

SOME REMARKS ON ROUGH STATISTICAL Λ -CONVERGENCE OF ORDER α

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Abstract: The main purpose of this work is to define Rough Statistical Λ -Convergence of order α ($0 < \alpha \leq 1$) in normed linear spaces. We have proved some basic properties and also provided some examples to show that this method of convergence is more generalized than the rough statistical convergence. Further, we have shown the results related to statistically Λ -bounded sets of order α and sets of rough statistically Λ -convergent sequences of order α .

Keywords: Statistical convergence, Rough statistical convergence, Rough statistical limit points.

1. Introduction

In 1951, Fast [5] presented a new idea of convergence named as statistical convergence that is more generalized than the usual convergence for the sequences.

Definition 1 [5]. A sequence $x = \{x_m\}$ of numbers is said to be statistically convergent to ξ if for every $\epsilon > 0$ we have $\lim_{n \rightarrow \infty} |M(x, \epsilon)|/n = 0$, where $|M(x, \epsilon)|$ represents the order of the enclosed set $M(x, \epsilon) = \{m \leq n : |x_m - \xi| \geq \epsilon\}$.

This idea has interesting applications in the field of Fourier Analysis [1], Measure Theory [16], Approximation Theory [7] etc. It has been studied by many researchers for various types of sequences in different setups like locally convex spaces [10], probabilistic normed spaces [8], random normed spaces [3], intuitionistic fuzzy normed spaces [9] etc.

An interesting generalization of usual convergence named as rough convergence was introduced by Phu [19] for the sequences in finite dimensional normed linear spaces and later on introduced on infinite dimensional normed linear spaces [20]. He mainly worked on rough limits, roughness degree, rough continuity of linear operators and also introduced rough Cauchy sequences.

Definition 2 [19]. A sequence $x = \{x_m\}$ in a normed linear space $(\mathbb{X}, \|\cdot\|)$ is said to be rough convergent to $\xi \in \mathbb{X}$ if for every $\epsilon > 0$ there exists a non-negative number r and $m_0 \in \mathbb{N}$ such that $\|x_m - \xi\| < r + \epsilon$, for all $m \geq m_0$.

Aytar [2] extended the rough convergence to rough statistical convergence like usual convergence is extended to statistical convergence with the help of natural density.

Definition 3 [2]. A sequence $x = \{x_m\}$ in a normed linear space $(\mathbb{X}, \|\cdot\|)$ is said to be rough statistically convergent to $\xi \in \mathbb{X}$ if for every $\epsilon > 0$ there exists a non-negative number r such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{m \leq n : \|x_m - \xi\| \geq r + \epsilon\}| = 0,$$

where ξ is known as r -St-limit of sequence $x = \{x_m\}$.

Aytar [2] also defined the rough statistical bounded sequence along with the set of rough statistical limit points of a sequence. Further, some criterion associated with the convexity and closeness of the set of rough statistical limit points of a sequence was investigated.

Inspired by the work of Aytar [2], Maity [12] presented the concept of rough statistical convergence of order α ($0 < \alpha \leq 1$) in normed linear spaces and explained some important results for the set of rough statistical limit points of order α . The idea of pointwise rough statistical convergence and rough statistical Cauchy sequences for real valued functions was introduced in [11]. The concept of rough convergence has been defined for double sequences by Malik and Maity in [13] and after that the authors extended this idea in [14] and defined rough statistical convergence for double sequences in normed linear spaces.

This idea has motivated many authors to use the concepts of ideals also. Pal et al. [18] introduced rough I -convergence with the help of ideals of \mathbb{N} . Later, Malik et al. in [15] extended this concept of rough I -convergence to rough I -statistical convergence and described some topological properties of the set of all rough I -statistical limits of sequences in normed linear spaces. A lot of work has been done on rough convergence and its generalizations. More investigations and applications of rough convergence can be revealed as it is an active area of research.

In this paper, we are introducing the concept of rough statistical Λ -convergence of order α ($0 < \alpha \leq 1$) in the normed linear spaces.

