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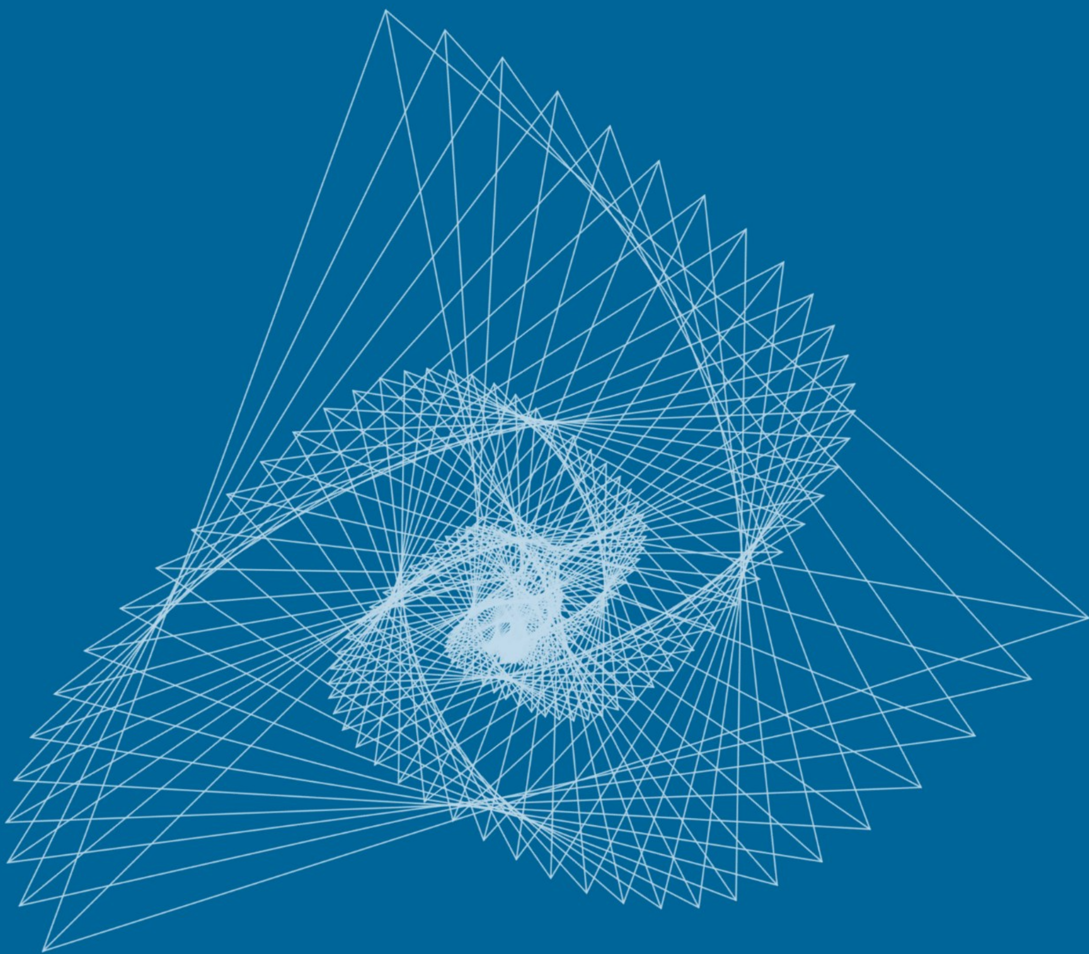

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N.N. Krasovskii Institute of Mathematics and Mechanics of the Ural Branch of Russian Academy of Sciences

Ural Federal University named after the first President of Russia B.N. Yeltsin

Contact Information

16 S. Kovalevskaya str., Ekaterinburg, Russia, 620990
Phone: +7 (343) 375-34-73 Fax: +7 (343) 374-25-81
Email: secretary@umjuran.ru
Web-site: https://umjuran.ru

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PRODUCTS OF ULTRAFILTERS AND MAXIMAL LINKED SYSTEMS ON WIDELY UNDERSTOOD MEASURABLE SPACES

Alexander G. Chentsov

Krasovskii Institute of Mathematics and Mechanics,
Ural Branch of the Russian Academy of Sciences,
16 S. Kovalevskaya Str., Ekaterinburg, 620108, Russia

chentsov@imm.uran.ru

Abstract: Constructions related to products of maximal linked systems (MLSs) and MLSs on the product of widely understood measurable spaces are considered (these measurable spaces are defined as sets equipped with π-systems of their subsets; a π-system is a family closed with respect to finite intersections). We compare families of MLSs on initial spaces and MLSs on the product. Separately, we consider the case of ultrafilters. Equipping set-products with topologies, we use the box-topology and the Tychonoff product of Stone-type topologies. The properties of compaction and homeomorphism hold, respectively.

Keywords: Maximal linked system, Topology, Ultrafilter.

1. Introduction

In this investigation, properties of maximal linked systems (MLSs) and ultrafilters on widely understood measurable spaces (MSs) are considered. Every such MS is realized by equipment of a nonempty set with π-system of subsets of this set with "zero" and "unit" (the "zero" is an empty set, and the "unit" is our original set); a π-system is a family closed with respect to finite intersections. Of course, algebras, semi-algebras, topologies, and families of closed sets in topological spaces (TSs) are π-systems. An important variant of a π-system is realized by a lattice of subsets of a fixed nonempty set. A semi-algebra of sets is a π-system but, generally speaking, not a lattice.

We note that MLSs were considered in connection with the superextension and supercompactness problem, see [2, 16, 17, 20, 21]. In addition, MLSs on the lattice of closed sets in a TS were studied. The nonempty set of all MLSs of such type is equipped with Wallman-type topology. The supercompactness property was implemented.

In [5–7, 9, 10, 12], an analog of the superextension and supercompactness property for the space of MLSs on a π-system was investigated. Moreover, a Stone-type topology was also used. In addition, a bitopological space was implemented. The present study continues the above works. But here the focus is on spaces of MLSs with Stone-type topology. We consider questions related to the products of widely understood measurable spaces. In addition, representations of MLSs on the product of these MSs in terms of analogous MLSs on spaces-factors are indicated. Namely, MLSs on the product of (widely understood) MSs are limited to products of MLSs on initial spaces. This important property is complemented by a proposition of a topological nature: the properties of compaction and homeomorphism hold. In addition, the box and Tychonoff variants of topology product are considered (similar variants are used for the product of MSs). In connection with the above assumptions, we use constructions of [11, 13, 14].

2. General notions and notation

We use standard set-theoretic notation, including quantifiers and propositional connectives; $\varnothing$ stands for an empty set) and $\triangleq$ for an equality by definition. A family is a set such that all its elements are sets themselves. We adopt the axiom of choice. For every objects x and y, denote by $\{x; y\}$ an unordered pair of x and y: $x \in \{x; y\}$, $y \in \{x; y\}$, and $(z = x) \vee (z = y)$ for every $z \in \{x; y\}$. For every object s, denote by $\{s\} \triangleq \{s; s\}$ a singleton containing s : $s \in \{s\}$. In addition, sets are objects. Then, for every objects x and y, the family $(x, y) \triangleq \{\{x\}; \{x; y\}\}$ is (see [12, Ch. II, Section 2]) the ordered pair with x as the first element and y as the second. For every ordered pair h, denote by $\mathrm{pr}_1(h)$ and $\mathrm{pr}_2(h)$ its first and second elements, respectively; thus, $h = (\mathrm{pr}_1(h), \mathrm{pr}_2(h))$.

Denote by $\mathcal{P}(H)$ the family of all subsets of H. Let $\mathcal{P}'(H) \triangleq \mathcal{P}(H) \setminus \{\varnothing\}$ be the family of all nonempty subsets of H. Denote by $\mathrm{Fin}(H)$ the family of all finite nonempty subsets of H. If $\mathcal{H}$ is a family and S is a set, then

$$[\mathcal{H}](S) \triangleq \big\{H \in \mathcal{H} |\ S \subset H\big\} \in \mathcal{P}(\mathcal{H}).$$

For every set $\mathbb{M}$ and a family $\mathcal{M} \in \mathcal{P}'(\mathcal{P}(\mathbb{M}))$, the dual family

$$\mathbf{C}_{\mathbb{M}}[\mathcal{M}] \triangleq \big\{\mathbb{M} \setminus M :\ M \in \mathcal{M}\big\} \in \mathcal{P}'(\mathcal{P}(\mathbb{M}))$$

is realized. If $\mathcal{A}$ is a nonempty family and B is a set, then

$$\mathcal{A}|_B \triangleq \big\{A \cap B :\ A \in \mathcal{A}\big\} \in \mathcal{P}'(\mathcal{P}(B))$$

is the trace of A onto the set B. Following to [7, Section 1], if $\mathfrak{X}$ is a nonempty family, then $\{\cup\}(\mathfrak{X})$, $\{\cap\}(\mathfrak{X})$, $\{\cup\}_\sharp(\mathfrak{X})$, and $\{\cap\}_\sharp(\mathfrak{X})$ stand for the families of arbitrary unions, arbitrary intersections of nonempty subfamilies of $\mathfrak{X}$, finite unions, and finite intersections of sets from $\mathfrak{X}$, respectively.

Remark 1. In what follows, we use two types of formulas. Namely, we use expressions of type $\{x \in X |\ \ldots\}$ and expressions of type $\{f(z) :\ z \in \ldots\}$. In function theory, the former is used for the preimage of a set; we have a formula corresponding to Zermelo–Fraenkel axiomatic (we first select a set X, for points of which some property $\ldots$ is postulated). The second expression corresponds logically to the image of a set. This difference is essential from point of view of bibliographic references to earlier publications of the author. Therefore, we use two variants of separator character: $|$ (vertical line) in the first case and : (colon) in the second. This stipulation is important for the constructions that follow.

