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AUTOMORPHISMS OF A DISTANCE-REGULAR GRAPH WITH INTERSECTION ARRAY $\{39, 36, 4; 1, 1, 36\}$[1].

Konstantin S. Efimov

Ural State University of Economics,
62 March 8th Str., Ekaterinburg, Russia, 620144

konstantin.s.efimov@gmail.com

Alexander A. Makhnev

Krasovskii Institute of Mathematics and Mechanics,
Ural Branch of the Russian Academy of Sciences,
16 S. Kovalevskaya Str., Ekaterinburg, Russia, 620990

makhnev@imm.uran.ru

Abstract: Makhnev and Nirova have found intersection arrays of distance-regular graphs with no more than 4096 vertices, in which $\lambda = 2$ and $\mu = 1$. They proposed the program of investigation of distance-regular graphs with $\lambda = 2$ and $\mu = 1$. In this paper the automorphisms of a distance-regular graph with intersection array $\{39, 36, 4; 1, 1, 36\}$ are studied.

Keywords: Strongly regular graph, Distance-regular graph.

Introduction

We consider undirected graphs without loops and multiple edges. Our terminology and notation are mostly standard and could be found in [1]. Given a vertex a in a graph Γ, we denote by $\Gamma_i(a)$ the subgraph induced by Γ on the set of all the vertices of Γ, that are at the distance i from a. The subgraph $[a] = \Gamma_1(a)$ is called the *neighbourhood of a vertex* a. Let $\Gamma(a) = \Gamma_1(a)$, $a^\perp = \{a\} \cup \Gamma(a)$. If graph Γ is fixed, then we write $[a]$ instead of $\Gamma(a)$.

The incidence system with the set of points P and the set of lines $\mathcal{L}$ is called *α-partial geometry of order* (s,t) if each line contains exactly $s+1$ points, each point lies exactly on $t+1$ lines, any two points lie on no more than one line, and for any antiflag $(a,l) \in (P,\mathcal{L})$ there are exactly α lines passing through a and intersecting l. This geometry is denoted by $pG_\alpha(s,t)$.

In the case $\alpha = 1$, the geometry $pG_\alpha(s,t)$ is called a *generalized quadrangle* and is denoted by $GQ(s,t)$. A point graph of this geometry is defined on the set of points P and two points are adjacent if they lie on a line. The point graph of a geometry $pG_\alpha(s,t)$ is strongly regular with parameters $v = (s+1)(1+st/\alpha)$, $k = s(t+1)$, $\lambda = s-1+t(\alpha-1)$, $\mu = \alpha(t+1)$. A strongly regular graph with such parameters for some natural numbers α, s, t is called a *pseudo-geometric graph* for $pG_\alpha(s,t)$.

If vertices u, w are at distance i in Γ, then by $b_i(u,w)$ (respectively, $c_i(u,w)$) we denote the number of vertices in $\Gamma_{i+1}(u) \cap [w]$ (respectively, $\Gamma_{i-1}(u) \cap [w]$). A graph Γ of diameter d is called *distance-regular with intersection array* $\{b_0, b_1, \ldots, b_{d-1}; c_1, \ldots, c_d\}$ if the values $b_i(u,w)$ and $c_i(u,w)$ do not depend on the choice of vertices u, w at distance i in Γ for each $i = 0, ..., d$. Note that, for a

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distance-regular graph, b_0 is the degree of the graph and $c_1 = 1$. For a subset X of automorphisms of a graph Γ, $\mathrm{Fix}(X)$ denotes the set of all vertices of Γ, fixed with respect to any automorphism of X. Further, by $p^l_{ij}(x, y)$ we denote the number of vertices in a subgraph $\Gamma_i(x) \cap \Gamma_j(y)$ for vertices x, y at distance l in Γ.

A graph is said to be vertex-symmetric if its automorphism group acts transitively on the set of its vertices.

In [2], intersection arrays of distance-regular graphs with $\lambda = 2$, $\mu = 1$ and with the number of vertices at most 4096 were found. A.A. Makhnev and M.S. Nirova proposed an investigation program of automorphisms of distance-regular graphs from the obtained list.

Proposition 1. [2] *Let Γ be a distance-regular graph with $\lambda = 2$, $\mu = 1$, which has at most 4096 vertices. Then Γ has one of the following intersection arrays:*

(1) $\{21, 18; 1, 1\}(v = 400)$;

(2) $\{6, 3, 3, 3; 1, 1, 1, 2\}$ (Γ *is a generalized octagon of order* $(3, 1)$, $v = 160$), $\{6, 3, 3; 1, 1, 2\}$ (Γ *is a generalized hexagon of order* $(3, 1)$, $v = 52$), $\{12, 9, 9; 1, 1, 4\}$ (Γ *is a generalized hexagon of order* $(3, 3)$, $v = 364$), $\{6, 3, 3, 3, 3, 3; 1, 1, 1, 1, 1, 2\}$ (Γ *is a generalized dodecagon of order* $(3, 1)$, $v = 1456$);

(3) $\{18, 15, 9; 1, 1, 10\}$($v = 1 + 18 + 270 + 243 = 532$, Γ_3 *is a strongly regular graph*); $\{33, 30, 8; 1, 1, 30\}$, $\{39, 36, 4; 1, 1, 36\}$, $\{21, 18, 12, 4; 1, 1, 6, 21\}$.

In this paper we study automorphisms of a hypothetical distance-regular graph Γ with intersection array $\{39, 36, 4; 1, 1, 36\}$. The maximal order of a clique C in Γ is not more than 4. A graph with intersection array $\{39, 36, 4; 1, 1, 36\}$ has $v = 1 + 39 + 1404 + 156 = 1600$ vertices and the spectrum $39^1, 7^{675}, -1^{156}, -6^{768}$.

