ONE-SIDED WIDTHS OF CLASSES OF SMOOTH FUNCTIONS[1]

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One-sided widths of the classes of functions $W_p^r[0,1]$ in the metric $L_q[0,1]$, $1\le p,q\le\infty$, $r\ge 1$, are studied. Such widths are defined similarly to Kolmogorov widths with additional constraints on the approximating functions.

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Let us introduce some definitions. The Kolmogorov width (see [1]) is, by definition, the value

$$d_n(W_p^r, L_q) = \inf_{L_n\subset L_q}\ \sup_{f\in W_p^r}\ \inf_{g(x)\in L_n} \|f-g\|_{L_q}, \tag{1}$$

where L_n is an n-dimensional subspace of the space $L_q[0,1]$; W_p^r is the class of functions $f(x)$ representable in the form

$$f(x) = P_{r-1}(x) + \frac{1}{(r-1)!}\int\limits_0^x (x-t)^{r-1} f^{(r)}(t)\,dt.$$

Here, $P_{r-1}(x)$ is a polynomial of degree at most $r-1$, r is a positive integer, and $r\ge 1$; $f^{(r-1)}(x)$ is absolutely continuous and $\|f^{(r)}\|_{L_p} = \Big(\int_0^1 |f^{(r)}(x)|^p\,dx\Big)^{1/p} \le 1$, $1\le p\le\infty$; by $\|f^{(r)}\|_{L_\infty}$ we mean $\operatorname{ess\,sup}\{|f^{(r)}(x)|\colon 0\le x\le 1\}$.

The corresponding one-sided width is defined as follows (see [2]):

$$d_n^+(W_p^r, L_q) = \inf_{L_n\subset L_q}\ \sup_{f\in W_p^r}\ \inf_{\substack{g(x)\in L_n\\ g(x)\ge f(x)}} \|f-g\|_{L_q}.$$

Orders of widths $d_n(W_p^r, L_q)$ (1) with respect to n were studied by many authors. Detailed information on this subject is given quite completely in [3], where the final results in this direction were obtained. The following final order result is valid:

$$d_n(W_p^r, L_q) \asymp \begin{cases} n^{-r}, & \text{if}\quad 1\le q\le p\le\infty \quad\text{or}\quad 2<p\le q\le\infty,\\ n^{-r-\frac12+\frac1p}, & \text{if}\quad 1\le p\le 2\le q\le\infty,\\ n^{-r-\frac1q+\frac1p}, & \text{if}\quad 1\le p<q\le 2, \end{cases} \tag{2}$$

where the symbol $\asymp$ means that the upper and lower bounds hold for $d_n(W_p^r, L_q)$ with the given orders with respect to n accurately to the constants that depend only on r, p and q.

In the present paper, we show that one-sided widths $d_n^+(W_p^r, L_q)$ have the same orders (2) with respect to n.

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Theorem. *For all positive integers $r \ge 1$ and $1 \le p,\, q \le \infty$, the following order equalities are valid*:

$$d_n^+(W_p^r, L_q) \asymp \begin{cases} n^{-r}, & \text{if} \quad 1 \le q \le p \le \infty \quad \text{or} \quad 2 < p \le q \le \infty, \\ n^{-r-\frac{1}{2}+\frac{1}{p}}, & \text{if} \quad 1 \le p \le 2 \le q \le \infty, \\ n^{-r-\frac{1}{q}+\frac{1}{p}}, & \text{if} \quad 1 \le p < q \le 2. \end{cases}$$

P r o o f. Since, by definition, $d_n^+(W_p^r, L_q) \ge d_n(W_p^r, L_q)$ and (2) is valid, the lower bounds follow immediately.

Estimating the widths from above, we consider several cases. Divide the interval $[0,1]$ into n equal intervals $[x_i, x_{i+1}]$ $(i = 0, 1, \ldots, n-1)$, $x_i = i/n$. On each interval, we will approximate a function $f(x)$ from W_p^r by the Taylor partial sum

$$\varphi_{i,r}(x) = f(\overline{x}_i)(x - \overline{x}_i) + \cdots + f^{(r-1)}(\overline{x}_i)\frac{(x-\overline{x}_i)^{r-1}}{(r-1)!}, \quad \overline{x}_i = \frac{x_i + x_{i+1}}{2}.$$