For sets A and B, we denote by B^A (see [19, Ch. II, § 6]) the set of all mappings (functions) from A to B; values of mappings are denoted in traditionalway. If A and B are sets, $f \in B^A$, and $C \in \mathcal{P}(A)$, then $f^1(C) \triangleq \{f(x) :\ x \in C\} \in \mathcal{P}(B)$ and $(f|C) \in B^C$ is, by definition, the restriction of the mapping f to the set C : $(f|C)(x) \triangleq f(x)\ \ \forall x \in C$. For mappings, index form is often used (a family with index, see [22, Ch. I, I.1]).

In what follows, $\mathbb{R}$ is the real line, $\mathbb{N} \triangleq \{1; 2; \ldots\} \in \mathcal{P}'(\mathbb{R})$, and $\overline{1, n} \triangleq \{k \in \mathbb{N} |\ k \le n\}$ for $n \in \mathbb{N}$. We assume that elements of $\mathbb{N}$, i.e., positive integer natural numbers are not sets. Therefore, for every set H and $n \in \mathbb{N}$, instead of $H^{\overline{1,n}}$, we use the more traditional notation H^n for the set of all mappings from $\overline{1, n}$ to H; thus, H^n is the set of all processions $(h_i)_{i \in \overline{1,n}} :\ \overline{1, n} \longrightarrow H$.

Special families. Until the end of this section, we fix a nonempty set $\mathbf{I}$. The elements of $\mathcal{P}'(\mathcal{P}(\mathbf{I}))$ are nonempty families of subsets of $\mathbf{I}$. Define the family of all π-systems of subsets of $\mathbf{I}$

with "zero" and "unit":

$$\pi[\mathbf{I}] \triangleq \big\{\mathcal{I} \in \mathcal{P}'(\mathcal{P}(\mathbf{I}))|\ (\varnothing \in \mathcal{I})\&(\mathbf{I} \in \mathcal{I})\&(A \cap B \in \mathcal{I}\ \ \forall A \in \mathcal{I}\ \forall B \in \mathcal{I})\big\}. \tag{2.1}$$

Of course, $\mathcal{P}(\mathbf{I}) \in \pi[\mathbf{I}]$. Consider a very useful notion of semi-algebra of sets. For $\mathcal{L} \in \pi[\mathbf{I}]$, $A \in \mathcal{P}(\mathbf{I})$, and $n \in \mathbb{N}$, we introduce finite partitions of A by sets of $\mathcal{L}$:

$$\Delta_n(A, \mathcal{L}) \triangleq \Big\{(L_i)_{i\in\overline{1,n}} \in \mathcal{L}^n|\ \big(A = \bigcup_{i=1}^{n} L_i\big)\&\big(L_p \cap L_q = \varnothing\ \ \forall p \in \overline{1,n}\ \forall q \in \overline{1,n} \setminus \{p\}\big)\Big\}.$$

The family of all semi-algebras of subsets of $\mathbf{I}$ is defined as follows:

$$\Pi[\mathbf{I}] \triangleq \big\{\mathcal{L} \in \pi[\mathbf{I}]|\ \forall L \in \mathcal{L}\ \ \exists n \in \mathbb{N}:\ \Delta_n(\mathbf{I} \setminus L, \mathcal{L}) \neq \varnothing\big\}. \tag{2.2}$$

In addition, we introduce yet another type of π-systems; this type is important in questions of interconnection between ultrafilters and MLSs. Namely,

$$\pi_*^\natural[\mathbf{I}] \triangleq \big\{\mathcal{I} \in \pi[\mathbf{I}]|\ \forall I_1 \in \mathcal{I}\ \ \forall I_2 \in \mathcal{I}\ \ \forall I_3 \in \mathcal{I}$$
$$((I_1 \cap I_2 \neq \varnothing)\&(I_2 \cap I_3 \neq \varnothing)\&(I_1 \cap I_3 \neq \varnothing)) \Longrightarrow (I_1 \cap I_2 \cap I_3 \neq \varnothing)\big\}.$$

Of course, very general constructions are connected with lattices. The family of all lattices of subsets of $\mathbf{I}$ with "zero" and "unit" is

$$(\mathrm{LAT})_0[\mathbf{I}] \triangleq \big\{\mathcal{I} \in \pi[\mathbf{I}]|\ A \cup B \in \mathcal{I}\ \ \forall A \in \mathcal{I}\ \forall B \in \mathcal{I}\big\}, \tag{2.3}$$

We introduce the family

$$(\mathrm{alg})[\mathbf{I}] \triangleq \big\{\mathcal{A} \in (\mathrm{LAT})_0[\mathbf{I}]|\ \mathbf{I} \setminus A \in \mathcal{A}\ \ \forall A \in \mathcal{A}\big\} \tag{2.4}$$

of all algebras of subsets of $\mathbf{I}$. For $\mathcal{A} \in (\mathrm{alg})[\mathbf{I}]$, $(\mathbf{I}, \mathcal{A})$ is an MS with algebra of sets. Moreover,

$$(\mathrm{top})[\mathbf{I}] \triangleq \big\{\tau \in (\mathrm{LAT})_0[\mathbf{I}]|\ \bigcup_{G\in\mathcal{G}} G \in \tau\ \ \forall \mathcal{G} \in \mathcal{P}'(\tau)\big\} \tag{2.5}$$

is the family of all topologies on $\mathbf{I}$ and

$$(\mathrm{clos})[\mathbf{I}] \triangleq \big\{\mathbf{C}_{\mathbf{I}}[\tau]:\ \ \tau \in (\mathrm{top})[\mathbf{I}]\big\} \in \mathcal{P}'((\mathrm{LAT})_0[\mathbf{I}]) \tag{2.6}$$

is the family of all closed topologies [1, Ch.4, § 1] on $\mathbf{I}$. So, (2.4)–(2.6) are important types of lattices (see (2.3)). Semi-algebras (see (2.2)) are, generally speaking, not lattices: if $\mathcal{L} \in \Pi[\mathbf{I}]$, then it is possible that $\mathcal{L} \notin (\mathrm{LAT})_0[\mathbf{I}]$.

Elements of topology. We consider the families $(\mathrm{BAS})[\mathbf{I}]$ and $(\mathrm{p}-\mathrm{BAS})[\mathbf{I}]$ of all open bases and subbases on $\mathbf{I}$, respectively; this notation correspond to [9, Section 2] (see also [7, Section 2]). Of course, $\{\cup\}(\beta) \in (\mathrm{top})[\mathbf{I}]$ for $\beta \in (\mathrm{BAS})[\mathbf{I}]$; moreover, $\{\cap\}_\sharp(\chi) \in (\mathrm{BAS})[\mathbf{I}]$ for $\chi \in (\mathrm{p}-\mathrm{BAS})[\mathbf{I}]$. Note that (see (2.1))

$$\pi[\mathbf{I}] \subset (\mathrm{BAS})[\mathbf{I}]; \tag{2.7}$$

therefore, $\{\cup\}(\mathcal{L}) \in (\mathrm{top})[\mathbf{I}]$ is defined for $\mathcal{L} \in \pi[\mathbf{I}]$. If $\tau \in (\mathrm{top})[\mathbf{I}]$, then

$$(\tau-\mathrm{BAS})_0[\mathbf{I}] \triangleq \big\{\beta \in (\mathrm{BAS})[\mathbf{I}]|\tau = \{\cup\}(\beta)\big\}.$$

Moreover,

$$(\mathrm{p}-\mathrm{BAS})_0[\mathbf{I}; \tau] \triangleq \big\{\chi \in (\mathrm{p}-\mathrm{BAS})[\mathbf{I}]|\{\cap\}_\sharp(\chi) \in (\tau-\mathrm{BAS})_0[\mathbf{I}]\big\}.$$

Thus, we have introduced open bases and subbases of the specific TS $(\mathbf{I}, \tau)$.

Linkedness. If $\mathcal{J} \in \mathcal{P}'(\mathcal{P}(\mathbf{I}))$, then we suppose that

$$\langle \mathcal{J} - \text{link}\rangle[\mathbf{I}] \stackrel{\triangle}{=} \big\{\mathcal{E} \in \mathcal{P}'(\mathcal{J}) |\ \Sigma_1 \cap \Sigma_2 \neq \varnothing \quad \forall \Sigma_1 \in \mathcal{E}\ \forall \Sigma_2 \in \mathcal{E}\big\}. \tag{2.8}$$

Elements of the family (2.8 and only they are linked subfamilies of $\mathcal{J}$. As a corollary,

$$\langle \mathcal{J} - \text{link}\rangle_0[\mathbf{I}] \stackrel{\triangle}{=} \big\{\mathcal{E} \in \langle \mathcal{J} - \text{link}\rangle[\mathbf{I}] |\ \forall \mathcal{S} \in \langle \mathcal{J} - \text{link}\rangle[\mathbf{I}]\ \ (\mathcal{E} \subset \mathcal{S}) \Longrightarrow (\mathcal{E} = \mathcal{S})\big\} \tag{2.9}$$

is the family of all maximal linked subfamilies of $\mathcal{J}$. We call every family of (2.9) an MLS (on $\mathcal{J}$). In what follows, for our goals, it suffices to consider the case $\mathcal{J} \in \pi[\mathbf{I}]$. Therefore, until the end of this section, suppose that $\mathcal{J} \in \pi[\mathbf{I}]$. Now, we note only several simple properties. So, $\{\Sigma\} \in \langle \mathcal{J} - \text{link}\rangle[\mathbf{I}]$ for $\Sigma \in \mathcal{J} \setminus \{\varnothing\}$. Then, by the Zorn lemma, $\langle \mathcal{J} - \text{link}\rangle_0[\mathbf{I}] \neq \varnothing$. Moreover,

$$\langle \mathcal{J} - \text{link}\rangle_0[\mathbf{I}] = \big\{\mathcal{E} \in \langle \mathcal{J} - \text{link}\rangle[\mathbf{I}] |\ \forall J \in \mathcal{J}\ \ (J \cap \Sigma \neq \varnothing \quad \forall \Sigma \in \mathcal{E}) \Longrightarrow (J \in \mathcal{E})\big\}.$$

