_0}) = \sum_{d_0\in\mathcal{D}_0} \langle l_{d_0}, F_2(\vartheta, t_0)d_0\rangle$$

$$+ \sum_{k=1}^{K_2} \int_{\tau_{k-1}^2}^{\tau_k^2} \sum_{D_{0k}\in\mathcal{D}_0(\tau_k^2)} \rho^-\Big(\sum_{d_0\in D_{0k}} \bar{B}^*(s)F_2^*(\vartheta, s)l_{d_0}\big|S_v\Big)\, ds - \sum_{d_0\in\mathcal{D}_0} \rho^+(l_{d_0}|\mathcal{D}). \tag{5.3}$$

Here, $(l_{m_0})_{m_0\in\mathcal{M}_0}$, $(l_{d_0})_{d_0\in\mathcal{D}_0}$ are families of vectors parameterized by the corresponding initial states; $\mathcal{S}_1$, $\mathcal{S}_2$ are the set of families $(l_{m_0})_{m_0\in\mathcal{M}_0}$, $\sum_{m_0\in\mathcal{M}_0} \|l_{m_0}\|^2_{\mathbb{R}^n} = 1$ and $(l_{d_0})_{d_0\in\mathcal{D}_0}$ $\sum_{d_0\in\mathcal{D}_0} \|l_{d_0}\|^2_{\mathbb{R}^{n_d}} = 1$; $F_1(\cdot,\cdot)$ and $F_2(\cdot,\cdot)$ are the fundamental matrices of systems (2.1) and (2.6). Let us prove the following statement actually being the main result of the paper.

Theorem 3. *Let conditions* (5.2) *and* (5.3) *be fulfilled, let the information on N trajectories of* SDE (1.1) *be received at the times belonging to the set T^** (5.1)*, let the constants C_h, C_g, and γ be taken from the Lemma 1 (see* (4.12)*) and*

$$N > (2C_h/\rho(\varepsilon))^{2/(1-2\gamma)}, \tag{5.4}$$

$$\rho(\varepsilon) = \min\Big\{\min_{\tau_k^{1*}\in T^{1*},\,(g_i^1, g_j^1)\in G_k^{1*}} \|g_i^1(\tau_k^{1*}) - g_j^1(\tau_k^{1*})\|_{\mathbb{R}^q},\ \min_{\tau_k^{2*}\in T^{2*},\,(g_i^2, g_j^2)\in G_k^{2*}} \|g_i^2(\tau_k^{2*}) - g_j^2(\tau_k^{2*})\|_{\mathbb{R}^{n_q}}\Big\}.$$

Then, problem 1 *of ε-guidance is solvable with the probability $1 - C_g N^{\max\{-1,-1/2-3\gamma\}}$ and there exists an ε-guiding control in equation (1.1) based on open-loop control packages for systems (2.1) and (2.6).*

P r o o f. Conditions (5.2) and (5.3) provide the solvability of the idealized package guidance problems corresponding to the problems of guaranteed closed-loop control composing problem 2. Thus, there exist open-loop control packages $(u^*_{1m_0}(\cdot))_{m_0\in\mathcal{M}_0}$ and $(v^*_{d_0}(\cdot))_{d_0\in\mathcal{D}_0}$ parameterized by the initial states and solving the idealized package guidance problems for systems (2.1) and (2.6). The solution of the problems of guaranteed ε-guidance implies the possibility of synthesis (by $u^*_{1m_0}$ and $v^*_{d_0}$) of the corresponding closed-loop controls bringing the trajectories to ε-neighborhoods of the target sets $\mathcal{M}$ and $\mathcal{D}$, respectively. The input information is estimates (2.2) and (2.7) constructed by realizations of signal (1.3). Thereby, according to Theorem 1, we can design an ε-guiding control solving problem 1 for system (1.1).