We have

$$|f(x) - \varphi_{i,r}(x)| = \left| \frac{1}{(r-1)!} \int_{\overline{x}_i}^{x} (x-t)^{r-1} f^{(r)}(t)\, dt \right|, \quad x \in [x_i, x_{i+1}]. \tag{3}$$

The following estimates hold $(1/p + 1/p_1 = 1)$:

$$|f(x) - \varphi_{i,r}(x)| \le \frac{1}{(r-1)!} \left| \int_{\overline{x}_i}^{x} (x-t)^{r-1} f^{(r)}(t)\, dt \right|$$

$$\le \frac{1}{(r-1)!} \left| \int_{\overline{x}_i}^{x} |x-t|^{(r-1)p_1}\, dt \right|^{\frac{1}{p_1}} \left| \int_{\overline{x}_i}^{x} |f^{(r)}(t)|^p\, dt \right|^{\frac{1}{p}}$$

$$\le \frac{1}{(r-1)!} |x - \overline{x}_i|^{\frac{(r-1)p_1+1}{p_1}} \left(\int_{x_i}^{x_{i+1}} |f^{(r)}(t)|^p\, dt \right)^{\frac{1}{p}}$$

$$\le \frac{1}{(r-1)!} \frac{(x_{i+1} - x_i)^{r-1+\frac{1}{p_1}}}{2^{r-1+\frac{1}{p_1}}} \left(\int_{x_i}^{x_{i+1}} |f^{(r)}(t)|^p\, dt \right)^{\frac{1}{p}} = C_i. \tag{4}$$

Thus, the following inequalities are valid:

$$f(x) - \varphi_{i,r}(x) + C_i \ge 0 \quad (i = 0, 1, \ldots, n-1), \tag{5}$$

$$0 \le f(x) - \varphi_{i,r}(x) + C_i \le 2C_i = \frac{1}{(r-1)!} \frac{(x_{i+1} - x_i)^{r-1+\frac{1}{p_1}}}{2^{r+\frac{1}{p_1}}} \left(\int_{x_i}^{x_{i+1}} |f^{(r)}(t)|^p\, dt \right)^{\frac{1}{p}}. \tag{6}$$

Denote by L_{nr} the nr-dimensional subspace of functions $g(x)$ of the form

$$g(x) = P_{r-1,i}(x), \quad x \in [x_i, x_{i+1}] \quad (i = 0, 1, \ldots, n-1),$$

where $P_{r-1,i}(x)$ is a polynomial of degree at most $r-1$. Then, for the functions from (3)–(6), which belong to L_{nr}, we have

$$d_{nr_1}^+(W_p^r, L_q) \le \left(\sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} |f(x) - \varphi_{i,r}(x) + C_i|^q\, dx \right)^{\frac{1}{q}}$$

$$\leq \left[\sum_{i=0}^{n-1}(x_{i+1}-x_i)(2C_i)^q\right]^{\frac{1}{q}} \leq \left(\frac{1}{n}\right)^{r-\frac{1}{p}+\frac{1}{q}} \frac{1}{(r-1)!2^{r-\frac{1}{p}}} \left[\sum_{i=0}^{n-1}\left(\int\limits_{x_i}^{x_{i+1}} |f^{(r)}(t)|^p\,dt\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}. \qquad (7)$$

Denote $\alpha_i = \displaystyle\int_{x_i}^{x_{i+1}} |f^{(r)}(t)|^p\,dt \geq 0$. Since $f \in W_p^r$, we have $\sum_{i=0}^{n-1}\alpha_i = 1$. This and (7) imply that $\sum_{i=0}^{n-1}\alpha_i^{\frac{q}{p}}$ achieves the largest value for $q/p > 1$ if one of α_i is equal to 1 and all the other are zero; i. e., in this case,

$$d_{nr}^{+}(W_p^r, L_q) \leq \frac{1}{(r-1)!2^{r-\frac{1}{p}}}\left(\frac{1}{n}\right)^{r-\frac{1}{p}+\frac{1}{q}}, \quad q > p.$$

