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A MODEL OF AGE–STRUCTURED POPULATION UNDER STOCHASTIC PERTURBATION OF DEATH AND BIRTH RATES[1]

Maxim A. Alshanskiy

Ural Federal University, Ekaterinburg, Russia
m.a.alshansky@urfu.ru

Abstract: Under consideration is construction of a model of age-structured population reflecting random oscillations of the death and birth rate functions. We arrive at an Itô-type difference equation in a Hilbert space of functions which can not be transformed into a proper Itô equation via passing to the limit procedure due to the properties of the operator coefficients. We suggest overcoming the obstacle by building the model in a space of Hilbert space valued generalized random variables where it has the form of an operator-differential equation with multiplicative noise. The result on existence and uniqueness of the solution to the obtained equation is stated.

Key words: Brownian sheet, Cylindrical Wiener process, Gaussian white noise, Stochastic differential equation, Age-structured population model.

Introduction

A well known model of an age-structured population dynamics is the famous McKendrick–von Foerster equation

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial u(x,t)}{\partial x} = -m(x)u(x,t), \tag{0.1}$$

where $u(x,t)$ is density of the population at age x at time t (so, that $\int\limits_{x_1}^{x_2} u(s,t)ds$ is the number of individuals with the age belonging to $[x_1; x_2]$ at the time t) and $m(x)$ is the death rate. The usual assumption is that the age of individuals is limited, say $x \in [0;1]$. The process of reproduction is modeled by the boundary condition

$$u(0,t) = \int_0^1 b(x)u(x,t)\,dx. \tag{0.2}$$

Here $b(x)$ is the birth rate which describes the reproductive capacity of the population with respect to age. The model would be more realistic if it reflected random oscillations of the rates of death and birth. Presence of these oscillations can be considered as the result of superposition of multitude of factors connected with different aspects of vital activity of the individuals in the population as well as with unpredictable changes in the environment connected with its physical nature, with food supply, vital activity of competing populations, predators and so on. The assumption of randomness of the oscillations is the way of avoiding unnecessary complication of the model that

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occurs when one tries to reflect the interaction of all these factors which are often hardly subject to formalization.

Stochastically perturbed McKendrick–von Furster equation was for the first time introduced in [10] in a straightforward way by adding a term containing Gaussian white noise and having form $g(t,u)\dot{W}(t)$ (or equivalently $g(t,u)dW(t)$ in the corresponding Itô equation), where $W(t)$ is a Hilbert space valued Wiener process and $g(t,\cdot)$ maps the Hilbert space H, where u considered as a function of t takes values, onto the space of linear bounded operators acting from the separable Hilbert space K, where the values of $W(t)$ lie, to H. In an analogous fashion in [7] was introduced the McKendrick–von Furster equation perturbed with the Levy noise. However both works do not consider the question of choice of appropriate mapping $g(t,\cdot)$.

The aim of our work is clarification of this question in order consistent with the desired properties of the noisy influence on the population.

Since both of the rates are described by functions m and b of age $x \in [0;1]$, it seems natural to model these oscillations by appropriate random processes taking values in spaces of functions of x and to build a model having form of a stochastic equation in such a space. In the present work we discuss problems that arise in building such a model.

We start with a difference equation for the increment of the number of individuals belonging to a small segment of length Δx of the age scale during a small period of time Δt. In section 1 we show that a Brownian sheet naturally arises in modelling the random fluctuations of the death rate. Crucial assumption here is independence between fluctuations of per capita amounts of dead individuals at disjoint segments of the age scale or the time line.

In section 2 we consider passage to limit in the obtained difference equation when Δx tends to zero. We show that the obstacle connected with non-differentiability of the Brownian sheet can be overcome with the help of the concept of a cylindrical random variable on a Hilbert space. Thus, we obtain a difference equation for the increments of the density of the population in a Hilbert space H of functions of $x \in [0;1]$. We show that the random fluctuations of the death rate can be modeled by increments of a cylindrical Wiener process. We also show how this idea can be implemented in modeling the random fluctuations of the rate of birth.