Let make the aforesaid more precise using system (2.1) as an example. This system starts at the time t_0 from some unknown initial state $\widehat{m}_0 \in \mathcal{M}_0$; the guiding open-loop control for this state is denoted by $u^*_{1\widehat{m}_0}(\cdot)$. Let T^{1*} contain at least one distinguishing moment. We construct the closed-loop control $\hat{u}^*_1(\cdot)$ as follows. Before the motion starts, we decide to apply to system (2.1) the test control $\hat{u}^*_1(t) = \hat{u}^*_{10} \in S_{u1}$, $t \in [t_0, \tau^{1*}_0)$. At the moment τ^{1*}_0 of initial distinguishing of the signals, we measure N trajectories of (1.1) and, using them, construct the estimate $y^N_m(\tau^{1*}_0)$ of the realized signal $y_m(\tau^{1*}_0) = Q_m(\tau^{1*}_0)m(\tau^{1*}_0)$. Using the Cauchy formula, we obtain the homogeneous signal

$$Q_m(\tau^{1*}_0)F_1(\tau^{1*}_0, t_0)\widehat{m}_0 = y_m(\tau^{1*}_0) - Q_m(\tau^{1*}_0)\int_{t_0}^{\tau^{1*}_0} F_1(\tau^{1*}_0, \tau)(B_1(\tau)\hat{u}^*_{10} + f(\tau))\,d\tau.$$

Now, basing on the estimate $y^N_m(\tau^{1*}_0)$, we find the cluster of initial states M_{0k} containing $\widehat{m}_0$. Taking into account (2.3) and (4.12), we have $\left\|y^N_m(\tau^{1*}_0) - y_m(\tau^{1*}_0)\right\|_{\mathbb{R}^q} \le h(N)$. Further, if

$$h(N) < \rho^1_0(\varepsilon)/2 = \min_{(g^1_i, g^1_j)\in G^{1*}_0} \|g^1_i(\tau^{1*}_0) - g^1_j(\tau^{1*}_0)\|_{\mathbb{R}^q}/2, \tag{5.5}$$

then the cluster $M_{01} = \{m_0 \in \mathcal{M}_0 : g_{m_0}(\cdot)|_{[t_0,\tau^1_1]} = g_{\widehat{m}_0}(\cdot)|_{[t_0,\tau^1_1]}\}$ is defined uniquely (since there is no any splitting of the homogeneous signal till the moment τ^1_1). Consequently, at every time $t \in [\tau^{1*}_0, \tau^{1*}_1)$, we apply the control $u^*_{1M_{01}}(t)$ from the package $(u^*_{1m_0}(\cdot))_{m_0\in\mathcal{M}_0}$ corresponding to the cluster M_{01}, i.e., the control $u^*_{1\widehat{m}_0}(t)$. A similar procedure is applied for all the splitting moments τ^{1*}_k of the signals. Obviously, the control $\hat{u}^*_1(t)$ constructed in such a way differs from the guiding open-loop control $u^*_{1\widehat{m}_0}(t)$ only on at most K_1 intervals, each of which has a length not more than $C\varepsilon$. Denote the solutions of (2.1) for the initial state $\widehat{m}_0$ corresponding to the controls $u^*_{1\widehat{m}_0}(t)$ and $\hat{u}^*_1(t)$ by $m^*(\cdot)$ and $\widehat{m}^*(\cdot)$, respectively. Note that $m^*(\vartheta) \in \mathcal{M}$. In view of the boundedness of all the functions and the control in (2.1), we estimate the value

$$\|\widehat{m}^*(\vartheta) - m^*(\vartheta)\|_{\mathbb{R}^n} = \left\|\int_{t_0}^{\vartheta} F_1(\vartheta,\tau)B_1(\tau)(\hat{u}^*_1(\tau) - u^*_{1\widehat{m}_0}(\tau))\,d\tau\right\|_{\mathbb{R}^n} \le \overline{K}_m C\varepsilon, \tag{5.6}$$

where the constant $\overline{K}_m$ can be written explicitly. Hence, choosing $C < 1/\overline{K}_m$, we guarantee $\|\widehat{m}^*(\vartheta) - m^*(\vartheta)\| < \varepsilon$, i.e., the value $\widehat{m}^*(\vartheta)$ reaches the ε-neighborhood of the target set $\mathcal{M}$.