Finally, note that, for $\mathcal{E} \in \langle \mathcal{J} - \text{link}\rangle_0[\mathbf{I}]$, we have

$$([\mathcal{J}](\Sigma) \subset \mathcal{E} \quad \forall \Sigma \in \mathcal{E}) \& (\mathbf{I} \in \mathcal{E}). \tag{2.10}$$

More detailed information on the properties of MLSs can be found in [5–7, 9–12]. Now we introduce some constructions for a Stone-type topology. If $J \in \mathcal{J}$, then

$$\langle \mathcal{J} - \text{link}\rangle^0[\mathbf{I}|J] \stackrel{\triangle}{=} \big\{\mathcal{E} \in \langle \mathcal{J} - \text{link}\rangle_0[\mathbf{I}] |\ J \in \mathcal{E}\big\} \in \mathcal{P}(\langle \mathcal{J} - \text{link}\rangle_0[\mathbf{I}]). \tag{2.11}$$

The sets (2.11) define an open subbase. More precisely, the subbase

$$\hat{\mathfrak{C}}^*_0[\mathbf{I}; \mathcal{J}] \stackrel{\triangle}{=} \big\{\langle \mathcal{J} - \text{link}\rangle^0[\mathbf{I}|J] :\ J \in \mathcal{J}\big\} \in (\text{p} - \text{BAS})[\langle \mathcal{J} - \text{link}\rangle_0[\mathbf{I}]]$$

generates the following topology of Stone type:

$$\mathbb{T}_*\langle \mathbf{I}|\mathcal{J}\rangle \stackrel{\triangle}{=} \{\cup\}\big(\{\cap\}_\sharp(\hat{\mathfrak{C}}^*_0[\mathbf{I}; \mathcal{J}])\big) \in (\text{top})[\langle \mathcal{J} - \text{link}\rangle_0[\mathbf{I}]]. \tag{2.12}$$

In addition, $(\langle \mathcal{J} - \text{link}\rangle_0[\mathbf{I}], \mathbb{T}_*\langle \mathbf{I}|\mathcal{J}\rangle)$ is a zero-dimensional T_2-space.

3. Generalized Cartesian products

In this sections, we recall some constructions connected with Cartesian products and generalized Cartesian products. We note also some notions connected with family products.

If X and Y are nonempty sets, $\mathcal{X} \in \mathcal{P}'(\mathcal{P}(X))$, and $\mathcal{Y} \in \mathcal{P}'(\mathcal{P}(Y))$, then

$$\mathcal{X}\{\times\}\mathcal{Y} \stackrel{\triangle}{=} \{\text{pr}_1(z) \times \text{pr}_2(z) :\ z \in \mathcal{X} \times \mathcal{Y}\} \in \mathcal{P}'(\mathcal{P}(X \times Y)) \tag{3.1}$$

($\mathcal{X} \times \mathcal{Y}$ is the usual product of $\mathcal{X}$ and $\mathcal{Y}$, i.e., the set of ordered pairs); (3.1) is the simplest variant of the constructions used below. It is easy to verify the property

$$\mathcal{X}\{\times\}\mathcal{Y} \in \pi[X \times Y] \quad \forall \mathcal{X} \in \pi[X] \quad \forall \mathcal{Y} \in \pi[Y]. \tag{3.2}$$

We consider $(X \times Y, \mathcal{X}\{\times\}\mathcal{Y})$ as the product of the MSs $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$.

Now we recall notions connected with generalized Cartesian products. If $\mathbb{X}$ and $\mathbb{Y}$ are nonempty sets and $(Y_x)_{x\in\mathbb{X}} \in \mathcal{P}'(\mathbb{Y})^{\mathbb{X}}$, then (by the axiom of choice)

$$\prod_{x\in\mathbb{X}} Y_x \overset{\triangle}{=} \big\{f \in \mathbb{Y}^{\mathbb{X}} | f(x) \in Y_x \;\; \forall x \in \mathbb{X}\big\} \in \mathcal{P}'(\mathbb{Y}^{\mathbb{X}}). \tag{3.3}$$

In connection with (3.3), note that, for every nonempty sets $\mathbb{X}$, $\tilde{\mathbb{Y}}$, and $\hat{\mathbb{Y}}$ and a mapping $(Y_x)_{x\in\mathbb{X}} \in \mathcal{P}'(\tilde{\mathbb{Y}})^{\mathbb{X}} \cap \mathcal{P}'(\hat{\mathbb{Y}})^{\mathbb{X}}$, we have

$$\big\{f \in \tilde{\mathbb{Y}}^{\mathbb{X}} |\; f(x) \in Y_x \;\; \forall x \in \mathbb{X}\big\} = \big\{f \in \hat{\mathbb{Y}}^{\mathbb{X}} |\; f(x) \in Y_x \;\; \forall x \in \mathbb{X}\big\}. \tag{3.4}$$

In what follows, in constructions of type (3.3) we take into account (3.4). If $\mathbb{X}$ and $\mathbb{Y}$ are nonempty sets and $(Y_x)_{x\in\mathbb{X}} \in \mathcal{P}'(\mathbb{Y})^{\mathbb{X}}$, then

$$\prod_{x\in\mathbb{X}} \mathcal{P}'(\mathcal{P}(Y_x)) = \big\{(\mathcal{Y}_x)_{x\in\mathbb{X}} \in \mathcal{P}'(\mathcal{P}(\mathbb{Y}))^{\mathbb{X}} |\; \mathcal{Y}_s \in \mathcal{P}'(\mathcal{P}(Y_s)) \;\; \forall s \in \mathbb{X}\big\};$$

moreover, if $(\mathcal{E}_x)_{x\in\mathbb{X}} \in \prod_{x\in\mathbb{X}} \mathcal{P}'(\mathcal{P}(Y_x))$, then

$$\bigodot_{x\in\mathbb{X}} \mathcal{E}_x \overset{\triangle}{=} \Big\{ \prod_{x\in\mathbb{X}} \Sigma_x : \; (\Sigma_x)_{x\in\mathbb{X}} \in \prod_{x\in\mathbb{X}} \mathcal{E}_x \Big\}. \tag{3.5}$$

We consider the family (3.5) as a box product of the families $\mathcal{E}_x$, $x \in \mathbb{X}$. Here, we note the natural analogy with the base of the known box topology (see [18, Ch. 3]).

If $\mathbb{H}$ is a set, then we suppose that

$$(\mathrm{Fam})[\mathbb{H}] \overset{\triangle}{=} \{\mathcal{H} \in \mathcal{P}'(\mathcal{P}(\mathbb{H})) | \mathbb{H} \in \mathcal{H}\};$$

of course, $\mathcal{P}(\mathbb{H}) \in (\mathrm{Fam})[\mathbb{H}]$; moreover, $(\mathrm{alg})[\mathbb{H}] \subset \Pi[\mathbb{H}] \subset \pi[\mathbb{H}] \subset (\mathrm{Fam})[\mathbb{H}]$ and, by (2.5), $(\mathrm{top})[\mathbb{H}] \subset (\mathrm{Fam})[\mathbb{H}]$. As a corollary, for nonempty sets $\mathbb{X}$ and $\mathbb{Y}$, a mapping $(Y_x)_{x\in\mathbb{X}} \in \mathcal{P}'(\mathbb{Y})^{\mathbb{X}}$, and a mapping $(\mathcal{F}_x)_{x\in\mathbb{X}} \in \prod\limits_{x\in\mathbb{X}} (\mathrm{Fam})[Y_x]$, we obtain

$$\begin{gathered}\bigotimes_{x\in\mathbb{X}} \mathcal{F}_x \overset{\triangle}{=} \Big\{H \in \mathcal{P}(\prod_{x\in\mathbb{X}} Y_x) |\; \exists (\mathbb{F}_x)_{x\in\mathbb{X}} \in \prod_{x\in\mathbb{X}} \mathcal{F}_x : \\ (H = \prod_{x\in\mathbb{X}} F_x) \& (\exists K \in \mathrm{Fin}(\mathbb{X}) : \; \mathbb{F}_s = Y_s \;\; \forall s \in \mathbb{X} \setminus K)\Big\}.\end{gathered} \tag{3.6}$$

In connection with (3.5), note that, for every nonempty sets $\mathbb{X}$ and $\mathbb{Y}$, a mapping $(Y_x)_{x\in\mathbb{X}} \in \mathcal{P}'(\mathbb{Y})^{\mathbb{X}}$, and a mapping $(\mathcal{Y}_x)_{x\in\mathbb{X}} \in \prod\limits_{x\in\mathbb{X}} \pi[Y_x]$, we have

$$\bigodot_{x\in\mathbb{X}} \mathcal{Y}_x = \Big\{ \prod_{x\in\mathbb{X}} \Sigma_x : \;\; (\Sigma_x)_{x\in\mathbb{X}} \in \prod_{x\in\mathbb{X}} \mathcal{Y}_x \Big\} \in \pi[\prod_{x\in\mathbb{X}} Y_x]. \tag{3.7}$$

In connection with (3.6), note that, for the above $\mathbb{X}$, $\mathbb{Y}$, $(Y_x)_{x\in\mathbb{X}}$, and $(\mathcal{Y}_x)_{x\in\mathbb{X}}$, we have

$$\begin{gathered}\bigotimes_{x\in\mathbb{X}} \mathcal{Y}_x = \Big\{H \in \mathcal{P}(\prod_{x\in\mathbb{X}} Y_x) |\; \exists (\mathbb{F}_x)_{x\in\mathbb{X}} \in \prod_{x\in\mathbb{X}} \mathcal{Y}_x : \; (H = \prod_{x\in\mathbb{X}} \mathbb{F}_x) \\ \& (\exists K \in \mathrm{Fin}(\mathbb{X}) : \; \mathbb{F}_s = Y_s \, \forall s \in \mathbb{X} \setminus K)\Big\} \in \pi[\prod_{x\in\mathbb{X}} Y_x].\end{gathered} \tag{3.8}$$