Theorem 1. *Let Γ be a distance-regular graph with intersection array $\{39, 36, 4; 1, 1, 36\}$, $G = \mathrm{Aut}(\Gamma)$, g is an element of prime order p in G and $\Omega = \mathrm{Fix}(g)$ contains exactly s vertices in t antipodal classes. Then $\pi(G) \subseteq \{2, 3, 5\}$ and one of the following statements holds:*

(1) *Ω is an empty graph and either $p = 2$, $\alpha_1(g) = 10r + 26m + 12$ and $\alpha_3(g) = 80r$ or $p = 5$, $\alpha_1(g) = 65n + 10l + 10$ and $\alpha_3(g) = 200l$;*

(2) *Ω is an n-clique and one of the following statements holds:*

(*i*) *$n = 1$, $p = 3$, $\alpha_1(g) = 15l + 24 + 39m$ and $\alpha_3(g) = 120l + 36$,*

(*ii*) *$n = 2$, $p = 2$, $\alpha_1(g) = 10l + 26m$ and $\alpha_3(g) = 80l - 8$,*

(*iii*) *$n = 4$, $p = 2$, $\alpha_1(g) = 10l + 26m + 14$ and $\alpha_3(g) = 80l - 16$ or $p = 3$, $\alpha_1(g) = 10l + 39m + 1$, l is congruent to -1 modulo 3 and $\alpha_3(g) = 120l + 24$;*

(3) *Ω consists of n vertices pairwise at distance 3 in Γ, $p = 3$, $n \in \{4, 7, ..., 40\}$, $\alpha_3(g) = 120l + 40 - 4n$ and $\alpha_1(g) = 15l + 30 + 39m - 6n$;*

(4) *Ω contains an edge and is a union of isolated cliques, any two vertices of different cliques are at distance 3 in Γ, and either $p = 3$ and the orders of these cliques are 1 or 4, or $p = 2$ and the orders of these cliques are 2 or 4;*

(5) *Ω contains vertices that are at distance 2 in Γ and $p \leq 3$.*

If Γ is a distance-regular graph with the intersection array $\{39, 36, 4; 1, 1, 36\}$ then Γ_3 is a pseudo-geometric for $pG_3(39, 3)$.

Theorem 2. *Let Γ be a strongly regular graph with parameters $(1600, 156, 44, 12)$, $G = \mathrm{Aut}(\Gamma)$, g is an element of prime order p in G and $\Delta = \mathrm{Fix}(g)$. Then $p \leq 43$ and the following statements hold:*

(1) *if Δ is an empty graph, then $p = 2$ and $\alpha_1(g) = 80s$ or $p = 5$ and $\alpha_1(g) = 200t$;*

(2) *if Δ is an n-clique, then one of the following statements holds:*

(i) $n = 1$, $p = 2$ *and* $\alpha_1(g) = 80s - 4$, *or* $p = 3$ *and* $\alpha_1(g) = 120t + 36$, *or* $p = 13$ *and* $\alpha_1(g) = 520l + 156$,

(ii) $n \in \{4, 7, 10, ..., 40\}$, $p = 3$ *and* $\alpha_1(g) = 120t + 40 - 4n$,

(iii) $n = 9$, $p = 37$ *and* $\alpha_1(g) = 444$;

(3) *if* Δ *is an* m*-coclique, where* $m > 1$, *then either* $p = 2$, $m \in \{4, 6, 8, ..., 40\}$ *and* $\alpha_1(g) = 80s - 4m$ *or* $p = 3$, $m \in \{4, 7, 10, ..., 40\}$ *and* $\alpha_1(g) = 120t + 40 - 4m$;

(4) *if* Δ *contains an edge and is an union of isolated cliques, then* $p = 3$;

(5) *if* Δ *contains a geodesic* 2*-path, then* $p \leq 43$.

Corollary 1. *Let* Γ *be a distance-regular graph with intersection array* $\{39, 36, 4; 1, 1, 36\}$ *and nonsolvable group* $G = \mathrm{Aut}(\Gamma)$ *acts transitively on the set of vertices of* Γ. *If* a *is a vertex of* Γ, $\bar{T}$ *is the socle of the group* $\bar{G} = G/O_{5'}(G)$, *then* $\bar{T} = L \times M$, *and each of subgroups* L, M *is isomorphic to one of the following groups:* Z_5, A_5, A_6 *or* $PSp(4, 3)$.

If $|\bar{T} : \bar{T}_a| = 40^2$, then $O_{5'}(G) = 1$ and this case is realized if one of the following statements holds:

(1) $L \cong M \cong PSp(4, 3)$, $|L : L_a| = |M : M_a| = 40$,

(2) $L \cong PSp(4, 3)$, $|L : L_a| = 40$, $M \cong A_6$ and $|M_a| = 9$,

(3) $L \cong M \cong A_6$ and $|L_a| = |M_a| = 9$.

1. Proof of Theorem 2

First we give auxiliary results.

Lemma 1. [2, Theorem 3.2] *Let* Γ *be a strongly regular graph with parameters* (v, k, λ, μ) *and with the second eigenvalue* r. *If* g *is an automorphism of* Γ *and* $\Delta = \mathrm{Fix}(g)$, *then*

$$|\Delta| \leq v \cdot \max\{\lambda, \mu\}/(k - r).$$

By Lemma 1, for a strongly regular graph with parameters $(1600, 156, 44, 12)$ we have $|\Delta| \leq 1600 \cdot \max\{44, 12\}/(156 - 36)$, $|\Delta| \leq 586$.