2. Main results

In order to study the basic concept of rough statistical Λ -convergence, we first consider a sequence $\lambda = \{\lambda_j\}$ of real numbers such that $0 < \lambda_0 < \lambda_1 < \dots < \lambda_j < \dots$ and $\lambda_j \rightarrow \infty$ as $j \rightarrow \infty$. The concept of Λ -convergence for real sequences have been defined by Mursaleen[17] as given below: a sequence $x = \{x_m\}$ of real numbers is Λ -convergent to a number L if $\Lambda x_m \rightarrow L$ as $m \rightarrow \infty$ where

$$\Lambda x_m = \frac{1}{\lambda_m} \sum_{j=0}^m (\lambda_j - \lambda_{j-1}) x_j.$$

Here, without loss of generality we take all the terms with negative subscripts equal to zero.

Using this concept, we are defining the notion of the rough Λ -convergence and rough statistical Λ -convergence as follows:

Definition 4. A sequence $x = \{x_m\}$ in a normed linear space $(\mathbb{X}, \|\cdot\|)$ is said to be rough Λ -convergent to $\xi \in \mathbb{X}$ if for every $\epsilon > 0$ there exist a non-negative number r and $m_0 \in \mathbb{N}$ such that $\|\Lambda x_m - \xi\| < r + \epsilon$, for all $m \geq m_0$.

Definition 5. A sequence $x = \{x_m\}$ in a normed linear space $(\mathbb{X}, \|\cdot\|)$ is said to be rough statistically Λ -convergent to ξ if for every $\epsilon > 0$ there exists some non-negative number r such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{m \leq n : \|\Lambda x_m - \xi\| \geq r + \epsilon\}| = 0,$$

where ξ is known as r - St_Λ -limit of sequence $x = \{x_m\}$.

Remark 1. For the case $r = 0$, the notion of rough statistical Λ -convergence agrees with the statistical Λ -convergence.

Çolak [4] has given an interesting idea related to the statistical convergence of order α ($0 < \alpha \leq 1$) with the help of α -density. Motivated by his idea, now we are defining a rough statistical Λ -convergence of order α ($0 < \alpha \leq 1$) as follows:

Definition 6. A sequence $x = \{x_m\}$ in a normed linear space $(\mathbb{X}, \|\cdot\|)$ is said to be rough statistically Λ -convergent of order α ($0 < \alpha \leq 1$) to the number $\xi \in \mathbb{X}$ if for every $\epsilon > 0$ there exists some non-negative number r such that

$$\lim_{n \rightarrow \infty} \frac{1}{n^\alpha} |\{m \leq n : \|\Lambda x_m - \xi\| \geq r + \epsilon\}| = 0,$$

where ξ is known as r - St_Λ^α -limit of sequence $x = \{x_m\}$. It is denoted by

$$x_m \xrightarrow{r-St_\Lambda^\alpha} \xi.$$

The set of all the rough statistically Λ -convergent sequences of order α ($0 < \alpha \leq 1$) is denoted by rSt_Λ^α for fixed r .

In general, the r - St_Λ^α -limit of a sequence may be not unique. So we consider r - St_Λ^α -limit set of a sequence $x = \{x_m\}$ as

$$r-St_\Lambda^\alpha-LT_x = \{\xi : x_m \xrightarrow{r-St_\Lambda^\alpha} \xi\}.$$

The sequence $x = \{x_m\}$ is said to be r - St_Λ^α -convergent such that $r-St_\Lambda^\alpha-LT_x \neq \phi$. For unbounded sequence the rough limit set is always empty.

But in case of rough statistical Λ -convergence of order α , we have $r-St_\Lambda^\alpha-LT_x \neq \phi$ even though sequence may be unbounded. For this we have given the next example.

Example 1. Let $\mathbb{X} = \mathbb{R}$. Then, define a sequence

$$\Lambda x_m = \begin{cases} (-1)^m, & m \neq n^2, \\ m, & \text{otherwise.} \end{cases}$$

Take $\alpha = 1$, then

$$r-St_\Lambda^\alpha-LT_x = \begin{cases} \phi, & r < 1, \\ [1-r, r-1], & \text{otherwise} \end{cases}$$

and r - $\Lambda-LT_x = \phi$ for all $r \geq 0$. Thus, this sequence is divergent in ordinary sense as it is unbounded. Also, the sequence is not statistically Λ -convergent for any r .