Note useful particular cases of (3.7) and (3.8): for nonempty sets $\mathbb{X}$ and $\mathbb{Y}$ and mappings $(Y_x)_{x\in\mathbb{X}} \in \mathcal{P}'(\mathbb{Y})^{\mathbb{X}}$ and $(\tau_x)_{x\in\mathbb{X}} \in \prod_{x\in\mathbb{X}}(\mathrm{top})[Y_x]$, we have

$$\Big(\bigodot_{x\in\mathbb{X}} \tau_x \in \pi[\prod_{x\in\mathbb{X}} Y_x]\Big)\&\Big(\bigotimes_{x\in\mathbb{X}} \tau_x \in \pi[\prod_{x\in\mathbb{X}} Y_x]\Big). \tag{3.9}$$

Using (2.7) in (3.9), we obtain two variants of topological equipment:

$$\Big(\mathbf{t}_{\odot}[(\tau_x)_{x\in\mathbb{X}}] \triangleq \{\cup\}(\bigodot_{x\in\mathbb{X}} \tau_x)\in(\mathrm{top})[\prod_{x\in\mathbb{X}} Y_x]\Big)\&\Big(\mathbf{t}_{\otimes}[(\tau_x)_{x\in\mathbb{X}}] \triangleq \{\cup\}(\bigotimes_{x\in\mathbb{X}} \tau_x)\in(\mathrm{top})[\prod_{x\in\mathbb{X}} Y_x]\Big). \tag{3.10}$$

Namely, by (3.10), we obtain the following two TSs:

$$\Big(\prod_{x\in\mathbb{X}} Y_x, \mathbf{t}_{\odot}[(\tau_x)_{x\in\mathbb{X}}]\Big), \quad \Big(\prod_{x\in\mathbb{X}} Y_x, \mathbf{t}_{\otimes}[(\tau_x)_{x\in\mathbb{X}}]\Big);$$

thus, we obtain the box TS and the Tychonoff product. Of course, topologies (3.10) are comparable. Moreover, for every nonempty sets $\mathbb{X}$ and $\mathbb{Y}$ and mappings $(Y_x)_{x\in\mathbb{X}} \in \mathcal{P}'(\mathbb{Y})^{\mathbb{X}}$ and $(\mathcal{I}_x)_{x\in X} \in \prod_{x\in\mathbb{X}} \pi[Y_x]$, we have

$$\bigotimes_{x\in\mathbb{X}} \mathcal{I}_x \subset \bigodot_{x\in\mathbb{X}} \mathcal{I}_x. \tag{3.11}$$

From (3.11), the comparability of topologies (3.10) follows, since

$$\prod_{x\in\mathbb{X}} (\mathrm{top})[Y_x] \subset \prod_{x\in\mathbb{X}} \pi[Y_x].$$

Thus, for every nonempty sets $\mathbb{X}$ and $\mathbb{Y}$ and mappings $(Y_x)_{x\in\mathbb{X}} \in \mathcal{P}'(\mathbb{Y})^{\mathbb{X}}$ and $(\tau_x)_{x\in\mathbb{X}} \in \prod_{x\in\mathbb{X}}(\mathrm{top})[Y_x]$, we have

$$\mathbf{t}_{\otimes}[(\tau_x)_{x\in\mathbb{X}}] \subset \mathbf{t}_{\odot}[(\tau_x)_{x\in\mathbb{X}}].$$

4. Ultrafilters and maximal linked systems

In this section, we fix a nonempty set E and a π-system $\mathcal{L} \in \pi[E]$. Recall the notions of filter and ultrafilter on this π-system. So,

$$\mathbb{F}^*(\mathcal{L}) \triangleq \big\{\mathcal{F} \in \mathcal{P}'(\mathcal{L} \setminus \{\varnothing\})|\ (A \cap B \in \mathcal{F} \ \ \forall A \in \mathcal{F} \ \ \forall B \in \mathcal{F})\&([\mathcal{L}](F) \subset \mathcal{F} \ \ \forall F \in \mathcal{F})\big\}$$

is the set of all filters on $\mathcal{L}$. Hence (see [7, Section 2]),

$$\begin{aligned}\mathbb{F}_0^*(\mathcal{L}) &\triangleq \big\{\mathcal{U} \in \mathbb{F}^*(\mathcal{L})|\ \forall \mathcal{F} \in \mathbb{F}^*(\mathcal{L}) \ \ (\mathcal{U} \subset \mathcal{F}) \Longrightarrow (\mathcal{U} = \mathcal{F})\big\}\\ &= \big\{\mathcal{U} \in \mathbb{F}^*(\mathcal{L})|\ \forall L \in \mathcal{L} \ \ (L \cap U \neq \varnothing \ \forall U \in \mathcal{U}) \Longrightarrow (L \in \mathcal{U})\big\}.\end{aligned}$$

We recall that $\mathbb{F}_0^*(\mathcal{L}) \neq \varnothing$ (this is a simplest corollary of the Zorn Lemma). If $L \in \mathcal{L}$, then

$$\Phi_{\mathcal{L}}(L) \triangleq \big\{\mathcal{U} \in \mathbb{F}_0^*(\mathcal{L})|\ L \in \mathcal{U}\big\} = \big\{\mathcal{U} \in \mathbb{F}_0^*(\mathcal{L})|\ L \cap U \neq \varnothing \ \ \forall U \in \mathcal{U}\big\}. \tag{4.1}$$

Using (4.1), we introduce the following π-system:

$$(\mathbb{UF})[E; \mathcal{L}] \triangleq \{\Phi_{\mathcal{L}}(L) :\ L \in \mathcal{L}\} \in \pi[\mathbb{F}_0^*(\mathcal{L})]. \tag{4.2}$$

From (2.7) and (4.2), the property $(\mathbb{UF})[E;\mathcal{L}] \in (\mathrm{BAS})[\mathbb{F}_0^*(\mathcal{L})]$ follows and, as a corollary,

$$\mathbf{T}_{\mathcal{L}}^*[E] \triangleq \{\cup\}\big((\mathbb{UF})[E;\mathcal{L}]\big) \in (\mathrm{top})[\mathbb{F}_0^*(\mathcal{L})]. \tag{4.3}$$

In connection with (4.3), note that $(\mathbb{F}_0^*(\mathcal{L}), \mathbf{T}_{\mathcal{L}}^*[E])$ is a zero-dimensional T_2-space, see [3]. Thus,

$$(\mathbb{UF})[E;\mathcal{L}] \in (\mathbf{T}_{\mathcal{L}}^*[E] - \mathrm{BAS})_0[\mathbb{F}_0^*(\mathcal{L})].$$

In what follows, we use the inclusion $\mathbb{F}_0^*(\mathcal{L}) \subset \langle \mathcal{L} - \mathrm{link}\rangle_0[E]$, see [8, (3.2)]. Now, we recall one general property (see [8, (4.2)]):

$$\big(\langle \mathcal{L} - \mathrm{link}\rangle_0[E] = \mathbb{F}_0^*(\mathcal{L})\big) \Longleftrightarrow \big(\mathcal{L} \in \pi_*^{\natural}[E]\big). \tag{4.4}$$

In this connection, note that (see [8, (3.12)]), in the general case of $\mathcal{L}$, we have

$$\mathbf{T}_{\mathcal{L}}^*[E] = \mathbb{T}_*\langle E|\mathcal{L}\rangle|_{\mathbb{F}_0^*(\mathcal{L})}. \tag{4.5}$$

In connection with (4.4), we note [8, (4.3)] where supercompactness conditions for a topology of Wallman type were considered. Moreover, in the general case of $\mathcal{L} \in \pi[E]$, we have the following representation [8, (4.1)]:

$$\langle \mathcal{L} - \mathrm{link}\rangle_0[E] \setminus \mathbb{F}_0^*(\mathcal{L}) = \big\{\mathcal{E} \in \langle \mathcal{L} - \mathrm{link}\rangle_0[E]|\ \exists \Sigma_1 \in \mathcal{E}\ \exists \Sigma_2 \in \mathcal{E}\ \exists \Sigma_3 \in \mathcal{E}:\ \Sigma_1 \cap \Sigma_2 \cap \Sigma_3 = \varnothing\big\}.$$

Therefore, we obtain the following useful equality:

$$\mathbb{F}_0^*(\mathcal{L}) = \big\{\mathcal{E} \in \langle \mathcal{L} - \mathrm{link}\rangle_0[E]|\ \Sigma_1 \cap \Sigma_2 \cap \Sigma_3 \neq \varnothing\quad \forall \Sigma_1 \in \mathcal{E}\quad \forall \Sigma_2 \in \mathcal{E}\quad \forall \Sigma_3 \in \mathcal{E}\big\}. \tag{4.6}$$

It is easily to verify that

$$\mathbb{F}_0^*(\mathcal{L}) \in \mathbf{C}_{\langle \mathcal{L}-\mathrm{link}\rangle_0[E]}[\mathbb{T}_*\langle E|\mathcal{L}\rangle]. \tag{4.7}$$

By (4.5) and (4.7), we conclude that $(\mathbb{F}_0^*(\mathcal{L}), \mathbf{T}_{\mathcal{L}}^*[E])$ is a closed subspace of $(\langle \mathcal{L}-\mathrm{link}\rangle_0[E], \mathbb{T}_*\langle E|\mathcal{L}\rangle)$.