In section 3 we discuss difficulties that arise when we attempt to convert the difference equation into a stochastic differential equation in the Hilbert space H. We show that the use of the theory of Itô-type stochastic differential equations in infinite dimensional Hilbert spaces (see the review of the theory in [5, 6]) is limited due to the properties of the operator coefficients in the difference equation obtained on the previous step. The necessary requirement for the operator-valued integrand of a well defined Itô integral with respect to a cylindrical Wiener process is the condition of being a Hilbert–Schmidt operator, which is not the case here. The way out can be found in setting the equation in the space $(\mathcal{S})_{-\rho}(H)$ of H-valued generalized random variables introduced and studied in [2, 8, 9]. Cylindrical Wiener process $W(t)$ considered a function of t with values in this space happens to be differentiable with the derivative $W'(t) = \mathbb{W}(t)$ being the cylindrical H-valued white noise. We use the established in [3] connection between the Itô integral with respect to a cylindrical Wiener process and the Hitsuda–Skorohod integral. Thus, we finally arrive at a model having form of an operator-differential equation in $(\mathcal{S})_{-\rho}(H)$ and formulate the existence and uniqueness result for the Cauchy problem for this equation.

1. Difference equation

Consider evolution of the population density $u(x,t)$ of an age-structured population, where $x \in [0;1]$ is age, $t \geq 0$ is time. Given $x \in [0;1]$ and $t \geq 0$ we consider the change of the number of individuals that belong to a small segment $[x; x+\Delta x]$ at the moment t during a small time interval

$[t; t + \Delta t]$:

$$\int\limits_{x+\Delta t}^{x+\Delta x+\Delta t} u(s, t + \Delta t)ds - \int\limits_{x}^{x+\Delta x} u(s, t)ds = u(x + \Delta x, t + \Delta t)\Delta x - u(x, t)\Delta x + o(\Delta x).$$

Suppose the change is due to death of individuals and $m(x)$ is the expected rate of death at age x, i.e. $m(x)\Delta x + o(\Delta x)$ is the mean number of dead in the age segment $[x; x + \Delta x]$ in a unit time under constant unit density with respect to age. Suppose also that the population replenishment is due to reproduction which is characterized by the birth rate function $b(x)$ and is described by the boundary condition (0.2). Now let the death rate be subject to random fluctuations, so that omitting the $o(\Delta x)$'s we arrive at the following equation:

$$u(x + \Delta t, t + \Delta t)\Delta x - u(x, t)\Delta x = -u(x, t)m(x)\Delta x\Delta t + u(x, t)\Delta\eta, \tag{1.1}$$

where $\Delta\eta = \Delta\eta^{x,t}_{\Delta x,\Delta t}$ is the random increment of the number of dead individuals in an arbitrary age segment $[x; x + \Delta x]$ during the time $[t; t + \Delta t]$ under constant unit density of population.

The individuals belonging to the age segment $[x; x + \Delta x]$ at the moment t move along the age scale as the time goes and get into the segment $[x + \Delta t; x + \Delta t + \Delta x]$ at the time $t + \Delta t$. This suggests a natural parametrization of the introduced family of random variables by means of parallelograms $\Pi^{x,t}_{\Delta x,\Delta t}$ (the upper and the right sides are supposed to be excluded, see figure 1):

$$\Delta\eta = \Delta\eta^{x,t}_{\Delta x,\Delta t} = \Delta\eta\left(\Pi^{x,t}_{\Delta x,\Delta t}\right). \tag{1.2}$$

We will suppose that the following hypothesis holds.

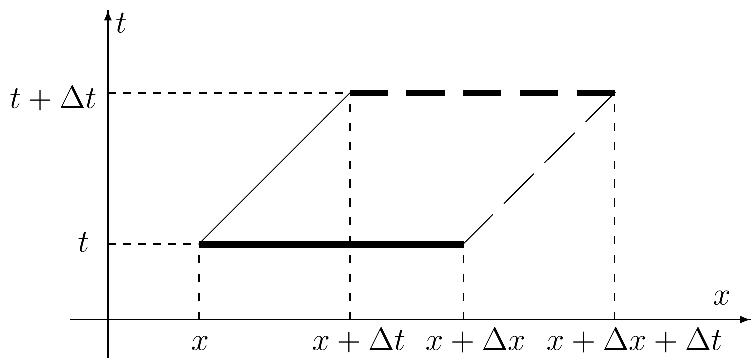

Figure 1. Parallelogram $\Pi^{x,t}_{\Delta x,\Delta t}$.