With the help of statistically cluster points defined by Fridy [6], we are giving the following definition as follows:

Definition 7. A point ξ is said to be rough statistically Λ -cluster point of order α ($0 < \alpha \leq 1$) of a sequence $x = \{x_m\}$ in a normed linear space $(\mathbb{X}, \|\cdot\|)$ if for every $\epsilon > 0$ there exists some non-negative number r such that

$$\lim_{n \rightarrow \infty} \frac{1}{n^\alpha} |\{m \leq n : \|\Lambda x_m - \xi\| \geq r + \epsilon\}| \neq 0.$$

Definition 8. A sequence $x = \{x_m\}$ is said to be statistically Λ -bounded if there exists a real number $M_0 > 0$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{m \leq n : \|\Lambda x_m\| \geq M_0\}| = 0.$$

Definition 9. A sequence $x = \{x_m\}$ is said to be statistically Λ -bounded of order α ($0 < \alpha \leq 1$) if there exists a real number $M_0 > 0$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{n^\alpha} |\{m \leq n : \|\Lambda x_m\| \geq M_0\}| = 0.$$

In view of above definitions, we obtained the following interesting results on rough statistical Λ -convergence.

Theorem 1. Every rough Λ -convergent sequence is also rough statistically Λ -convergent of order α ($0 < \alpha \leq 1$), but converse may be not true.

P r o o f. Let the sequence $x = \{x_m\}$ be rough Λ -convergent in a normed linear space $(\mathbb{X}, \|\cdot\|)$. Then, for every $\epsilon > 0$ and some $r > 0$ there exists a real number $M_0 > 0$ such that $\|\Lambda x_m - \xi\| \geq r + \epsilon$ for all $m \geq M_0$.

The set $\{m \leq n : \|\Lambda x_m - \xi\| \geq r + \epsilon\}$ has finitely many terms. Thus,

$$\lim_{n \rightarrow \infty} \frac{1}{n^\alpha} |\{m \leq n : \|\Lambda x_m - \xi\| \geq r + \epsilon\}| = 0.$$

Hence, the sequence $x = \{x_m\}$ is rough statistically Λ -convergent of order α ($0 < \alpha \leq 1$).

But the contrary part is not true which can be justified by the next example.

Example 2. Consider the normed space $(\mathbb{R}, \|\cdot\|)$ under the usual norm. Define a sequence

$$\Lambda x_m = \begin{cases} 1, & m \text{ is a square,} \\ 0, & \text{otherwise.} \end{cases}$$

For $\epsilon > 0$ and some $r \geq 0$ we have

$$\begin{aligned} M(r, \epsilon) &= \{m \leq n : \|\Lambda x_m - \xi\| \geq r + \epsilon\}; \quad \xi = 0 \\ &= \{m \leq n : \|\Lambda x_m\| \geq r + \epsilon > 0\} \\ &= \{m \leq n : \|\Lambda x_m\| = 1\} \\ &= \{m \leq n : m \text{ is a square}\}. \end{aligned}$$

Thus,

$$\lim_{n \rightarrow \infty} \frac{1}{n^\alpha} |M(r, \epsilon)| \leq \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{n^\alpha} = 0.$$

Therefore, $x = \{x_m\}$ is rough statistically Λ -convergent of order α to 0 for $\alpha > 1/2$.

□

In the next theorem we discuss the algebraic characterization of rough statistically Λ -convergent sequences of order α ($0 < \alpha \leq 1$).

Theorem 2. Let $x = \{x_m\}$ and $y = \{y_m\}$ be two sequences in a normed linear space $(\mathbb{X}, \|\cdot\|)$ and α ($0 < \alpha \leq 1$) be given. Then for some non-negative number r the following holds:

- (1) if $x_m \xrightarrow{r-St_\Lambda^\alpha} x_0$ and $k \in \mathbb{N}$ then $kx_m \xrightarrow{r-St_\Lambda^\alpha} kx_0$;
- (2) if $x_m \xrightarrow{r-St_\Lambda^\alpha} x_0$ and $y_m \xrightarrow{r-St_\Lambda^\alpha} y_0$ then $(x_m + y_m) \xrightarrow{r-St_\Lambda^\alpha} (x_0 + y_0)$.

P r o o f. (1) If $k = 0$ then there is nothing to prove.