Lemma 2. *Let* Γ *be a distance regular graph with intersection array* $\{39, 36, 4; 1, 1, 36\}$. *Then for intersection numbers of* Γ *the following statements hold*:

(1) $p^1_{11} = 2$, $p^1_{12} = 36$, $p^1_{22} = 1224$, $p^1_{23} = 144$, $p^1_{33} = 12$;

(2) $p^2_{11} = 1$, $p^2_{12} = 34$, $p^2_{13} = 4$, $p^2_{22} = 1229$, $p^2_{23} = 140$, $p^2_{33} = 12$;

(3) $p^3_{12} = 36$, $p^3_{13} = 3$, $p^3_{22} = 1260$, $p^3_{23} = 108$, $p^3_{33} = 44$.

P r o o f. This follows from [1, Lemma 4.1.7]. □

The proofs of Theorems 1 and 2 are based on Higman's method of working with automorphisms of a distance-regular graph, presented in the third chapter of Cameron's book [4].

Let Γ be a distance-regular graph of diameter d with v vertices. Then we have a symmetric association scheme $(X, \mathcal{R})$ with d classes, where X is the set of vertices of Γ and $R_i = \{(u, w) \in X^2 | d(u, w) = i\}$. For a vertex $u \in X$ we set $k_i = |\Gamma_i(u)|$. Let A_i be an adjacency matrix of graph Γ_i. Then $A_i A_j = \sum p^l_{ij} A_l$ for some integer numbers $p^l_{ij} \geq 0$, which are called the intersection numbers. Note that $p^l_{ij} = |\Gamma_i(u) \cap \Gamma_j(w)|$ for any vertices u, w with $d(u, w) = l$.

Let P_i be a matrix in which in the (j, l)-th entry is p^l_{ij}. Then eigenvalues $k = p_1(0), ..., p_1(d)$ of the matrix P_1 are eigenvalues of Γ with multiplicities $m_0 = 1, ..., m_d$, respectively. The matrices P

and Q with $P_{ij} = p_j(i)$ and $Q_{ji} = m_j p_i(j)/k_i$ are called the first and the second eigenmatrices of Γ, respectively, and $PQ = QP = vI$, where I is an identity matrix of order $d+1$.

The permutation representation of the group $G = \mathrm{Aut}(\Gamma)$ on the vertex set of Γ naturally gives the monomial matrix representation ψ of a group G in $GL(v, \mathbb{C})$. The space $\mathbb{C}^v$ is an orthogonal direct sum of the eigenspaces $W_0, W_1, ..., W_d$ of the adjacent matrix $A = A_1$ of Γ. For every $g \in G$, we have $\psi(g)A = A\psi(g)$, so each subspace W_i is $\psi(G)$-invariant. Let χ_i be the character of a representation ψ_{W_i}. Then for $g \in G$ we obtain $\chi_i(g) = v^{-1}\sum_{j=0}^{d} Q_{ij}\alpha_j(g)$, where $\alpha_j(g)$ is the number of vertices x of X such that $d(x, x^g) = j$.

Lemma 3. *Let Γ be a strongly regular graph with parameters $(1600, 156, 44, 12)$ and with the spectrum $156^1, 36^{156}, -4^{1443}$, $G = \mathrm{Aut}(\Gamma)$. If $g \in G$, χ_1 is the character of ψ_{W_1}, where $\dim(W_1) = 156$, then $\alpha_i(g) = \alpha_i(g^l)$ for any natural number l, coprime to $|g|$, $\chi_1(g) = (4\alpha_0(g) + \alpha_1(g))/40 - 4$. Moreover, if $|g| = p$ is a prime, then $\chi_1(g) - 156$ is divisible by p.*

P r o o f. We have

$$Q = \begin{pmatrix} 1 & 1 & 1 \\ 156 & 36 & -4 \\ 1443 & -37 & 3 \end{pmatrix}.$$

So, $\chi_1(g) = (39\alpha_0(g) + 9\alpha_1(g) - \alpha_2(g))/400$. Note that $\alpha_2(g) = 1600 - \alpha_0(g) - \alpha_1(g)$, so $\chi_1(g) = (4\alpha_0(g) + \alpha_1(g))/40 - 4$. The remaining statements of the lemma follow from Lemma 2 [5]. □

Lemma 4. *Let Γ be a distance-regular graph with intersection array $\{39, 36, 4; 1, 1, 36\}$, $G = \mathrm{Aut}(\Gamma)$. If $g \in G$, χ_1 is the character of ψ_{W_1}, where $\dim(W_1) = 675$, χ_2 is the character of ψ_{W_2}, where $\dim(W_2) = 156$, then $\alpha_i(g) = \alpha_i(g^l)$ for any natural number l coprime to $|g|$, $\chi_1(g) = (44\alpha_0(g) + 8\alpha_1(g) - \alpha_3(g))/104 - 25/13$ and $\chi_2(g) = (4\alpha_0(g) + \alpha_3(g))/40 - 4$. Moreover, if $|g| = p$ is a prime, then $\chi_1(g) - 675$ and $\chi_2(g) - 156$ are divisible by p.*

P r o o f. We have

$$Q = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 675 & 1575/13 & -25/13 & -225/13 \\ 156 & -4 & -4 & 36 \\ 768 & -1536/13 & 64/13 & -256/13 \end{pmatrix}.$$

This means $\chi_1(g) = (351\alpha_0(g) + 63\alpha_1(g) - \alpha_2(g) - 9\alpha_3(g))/832$. Note that $\alpha_2(g) = 1600 - \alpha_0(g) - \alpha_1(g) - \alpha_3(g)$, so $\chi_1(g) = (44\alpha_0(g) + 8\alpha_1(g) - \alpha_3(g))/104 - 25/13$.

Similarly, $\chi_2(g) = (39\alpha_0(g) - \alpha_1(g) - \alpha_2(g) + 9\alpha_3(g))/400$. Note that $\alpha_1(g) + \alpha_2(g) = 1600 - \alpha_0(g) - \alpha_3(g)$, so $\chi_2(g) = (4\alpha_0(g) + \alpha_3(g))/40 - 4$.