Hypothesis 1. $\Delta\eta\left(\Pi^{x_k,t_k}_{\Delta x_k,\Delta t_k}\right)$, $k = 1, \ldots, n$, $n \in \mathbb{N}$, *are independent if the parallelograms* $\Pi^{x_k,t_k}_{\Delta x_k,\Delta t_k}$ *are pairwise disjoint.*

Given arbitrary segments $[x; x + \Delta x]$ and $[t; t + \Delta t]$ consider the uniform partition $\{x_k\}$ of $[x; x + \Delta x]$, where $x_k = x + k\Delta x/n$, $k = 0, 1, \ldots, n$ and the corresponding decomposition $\Pi^{x,t}_{\Delta x,\Delta t} = \bigcup_{k=0}^{n-1} \Pi^{x_k,t}_{\Delta x/n,\Delta t}$. The definition of $\Delta\eta$'s implies the following "additivity" for them:

$$\Delta\eta\left(\Pi^{x,t}_{\Delta x,\Delta t}\right) = \sum_{k=0}^{n-1} \Delta\eta\left(\Pi^{x_k,t}_{\Delta x/n,\Delta t}\right).$$

Due to the Hypothesis 1 it follows

$$\mathrm{Var}\left[\Delta\eta\left(\Pi^{x,t}_{\Delta x,\Delta t}\right)\right]=\sum_{k=0}^{n-1}\mathrm{Var}\left[\Delta\eta\left(\Pi^{x_k,t}_{\Delta x/n,\Delta t}\right)\right].$$

This condition will be fulfilled if we let

$$\Delta\eta\left(\Pi^{x,t}_{\Delta x,\Delta t}\right)=\begin{cases}\gamma\sqrt{\Delta x},\ \text{with probability } \lambda\Delta t,\\ 0,\ \text{with probability } 1-2\lambda\Delta t,\\ -\gamma\sqrt{\Delta x},\ \text{with probability } \lambda\Delta t,\end{cases}\tag{1.3}$$

for any $x, t, \Delta x, \Delta t$. Here γ and λ are some proportionality factors. This is true since we have

$$\Delta\eta\left(\Pi^{x_k,t}_{\Delta x/n,\Delta t}\right)=\begin{cases}\gamma\sqrt{\dfrac{\Delta x}{n}},\ \text{with probability } \lambda\Delta t,\\ 0,\ \text{with probability } 1-2\lambda\Delta t,\\ -\gamma\sqrt{\dfrac{\Delta x}{n}},\ \text{with probability } \lambda\Delta t,\end{cases}$$

and therefore

$$\mathrm{Var}\left[\Delta\eta\left(\Pi^{x_k,t}_{\Delta x/n,\Delta t}\right)\right]=\gamma^2\frac{\Delta x}{n}2\lambda\Delta t.\tag{1.4}$$

Note that the Central Limit Theorem holds for the sequence of series of random variables $\{\xi_k^{(n)}\}_{k=1}^n$, $n=1,2,\dots$, where $\xi_k^{(n)}=\Delta\eta\left(\Pi^{x_k,t}_{\Delta x/n,\Delta t}\right)$, since $\xi_k^{(n)}$ are independent and identically distributed with $E\xi_k^{(n)}=0$ and $\mathrm{Var}\,\xi_k^{(n)}$ given by (1.4). By the Central Limit Theorem we conclude that the distribution of $\dfrac{1}{\gamma\sqrt{2\lambda\Delta x\Delta t}}\displaystyle\sum_{i=1}^{n}\xi_i^{(n)}$ converges to standard Gaussian when $n\to\infty$. So, the Hypothesis 1 together with the additivity property (1) makes it natural to impose the following hypothesis.

Hypothesis 2. $\Delta\eta\left(\Pi^{x,t}_{\Delta x,\Delta t}\right)\sim N\left(0,2\lambda\gamma^2\Delta x\Delta t\right)$.

Definition 1. *The collection of random variables $\{\Theta(B), B\in\mathcal{B}(\mathbb{R}^2)\}$ is called a Gaussian orthogonal measure on the Borel σ-field $\mathcal{B}(\mathbb{R}^2)$ if the following holds:*

1. *$\Theta(B)\sim N(0,\mu_L(B))$ for all $B\in\mathcal{B}(\mathbb{R}^2)$, where μ_L is the Lebesque measure of B;*
2. *$B_1\cap B_2=\varnothing$ implies $\Theta(B_1)$ and $\Theta(B_2)$ are independent for all $B_1,B_2\in\mathcal{B}(\mathbb{R}^2)$;*
3. *$\Theta(\cup_{k=1}^\infty B_k)=\sum_{k=1}^\infty\Theta(B_k)$ (the series is mean square convergent) for any sequence $\{B_k\}\subset\mathcal{B}(\mathbb{R}^2)$ of pairwise disjoint sets.*

Hypotheses 1 and 2 imply that $\Delta\eta=\gamma\sqrt{2\lambda}\Theta$, where Θ is a Gaussian orthogonal measure on $\mathcal{B}(\mathbb{R}^2)$. Thus, equation (1.1) turns into

$$u(x+\Delta t,t+\Delta t)\Delta x-u(x,t)\Delta x=-u(x,t)m(x)\Delta x\Delta t+\alpha_0u(x,t)\Theta\left(\Pi^{x_k,t}_{\Delta x,\Delta t}\right),\tag{1.5}$$

where $\alpha_0\in\mathbb{R}$ is a constant.