If $k \neq 0$. Since $x_m \xrightarrow{r-St_\Lambda^\alpha} x_0$ then for given $\epsilon > 0$ and some $r \geq 0$, we have the set

$$M(r, \epsilon) = \left\{ m \leq n : \|\Lambda x_m - x_0\| \geq \frac{r + \epsilon}{|k|} \right\} \quad \text{with} \quad \lim_{n \rightarrow \infty} \frac{1}{n^\alpha} |M(r, \epsilon)| = 0.$$

Let $m \in M^c(r, \epsilon)$. Then

$$\|\Lambda kx_m - kx_0\| = |k| \|\Lambda x_m - x_0\| < |k| \left(\frac{r + \epsilon}{|k|} \right) < r + \epsilon.$$

This implies that

$$\lim_{n \rightarrow \infty} \frac{1}{n^\alpha} \left| \left\{ m \leq n : \|\Lambda kx_m - kx_0\| < \frac{r + \epsilon}{|k|} \right\} \right| = 1,$$

i. e.

$$\lim_{n \rightarrow \infty} \frac{1}{n^\alpha} \left| \left\{ m \leq n : \|\Lambda kx_m - kx_0\| \geq \frac{r + \epsilon}{|k|} \right\} \right| = 0.$$

Therefore, $kx_m \xrightarrow{r-St_\Lambda^\alpha} kx_0$.

- (2) Since $x_m \xrightarrow{r-St_\Lambda^\alpha} x_0$ and $y_m \xrightarrow{r-St_\Lambda^\alpha} y_0$ then for given $\epsilon > 0$ and some $r \geq 0$, we have sets

$$M_x(r, \epsilon) = \left\{ m \leq n : \|\Lambda x_m - x_0\| \geq \frac{r + \epsilon}{2} \right\} \quad \text{with} \quad \lim_{n \rightarrow \infty} \frac{1}{n^\alpha} |M_x(r, \epsilon)| = 0,$$

$$M_y(r, \epsilon) = \left\{ m \leq n : \|\Lambda y_m - y_0\| \geq \frac{r + \epsilon}{2} \right\} \quad \text{with} \quad \lim_{n \rightarrow \infty} \frac{1}{n^\alpha} |M_y(r, \epsilon)| = 0.$$

Let $m \in M_x^c(r, \epsilon) \cap M_y^c(r, \epsilon)$. Then

$$\|\Lambda(x_m + y_m) - (x_0 + y_0)\| \leq \|\Lambda x_m - x_0\| + \|\Lambda y_m - y_0\| < \frac{r + \epsilon}{2} + \frac{r + \epsilon}{2} = r + \epsilon.$$

This implies that

$$\lim_{n \rightarrow \infty} \frac{1}{n^\alpha} \left| \left\{ m \leq n : \|\Lambda(x_m + y_m) - (x_0 + y_0)\| < r + \epsilon \right\} \right| = 1,$$

i. e.

$$\lim_{n \rightarrow \infty} \frac{1}{n^\alpha} \left| \left\{ m \leq n : \|\Lambda(x_m + y_m) - (x_0 + y_0)\| \geq r + \epsilon \right\} \right| = 0.$$

Therefore, $(x_m + y_m) \xrightarrow{r-St_\Lambda^\alpha} (x_0 + y_0)$. □

Theorem 3. Let $0 < \alpha \leq \beta \leq 1$ then $rSt_\Lambda^\alpha \subseteq rSt_\Lambda^\beta$ where rSt_Λ^α and rSt_Λ^β represent the sets of all rough statistically Λ -convergent of order α and β respectively.

P r o o f. Let $x = \{x_m\}$ be a sequence in a normed linear space $(\mathbb{X}, \|\cdot\|)$. If $0 < \alpha \leq \beta \leq 1$ then for every $\epsilon > 0$ and some $r > 0$ with the limit point ξ , we have

$$\frac{1}{n^\beta} \left| \left\{ m \leq n : \|\Lambda x_m - \xi\| \geq r + \epsilon \right\} \right| \leq \frac{1}{n^\alpha} \left| \left\{ m \leq n : \|\Lambda x_m - \xi\| \geq r + \epsilon \right\} \right|.$$

Therefore, we get $rSt_\Lambda^\alpha \subseteq rSt_\Lambda^\beta$. □

Theorem 4. A sequence $x = \{x_m\}$ in a normed linear space $(\mathbb{X}, \|\cdot\|)$ is statistically Λ -bounded of order α ($0 < \alpha \leq 1$) if and only if $r\text{-St}_\Lambda^\alpha\text{-LT}_x \neq \phi$, for some non-negative number r .