The remaining statements of this lemma follow from Lemma 2 of [5]. □.

In Lemmas 5–7 we suppose that Γ is a strongly regular graph with parameters $(1600, 156, 44, 12)$, $G = \mathrm{Aut}(\Gamma)$, g is an element of prime order p from G, $\alpha_i(g) = pw_i$ for $i > 0$ and $\Delta = \mathrm{Fix}(g)$. By Delsarts's boundary the maximal order of a clique K in Γ is not greater than $1 - k/\theta_d$, so $|K| \leq 40$. Due to Hoffman's boundary the maximum order of a coclique C in Γ is not greater than $-v\theta_d/(k - \theta_d)$, so $|C| \leq 40$.

Lemma 5. *The following statements hold*:

(1) *if Δ is an empty graph, then either $p = 2$ and $\alpha_1(g) = 80s$ or $p = 5$ and $\alpha_1(g) = 200t$;*

(2) *if Δ is an n-clique, then one of the following statements holds*:

(i) $n = 1$, $p = 2$ *and* $\alpha_1(g) = 80s - 4$, *or* $p = 3$ *and* $\alpha_1(g) = 120t + 36$, *or* $p = 13$ *and* $\alpha_1(g) = 520l + 156$,

(ii) $n \in \{4, 7, 10, ..., 40\}$, $p = 3$ *and* $\alpha_1(g) = 120t + 40 - 4n$,

(iii) $n = 9$, $p = 37$ *and* $\alpha_1(g) = 444$;

(3) *if* Δ *is an* m*-coclique, where* $m > 1$, *then* $p = 2$, $m \in \{4, 6, 8, ..., 40\}$ *and* $\alpha_1(g) = 80s - 4m$ *or* $p = 3$, $m \in \{4, 7, 10, ..., 40\}$ *and* $\alpha_1(g) = 120t + 40 - 4m$;

(4) *if* Δ *contains an edge and is an union of isolated cliques, then* $p = 3$.

P r o o f. Let Δ be an empty graph. As $v = 2^6 \cdot 25$, then p is equal to 2 or 5.

In the case $p = 2$ we have $\chi_1(g) = \alpha_1(g)/40 - 4$ and $\alpha_1(g) = 80s$.

In the case $p = 5$ we have $\chi_1(g) = \alpha_1(g)/40 - 4$ and $\alpha_1(g) = 200t$.

Let Δ be an n-clique. If $n = 1$, then p divides 156 and 1443, therefore $p \in \{2, 3, 13\}$. In the case $p = 2$ we have $\chi_1(g) = (4 + \alpha_1(g))/40 - 4$ and $\alpha_1(g) = 80s - 4$.

In the case $p = 3$ we have $\chi_1(g) = (4+\alpha_1(g))/40-4$ and the number $(4+3w_1)/40$ is congruent to 1 modulo 3. Hence, $4 + 3w_1 = 120t + 40$ and $\alpha_1(g) = 120t + 36$.

In the case $p = 13$ we have $\chi_1(g) = (4+13w_1)/40-4$ and the number $(4+13w_1)/40$ is congruent to 4 modulo 13. Hence, $4 + 13w_1 = 520l + 160$ and $\alpha_1(g) = 520l + 156$.

If $n > 1$, then for any two vertices $a, b \in \Delta$ the element g acts without fixed points on $[a] \cap [b] - \Delta$, on $[a] - b^\perp$ and on $\Gamma - (a^\perp \cup b^\perp)$. Hence, p divides $46 - n$, 111 and 1332, therefore $p \in \{3, 37\}$.

In the case $p = 3$ we have $n \in \{4, 7, 10, ..., 40\}$. Further, $\chi_1(g) = (4n + \alpha_1(g))/40 - 4$ and the number $(4n + \alpha_1(g))/40$ is congruent to 1 modulo 3. Hence, $4n + 3w_1 = 120t + 40$ and $\alpha_1(g) = 120t + 40 - 4n$.

In the case $p = 37$ we have $n = 9$. Further, $\chi_1(g) = (36 + \alpha_1(g))/40 - 4$ and the number $(36 + \alpha_1(g))/40$ is congruent to 12 modulo 37. Hence, $\alpha_1(g) = 444$.

Let Δ be an m-coclique, where $m > 1$. Then for any two vertices $a, b \in \Delta$ the element g acts without fixed points on $[a] \cap [b]$, on $[a] - b^\perp$ and on $\Gamma - (a^\perp \cup b^\perp \cup \Delta)$. Hence, p divides 12, 144 and $1300 - m$, therefore $p \in \{2, 3\}$.

In the case $p = 2$ we have $m \in \{4, 6, 8, ..., 40\}$. Further, $\chi_1(g) = (4m + \alpha_1(g))/40 - 4$ and the number $(4m + \alpha_1(g))/40$ is even. Hence, $\alpha_1(g) = 80s - 4m$.

In the case $p = 3$ we have $m \in \{4, 7, 10, ..., 40\}$. Further, $\chi_1(g) = (4m + \alpha_1(g))/40 - 4$ and the number $(4m + \alpha_1(g))/40$ is congruent to 1 modulo 3. Hence, $\alpha_1(g) = 120t + 40 - 4m$.