Definition 2. [1, p. 649] *A two-parameter Gaussian random process $\{\mathbb{B}(x,t)\,, x \geq 0, t \geq 0\}$ is called a Brownian sheet if it satisfies the following conditions:*

1. $E\left[\mathbb{B}(x,t)\right] = 0\,,$ *for all* $x, t \geq 0$*;*

2. $\operatorname{Cov}\left(\mathbb{B}(x_1,t_1), \mathbb{B}(x_2,t_2)\right) = \min\{x_1; x_2\} \cdot \min\{t_1; t_2\}$ *for all* $x_1, x_2, t_1, t_2 \geq 0$.

In [4, Definition 12, p. 107] a random process, satisfying the conditions of Definition 2 is called a Wiener–Chentsov random field.

It is easy to see that the random process defined by

$$\mathbb{B}(x,t) := \Theta\left(\Pi^{0,0}_{x,t}\right), \quad x,t \geq 0 \tag{1.6}$$

is a Brownian sheet.

Note that a Brownian sheet on $[0;1] \times [0;T]$ admits the following decomposition

$$\mathbb{B}(x,t) = \sum_{n,k=0}^{n} \theta_{n,k} \frac{8\sqrt{T}}{\pi^2(2n+1)(2k+1)} \sin\frac{\pi(2n+1)t}{2T} \sin\frac{\pi(2k+1)x}{2}, \tag{1.7}$$

where $\theta_{n,k}$ are independent standard Gaussian random variables, defined on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$. Replacing $\Theta\left(\Pi^{x_k,t}_{\Delta x,\Delta t}\right)$ in (1.5) by the increment of the Brownian sheet, defined by (1.6) we obtain

$$\begin{aligned} u(x+\Delta t, t+\Delta t)\Delta x - u(x,t)\Delta x = -u(x,t)\Delta x \mu(x)\Delta t + \\ +\alpha_0 u(x,t)\left[\mathbb{B}(x+\Delta x, t+\Delta t) - \mathbb{B}(x, t+\Delta t) - \mathbb{B}(x+\Delta x, t) + \mathbb{B}(x,t)\right]. \end{aligned}$$

Let $u(x,t)$ be continuously differentiable with respect to x. Then we have

$$\begin{aligned} u(x+\Delta t, t+\Delta t) - u(x,t) &= u(x, t+\Delta t) + \frac{\partial u}{\partial x}(x, t+\Delta t)\Delta t + o(\Delta t) - u(x,t) = \\ &= u(x, t+\Delta t) - u(x,t) + \left[\frac{\partial u}{\partial x}(x,t) + o(1)\right]\Delta t + o(\Delta t) = \\ &= u(x, t+\Delta t) - u(x,t) + \frac{\partial u}{\partial x}(x,t)\Delta t + o(\Delta t). \end{aligned}$$

Omitting the $o(\Delta t)$'s and dividing both sides of the equation by Δx, we obtain the equation

$$\begin{aligned} u(x, t+\Delta t) - u(x,t) = \left(-\frac{\partial u}{\partial x}(x,t) - \mu(x)u(x,t)\right)\Delta t + \\ +\alpha_0 u(x,t)\left[\frac{\mathbb{B}(x+\Delta x, t+\Delta t) - \mathbb{B}(x, t+\Delta t)}{\Delta x} - \frac{\mathbb{B}(x+\Delta x, t) - \mathbb{B}(x,t)}{\Delta x}\right]. \end{aligned} \tag{1.8}$$

Brownian sheet is nowhere differentiable in both variables. Therefore we can not pass to limit in this difference equation letting $\Delta x \to 0$. In the next section we consider this equation in a Hilbert space of functions of $x \in [0;1]$ and justify this passage to limit with the help of the concept of cylindrical random variable.