For $q \leq p$, the largest value on the right-hand side of (7) is achieved for $\alpha_i = (1/n)$; i. e., in this case,

$$d_{nr}^{+}(W_p^r, L_q) \leq \frac{1}{(r-1)!2^{r-\frac{1}{p}}}\left(\frac{1}{n}\right)^{r-\frac{1}{p}+\frac{1}{q}}\left[\sum_{i=0}^{n-1}\left(\frac{1}{n}\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}$$

$$= \frac{1}{(r-1)!2^{r-\frac{1}{p}}}\left(\frac{1}{n}\right)^{r-\frac{1}{p}+\frac{1}{q}}\left(\frac{1}{n}\right)^{\frac{1}{p}} n^{\frac{1}{q}} = \frac{1}{(r-1)!2^{r-\frac{1}{p}}}\left(\frac{1}{n}\right)^{r} \quad (q \leq p). \qquad (8)$$

Further, consider the case $2 < p \leq q \leq \infty$. Here, we use a fact mentioned in [3]. The following inequalities are valid:

$$d_n^{+}(W_p^r, L_q) \leq d_n^{+}(W_p^r, L_\infty) \leq d_n^{+}(W_2^r, L_\infty). \qquad (9)$$

The former inequality in (9) follows from the inequality $\|f\|_{L_q} \leq \|f\|_{L_\infty}$, and the latter inequality follows from the embedding $W_p^r \subset W_2^r$ because

$$\left(\int\limits_0^1 |f^{(r)}(x)|^2\,dx\right)^{\frac{1}{2}} \leq \left(\int\limits_0^1 |f^{(r)}(x)|^{2\cdot\frac{p}{2}}\,dx\right)^{\frac{1}{p}}\left(\int\limits_0^1 (1)^{\frac{p}{p-2}}\,dx\right)^{\frac{p-2}{p}} = \left(\int\limits_0^1 |f^{(r)}(x)|^p\,dx\right)^{\frac{1}{p}}.$$

From inequality (9) for $2 < p \leq q \leq \infty$, we deduce that

$$d_n(W_p^r, L_q) \leq d_n^{+}(W_p^r, L_q) \leq d_n^{+}(W_2^r, L_\infty) \leq 2d_n(W_2^r, L_\infty) \asymp n^{-r},$$

$$2 < p \leq q \leq \infty;$$

i. e., in this case,

$$d_n^{+}(W_2^r, L_\infty) \asymp n^{-r}, \quad 2 \leq p \leq q \leq \infty.$$

It remains to prove that $d_n^{+}(W_p^r, L_q) \asymp n^{-r-\frac{1}{2}+\frac{1}{p}}$ for $1 \leq p \leq 2 \leq q \leq \infty$. Taking into account the former inequality in (9), we have

$$d_n^{+}(W_p^r, L_q) \leq d_n^{+}(W_p^r, L_\infty).$$

Note the following fact. If a set $W[0,1]$ from L_∞ contains an arbitrary constant, then approximating subspaces must also contain this constant. Otherwise, $d_n(W, L_\infty) = \infty$. Therefore,

$$d_n(W_p^r, L_\infty) \leq d_n^{+}(W_p^r, L_\infty) = \inf_{L_n}\sup_{f\in W_p^r}\inf_{\substack{g(x)\in L_n\\ g(x)\geq f(x)}}\|f-g\|_{L_q}$$

$$\leq \inf_{L_n}\sup_{f\in W_p^r}\inf_{g(x)\in L_n}\|f-g+d_n(W_p^r, L_\infty)\|_{L_\infty} \leq 2d_n(W_p^r, L_\infty) \asymp n^{-r+\frac{1}{2}+\frac{1}{p}} \quad (1 \leq p \leq 2 \leq q \leq \infty).$$

For the latter equivalence, see the case $p \leq 2 \leq q \leq \infty$ in (2).

For a given m, we find $[m/r]$, where $[m/r]$ is the integer part of the number m/r. Then, $[m/r]r \le m \le ([m/r]+1)r$. In this case,

$$d^+_{[\frac{m}{r}]r+1}(W^r_p, L_q) \le d^+_m(W^r_p, L_q) \le d^+_{[\frac{m}{r}]r}(W^r_p, L_q)$$

and, from the foregoing, we obtain the exact order of behavior of the one-sided widths with respect to m ($m \to \infty$) for all m, not only for m that are multiples of r. Moreover, the equivalence constants are finite and depend only on r, p, and q; $r \in \mathbb{N}$ and $1 \le p, q \le \infty$. □

For an even positive integer r, we can also use the results from [4]. Then, in a number of cases, estimating from above, we can obtain the constants independent of n that may be less than the constants in the present paper; however, the order of their behavior with respect to n ($n \to \infty$) will be the same.

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