Let Δ contains an edge and is a union of isolated cliques. Then p divides 12 and 111, therefore $p = 3$. □

Lemma 6. *If* $[a] \subset \Delta$ *for some vertex* a, *then for any vertex* $u \in \Gamma_2(a) - \Delta$ *the orbit of* $u^{\langle g \rangle}$ *is a clique or a coclique, and one of the following statements holds:*

(1) *if on* $\Gamma - \Delta$ *there are no coclique orbits, then* $\alpha_1(g) = 1600 - \alpha_0(g)$, $\alpha_0(g) = 40l$ *and either*

(i) $l = 4$, $p = 2, 3$ *or*

(ii) $l = 5$, $p = 5, 7$, *or*

(iii) $l = 6$, $p = 2$, *or*

(iv) $l = 7$, $p = 3, 11$, *or*

(v) $l = 8$, $p = 2$, *or*

(vi) $l = 10$, $p = 2, 3, 5$, *or*

(vii) $l = 12$, $p = 2, 7$, *or*

(viii) $l = 13$, $p = 3$, *or*

(ix) $l = 14$, $p = 2$;

(2) *if on* $\Gamma - \Delta$ *there is a coclique orbit, then* $p \leq 3$, *and if* $a^\perp = \Delta$, *then* $p = 3$ *and* $\alpha_1(g) = 120l + 12$.

P r o o f. Let $[a] \subset \Delta$ for some vertex a. Then for any vertex $u \in \Gamma_2(a) - \Delta$ the orbit $u^{\langle g \rangle}$ doesn't contain a geodesic 2-pathes and is a clique or a coclique.

In the case $p \geq 13$ a subgraph $[a] \cap [u]$ is a 12-clique and for two vertices $b, c \in [a] \cap [u]$ a subgraph $[b] \cap [c]$ contains a, 10 vertices from $[a] \cap [u]$ and p vertices from $u^{\langle g \rangle}$, so $11 + p \leq 44$, therefore $p \leq 31$.

If on $\Gamma - \Delta$ there are no coclique orbits, then $\alpha_1(g) = v - |\Delta|$ and for a vertex $u' \in u^{\langle g \rangle} - \{u\}$ a subgraph $[u] \cap [u']$ contains $p - 2$ vertices from $u^{\langle g \rangle}$ and 12 vertices from Δ. Further, $\chi_1(g) = (3\alpha_0(g) + 1600)/40 - 4$, $\chi_1(g) - 156$ is divisible by p and p divides $3\alpha_0(g)/40 - 120$. We denote $\alpha_0(g) = 40l$. Then $4 \leq l \leq 14$, p divides $40(40 - l)$ and $3(40 - l)$. Thus, either $l = 4$, $p = 2, 3$, or $l = 5$, $p = 5, 7$, or $l = 6$, $p = 2, 17$, or $l = 7$, $p = 3, 11$, or $l = 8$, $p = 2$, or $l = 9$, $p = 31$, or $l = 10$, $p = 2, 3, 5$, or $l = 11$, $p = 29$, or $l = 12$, $p = 2, 7$, or $l = 13$, $p = 3$, or $l = 14$, $p = 2, 13$. In the case $p \geq 13$ a subgraph $[a] \cap [u]$ is a 12-clique and $p \leq 23$.

Let $p = 17$ and $b \in \Delta - a^\perp$. Then $|\Delta(b) - a^\perp| \leq 82$ and $|[b] - \Delta| \geq 68$. For $w \in [b] - \Delta$ we have $[a] \cap [w] = [a] \cap [b]$ (otherwise $w^{\langle g \rangle}$ is contained in $[b] \cap [c]$ for $c \in [a] \cap [w] - [b]$). A contradiction with a fact that for two vertices $c, d \in [a] \cap [w]$ a subgraph $[c] \cap [d]$ contains 68 vertices from $[b] - \Delta$.

Let $p = 13$. Then $|\Delta - a^\perp| = 403$. If $b \in \Delta - a^\perp$ and $|[b] - \Delta| = 13$, then for any $w \in [b] - \Delta$ we have $[a] \cap [w] = [a] \cap [b]$ (otherwise $w^{\langle g \rangle}$ is contained in $[b] \cap [c]$ for a vertex $c \in [a] \cap [w] - [b]$). Further, $[b] \cap [w]$ contains 12 vertices from $w^{\langle g \rangle}$ and 32 vertices from $\Delta(b)$. Hence, for $w' \in w^{\langle g \rangle} - \{w\}$ a subgraph $[w] \cap [w']$ contains b, 32-clique from $\Delta(b)$ and 11 vertices from $w^{\langle g \rangle}$. A contradiction with a fact that the order of a clique in Γ is not greater than 40.

If $b \in \Delta - a^\perp$ and $|[b] - \Delta| = 26$, then $[b] - \Delta = u^{\langle g \rangle} \cup w^{\langle g \rangle}$. As above, $[a] \cap [u] = [a] \cap [w] = [a] \cap [b]$, therefore a subgraph $\{b\} \cup ([a] \cap [b]) \cup u^{\langle g \rangle} \cup w^{\langle g \rangle}$ is a 39-clique. If $e \in [u] \cap \Delta(b) - [w]$, then $[e] \cap [w]$ contains 13 vertices from $u^{\langle g \rangle}$, a contradiction. So, $\{b\} \cup ([u] \cap \Delta(b)) \cup u^{\langle g \rangle} \cup w^{\langle g \rangle}$ is a 46-coclique, a contradiction. If $b \in \Delta - a^\perp$ and $|[b] - \Delta| \geq 39$, then for any two vertices $c, d \in [a] \cap [b]$ a subgraph $[c] \cap [d]$ contains a, b and 39 vertices from $[b] - \Delta$, a contradiction. Statement (1) is proved.

Let on $\Gamma - \Delta$ there is a coclique orbit $u^{\langle g \rangle}$. Then $[u^{g_i}] \cap [u^{g_j}]$ does not intersect $\Gamma - \Delta$ for distinct vertices u^{g_i}, u^{g_j}, so $145p \leq |\Gamma - \Delta| \leq 1443$, therefore $p \leq 7$.

Let us show that $p \leq 3$.