2. Difference equation in a Hilbert space

The set of functions $e_k(x) = \sqrt{2}\sin\dfrac{x}{\lambda_k}$, $k = 0, 1, \ldots$, used in expansion (1.7), where $\lambda_k = \dfrac{2}{\pi(2k+1)}$ is an orthonormal basis in the space $H = L^2[0;1]$. Note that the random processes $\beta_k(t)$, defined by the series

$$\beta_k(t) = \sum_{n=0}^{\infty} \theta_{n,k} \frac{2\sqrt{2T}}{\pi(2n+1)} \sin\frac{\pi(2n+1)t}{2T}, \quad t \in [0;T], \tag{2.1}$$

are independent Brownian motions (here, as in (1.7), $\theta_{n,k}$ are independent standard Gaussian random variables). Thus, we can rewrite the expansion (1.7) as

$$\mathbb{B}(x,t) = \sum_{k=0}^{\infty} \lambda_k \beta_k(t) e_k(x) \tag{2.2}$$

and consider $\mathbb{B}(t) = \mathbb{B}(\cdot, t)$ as a random process in H. It is easy to see that the series (2.2) is convergent in $L^2(\Omega, \mathcal{F}, \mathrm{P}; H)$ for any t.

Let us introduce the shift operator $\tau_{\Delta x} : H \to H$, defining it on the elements of the basis $\{e_k\}$ by

$$\tau_{\Delta x} e_k = \sin\frac{\Delta x}{\lambda_k}\tilde{e}_k + \cos\frac{\Delta x}{\lambda_k} e_k\,, \tag{2.3}$$

where $\tilde{e}_k(x) := \lambda_k e_k'(x) = \sqrt{2}\cos\frac{x}{\lambda_k}$, $k = 0, 1, \ldots$. Note that the set $\{\tilde{e}_k\}$ is also an orthonormal basis in $L^2[0;1]$. Equation (1.8) can be written as the following difference equation in H:

$$\begin{gathered} u(t+\Delta t) - u(t) = \left(-\frac{\partial}{\partial x}u(t) - mu(t)\right)\Delta t + \\ +\alpha_0 u(t)\left[\frac{\tau_{\Delta x}\mathbb{B}(t+\Delta t) - \mathbb{B}(t+\Delta t)}{\Delta x} - \frac{\tau_{\Delta x}\mathbb{B}(t) - \mathbb{B}(t)}{\Delta x}\right], \end{gathered}$$

where $u(t) = u(\cdot, t)$.

Definition 3. [6, p. 17] *Let H be a Hilbert space. A linear operator $X : H \to L^2(\Omega, \mathcal{F}, \mathrm{P})$ with the properties:*

1. *$X[h] \sim N(0, \|h\|^2)$ for any $h \in H$,*
2. *$X(h_1)$ and $X(h_2)$ are independent if $(h_1, h_2)_H = 0$,*

is called a cylindrical standard Gaussian random variable on H.

It follows from the definition that any cylinder standard Gaussian random variable X is a bounded operator: $X \in \mathcal{L}\left(H; L^2(\Omega, \mathcal{F}, \mathrm{P})\right)$ with $\|X\| = 1$.

Definition 4. [6, p. 19] *A family $\{W(t), t \in \mathbb{R}\}$ is called a cylindrical Wiener process if*

1. *$W(t) : H \to L^2(\Omega, \mathcal{F}, \mathrm{P})$ is a linear operator;*
2. *$W(t)[h]$ is a brownian motion for any $h \in H$;*
3. *$E\left(W(t)[h_1]W(t)[h_2]\right) = t(h_1, h_2)_H$ for any $h_1, h_2 \in H$.*

Let $W(t)$ be a cylindrical Wiener process on a Hilbert space H. It follows from the definition that $\frac{1}{\sqrt{t}}W(t)$ is a cylindrical standard Gaussian random variable on H for any $t>0$. We also have that for any orthonormal basis $\{g_k\}_{k=0}^{\infty}$ in H $\beta_k(t) := W(t)[g_k]$ are independent Brownian motions, therefore one can identify $W(t)$ with the expansion

$$W(t) = \sum_{k=0}^{\infty} \beta_k(t) g_k \tag{2.4}$$

by letting

$$W(t)[h] := \sum_{k=0}^{\infty} h_k \beta_k(t)\,, \quad h = \sum_{k=0}^{\infty} h_k g_k \in H\,. \tag{2.5}$$

Although the series (2.4) is divergent in $L^2(\Omega, \mathcal{F}, \mathrm{P}; H)$, the right hand side of the equality (2.5) defines a random variable belonging to $L^2(\Omega, \mathcal{F}, \mathrm{P})$ which can be thought of as a scalar product $(W(t), h)_H$. Conversely, any sequence of independent Brownian motions $\{\beta_k(t)\}_{k=0}^{\infty}$ and an orthonormal basis $\{g_k\}_{k=0}^{\infty}$ in H generate a cylindrical Wiener process on H, defined by (2.5).