5. The case of product of two widely understood measurable spaces

In this section, we fix nonempty sets X and Y. In addition, we fix two π-systems $\mathcal{X} \in \pi[X]$ and $\mathcal{Y} \in \pi[Y]$. We recall that (see (3.1))

$$\mathcal{A}\{\times\}\mathcal{B} \triangleq \big\{\mathrm{pr}_1(z) \times \mathrm{pr}_2(z):\ z \in \mathcal{A} \times \mathcal{B}\big\}$$

for $\mathcal{A} \in \mathcal{P}'(\mathcal{P}(X))$ and $\mathcal{B} \in \mathcal{P}'(\mathcal{P}(Y))$; of course, $\mathcal{A}\{\times\}\mathcal{B} \in \mathcal{P}'(\mathcal{P}(X \times Y))$. Note that $X \times Y \neq \varnothing$ and

$$\mathcal{X}\{\times\}\mathcal{Y} = \big\{\mathrm{pr}_1(z) \times \mathrm{pr}_2(z):\ z \in \mathcal{X} \times \mathcal{Y}\big\} \in \pi[X \times Y]. \tag{5.1}$$

Proposition 1. *For $\mathcal{A} \in \langle \mathcal{X} - \mathrm{link}\rangle[X]$ and $\mathcal{B} \in \langle \mathcal{Y} - \mathrm{link}\rangle[Y]$, we have*

$$\mathcal{A}\{\times\}\mathcal{B} \in \langle \mathcal{X}\{\times\}\mathcal{Y} - \mathrm{link}\rangle[X \times Y].$$

The proof follows from the definitions.

Below, we use constructions of [11, § 7]. We recall these constructions very briefly. So (see [11, Proposition 17]),

$$\forall H \in (\mathcal{X}\{\times\}\mathcal{Y}) \setminus \{\varnothing\}\quad \exists! z \in (\mathcal{X} \setminus \{\varnothing\}) \times (\mathcal{Y} \setminus \{\varnothing\}):\quad H = \mathrm{pr}_1(z) \times \mathrm{pr}_2(z). \tag{5.2}$$

Using (5.2), we introduce the mappings

$$\big(\varphi_1 \in (\mathcal{X} \setminus \{\varnothing\})^{(\mathcal{X}\{\times\}\mathcal{Y})\setminus\{\varnothing\}}\big)\&\big(\varphi_2 \in (\mathcal{Y} \setminus \{\varnothing\})^{(\mathcal{X}\{\times\}\mathcal{Y})\setminus\{\varnothing\}}\big),$$

for which $S = \varphi_1(S) \times \varphi_2(S)\ \ \forall S \in (\mathcal{X}\{\times\}\mathcal{Y}) \setminus \{\varnothing\}$. By (2.8), we obtain

$$\langle \mathcal{J} - \mathrm{link}\rangle[\mathbf{I}] \subset \mathcal{P}'(\mathcal{J} \setminus \{\varnothing\}) \tag{5.3}$$

for every nonempty set $\mathbf{I}$ and $\mathcal{J} \in \pi[\mathbf{I}]$. Then, by (5.1) and (5.3), we have

$$\big\langle \mathcal{X}\{\times\}\mathcal{Y} - \mathrm{link}\big\rangle[X \times Y] \subset \mathcal{P}'\big((\mathcal{X}\{\times\}\mathcal{Y}) \setminus \{\varnothing\}\big).$$

Then, by [11, Proposition 18], for $\mathcal{E} \in \langle \mathcal{X}\{\times\}\mathcal{Y} - \mathrm{link}\rangle[X \times Y]$, we obtain

$$\Big((\varphi_1)^1(\mathcal{E}) \in \big\langle \mathcal{X} - \mathrm{link}\big\rangle[X]\Big)\&\Big((\varphi_2)^1(\mathcal{E}) \in \big\langle \mathcal{Y} - \mathrm{link}\big\rangle[Y]\Big) \tag{5.4}$$

and, by [11, Proposition 19], the following inclusion holds:

$$\mathcal{E} \subset (\varphi_1)^1(\mathcal{E})\{\times\}(\varphi_2)^1(\mathcal{E}). \tag{5.5}$$

From (2.9), (5.4), (5.5), and Proposition 1, we find (see [11, Propositions 20–21]) that, for

$$\begin{gathered} \mathcal{E} \in \langle \mathcal{X}\{\times\}\mathcal{Y} - \mathrm{link}\rangle_0[X \times Y], \\ \mathcal{E} = (\varphi_1)^1(\mathcal{E})\{\times\}(\varphi_2)^1(\mathcal{E}), \end{gathered} \tag{5.6}$$

where $(\varphi_1)^1(\mathcal{E}) \in \langle \mathcal{X} - \mathrm{link}\rangle_0[X]$ and $(\varphi_2)^1(\mathcal{E}) \in \langle \mathcal{Y} - \mathrm{link}\rangle_0[Y]$. Moreover,

$$\begin{gathered} \forall \mathcal{A} \in \langle \mathcal{X} - \mathrm{link}\rangle_0[X] \quad \forall \mathcal{B} \in \langle \mathcal{Y} - \mathrm{link}\rangle_0[Y] \\ \mathcal{A}\{\times\}\mathcal{B} \in \langle \mathcal{X}\{\times\}\mathcal{Y} - \mathrm{link}\rangle_0[X \times Y], \end{gathered} \tag{5.7}$$

see [11, Proposition 22]. As a corollary, from (5.6) and (5.7), we obtain

$$\langle \mathcal{X}\{\times\}\mathcal{Y} - \mathrm{link}\rangle_0[X \times Y] = \big\{\mathrm{pr}_1(z)\{\times\}\mathrm{pr}_2(z) :\ z \in \langle \mathcal{X} - \mathrm{link}\rangle_0[X] \times \langle \mathcal{Y} - \mathrm{link}\rangle_0[Y]\big\} \tag{5.8}$$

(see [11, Theorem 2]). So, MLSs on the product $(X \times Y, \mathcal{X}\{\times\}\mathcal{Y})$ are exhausted by products of MLSs on $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$. Note that it is possible to use that MLSs in (5.8). For arbitrary linked families, the property similar to (5.8) is, generally speaking, incorrect.

Example 1. Assume that $X = Y = \overline{1,3}$; thus, $X = Y$ is a three-element set: $1 \in X$, $2 \in X$, and $3 \in X$. Suppose that $\mathcal{X} = \mathcal{P}(X)$ and $\mathcal{Y} = \mathcal{P}(Y)$; of course, $\mathcal{X} = \mathcal{Y}$. Now, we introduce the linked family $\mathcal{E}$ by the rule $X \times \{2\} \in \mathcal{E}$, $\{2\} \times Y \in \mathcal{E}$, $\{(2,2)\} \in \mathcal{E}$, and the family $\mathcal{E}$ does not contain any other sets. So, $\mathcal{E}$ is a specific three-element family. Of course, $\{(2,2)\} = \{2\} \times \{2\}$. We have the obvious inclusion

$$\mathcal{E} \in \langle \mathcal{X}\{\times\}\mathcal{Y} - \mathrm{link}\rangle[X \times Y].$$

However,

$$\mathcal{E} \neq \mathcal{A}\{\times\}\mathcal{B} \quad \forall \mathcal{A} \in \langle \mathcal{X} - \mathrm{link}\rangle[X] \quad \forall \mathcal{B} \in \langle \mathcal{Y} - \mathrm{link}\rangle[Y].$$

Indeed, let $\mathcal{E} = \mathcal{A}\{\times\}\mathcal{B}$ for some $\mathcal{A} \in \langle \mathcal{X} - \mathrm{link}\rangle[X]$ and $\mathcal{B} \in \langle \mathcal{Y} - \mathrm{link}\rangle[Y]$. Then

$$(X \times \{2\} \in \mathcal{A}\{\times\}\mathcal{B})\&(\{2\} \times Y \in \mathcal{A}\{\times\}\mathcal{B}).$$

Using (5.2), we find that $X \in \mathcal{A}$ and $Y \in \mathcal{B}$. Then, $X \times Y \in \mathcal{A}\{\times\}\mathcal{B}$. But $X \times Y \notin \mathcal{E}$. The obtained contradiction proves the required property: $\mathcal{E}$ does not have a rectangular structure. □

Note that, by (5.7), we have

$$\mathcal{U}_1\{\times\}\mathcal{U}_2 \in \langle \mathcal{X}\{\times\}\mathcal{Y} - \mathrm{link}\rangle_0[X \times Y] \quad \forall \mathcal{U}_1 \in \mathbb{F}_0^*(\mathcal{X}) \quad \forall \mathcal{U}_2 \in \mathbb{F}_0^*(\mathcal{Y}).$$

Proposition 2. *If* $\mathcal{U}_1 \in \mathbb{F}_0^*(\mathcal{X})$ *and* $\mathcal{U}_2 \in \mathbb{F}_0^*(\mathcal{Y})$, *then* $\mathcal{U}_1\{\times\}\mathcal{U}_2 \in \mathbb{F}_0^*(\mathcal{X}\{\times\}\mathcal{Y})$.