Let $c \in [a] \cap [u]$ and $[c] \cap [u]$ contains exactly γ vertices from $[a] \cap [u]$. Then $[c] \cap [u]$ contains $44 - \gamma$ vertices outside of Δ (lying in distinct $\langle g \rangle$-orbits) and $p(44 - \gamma) \leq |[c] - \Delta| \leq 156 - 45 = 111$. Hence, $32p \leq 111$.

If $a^\perp = \Delta$, then $\alpha_0(g) = 157$, p divides 1443 and $p = 3$. Further, $\chi_1(g) = (628 + \alpha_1(g))/40 - 4$, $(628 + \alpha_1(g))/40$ is congruent to 1 modulo 3 and $\alpha_1(g) = 120l + 12$. □

Lemma 7. *The following statements hold*:

(1) Γ *does not contain proper strongly regular subgraphs with parameters* $(v', k', 44, 12)$;

(2) $p \leq 43$.

P r o o f. Assume that Γ contains proper strongly regular subgraph Σ with parameters $(v', k', 44, 12)$. Then $4(k' - 12) + 32^2 = n^2$, therefore $n = 2l$, $k' = l^2 - 244$, $l \geq 16$, Σ has nonprincipal eigenvalues $16 + l, 16 - l$ and multiplicity of $16 + l$ is equal to $(l - 17)(l^2 - 244)(l^2 + l - 260)/24l$. If l is odd, then 8 divides $(l - 17)(l^2 + l - 20)$, l divides $17 \cdot 61 \cdot 65$ and $l \in \{5, 13\}$. If l is even, then 3 divides $(l - 2)(l^2 - 1)(l^2 + l - 2)$ and $l = 16$. In all cases we have contradictions.

If $p \geq 47$, then Δ is a strongly regular graph with parameters $(v', k', 44, 12)$, so $\Delta = \Gamma$, a contradiction. □

Theorem 2 follows from Lemmas 5–7.

2. Proof of Theorem 1

In Lemmas 8–9 it is assumed that Γ is a distance-regular graph with intersection array $\{39,36,4;1,1,36\}$, $G=\mathrm{Aut}(\Gamma)$, g is an element of prime order p from G, $\alpha_i(g)=pw_i$ for $i>0$ and $\Omega=\mathrm{Fix}(g)$.

Lemma 8. *The following statements hold*:

(1) *if Ω is an empty graph, then either $p=2$, $\alpha_1(g)=10r+26m+12$ and $\alpha_3(g)=80r=1600-\alpha_1(g)$ or $p=5$, $\alpha_1(g)=65n+10l+10$ and $\alpha_3(g)=200l$;*

(2) *if Ω is an n-clique, then one of the following statements holds*:

(*i*) *$n=1$, $p=3$, $\alpha_1(g)=15l+24+39m$ and $\alpha_3(g)=120l+36$,*

(*ii*) *$n=2$, $p=2$, $\alpha_1(g)=10l+26m$ and $\alpha_3(g)=80l-8$,*

(*iii*) *or $n=4$, $p=2$, $\alpha_1(g)=10l+26m+14$ and $\alpha_3(g)=80l-16$ or $p=3$, $\alpha_1(g)=10l+39m+1$, l is congruent to -1 modulo 3 and $\alpha_3(g)=120l+24$;*

(3) *if Ω consists of n vertices at distance 3 in Γ, then $p=3$, $n\in\{4,7,10,...,40\}$, $\alpha_3(g)=120l+40-4n$ and $\alpha_1(g)=15l+30+39m-6n$;*

(4) *if Ω contains an edge and doesn't contain vertices at distance 2 in Γ, then Ω is an union of isolated cliques and any two vertices from different cliques are at distance 3 in Γ, either $p=3$ and the orders of these cliques are equal to 1 or 4, or $p=2$ and the orders of these cliques are equal to 2 or 4.*

P r o o f. Let Ω be an empty graph and $\alpha_i(g)=pw_i$ for $i\geq 1$. As $v=1600$, then p is equal to 2 or 5.

Let $p=2$. Then $w_1+w_2+w_3=800$ and $\chi_2(g)=w_3/20-4$. Hence, $w_3=40r$. Further, the number $\chi_1(g)=(2w_1-10r-25)/13$ is odd, therefore $w_1=13m+6+5r$. Finally, $\alpha_2(g)=0$ (if $d(u,u^g)=2$, then the only vertex from $[u]\cap[u^g]$ belongs to Ω, a contradiction). Therefore $\alpha_1(g)=10r+26m+12=1600-80r$.

Let $p=5$. Then $w_1+w_2+w_3=320$ and $\chi_2(g)=w_3/8-4$. Hence, $w_3=40l$. Finally, $\chi_1(g)=(5w_1-25l-25)/13$, therefore $w_1=13n+5l+5$. Statement (1) is proved.

Let Ω be an n-clique. If $n=1$, then p divides 39 and 315, therefore $p=3$. We have $\chi_1(g)=(8\alpha_1(g)-\alpha_3(g)-156)/104$, $\chi_2(g)=(4+\alpha_3(g))/40-4$. Therefore the number $(4+\alpha_3(g))/40$ is congruent to 1 modulo 3, $\alpha_3(g)=120l+36$ and the number $\chi_1(g)=(\alpha_1(g)-15l-24)/13$ is divisible by 3. Hence, $\alpha_1(g)=15l+24+39m$.

If $n>1$, then p divides $4-n$ and 36, therefore either $n=2$, $p=2$, or $n=4$, $p=2,3$. In the first case the number $\chi_2(g)=(8+\alpha_3(g))/40-4$ is even and $\alpha_3(g)=80l-8$. Further, the number $\chi_1(g)=(\alpha_1(g)-10l)/13-1$ is odd and $\alpha_1(g)=10l+26m$. In the second case $\chi_2(g)=(16+\alpha_3(g))/40-4$ and either $p=2$, $\alpha_3(g)=80l-16$, or $p=3$ and $\alpha_3(g)=120l+24$. Further, $\chi_1(g)=(176+8\alpha_1(g)-\alpha_3(g))/104-25/13$ and either $p=2$, $\alpha_1(g)=10l+26m+14$, or $p=3$ and $\alpha_1(g)=10l+39m+1$, l is congruent to -1 modulo 3.