The next proposition states that when $\Delta x \to 0$, the difference quotients $\dfrac{\tau_{\Delta x}\mathbb{B}(\cdot,t) - \mathbb{B}(\cdot,t)}{\Delta x}$ converge to a cylindrical Wiener process as cylindrical random variables on the Hilbert space $H = L^2[0;1]$.

Proposition 1. *For any $h \in H$*

$$\lim_{\Delta x \to 0} E\left(\frac{\tau_{\Delta x}\mathbb{B}(t) - \mathbb{B}(t)}{\Delta x} - W_0(t), h\right)_H^2 = 0\,, \tag{2.6}$$

where $W_0(t)$ is the cylindrical Wiener process, defined by the expansion

$$W_0(t) = \sum_{k=0}^{\infty} \beta_k(t) \tilde{e}_k.$$

P r o o f. Let $h = \sum_{k=1}^{\infty} h_k e_k = \sum_{k=1}^{\infty} \tilde{h}_k \tilde{e}_k \in H$. Using the expansion (2.2) and the equality (2.3), we obtain

$$\left(\frac{\tau_{\Delta x}\mathbb{B}(t) - \mathbb{B}(t)}{\Delta x} - W_0(t), h\right)_H = \sum_{k=0}^{\infty} \beta_k(t) \left[\zeta_k(\Delta x)\tilde{h}_k + \gamma_k(\Delta x) h_k\right],$$

where

$$\zeta_k(\Delta x) = \frac{\sin \Delta x/\lambda_k}{\Delta x/\lambda_k} - 1, \quad \gamma_k(\Delta x) = \frac{\cos \Delta x/\lambda_k - 1}{\Delta x/\lambda_k}.$$

We have

$$E\left(\frac{\tau_{\Delta x}\mathbb{B}(t) - \mathbb{B}(t)}{\Delta x} - W_0(t), h\right)_H^2 = t\sum_{k=0}^{\infty} \left[\zeta_k(\Delta x)\tilde{h}_k + \gamma_k(\Delta x) h_k\right]^2 \tag{2.7}$$

and due to the estimate

$$\left[\zeta_k(\Delta x)\tilde{h}_k + \gamma_k(\Delta x) h_k\right]^2 \le 2\left[\zeta_k^2(\Delta x)\tilde{h}_k^2 + \gamma_k^2(\Delta x) h_k^2\right] \le 4\left[\tilde{h}_k^2 + h_k^2\right],$$

we conclude that the series in the right hand side of (2.7) is uniformly convergent with respect to $\Delta x \in \mathbb{R}$. Since $\lim\limits_{\Delta x \to 0} \zeta_k(\Delta x) = \lim\limits_{\Delta x \to 0} \gamma_k(\Delta x) = 0$ for any k, it follows (2.6). $\square$

Thus, letting $\Delta x \to 0$ in (1.8), we arrive at the following difference equation in H:

$$u(t+\Delta t) - u(t) = \left(-\frac{\partial}{\partial x}u(t) - mu(t)\right)\Delta t + \alpha_0 u(t)\left(W_0(t+\Delta t) - W_0(t)\right).$$

Note, that the last term in the right hand side of this equation can not be thought of as a product of functions of x. This is due to the fact that the increments of the cylindrical Wiener process are cylindrical Gaussian random variables on H and do not belong to $H = L^2[0;1]$ with probability one. In order to give meaning to the product we rewrite the equation in the following form:

$$u(t+\Delta t) - u(t) = Au(t)\Delta t + \alpha_0 B_0(u(t))\left(W_0(t+\Delta t) - W_0(t)\right), \tag{2.8}$$

where $A = -\frac{d}{dx} - m(x) : H \to H$ with the domain

$$D(A) = \left\{ u \in H^1[0;1] \,\middle|\, u(0) = \int_0^1 b(x)u(x)\,dx \right\},$$

and $B_0 : H \to L(H)$ is the operator, defined by $B_0 : u \mapsto B_0(u)$, where $B_0(u)$ is the operator of multiplication by u.