For system (2.6), all the argument is similar. The attainability of the ε-neighborhood of the target set $\mathcal{D}$ by the corresponding trajectory is guaranteed by the choice of $C < 1/\overline{K}_d$ in the relation similar to (5.6). The form of the function $\rho(\varepsilon)$ (all the minimums in it exist in view of the finiteness of all the sets involved) is explained by the necessity to require estimates similar to (5.5) for any $\tau^*_k \in T^*$. Thus, we obtain $h(N) < \rho(\varepsilon)/2$. This and (4.12) imply relation (5.4) connecting N and ε. Finally, by (4.12), all the actions described above, including the ε-guidance of solutions (2.1) and (2.6), consequently, the solution of (1.1), hold with the probability $1 - C_g N^{\max\{-1,-1/2-3\gamma\}}$ converging to 1 as $N \to \infty$.

The theorem is proved. □

6. Illustrative example

Consider the linear SDE of the first order:

$$dx(t) = -x(t)dt + u_1(t)dt + u_2(t)d\xi(t), \quad t \in T = [0,2], \quad u_1, u_2 \in [0,1], \tag{6.1}$$

with the unknown initial state $x_0 \in X_0$, X_0 consists of four normally distributed random variables with numerical parameters (m_0, d_0), where the mathematical expectation $m_0 \in \mathcal{M}_0 = \{m_0^1, m_0^2\}$, $m_0^1 = (3-e)e$, $m_0^2 = e^2$, and the dispersion $d_0 \in \mathcal{D}_0 = \{d_0^1, d_0^2\}$, $d_0^1 = e^2/2$, $d_0^2 = e^4$. Incomplete observations have the following form:

$$y(t) = Q(t)x(t), \quad Q(t) = \begin{cases} 0, & t \in [0,1] \\ t-1, & t \in (1,2] \end{cases}. \tag{6.2}$$

Let us write ODEs for the mathematical expectation and dispersion, as well as the observed signals, using the formulas from Section 2:

$$\dot{m}(t) = -m(t) + u_1(t), \quad m(0) = m_0 \in \{m_0^1, m_0^2\}, \quad y_m(t) = Q(t)m(t); \tag{6.3}$$

$$\dot{d}(t) = -2d(t) + u_2^2(t), \quad d(0) = d_0 \in \{d_0^1, d_0^2\}, \quad y_d(t) = Q^2(t)d(t). \tag{6.4}$$

Let the target sets for m and d be as follows: $\mathcal{M} = [2/e,\ 1]$, $\mathcal{D} = [1/2,\ 1]$.

The problem consists in forming, for an arbitrary small (in advance specified) $\varepsilon > 0$, an open-loop control (u_1, u_2) guaranteeing, whatever the initial states $m_0 \in \mathcal{M}_0$ and $d_0 \in \mathcal{D}_0$, by the information on N trajectories of equation (6.1), the attainability (with a probability close to 1) of ε-neighborhoods of the target sets $\mathcal{M}$ and $\mathcal{D}$ by the the mathematical expectation $m(2)$ and the dispersion $d(2)$, respectively. Let us present the solutions of equations (6.3) and (6.4):

$$m(t) = e^{-t}m_0 + e^{-t}\int_0^t e^{\tau}u_1(\tau)d\tau, \quad d(t) = e^{-2t}d_0 + e^{-2t}\int_0^t e^{2\tau}u_2^2(\tau)d\tau.$$

Obviously, the splitting moments of the homogeneous signals for equations (6.3) and (6.4) coincide. The form of the function $Q(t)$ and the structure of the sets $\mathcal{M}_0$ and $\mathcal{D}_0$ imply that the number K of the splitting moments is equal to 2, $\tau_1 = 1$ is the first splitting moment, after which it is possible to distinguish the homogeneous signals corresponding to the different initial states, the terminal time $\tau_2 = 2$ is, by definition, the second splitting moment. Note that, since the guiding open-loop control package is presented below, the solvability criterion for the original problem (written by the formulas form Section 5), obviously, holds.