P r o o f. Let the sequence $x = \{x_m\}$ is statistically Λ -bounded of order α ($0 < \alpha \leq 1$), then there exists a real number $M_0 > 0$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{n^\alpha} |\{m \leq n : \|\Lambda x_m\| \geq M_0\}| = 0.$$

Let $M = \{m \in \mathbb{N} : \|\Lambda x_m\| \geq M_0\}$. Define $r_0 = \sup \{\|\Lambda x_m\| : m \in M^c\}$. As

$$0 \in r_0\text{-St}_\Lambda^\alpha\text{-LT}_x \Rightarrow r_0\text{-St}_\Lambda^\alpha\text{-LT}_x \neq \phi.$$

Conversely, suppose that $r\text{-St}_\Lambda^\alpha\text{-LT}_x \neq \phi$ for some $r \geq 0$. Then, for each $\epsilon > 0$ there exists $\xi \in \mathbb{X}$ such that $\xi \in r\text{-St}_\Lambda^\alpha\text{-LT}_x$. Then

$$\lim_{n \rightarrow \infty} \frac{1}{n^\alpha} |\{m \leq n : \|\Lambda x_m - \xi\| \geq r + \epsilon\}| = 0.$$

Hence, the sequence $x = \{x_m\}$ is statistically Λ -bounded of order α . \square

Theorem 5. If $x' = \{x_{m_k}\}$ is a non-thin subsequence of a sequence $x = \{x_m\}$ then

$$r\text{-St}_\Lambda^\alpha\text{-LT}_x \subseteq r\text{-St}_\Lambda^\alpha\text{-LT}_{x'}.$$

P r o o f. The proof of above results is obvious, so we are omitting it. \square

Theorem 6. Let $x = \{x_m\}$ be a sequence in a normed linear space $(\mathbb{X}, \|\cdot\|)$. Then, the rough statistical limit set of order α ($0 < \alpha \leq 1$) is convex, i.e., $r\text{-St}_\Lambda^\alpha\text{-LT}_x$ is convex.

P r o o f. Let $\xi_1, \xi_2 \in r\text{-St}_\Lambda^\alpha\text{-LT}_x$ and $\epsilon > 0$ be given. For the convexity of the set $r\text{-St}_\Lambda^\alpha\text{-LT}_x$, we have to show that $[(1 - \beta)\xi_1 + \beta\xi_2] \in r\text{-St}_\Lambda^\alpha\text{-LT}_x$ for some $\beta \in (0, 1)$. Now, we define

$$M_1(r, \epsilon) = \left\{ m \in \mathbb{N} : \|\Lambda x_m - \xi_1\| \geq \frac{r + \epsilon}{2(1 - \beta)} \right\},$$

$$M_2(r, \epsilon) = \left\{ m \in \mathbb{N} : \|\Lambda x_m - \xi_2\| \geq \frac{r + \epsilon}{2\beta} \right\}.$$

As $\xi_1, \xi_2 \in r\text{-St}_\Lambda^\alpha\text{-LT}_x$, we have

$$\lim_{n \rightarrow \infty} \frac{1}{n^\alpha} |M_1(r, \epsilon)| = \lim_{n \rightarrow \infty} \frac{1}{n^\alpha} |M_2(r, \epsilon)| = 0.$$

Let $m \in M_1^c(r, \epsilon) \cap M_2^c(r, \epsilon)$. Then

$$\begin{aligned} \|\Lambda x_m - [(1 - \beta)\xi_1 + \beta\xi_2]\| &= \|(1 - \beta)(\Lambda x_m - \xi_1) + \beta(\Lambda x_m - \xi_2)\| \\ &\leq (1 - \beta)\|\Lambda x_m - \xi_1\| + \beta\|\Lambda x_m - \xi_2\| \\ &< r + \epsilon. \end{aligned}$$