P r o o f. Fix $\mathcal{U}_1 \in \mathbb{F}_0^*(\mathcal{X})$ and $\mathcal{U}_2 \in \mathbb{F}_0^*(\mathcal{Y})$. Then, in particular, $\mathcal{U}_1 \in \langle \mathcal{X} - \text{link}\rangle_0[X]$ and $\mathcal{U}_2 \in \langle \mathcal{Y} - \text{link}\rangle_0[Y]$. By (5.7), we have

$$\mathcal{U}_1\{\times\}\mathcal{U}_2 = \big\{\text{pr}_1(z) \times \text{pr}_2(z) : \ z \in \mathcal{U}_1 \times \mathcal{U}_2\big\} \in \langle \mathcal{X}\{\times\}\mathcal{Y} - \text{link}\rangle_0[X \times Y]. \tag{5.9}$$

Let $\Gamma \in \mathcal{U}_1\{\times\}\mathcal{U}_2$, $\Lambda \in \mathcal{U}_1\{\times\}\mathcal{U}_2$, and let $T \in \mathcal{U}_1\{\times\}\mathcal{U}_2$. Using (5.9), we choose

$$(\Gamma_1 \in \mathcal{U}_1)\&(\Gamma_2 \in \mathcal{U}_2)\&(\Lambda_1 \in \mathcal{U}_1)\&(\Lambda_2 \in \mathcal{U}_2)\&(T_1 \in \mathcal{U}_1)\&(T_2 \in \mathcal{U}_2)$$

with the following properties:

$$(\Gamma = \Gamma_1 \times \Gamma_2)\&(\Lambda = \Lambda_1 \times \Lambda_2)\&(T = T_1 \times T_2). \tag{5.10}$$

By (4.6), we obtain the following obvious statements:

$$(\Gamma_1 \cap \Lambda_1 \cap T_1 \neq \varnothing)\&(\Gamma_2 \cap \Lambda_2 \cap T_2 \neq \varnothing). \tag{5.11}$$

Let $\alpha \in \Gamma_1 \cap \Lambda_1 \cap T_1$ and $\beta \in \Gamma_2 \cap \Lambda_2 \cap T_2$ (we use (5.11)). Then, by (5.10), $(\alpha, \beta) \in \Gamma \cap \Lambda \cap T$. Since the choice of Γ, Λ, and T was arbitrary, the required inclusion $\mathcal{U}_1\{\times\}\mathcal{U}_2 \in \mathbb{F}_0^*(\mathcal{X}\{\times\}\mathcal{Y})$ follows from (4.6) and (5.9). $\square$

Proposition 3. *If* $\mathcal{U} \in \mathbb{F}_0^*(\mathcal{X}\{\times\}\mathcal{Y})$, *then* $\exists \mathcal{A} \in \mathbb{F}_0^*(\mathcal{X}) \quad \exists \mathcal{B} \in \mathbb{F}_0^*(\mathcal{Y}) : \ \mathcal{U} = \mathcal{A}\{\times\}\mathcal{B}$.

P r o o f. Fix $\mathcal{U} \in \mathbb{F}_0^*(\mathcal{X}\{\times\}\mathcal{Y})$. Then, by (4.6), we have

$$\mathcal{U} \in \langle \mathcal{X}\{\times\}\mathcal{Y} - \text{link}\rangle_0[X \times Y] \tag{5.12}$$

and the following property:

$$A \cap B \cap C \neq \varnothing \quad \forall A \in \mathcal{U} \quad \forall B \in \mathcal{U} \quad \forall C \in \mathcal{U}. \tag{5.13}$$

From (5.8) and (5.12), we conclude that $\mathcal{U} = \mathcal{U}_1\{\times\}\mathcal{U}_2$ for some $\mathcal{U}_1 \in \langle \mathcal{X} - \text{link}\rangle_0[X]$ and $\mathcal{U}_2 \in \langle \mathcal{Y} - \text{link}\rangle_0[Y]$. In addition (see (2.10)), $X \in \mathcal{U}_1$ and $Y \in \mathcal{U}_2$.

Consider an MLS $\mathcal{U}_1$. For this, we fix $M \in \mathcal{U}_1$, $N \in \mathcal{U}_1$, and $T \in \mathcal{U}_1$. Then, by the choice of $\mathcal{U}_1$ and $\mathcal{U}_2$, we have

$$(M \times Y \in \mathcal{U})\&(N \times Y \in \mathcal{U})\&(T \times Y \in \mathcal{U}), \tag{5.14}$$

(see (2.10)). From (5.13) and (5.14), we obtain $M \cap N \cap T \neq \varnothing$. Since the choice of M, N, and T was arbitrary, the inclusion $\mathcal{U}_1 \in \mathbb{F}_0^*(\mathcal{X})$ is obtained (see (4.6)). The inclusion $\mathcal{U}_2 \in \mathbb{F}_0^*(\mathcal{Y})$ is established similarly. $\square$

Theorem 1. *The following equality holds*:

$$\mathbb{F}_0^*(\mathcal{X}\{\times\}\mathcal{Y}) = \big\{\text{pr}_1(z)\{\times\}\text{pr}_2(z) : \ z \in \mathbb{F}_0^*(\mathcal{X}) \times \mathbb{F}_0^*(\mathcal{Y})\big\}.$$

The proof reduces to immediate combination of Propositions 2 and 3. Finally, we note an important property of topological character (see [13, Theorem 5.1]). We recall that, by (3.1) and (3.2),

$$\begin{gathered}\mathbb{T}_*\langle X|\mathcal{X}\rangle\{\times\}\mathbb{T}_*\langle Y|\mathcal{Y}\rangle = \big\{\text{pr}_1(z) \times \text{pr}_2(z) : \ z \in \mathbb{T}_*\langle X|\mathcal{X}\rangle \times \mathbb{T}_*\langle Y|\mathcal{Y}\rangle\big\} \\ \in \pi\big[\langle \mathcal{X} - \text{link}\rangle_0[X] \times \langle \mathcal{Y} - \text{link}\rangle_0[Y]\big];\end{gathered}$$

then, by (2.7), the natural topology

$$\mathbb{T}_*\langle X|\mathcal{X}\rangle \bigotimes \mathbb{T}_*\langle Y|\mathcal{Y}\rangle \triangleq \{\cup\}\big(\mathbb{T}_*\langle X|\mathcal{X}\rangle\{\times\}\mathbb{T}_*\langle Y|\mathcal{Y}\rangle\big) \in (\mathrm{top})\big[\langle \mathcal{X}-\mathrm{link}\rangle_0[X] \times \langle \mathcal{Y}-\mathrm{link}\rangle_0[Y]\big]$$

of the product of Stone-type TSs is realized. Moreover, the following Stone-type topology is defined:

$$\mathbb{T}_*\big\langle X \times Y|\mathcal{X}\{\times\}\mathcal{Y}\big\rangle \in (\mathrm{top})\big[\langle \mathcal{X}\{\times\}\mathcal{Y}-\mathrm{link}\rangle_0[X \times Y]\big].$$

Then, by [13, Theorem 5.1], the mapping

$$z \longmapsto \mathrm{pr}_1(z)\{\times\}\mathrm{pr}_2(z):\ \langle \mathcal{X}-\mathrm{link}\rangle_0[X] \times \langle \mathcal{Y}-\mathrm{link}\rangle_0[Y] \longrightarrow \langle \mathcal{X}\{\times\}\mathcal{Y}-\mathrm{link}\rangle_0[X \times Y] \tag{5.15}$$

is a homeomorphism from the TS

$$\big(\langle \mathcal{X}-\mathrm{link}\rangle_0[X] \times \langle \mathcal{Y}-\mathrm{link}\rangle_0[Y],\ \mathbb{T}_*\langle X|\mathcal{X}\rangle \bigotimes \mathbb{T}_*\langle Y|\mathcal{Y}\rangle\big) \tag{5.16}$$

onto the TS $(\langle \mathcal{X}\{\times\}\mathcal{Y}-\mathrm{link}\rangle_0[X \times Y],\ \mathbb{T}_*\langle X \times Y|\mathcal{X}\{\times\}\mathcal{Y}\rangle)$.

Note that, by (4.7), we have

$$\mathbb{F}_0^*(\mathcal{X}\{\times\}\mathcal{Y}) \in \mathbf{C}_{\langle \mathcal{X}\{\times\}\mathcal{Y}-\mathrm{link}\rangle_0[X \times Y]}[\mathbb{T}_*\langle X \times Y|\mathcal{X}\{\times\}\mathcal{Y}\rangle].$$

Moreover, using (4.5), we obtain

$$\mathbf{T}^*_{\mathcal{X}\{\times\}\mathcal{Y}}[X \times Y] = \mathbb{T}_*\langle X \times Y|\mathcal{X}\{\times\}\mathcal{Y}\rangle|_{\mathbb{F}_0^*(\mathcal{X}\{\times\}\mathcal{Y})}. \tag{5.17}$$

So, ultrafilters of π-system $\mathcal{X}\{\times\}\mathcal{Y}$ form a closed subspace of TSs homeomorphic to (5.16). Theorem 1 reveals the structure of this subspace.

6. Infinite products of maximal linked systems, 1

Unless otherwise stated, in what follows, nonempty sets X and $\mathbf{E}$ and a mapping $(E_x)_{x\in X} \in \mathcal{P}'(\mathbf{E})^X$ are fixed (for $x \in X$, we denote by E_x a nonempty subset of $\mathbf{E}$). Define the set

$$\mathbb{E} \triangleq \prod_{x\in X} E_x = \big\{f \in \mathbf{E}^X | f(x) \in E_x\ \forall x \in X\big\} \in \mathcal{P}'(\mathbf{E}^X) \tag{6.1}$$

(hereinafter, the axiom of choice is used). Finally, we fix

$$(\mathcal{L}_x)_{x\in X} \in \prod_{x\in X} \pi[E_x]. \tag{6.2}$$

We obtain (see (6.2)) the following two variants of π-systems:

$$\begin{gathered}\bigotimes_{x\in X} \mathcal{L}_x = \Big\{H \in \mathcal{P}(\mathbb{E})|\ \exists (L_x)_{x\in X} \in \prod_{x\in X} \mathcal{L}_x:\ (H = \prod_{x\in X} L_x)\&(\exists K \in \mathrm{Fin}(X):\\ L_s = E_s\ \forall s \in X \setminus K)\Big\} \in \pi[\mathbb{E}],\end{gathered} \tag{6.3}$$

$$\bigodot_{x\in X} \mathcal{L}_x = \Big\{\prod_{x\in X} L_x:\ (L_x)_{x\in X} \in \prod_{x\in X} \mathcal{L}_x\Big\} \in \pi[\mathbb{E}], \tag{6.4}$$

$$\bigotimes_{x\in X} \mathcal{L}_x \subset \bigodot_{x\in X} \mathcal{L}_x \tag{6.5}$$

(we use [8, (6.4)–(6.5)]); in connection with (6.3)–(6.5), we recall (3.6)–(3.8). So, we have two comparable π-systems on $\mathbb{E}$.