In the case when the controls u_1 and u_2^2 are in the form of piecewise constant functions

$$u_1(t) = \begin{cases} u_{[0,1]}, & t \in [0,1] \\ u_{(1,2]}, & t \in (1,2] \end{cases}, \quad u_2^2(t) = \begin{cases} v_{[0,1]}, & t \in [0,1] \\ v_{(1,2]}, & t \in (1,2] \end{cases},$$

we obtain

$$m(2) = e^{-2}m_0 + (e-1)e^{-1}(e^{-1}u_{[0,1]} + u_{(1,2]}), \quad d(2) = e^{-4}d_0 + (e^2-1)e^{-2}(e^{-2}v_{[0,1]} + v_{(1,2]})/2.$$

It follows from the form of $m(2)$ that the solution of equation (6.3), starting from the greater initial state $m_0^2 = e^2$, reaches (at $t = 2$) the set $\mathcal{M}$ (namely, its upper boundary $m = 1$) only under the action of zero control u_1 on the whole interval $[0,2]$, i.e., $u_{[0,1]} = u_{(1,2]} = 0$. At the same time, if the real initial state coincides with the smaller possible value $m_0^1 = (3-e)e$, then, after the necessary action of zero control till the splitting moment $t = 1$ of the homogeneous signal, only the choice of $u_{(1,2]} = 1$ forces the trajectory to reach the lower boundary $m = 2/e$ of the target set $\mathcal{M}$.

Obviously, for the set of parameters chosen in the example, the open-loop control of equation (6.3) guaranteeing the attainment of the trajectory to $\mathcal{M}$ at the time $t = 2$ is unique.

A similar argument is applicable to equation (6.4). Its solution, starting from the greater initial state $d_0^2 = e^4$, reaches (at $t = 2$) the set $\mathcal{D}$ (namely, its upper boundary $d = 2$) only under the action of zero control u_2^2 on the whole interval $[0, 2]$, i.e., $v_{[0,1]} = v_{(1,2]} = 0$. At the same time, if the real initial state coincides with the smaller possible value $d_0^1 = e^2/2$, then, after the necessary action of zero control till the splitting moment $t = 1$ of the homogeneous signal, only the choice of $v_{(1,2]} = 1$ forces the trajectory to reach the lower boundary $d = 1/2$ of the target set $\mathcal{D}$. The uniqueness of the open-loop control of equation (6.4) guaranteeing the attainment of the trajectory to $\mathcal{D}$ at the time $t = 2$ is also obvious.

We pass to constructing the guaranteeing closed-loop control using the guiding open-loop control package. Note that, in the given example, since the splitting moments of the homogeneous signals for equations (6.3) and (6.4) coincide (this may not be fulfilled in the general case), to find the real initial state of system (6.1), it is sufficient to perform measurements of N $(N > 1)$ trajectories $x^1(\tau_*), x^2(\tau_*), \ldots, x^N(\tau_*)$ of the original SDE at the unique distinguishing moment $\tau_* = 1 + \varepsilon$; the zero controls are fed onto equations (6.3) and (6.4) till this time. Using these measurements, we construct the estimates $y_m^N(\tau_*)$ and $y_d^N(\tau_*)$ of the signals $y_m(\tau_*)$ and $y_d(\tau_*)$ satisfying the relations like (2.3), (2.8), and (4.12):

$$y_m^N(\tau_*) = Q(\tau_*)m^N(\tau_*), \quad m^N(\tau_*) = \frac{1}{N}\sum_{r=1}^{N} x^r(\tau_*),$$

$$y_d^N(\tau_*) = Q^2(\tau_*)d^N(\tau_*), \quad d^N(\tau_*) = \frac{1}{N-1}\sum_{r=1}^{N} \left(x^r(\tau_*) - m^N(\tau_*)\right)^2,$$

$$P\left(\max\left\{\left|y_m^N(\tau_*) - y_m(\tau_*)\right|, \left|y_d^N(\tau_*) - y_d(\tau_*)\right|\right\} \le h(N)\right) = 1 - g(N).$$