Since

$$\lim_{n \rightarrow \infty} \frac{1}{n^\alpha} |M_1^c(r, \epsilon) \cap M_2^c(r, \epsilon)| = 1,$$

we get

$$\lim_{n \rightarrow \infty} \frac{1}{n^\alpha} |\{m \leq n : \|\Lambda x_m - [(1 - \beta)\xi_1 + \beta\xi_2]\| \geq r + \epsilon\}| = 0,$$

i. e.

$$[(1 - \beta)\xi_1 + \beta\xi_2] \in r\text{-St}_\Lambda^\alpha\text{-LT}_x.$$

Hence, $r\text{-St}_\Lambda^\alpha\text{-LT}_x$ is a convex set. \square

Theorem 7. A sequence $x = \{x_m\}$ in a normed linear space $(\mathbb{X}, \|\cdot\|)$ is rough statistically Λ -convergent of order α ($0 < \alpha \leq 1$) to $\xi \in \mathbb{X}$ for some non-negative number r if and only if there exists a sequence $y = \{y_m\}$ in \mathbb{X} which is rough statistically Λ -convergent of order α to ξ and $\|\Lambda x_m - \Lambda y_m\| \leq r$ for all $m \in \mathbb{N}$.

P r o o f. *Necessity.* Let $x_m \xrightarrow{r-St_\Lambda^\alpha} \xi$. Then, for each $\epsilon > 0$ and some $r > 0$ we have

$$\lim_{n \rightarrow \infty} \frac{1}{n^\alpha} \left| \{m \leq n : \|\Lambda x_m - \xi\| \geq r + \epsilon\} \right| = 0. \quad (2.1)$$

Now, we define the sequence as

$$\Lambda y_m = \begin{cases} \xi, & \|\Lambda x_m - \xi\| \leq r, \\ \Lambda x_m + r \frac{\xi - \Lambda x_m}{\|\Lambda x_m - \xi\|}, & \text{otherwise.} \end{cases}$$

Then, we have

$$\Lambda y_m - \xi = \begin{cases} 0, & \|\Lambda x_m - \xi\| \leq r, \\ \frac{\Lambda x_m - \xi}{\|\Lambda x_m - \xi\|} (\|\Lambda x_m - \xi\| - r), & \text{otherwise} \end{cases}$$

such that $\|\Lambda x_m - \Lambda y_m\| \leq r$ for all $m \in \mathbb{N}$. Further,

$$\|\Lambda y_m - \xi\| = \begin{cases} 0, & \|\Lambda x_m - \xi\| \leq r, \\ \|\Lambda x_m - \xi\| - r, & \text{otherwise.} \end{cases}$$

Hence, by the definition of Λy_m and (2.1), we have

$$\lim_{n \rightarrow \infty} \frac{1}{n^\alpha} \left| \{m \leq n : \|\Lambda y_m - \xi\| \geq r + \epsilon\} \right| = 0,$$

which prove that the sequence $y = \{y_m\}$ is rough statistically Λ -convergent of order α to ξ .

Sufficiency. Since the sequence $y = \{y_m\}$ is rough statistically Λ -convergent of order α ($0 < \alpha \leq 1$) to ξ then for $\epsilon > 0$ we have

$$\lim_{n \rightarrow \infty} \frac{1}{n^\alpha} \left| \{m \leq n : \|\Lambda y_m - \xi\| \geq r + \epsilon\} \right| = 0.$$

Now for some $r > 0$ and sequence $x = \{x_m\}$ with $\|\Lambda x_m - \Lambda y_m\| \leq r$, the following inclusion holds

$$\{m \leq n : \|\Lambda x_m - \xi\| \geq r + \epsilon\} \subseteq \{m \leq n : \|\Lambda y_m - \xi\| \geq r + \epsilon\}.$$

Hence, we get

$$\lim_{n \rightarrow \infty} \frac{1}{n^\alpha} \left| \{m \leq n : \|\Lambda x_m - \xi\| \geq r + \epsilon\} \right| = 0.$$

□

Theorem 8. The set $r-St_\Lambda^\alpha-LT_x$ of rough statistical Λ -limit set of order α ($0 < \alpha \leq 1$) is closed.

P r o o f. (i) If $r\text{-}St_{\Lambda}^{\alpha}\text{-}LT_x = \phi$, then we have to prove nothing.