Since for any $u \in H$ we have

$$\|B_0(u)e_k\|_H^2 = \int_0^1 |u(x)e_k(x)|^2 dx \le 2\|u\|_H^2$$

and $h = \sum_{k=0}^{\infty} h_k e_k \in H^1[0,1]$ iff $\|h\|_1^2 := \sum_{k=0}^{\infty}\left(\frac{h_k}{\lambda_k}\right)^2 < \infty$, the following estimate holds:

$$\|B_0(u)h\|_H^2 = \left\|\sum_{k=0}^{\infty} h_k B_0(u)e_k\right\|_H^2 \le \left(\sum_{k=0}^{\infty}|h_k|\|B_0(u)e_k\|\right)^2 \le$$
$$\le \sum_{k=0}^{\infty}\left(\frac{h_k}{\lambda_k}\right)^2 \sum_{k=0}^{\infty}\lambda_k^2\|B_0(u)e_k\|^2 \le \|h\|_1^2\|u\|_H^2 \cdot 2\sum_{k=0}^{\infty}\lambda_k^2 .$$

Since $\sum_{k=0}^{\infty}\lambda_k^2 < \infty$, it follows $B_0(u)h \in H$. Therefore the equation (2.8) can be understood in the following weak sense:

$$(u(t+\Delta t) - u(t), h)_H = (u(t), A^*h)_H \Delta t + \alpha_0 (W_0(t+\Delta t) - W_0(t))[B_0(u(t))h] \tag{2.9}$$

for any $h \in D(A^*)$. Here $A^*h(x) = h'(x) - m(x)h(x) + b(x)h(0)$ with the domain

$$D(A^*) = \left\{h \in H^1[0;1] \mid h(1) = 0\right\}.$$

Consider the first term in the right hand side of (2.9). We have

$$\begin{aligned}(u(t), A^*h)_H &= (u(t), h')_H - (u(t), mh)_H + (u(t), \langle h, \delta\rangle b)_H = \\ &= (u(t), h')_H - (m, B_0(u(t))h)_H + (b, B_1(u(t))h)_H ,\end{aligned} \tag{2.10}$$

where δ is the Dirac delta-function, considered as an element of the space$H^{-1}[0;1]$, the operator $B_1 : H \to \mathcal{L}(H^1[0;1]; H)$ is defined by $B_1(u)h = \langle h, \delta\rangle u$, $u \in H$, $h \in H^1[0;1]$. The second term in the right hand side of (2.9) has appeared as the result of stochastic perturbation of the operator of multiplication by $m(x)$ (the mean rate of death). This operator is represented by the second term in the right hand side of (2.10). Since the third term there corresponds to the operator of multiplication by $b(x)$ (the mean birth rate) initially contained in the boundary condition, it is natural to introduce an analogous stochastic perturbation of this factor by the term $\alpha_1(W_1(t+\Delta t)-W_1(t))[B_1(u(t))h]$, where $W_1(t)$ is a cylindrical Wiener process independent with $W_0(t)$ and α_1 is a constant.

Thus, we arrive at the following equation:

$$\begin{gathered} u(t+\Delta t)-u(t)=Au(t)\Delta t+\alpha_0 B_0(u(t))\left(W_0(t+\Delta t)-W_0(t)\right)+ \\ +\alpha_1 B_1(u(t))\left(W_1(t+\Delta t)-W_1(t)\right), \end{gathered} \tag{2.11}$$

which is understood in the weak sense, namely:

$$\begin{gathered} (u(t+\Delta t)-u(t),h)_H=(u(t),A^*h)_H\Delta t+\alpha_0(W_0(t+\Delta t)-W_0(t))[B_0(u(t))h] \\ +\alpha_1(W_1(t+\Delta t)-W_1(t))[B_1(u(t))h] \end{gathered}$$

for any $h\in D(A^*)$.

3. Differential equation

For any $t>0$ let $\{t_k\}_{k=0}^N$ be a partition of the segment $[0;t]$, where $t_k=k\Delta t$, $\Delta t=t/N$. Summing up the equality (2.11) written for the points t_k we obtain

$$\begin{gathered} u(t)-u(0)=\sum_{k=0}^{N-1}Au(t_k)\Delta t+\sum_{k=0}^{N-1}B_0(u(t_k))\left(W_0(t_{k+1})-W_0(t_k)\right)+ \\ +\sum_{k=0}^{N-1}B_1(u(t_k))\left(W_1(t_{k+1})-W_1(t_k)\right). \end{gathered}$$

Letting $N\to\infty$ we arrive at the following integral Itô equation

$$u(t)-u(0)=\int_0^t Au(s)\,ds+\int_0^t B_0(u(s))\,dW_0(s)+\int_0^t B_1(u(s))\,dW_1(s), \tag{3.1}$$

if the integrals in the right hand side exist. The equation is usually written in the following differential form:

$$du(t)=Au(t)\,dt+B_0(u(t))\,dW_0(t)+B_1(u(t))\,dW_1(t)\,,\quad u(0)=u_0. \tag{3.2}$$

The necessary condition of existence of the integrals in (3.1) is $B_0(u),B_1(u)\in\mathcal{L}_2(H;H)$ (the space of Hilbert–Schmidt operators acting in H) for any $u\in H$. It is not the case here, therefore it is impossible to obtain theorems on existence and uniqueness of solution (weak, or mild) for the problem (3.3) (see, for example, Theorem 6.7, p. 164 in [5], Theorem 3.3, p. 97 in [6]).