Now, we note one simple property:

$$\forall (A_x)_{x\in X} \in \mathcal{P}'(\mathbf{E})^X \quad \forall (B_x)_{x\in X} \in \mathcal{P}'(\mathbf{E})^X$$
$$\Big(\prod_{x\in X} A_x = \prod_{x\in X} B_x\Big) \Longleftrightarrow (A_x = B_x \quad \forall x \in X). \tag{6.6}$$

Moreover, we note that

$$\forall (H_x^{(1)})_{x\in X} \in \mathcal{P}(\mathbf{E})^X \quad \forall (H_x^{(2)})_{x\in X} \in \mathcal{P}(\mathbf{E})^X$$
$$\Big(\prod_{x\in X} H_x^{(1)}\Big) \cap \Big(\prod_{x\in X} H_x^{(2)}\Big) = \prod_{x\in X} (H_x^{(1)} \cap H_x^{(2)}). \tag{6.7}$$

The property (6.7) assumes a natural development; now, we note only that

$$\forall (H_x^{(1)})_{x\in X} \in \mathcal{P}(\mathbf{E})^X \quad \forall (H_x^{(2)})_{x\in X} \in \mathcal{P}(\mathbf{E})^X \quad \forall (H_x^{(3)})_{x\in X} \in \mathcal{P}(\mathbf{E})^X$$
$$\Big(\prod_{x\in X} H_x^{(1)}\Big) \cap \Big(\prod_{x\in X} H_x^{(2)}\Big) \cap \Big(\prod_{x\in X} H_x^{(3)}\Big) = \prod_{x\in X} (H_x^{(1)} \cap H_x^{(2)} \cap H_x^{(3)}). \tag{6.8}$$

By (6.6), an obvious corollary is realized; namely,

$$\forall H \in \Big(\bigodot_{x\in X} \mathcal{P}(E_x)\Big) \setminus \{\varnothing\} \; \exists! (\Sigma_x)_{x\in X} \in \prod_{x\in X} \mathcal{P}'(E_x)$$
$$H = \prod_{x\in X} \Sigma_x. \tag{6.9}$$

Using (6.9), we define a mapping

$$\mathbf{P} : \Big(\bigodot_{x\in X} \mathcal{P}(E_x)\Big) \setminus \{\varnothing\} \longrightarrow \prod_{x\in X} \mathcal{P}'(E_x)$$

by the following rule: if $H \in (\bigodot_{x\in X} \mathcal{P}(E_x)) \setminus \{\varnothing\}$, then $\mathbf{P}(H) \in \prod_{x\in X} \mathcal{P}'(E_x)$ is a mapping such that

$$H = \prod_{\chi\in X} \mathbf{P}(H)(\chi). \tag{6.10}$$

We can use the variant $H = \prod_{x\in X} \Sigma_x$, where $(\Sigma_x)_{x\in X} \in \prod_{x\in X} \mathcal{P}'(E_x)$. In addition, by (6.9), we have

$$\Sigma_\chi = \mathbf{P}\Big(\prod_{x\in X} \Sigma_x\Big)(\chi) \quad \forall (\Sigma_x)_{x\in X} \in \prod_{x\in X} \mathcal{P}'(E_x) \quad \forall \chi \in X. \tag{6.11}$$

Now, note the following obvious inclusions:

$$\Big(\prod_{x\in X} (\mathcal{L}_x \setminus \{\varnothing\}) \subset \prod_{x\in X} \mathcal{P}'(E_x)\Big) \& \Big(\big(\bigodot_{x\in X} \mathcal{L}_x\big) \setminus \{\varnothing\} \subset \big(\bigodot_{x\in X} \mathcal{P}(E_x)\big) \setminus \{\varnothing\}\Big). \tag{6.12}$$

Now, for $\chi \in X$, we define $\mathbf{P}_\chi : (\bigodot_{x\in X} \mathcal{P}(E_x)) \setminus \{\varnothing\} \longrightarrow \mathcal{P}'(E_\chi)$ by the natural rule

$$\mathbf{P}_\chi(H) \triangleq \mathbf{P}(H)(\chi) \quad \forall H \in \Big(\bigodot_{x\in X} \mathcal{P}(E_x)\Big) \setminus \{\varnothing\}. \tag{6.13}$$

Of course, (6.13) defines the corresponding projection mapping. From (6.11) and (6.13), for $\chi \in X$ and $(\Sigma_x)_{x\in X} \in \prod_{x\in X} \mathcal{P}'(E_x)$, we obtain

$$\mathbf{P}_\chi\Big(\prod_{x\in X} \Sigma_x\Big) = \Sigma_\chi. \tag{6.14}$$

From (6.12) and (6.14), we, in particular, obtain

$$\mathbf{P}_\chi\Big(\prod_{x\in X} L_x\Big) = L_\chi \quad \forall (L_x)_{x\in X} \in \prod_{x\in X}(\mathcal{L}_x \setminus \{\varnothing\}) \quad \forall \chi \in X.$$

Using the notion of the set image, we suppose that $\forall \mathcal{H} \in \mathcal{P}((\bigodot_{x\in X} \mathcal{P}(E_x)) \setminus \{\varnothing\}) \quad \forall \chi \in X$

$$\mathbf{P}^1_\chi(\mathcal{H}) \triangleq (\mathbf{P}_\chi)^1(\mathcal{H}). \tag{6.15}$$

Then, the following obvious property holds: if $\mathcal{H} \in \mathcal{P}((\bigodot_{x\in X} \mathcal{L}_x) \setminus \{\varnothing\})$ and $\chi \in X$, then

$$\mathbf{P}^1_\chi(\mathcal{H}) \in \mathcal{P}(\mathcal{L}_\chi \setminus \{\varnothing\}). \tag{6.16}$$

We can use a natural combination of (5.3) and (6.16): a linked system can be used as $\mathcal{H}$. In addition, by [13, Proposition 3.2], we have

$$\mathbf{P}^1_\chi(\mathcal{E}) \in \langle \mathcal{L}_\chi - \mathrm{link}\rangle[E_\chi] \quad \forall \mathcal{E} \in \Big\langle \bigodot_{x\in X} \mathcal{L}_x - \mathrm{link}\rangle[\mathbb{E}] \quad \forall \chi \in X.$$

As a corollary, for $\mathcal{E} \in \langle \bigodot_{x\in X} \mathcal{L}_x - \mathrm{link}\rangle[\mathbb{E}]$, we obtain the mapping

$$(\mathbf{P}^1_x(\mathcal{E}))_{x\in X} \in \prod_{x\in X} \langle \mathcal{L}_x - \mathrm{link}\rangle[E_x]. \tag{6.17}$$

Proposition 4. *If* $(\mathcal{E}_x)_{x\in X} \in \prod_{x\in X}\langle \mathcal{L}_x - \mathrm{link}\rangle[E_x]$, *then* $\bigodot_{x\in X} \mathcal{E}_x \in \langle \bigodot_{x\in X} \mathcal{L}_x - \mathrm{link}\rangle[\mathbb{E}]$.

This proposition corresponds to [13, Proposition 3.1]. To prove Proposition 4, it suffices to use (6.7) (and the axiom of choice). From (6.17) and Proposition 4, we obtain

$$\bigodot_{x\in X} \mathbf{P}^1_x(\mathcal{E}) \in \langle \bigodot_{x\in X} \mathcal{L}_x - \mathrm{link}\rangle[\mathbb{E}] \quad \forall \mathcal{E} \in \langle \bigodot_{x\in X} \mathcal{L}_x - \mathrm{link}\rangle[\mathbb{E}]. \tag{6.18}$$

Note an obvious analog of (5.5); namely, for $\mathcal{E} \in \langle \bigodot_{x\in X} \mathcal{L}_x - \mathrm{link}\rangle_0[\mathbb{E}]$, we have

$$\mathcal{E} \subset \bigodot_{x\in X} \mathbf{P}^1_x(\mathcal{E});$$

therefore (see (2.9) and (6.18)), by [13, Proposition 3.4], we obtain

$$\mathcal{E} = \bigodot_{x\in X} \mathbf{P}^1_x(\mathcal{E}). \tag{6.19}$$

In connection with (6.19), note that, by [13, Proposition 3.5], we have

$$\mathbf{P}^1_\chi(\mathcal{E}) \in \langle \mathcal{L}_\chi - \mathrm{link}\rangle_0[E_\chi] \quad \forall \mathcal{E} \in \langle \bigodot_{x\in X} \mathcal{L}_x - \mathrm{link}\rangle_0[\mathbb{E}] \quad \forall \chi \in X.$$

Then, (6.17) is supplemented by the following statement:

$$(\mathbf{P}^1_x(\mathcal{E}))_{x\in X} \in \prod_{x\in X} \langle \mathcal{L}_x - \mathrm{link}\rangle_0[E_x] \quad \forall \mathcal{E} \in \langle \bigodot_{x\in X} \mathcal{L}_x - \mathrm{link}\rangle_0[\mathbb{E}]. \tag{6.20}$$

Moreover, by [13, Proposition 3.6], we obtain the following property:

$$\bigodot_{x\in X} \mathcal{E}_x \in \Big\langle \bigodot_{x\in X} \mathcal{L}_x - \mathrm{link}\Big\rangle_0[\mathbb{E}] \quad \forall (\mathcal{E}_x)_{x\in X} \in \prod_{x\in X} \langle \mathcal{L}_x - \mathrm{link}\rangle_0[E_x].$$