Let Ω consists of n vertices at distance 3. As $p^3_{13}=3$, $p^3_{33}=44$, then p divides 3 and $46-n$. Hence, $p=3$ and $n\in\{4,7,10,...,40\}$. We have $\chi_2(g)=(4n+\alpha_3(g))/40-4$ and the number $(4n+\alpha_3(g))/40$ is congruent to 1 modulo 3, therefore $\alpha_3(g)=120l+40-4n$. Further, the number $\chi_1(g)=(6n+\alpha_1(g)-15l-30)/13$ is divisible by 3 and $\alpha_1(g)=15l+30+39m-6n$.

Let Ω contains an edge and does not contain vertices at distance 2 in Γ. Then Ω is an union of isolated cliques, any two vertices from distinct cliques are at distance 3 in Γ. As orders of these cliques are at most 4, then $p\leq 3$. If $p=3$, then the orders of these cliques are equal to 1 or 4. If $p=2$, then the orders of these cliques are equal to 2 or 4. □

Lemma 9. *If Ω contains vertices a, b at distance 2 in Γ, then $p \leq 3$.*

P r o o f. Let Ω contains vertices a, b at distance 2 in Γ and Ω_0 is a connected component of Ω containing a, b.

Assume that the diameter of graph Ω_0 is equal to 2. Then by [1, 1.17.1] one of the following statements holds:

(*i*) $\Omega_0 \subseteq a^\perp$ and $\Omega_0(a)$ is an union of isolated cliques;

(*ii*) Ω_0 ia a strongly regular graph;

(*iii*) Ω_0 is a biregular graph with degrees of vertices α, β, where $\alpha < \beta$, and if A and B are sets of vertices from Ω_0 with degrees α and β, then A is a coclique, the lines between A and B have order 2, the lines from B have order $l = \beta - \alpha + 2 > 2$, and $|\Omega_0| = \alpha\beta + 1$.

Last case is impossible because $c_2 = 1$ in Γ.

In the case (*i*) we have $p \in \{2, 3\}$ because of $p^1_{33} = 12$.

In the case (*ii*) either $p = 2$ and Ω_0 is the pentagon, Petersen graph or Hoffman-Singletone graph, or $p > 2$ and Ω_0 is a strongly regular graph with parameters $(v', k', 2, 1)$.

Let $p > 2$. Then $\Omega(a)$ consists of e isolated triangles and either $e = 1$, $p = 3$, or $e = 2$, $p = 3, 11$, or $e = 3$, $p = 3, 5$, or $e = 4$, $p = 3$, or $e = 5$, $p = 3$, or $e = 6$, $p = 3, 7$, or $e = 7$, $p = 3$, or $e = 8$, $p = 3, 5$, or $e \geq 9$, $p = 3$.

In case $p = 11$ graph Ω is a regular graph of degree 6, $|\Omega \cap \Gamma_2(a)| = 18$, $|\Omega \cap \Gamma_3(a)| = 24$ and $|\Gamma_3(a) - \Omega|$ is not divisible by 11.

In case $p = 7$ graph Ω is a regular graph of degree 18, $|\Omega \cap \Gamma_2(a)| = 270$, $|\Omega \cap \Gamma_3(a)| = 270 \cdot 4/15 = 64$ and $|\Gamma_3(a) - \Omega|$ is not divisible by 7.

In case $p = 5$ graph Ω contains vertices of degrees 9 and 24. Assume that $|\Omega(a)| = 24$, $\Omega(a)$ contains β vertices of degree 24 in Ω and $\Omega_3(a)$ contains γ vertices of degree 24 in Ω. Then the number $21\beta + 6(24 - \beta) = |\Omega \cap \Gamma_2(a)|$ is congruent to 4 modulo 5 and $4|\Omega \cap \Gamma_2(a)| = 21\gamma + 6(|\Omega \cap \Gamma_3(a)| - \gamma)$. Hence, $|\Gamma_2(a) \cap \Omega| = (144 + 15\beta)$ and $576 + 60\beta = 15\gamma + 6|\Omega \cap \Gamma_3(a)|$, a contradiction with the fact that $|\Omega \cap \Gamma_3(a)|$ is divisible by 5.

So, Ω is an amply regular graph with parameters $(v', 9, 2, 1)$, $54 = |\Omega \cap \Gamma_2(a)|$ and $|\Omega \cap \Gamma_3(a)| = 36$. Again we have a contradiction with the fact that $|\Omega \cap \Gamma_3(a)|$ is divisible by 5.

The lemma is proved. □

Theorem 1 follows from Lemmas 8–9.

3. Proof of Corollary 1

Until the end of the paper we will assume that Γ is a distance-regular graph with intersection array $\{39, 36, 4; 1, 1, 36\}$ and the nonsolvable group $G = \mathrm{Aut}(\Gamma)$ acts transitively on the set of vertices of this graph. For the vertex $a \in \Gamma$ we get $|G : G_a| = 1600$. In view of Theorem 1 we have $p \in \{2, 3, 5\}$. Let $\bar{T}$ be the socle of the group $\bar{G} = G/O_{5'}(G)$.