(ii) If $r\text{-}St_{\Lambda}^{\alpha}\text{-}LT_x \neq \phi$. Then, take a sequence $y = \{y_m\} \subseteq r\text{-}St_{\Lambda}^{\alpha}\text{-}LT_x$ such that $\Lambda y_m \rightarrow y_*$ for $m \rightarrow \infty$. It is sufficient to show that $y_* \in r\text{-}St_{\Lambda}^{\alpha}\text{-}LT_x$.

As $\Lambda y_m \rightarrow y_*$, then for given $\epsilon > 0$ there exists $m_{\epsilon} \in \mathbb{N}$ such that

$$\|\Lambda y_m - y_*\| < \frac{r + \epsilon}{3}$$

for $m > m_{\epsilon}$.

Now choose $m_0 \in \mathbb{N}$ such that $m_0 > m_{\epsilon}$. Then we have

$$\|\Lambda y_{m_0} - y_*\| < \frac{r + \epsilon}{3}.$$

Again as $y = \{y_m\} \subseteq r\text{-}St_{\Lambda}^{\alpha}\text{-}LT_x$, we have $y_{m_0} \in r\text{-}St_{\Lambda}^{\alpha}\text{-}LT_x$. Clearly,

$$\lim_{n \rightarrow \infty} \frac{1}{n^{\alpha}} \left| \left\{ m \leq n : \|\Lambda x_m - y_{m_0}\| \geq \frac{r + \epsilon}{3} \right\} \right| = 0. \quad (2.2)$$

Next we prove the inclusion

$$\left\{ m \leq n : \|\Lambda x_m - y_{m_0}\| < \frac{r + \epsilon}{3} \right\} \subseteq \left\{ m \leq n : \|\Lambda x_m - y_*\| < r + \epsilon \right\}. \quad (2.3)$$

Let

$$k \in \left\{ m \leq n : \|\Lambda x_m - y_{m_0}\| < \frac{r + \epsilon}{3} \right\} \Rightarrow \|\Lambda x_k - y_{m_0}\| < \frac{r + \epsilon}{3}.$$

Hence,

$$\|\Lambda x_k - y_*\| = \|\Lambda x_k - y_{m_0} + \Lambda y_m - y_* - \Lambda y_m + y_{m_0}\| \leq \|\Lambda x_k - y_{m_0}\| + \|\Lambda y_m - y_*\| + \|\Lambda y_k - y_{m_0}\|.$$

Using equation (2.2) and Theorem 7 we get

$$\|\Lambda y_k - y_{m_0}\| < \frac{r + \epsilon}{3}.$$

Thus,

$$\|\Lambda x_k - y_*\| < \frac{r + \epsilon}{3} + \frac{r + \epsilon}{3} + \frac{r + \epsilon}{3} = r + \epsilon.$$

This implies that

$$k \in \left\{ m \leq n : \|\Lambda x_m - y_*\| < r + \epsilon \right\}.$$

Hence the inclusion (2.3) is proved.

Thus,

$$\left\{ m \leq n : \|\Lambda x_m - y_*\| \geq r + \epsilon \right\} \subseteq \left\{ m \leq n : \|\Lambda x_m - y_{m_0}\| \geq \frac{r + \epsilon}{3} \right\}.$$

Now,

$$\lim_{n \rightarrow \infty} \frac{1}{n^{\alpha}} \left| \left\{ m \leq n : \|\Lambda x_m - y_*\| \geq r + \epsilon \right\} \right| \leq \lim_{n \rightarrow \infty} \frac{1}{n^{\alpha}} \left| \left\{ m \leq n : \|\Lambda x_m - y_{m_0}\| \geq \frac{r + \epsilon}{3} \right\} \right|. \quad (2.4)$$

Using equation (2.2), we obtained that the set on left side of (2.4) has density 0. Hence, we get

$$\lim_{n \rightarrow \infty} \frac{1}{n^{\alpha}} \left| \left\{ m \leq n : \|\Lambda x_m - y_*\| \geq r + \epsilon \right\} \right| = 0.$$

□

Theorem 9. Let $\Gamma_{\Lambda x}$ be the set of all rough statistical Λ -cluster points of order α ($0 < \alpha \leq 1$) for a sequence $x = \{x_m\}$ in the normed linear space $(\mathbb{X}, \|\cdot\|)$. Then for an arbitrary $c \in \Gamma_{\Lambda x}$ and a positive real number r , we have $\|\xi - c\| < r$ for all $\xi \in r\text{-St}_{\Lambda}^{\alpha}\text{-LT}_x$.