Let us derive the condition providing the determination at the time τ_* (using the estimates $y_m^N(\tau_*)$ and $y_d^N(\tau_*)$) of the real initial states of equations (6.3) and (6.4) (respectively, m_0^1 or m_0^2 and d_0^1 or d_0^2) and, consequently, of the initial state of equation (6.1). Actually, taking into account that $u_1(t) = 0$, $u_2^2(t) = 0$, $t \in [0, 1+\varepsilon]$, and the form of the observation function Q, we should distinguish the values $y_m^1(\tau_*) = \varepsilon e^{-(1+\varepsilon)}m_0^1$ and $y_m^2(\tau_*) = \varepsilon e^{-(1+\varepsilon)}m_0^2$, as well as $y_d^1(\tau_*) = \varepsilon^2 e^{-2(1+\varepsilon)}d_0^1$ and $y_d^2(\tau_*) = \varepsilon^2 e^{-2(1+\varepsilon)}d_0^2$. Therefore, N must be such that

$$h(N) < \min\left\{(\varepsilon e^{-(1+\varepsilon)}|m_0^1 - m_0^2|)/2,\ (\varepsilon^2 e^{-2(1+\varepsilon)}|d_0^1 - d_0^2|)/2\right\}. \tag{6.5}$$

Then, only one of the inequalities $\left|y_m^N(\tau_*) - y_m^1(\tau_*)\right| \le h(N)$ or $\left|y_m^N(\tau_*) - y_m^2(\tau_*)\right| \le h(N)$ holds with the probability $1 - g(N)$; the same is valid for the inequalities $\left|y_d^N(\tau_*) - y_d^1(\tau_*)\right| \le h(N)$ or $\left|y_d^N(\tau_*) - y_d^2(\tau_*)\right| \le h(N)$. In case equation (6.3) starts from the initial state m_0^1, we decide to apply the control $u_1(t) = 1$ on the interval $(1 + \varepsilon, 2)$; otherwise (from the state m_0^2), the control $u_1(t) = 0$. In the first variant, in view of the time delay in switching the control to optimal, $m(2)$ takes the value not $2/e$ but $2/e - (e^{\varepsilon} - 1)/e$; i.e., for small ε, $m(2)$ reaches the ε-neighborhood of the set $\mathcal{M}$. In the second variant, as a result, we have exactly $m(2) = 1$. By analogy, we proceed with equation (6.4): if the real initial state is d_0^1, then we apply the control $u_2^2(t) = 1$ on the interval $(1+\varepsilon, 2)$; if d_0^2, then $u_2^2(t) = 0$. In the first case, $d(2)$ takes the value not $1/2$ but $1/2 - (e^{2\varepsilon} - 1)/2e^2$; i.e., for small ε, $d(2)$ reaches the ε-neighborhood of the set $\mathcal{D}$. In the second case, we have exactly $d(2) = 1$.

Thus, the closed-loop control method with incomplete information described above solves the original ε-guidance problem: it guarantees the attaintment of the solution of equation (6.3) to the ε-neighborhood of the target set $\mathcal{M}$ and the attaintment of the solution of equation (6.4) to the ε-neighborhood of the target set $\mathcal{D}$ at the time $t = 2$ with a probability close to 1. The computations by formulas (4.12) and (6.5) showed that $N = 10^3$ guarantees the guidance accuracy $\varepsilon = 0.1$ with a probability $P \ge 0.96$, whereas $N = 10^5$ guarantees $\varepsilon = 0.01$ with $P \ge 0.996$.

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