Lemma 10. *If f is an element of order 5 of G, g is an element of order $p < 5$ of $C_G(f)$ and $\Omega = \mathrm{Fix}(g)$, then one of the following statements holds*:

(1) *Ω is an empty graph, $p = 2$, $\alpha_3(g) = 80r$, $r \leq 19$, $\alpha_1(g) = 10r + 26m + 12 = 1600 - 80r$, and $m \in \{-7, -2, 3, 8, ..., 58\}$;*

(2) *Ω consists of n vertices at distance 3 in Γ, $p = 3$, $n \in \{10, 25, 40\}$, $\alpha_3(g) = 120l + 40 - 4n$, $\alpha_1(g) + \alpha_3(g) = 135l - 10n + 39m + 70 \leq 1600$ and m is divisible by 5;*

(3) *$p = 3$, $\alpha_3(g) = 120s$, $\alpha_0(g) = 30t + 10$, $\alpha_1(g) = 39l - 165t + 15s - 30$ or $\alpha_3(g) = 120s + 60$, $\alpha_0(g) = 30t - 5$, $\alpha_1(g) = 195l - 165t + 15s + 60$;*

(4) *$p = 2$, $\alpha_3(g) = 80s - 4\alpha_0(g)$ and $\alpha_1(g) = 10s + 26l + 38 - 6\alpha_0(g)$, l is congruent to 2 modulo 5.*

P r o o f. In view of Theorem 1 Fix(f) is empty graph, $\alpha_1(f) = 65n+10l+10$ and $\alpha_3(f) = 200l$.

If Ω is an empty graph, then $p = 2$, $\alpha_3(g) = 80r$ and $\alpha_1(g) = 10r + 26m + 12 = 1600 - 80r$ is divisible by 5. Hence, $13m + 6$ is divisible by 5 and $m \in \{-7, -2, 3, 8, ..., 58\}$. Finally, $26m + 12 = 1600 - 90r$, therefore m is congruent to 2 modulo 3 and $m \in \{-7, 8, 23, 38, 53\}$.

If Ω is an n-clique, then n is divisible by 5, we have got a contradiction.

If Ω consists of n vertices at distance 3 in Γ, then $p = 3$, $n \in \{10, 25, 40\}$, the numbers $\alpha_3(g) = 120l + 40 - 4n$ and $\alpha_1(g) = 15l + 30 + 39m - 6n$ are divisible by 5. Hence, m is divisible by 5, $\alpha_1(g) + \alpha_3(g) = 135l - 10n + 39m + 70 \le 1600$.

If $p = 3$, then $\chi_2(g) = (4\alpha_0(g)+\alpha_3(g))/40-4$ and the number $(4\alpha_0(g)+\alpha_3(g))/40$ is congruent to 1 modulo 3. Further, the number $\chi_1(g) = (44\alpha_0(g) + 8\alpha_1(g) - \alpha_3(g))/104 - 25/13$ is divisible by 3, $\alpha_3(g)$ is divisible by 60. If $\alpha_3(g) = 120s$, then $\alpha_0(g) = 30t+10$, $\alpha_1(g) = 39l-165t+15s-30$. If $\alpha_3(g) = 120s + 60$, then $\alpha_0(g) = 30t - 5$, $\alpha_1(g) = 195l - 165t + 15s + 60$.

If $p = 2$, then $\chi_2(g) = (4\alpha_0(g) + \alpha_3(g))/40 - 4$, $4\alpha_0(g) + \alpha_3(g) = 80s$. Further, $\alpha_1(g) = -6\alpha_0(g) + 10s + 26l + 38$ and $13l + 19$ is divisible by 5, therefore $l \in \{2, 7, ...\}$. Finally, $1600 - 5\alpha_0(g) + 80s = -6\alpha_0(g) + 10s + 26l + 38$, $1600 = -70s - \alpha_0(g) + 26l + 38$. □

Lemma 11. *The following statements hold*:

(1) $\bar{T} = L \times M$, *and each of subgroups* L, M *is isomorphic to one of the following groups* Z_5, A_5, A_6 *or* $PSp(4, 3)$;

(2) *in case* $|\bar{T} : \bar{T}_a| = 40^2$ *we have* $O_{5'}(G) = 1$ *and this case is realized if one of the following statements holds*:

(*i*) $L \cong M \cong PSp(4, 3)$, *or*

(*ii*) $L \cong PSp(4, 3)$, $|L : L_a| = 40$, $M \cong A_6$ *and* $|M_a| = 9$, *or*

(*iii*) $L \cong M \cong A_6$ *and* $|L_a| = |M_a| = 9$.

P r o o f. Recall that a nonabelian simple $\{2, 3, 5\}$-group is isomorphic to A_5, A_6 or $PSp(4, 3)$ (see, [6, Table 1]). Hence, in view of Theorem 1 we have $\bar{T} = L \times M$, each of subgroups L, M is isomorphic to one of the following groups A_5, A_6 or $PSp(4, 3)$.

If $\bar{T} \cong PSp(4, 3)$, then the group $\bar{T}_a$ has an index 40 in $\bar{T}$ and is isomorphic to $E_9.SL_2(3)$ or $E_{27}.S_4$.

If $\bar{T} \cong A_6$, then the group $\bar{T}_a$ has an index in $\bar{T}$, divisible by 10, and dividing 40.

If $\bar{T} \cong A_5$, then the group $\bar{T}_a$ has an index in $\bar{T}$, divisible by 10, and dividing 20.

In case $|\bar{T} : \bar{T}_a| = 40^2$ we have $O_{5'}(G) = 1$ and this case is realized if one of the following statements holds: either $L \cong M \cong PSp(4, 3)$, or $L \cong PSp(4, 3)$, $M \cong A_6$ $|M_a| = 9$, or $L \cong M \cong A_6$ and $|L_a| = |M_a| = 9$. □

Corollary is proved.

4. Conclusion

We found possible automorphisms of a distance-regular graph with intersection array $\{39, 36, 4; 1, 1, 36\}$. In particular this graph is not arc-transitive.

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