P r o o f. We prove the result by contradiction. For given α ($0 < \alpha \leq 1$), we take a point $c \in \Gamma_{\Lambda x}$ and $\xi \in r\text{-St}_{\Lambda}^{\alpha}\text{-LT}_x$ such that $\|\xi - c\| > r$. By choosing $\epsilon = (\|\xi - c\| - r)/3$, we get the following inclusion

$$\{m \leq n : \|\Lambda x_m - \xi\| \geq r + \epsilon\} \supseteq \{m \leq n : \|\Lambda x_m - c\| < \epsilon\}. \quad (2.5)$$

Since $c \in \Gamma_{\Lambda x}$, then

$$\lim_{n \rightarrow \infty} \frac{1}{n^{\alpha}} |\{m \leq n : \|\Lambda x_m - c\| < \epsilon\}| \neq 0.$$

By (2.5), we get

$$\lim_{n \rightarrow \infty} \frac{1}{n^{\alpha}} |\{m \leq n : \|\Lambda x_m - \xi\| < r + \epsilon\}| \neq 0,$$

which is a contradiction to $\xi \in r\text{-St}_{\Lambda}^{\alpha}\text{-LT}_x$. \square

Theorem 10. Let $x = \{x_m\}$ be a sequence in a strictly convex normed linear space $(\mathbb{X}, \|\cdot\|)$. Let α and r be two positive real numbers. If any $\xi_0, \xi_1 \in r\text{-St}_{\Lambda}^{\alpha}\text{-LT}_x$ with $\|\xi_0 - \xi_1\| = 2r$, then $x = \{x_m\}$ is rough statistically Λ -convergent of order α ($0 < \alpha \leq 1$) to $(\xi_0 + \xi_1)/2$.

P r o o f. Let $z \in \Gamma_{\Lambda x}$ and $\xi_0, \xi_1 \in r\text{-St}_{\Lambda}^{\alpha}\text{-LT}_x$ such that $\|\xi_0 - \xi_1\| = 2r$. Then, we have

$$\|\xi_0 - z\| \leq r \quad \text{and} \quad \|\xi_1 - z\| \leq r, \quad (2.6)$$

and by triangle inequality, we get

$$\begin{aligned} \|\xi_0 - \xi_1\| &\leq \|\xi_0 - z\| + \|\xi_1 - z\| \\ &\Rightarrow 2r \leq \|\xi_0 - z\| + \|\xi_1 - z\|. \end{aligned} \quad (2.7)$$

We get from (2.6) and (2.7)

$$\|\xi_0 - z\| = \|\xi_1 - z\| = r.$$

Also

$$\frac{1}{2}(\xi_1 - \xi_0) = \frac{1}{2}[(z - \xi_0) + (\xi_1 - z)], \quad (2.8)$$

and using $\|\xi_0 - \xi_1\| = 2r$, we get $(\xi_1 - \xi_0)/2 = r$.

Now from equation (2.8) and from strict convexity of the normed linear space $(\mathbb{X}, \|\cdot\|)$, we have $(z - \xi_0) = (\xi_1 - z) = (\xi_1 - \xi_0)/2$ which implies that $z = (\xi_0 + \xi_1)/2$. Thus, z is a unique statistical Λ -cluster point of sequence $x = \{x_m\}$.

As $\xi_0, \xi_1 \in r\text{-St}_{\Lambda}^{\alpha}\text{-LT}_x \Rightarrow r\text{-St}_{\Lambda}^{\alpha}\text{-LT}_x \neq \phi$. Hence, by Theorem 4, the sequence $x = \{x_m\}$ is statistically Λ -bounded of order α .

Since z is the unique statistical Λ -cluster point to statistically Λ -bounded sequence $x = \{x_m\}$ of order α .

This implies that $x_m \xrightarrow{r\text{-St}_{\Lambda}^{\alpha}} z$, where $z = (\xi_0 + \xi_1)/2$. \square

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