The way out can be found in setting the problem in the space of generalized Hilbert-space-valued random variables $(\mathcal{S})_{-\rho}(H)\supset L^2(\Omega,\mathcal{F},P;H)$, $\rho\in[0;1]$ (see the definition and properties of this space in [9]). It turns out that a cylindrical Wiener process $W(t)$ on H is a differentiable $(\mathcal{S})_{-\rho}(H)$-valued function of t. Denote its derivative by $\mathbb{W}(t)$. It is called a cylindrical singular white noise. It was proved in [3] that for any predictable $\mathcal{L}_2(H,H)$-valued process $\Psi(t)$ it holds

$$\int_0^t\Psi(s)\,dW(s)=\int_0^t\Psi(s)\diamond\mathbb{W}(s)ds,$$

where $\diamond$ is the Wick product, if the Itô integral in the left hand side exists. The integral in the right hand side is often called the Hitsuda–Skorohod integral. It can be considered an extension of the Itô integral onto a wider class of integrands.

Thus, the problem (3.3) takes the form

$$\frac{du(t)}{dt} = Au(t) + B_0(u(t)) \diamond \mathbb{W}_0(t) + B_1(u(t)) \diamond \mathbb{W}_1(t), \quad u(0) = u_0. \tag{3.3}$$

in the space $(\mathcal{S})_{-\rho}(H)$. The following theorem is stated here without proof as it is a straightforward generalization of the Theorem 3 in [9].

Theorem 1. *Let A be the generator of a C_0-semigroup of operators in a Hilbert space H, $B_0(\cdot), B_1(\cdot) \in \mathcal{L}\big(H, \mathcal{L}_2(H_Q, H)\big)$, where Q is a positive trace class operator in H with the set of eigenvalues $\{\sigma_j^2\}_{j=1}^{\infty}$ satisfying the condition*

$$\sum_j \sigma_j^{-2} j^{-2p} < \infty\,, \quad \textit{for some } p \in \mathbb{N},$$

and $H_Q = Q^{1/2}(H)$ with the norm $\|x\|_Q = \|Q^{-1/2}x\|_H$. Then the problem (3.3) has a unique solution $u(t) \in (\mathcal{S})_{-0}(H)$ for any $u_0 \in (D(A))$, where $(D(A))$ denotes the domain of A in $(\mathcal{S})_{-0}(H)$.

Remark 1. Conditions of the theorem hold true for the operators, B and B_1 introduced in our model. To show this, note that the functions $\sigma_j \tilde{e}_j(x)$, $j = 0, 1, 2, \ldots$ form an orthonormal basis in H_Q and for any $u \in H = L^2[0; 1]$ we have:

$$\|B(u)\|^2_{\mathcal{L}_2(H_Q,H)} = \sum_j \|B(u)\sigma_j \tilde{e}_j\|^2 = \sum_j \sigma_j^2 \int_0^1 u^2(x)\tilde{e}_j(x)\,dx \le 2\sum_j \sigma_j^2 \|u\|^2,$$

$$\|B_1(u)\|^2_{\mathcal{L}_2(H_Q,H)} = \sum_j \|B_1(u)\sigma_j \tilde{e}_j\|^2 = \sum_j \sigma_j^2 \langle \tilde{e}_j, \delta\rangle^2 \|u\|^2 = 2\sum_j \sigma_j^2 \|u\|^2.$$

4. Conclusion

Introduction of stochastic perturbation into McKendrick–von Foerster model of an age-structured population requires taking into account certain properties of the oscillations of rates of death and birth. We have shown that the assumption of independence between the random fluctuations of per capita amounts of dead individuals in disjoint segments of the age scale or the time line together with the analogous assumption on the random fluctuations concerning the process of reproduction in the population lead to a difference equation in the Hilbert space $L^2[0; 1]$ with a cylindrical Wiener process. Due to nonregularity of the latter, we finally obtain a model which has the form of an operator-differential equation with cylindrical white noises in the space of generalized Hilbert space-valued random variables satisfying the conditions of the theorem on existence and uniqueness of solutions.

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