By (6.19) and (6.20), the following basic statement (see [13, Theorem 3.1]) holds:

$$\big\langle \bigodot_{x\in X} \mathcal{L}_x - \mathrm{link}\big\rangle_0[\mathbb{E}] = \Big\{ \bigodot_{x\in X} \mathcal{E}_x : \ (\mathcal{E}_x)_{x\in X} \in \prod_{x\in X} \langle \mathcal{L}_x - \mathrm{link}\rangle_0[E_x] \Big\}. \tag{6.21}$$

In (6.21), we have a natural analog of (5.8). In connection with (6.21), we note that

$$\prod_{x\in X} \mathbb{F}_0^*(\mathcal{L}_x) = \Big\{ (\mathcal{U}_x)_{x\in X} \in \mathcal{P}'(\mathcal{P}(\mathbf{E}))^X \,|\, \mathcal{U}_s \in \mathbb{F}_0^*(\mathcal{L}_s) \ \forall s \in X \Big\} \subset \prod_{x\in X} \langle \mathcal{L}_x - \mathrm{link}\rangle_0[E_x]. \tag{6.22}$$

Then, by (6.21) and (6.22), we obtain

$$\bigodot_{x\in X} \mathcal{U}_x \in \big\langle \bigodot_{x\in X} \mathcal{L}_x - \mathrm{link}\big\rangle_0[\mathbb{E}] \quad \forall (\mathcal{U}_x)_{x\in X} \in \prod_{x\in X} \mathbb{F}_0^*(\mathcal{L}_x). \tag{6.23}$$

Proposition 5. *If* $(\mathcal{U}_x)_{x\in X} \in \prod_{x\in X} \mathbb{F}_0^*(\mathcal{L}_x)$, *then* $\bigodot_{x\in X} \mathcal{U}_x \in \mathbb{F}_0^*(\bigodot_{x\in X} \mathcal{L}_x)$.

P r o o f. Fix $(\mathcal{U}_x)_{x\in X} \in \prod_{x\in X} \mathbb{F}_0^*(\mathcal{L}_x)$. Then, for $\chi \in X$, we obtain

$$\mathcal{U}_\chi \in \mathbb{F}_0^*(\mathcal{L}_\chi). \tag{6.24}$$

Recall (4.6) and (6.4). Then, by (4.6) and (6.23), we have

$$\begin{gathered}\Big(\Sigma_1 \cap \Sigma_2 \cap \Sigma_3 \neq \varnothing \quad \forall \Sigma_1 \in \bigodot_{x\in X} \mathcal{U}_x \quad \forall \Sigma_2 \in \bigodot_{x\in X} \mathcal{U}_x \quad \forall \Sigma_3 \in \bigodot_{x\in X} \mathcal{U}_x\Big) \\ \Longrightarrow \Big(\bigodot_{x\in X} \mathcal{U}_x \in \mathbb{F}_0^*\big(\bigodot_{x\in X} \mathcal{L}_x\big)\Big).\end{gathered} \tag{6.25}$$

Let $\mathbb{A} \in \bigodot_{x\in X} \mathcal{U}_x$, $\mathbb{B} \in \bigodot_{x\in X} \mathcal{U}_x$, and let $\mathbb{C} \in \bigodot_{x\in X} \mathcal{U}_x$. Then, by (3.7), for some

$$\Big((\mathbb{A}_x)_{x\in X} \in \prod_{x\in X} \mathcal{U}_x\Big) \& \Big((\mathbb{B}_x)_{x\in X} \in \prod_{x\in X} \mathcal{U}_x\Big) \& \Big((\mathbb{C}_x)_{x\in X} \in \prod_{x\in X} \mathcal{U}_x\Big),$$

we obtain the following equalities:

$$\Big(\mathbb{A} = \prod_{x\in X} \mathbb{A}_x\Big) \& \Big(\mathbb{B} = \prod_{x\in X} \mathbb{B}_x\Big) \& \Big(\mathbb{C} = \prod_{x\in X} \mathbb{C}_x\Big). \tag{6.26}$$

From (6.22), for $x \in X$, we obtain the inclusions $\mathbb{A}_x \in \mathcal{P}(\mathbf{E})$, $\mathbb{B}_x \in \mathcal{P}(\mathbf{E})$, and $\mathbb{C}_x \in \mathcal{P}(\mathbf{E})$. Then, by (6.8) and (6.26)

$$\mathbb{A} \cap \mathbb{B} \cap \mathbb{C} = \prod_{x\in X} (\mathbb{A}_x \cap \mathbb{B}_x \cap \mathbb{C}_x). \tag{6.27}$$

In addition, for $x \in X$, we obtain $\mathbb{A}_x \in \mathcal{U}_x$, $\mathbb{B}_x \in \mathcal{U}_x$, and $\mathbb{C}_x \in \mathcal{U}_x$; then, by (4.6) and (6.24) $\mathbb{A}_x \cap \mathbb{B}_x \cap \mathbb{C}_x \neq \varnothing$. So,

$$(\mathbb{A}_x \cap \mathbb{B}_x \cap \mathbb{C}_x)_{x\in X} \in \mathcal{P}'(\mathbf{E})^X.$$

Using (6.27) (and the axiom of choice), we obtain $\mathbb{A} \cap \mathbb{B} \cap \mathbb{C} \neq \varnothing$. Since the choice of $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$ was arbitrary, it is established that the premise of implication (6.25) is true. So, we obtain the required property

$$\bigodot_{x\in X} \mathcal{U}_x \in \mathbb{F}_0^*\big(\bigodot_{x\in X} \mathcal{L}_x\big).$$

$\square$

Proposition 6. *If* $\mathcal{U} \in \mathbb{F}_0^*(\bigodot_{x\in X} \mathcal{L}_x)$, *then* $\exists (\mathcal{U}_x)_{x\in X} \in \prod_{x\in X} \mathbb{F}_0^*(\mathcal{L}_x) : \mathcal{U} = \bigodot_{x\in X} \mathcal{U}_x$.

P r o o f. Fix $\mathcal{U} \in \mathbb{F}_0^*(\bigodot_{x\in X} \mathcal{L}_x)$. Then, in particular,

$$\mathcal{U} \in \Big\langle \bigodot_{x\in X} \mathcal{L}_x - \mathrm{link}\Big\rangle_0[\mathbb{E}].$$

By (6.21), $\mathcal{U} = \bigodot_{x\in X} \mathcal{E}_x$, where

$$(\mathcal{E}_x)_{x\in X} \in \prod_{x\in X} \langle \mathcal{L}_x - \mathrm{link}\rangle_0[E_x].$$

By formula (4.6), we get

$$\Sigma_1 \cap \Sigma_2 \cap \Sigma_3 \neq \varnothing \quad \forall \Sigma_1 \in \mathcal{U} \quad \forall \Sigma_2 \in \mathcal{U} \quad \forall \Sigma_3 \in \mathcal{U}. \tag{6.28}$$

Let $\chi \in X$. Then, $\mathcal{E}_\chi \in \langle \mathcal{L}_\chi - \mathrm{link}\rangle_0[E_\chi]$. Therefore, by formula (4.6), we get

$$\big(\Sigma_1 \cap \Sigma_2 \cap \Sigma_3 \neq \varnothing \quad \forall \Sigma_1 \in \mathcal{E}_\chi \quad \forall \Sigma_2 \in \mathcal{E}_\chi \quad \forall \Sigma_3 \in \mathcal{E}_\chi\big) \Longrightarrow (\,\mathcal{E}_\chi \in \mathbb{F}_0^*(\mathcal{L}_\chi)). \tag{6.29}$$

Choose arbitrary $A \in \mathcal{E}_\chi$, $B \in \mathcal{E}_\chi$, and $C \in \mathcal{E}_\chi$. By (5.3), $A \in \mathcal{P}'(\mathbf{E})$, $B \in \mathcal{P}'(\mathbf{E})$, and $C \in \mathcal{P}'(\mathbf{E})$. Now, we introduce $(\tilde{A}_x)_{x\in X} \in \mathcal{P}'(\mathbf{E})^X$ by the rule

$$(\tilde{A}_\chi \triangleq A)\&(\tilde{A}_x \triangleq E_x \quad \forall x \in X \setminus \{\chi\}).$$

Similarly, we introduce $(\tilde{B}_x)_{x\in X} \in \mathcal{P}'(\mathbf{E})^X$ by the rule

$$(\tilde{B}_\chi \triangleq B)\&(\tilde{B}_x \triangleq E_x \quad \forall x \in X \setminus \{\chi\}).$$

Finally, define $(\tilde{C}_x)_{x\in X} \in \mathcal{P}'(\mathbf{E})^X$ by the rule

$$(\tilde{C}_\chi \triangleq C)\&(\tilde{C}_x \triangleq E_x \quad \forall x \in X \setminus \{\chi\}).$$

Then, by (6.8), we obtain the following obvious equality:

$$\Big(\prod_{x\in X} \tilde{A}_x\Big) \cap \Big(\prod_{x\in X} \tilde{B}_x\Big) \cap \Big(\prod_{x\in X} \tilde{C}_x\Big) = \prod_{x\in X} (\tilde{A}_x \cap \tilde{B}_x \cap \tilde{C}_x). \tag{6.30}$$

Note that, by (2.10), $(\tilde{A}_x \in \mathcal{E}_x)\&(\tilde{B}_x \in \mathcal{E}_x)\&(\tilde{C}_x \in \mathcal{E}_x)$ for $x \in X$. Therefore,

$$\Big((\tilde{A}_x)_{x\in X} \in \prod_{x\in X} \mathcal{E}_x\Big)\&\Big((\tilde{B}_x)_{x\in X} \in \prod_{x\in X} \mathcal{E}_x\Big)\&\Big((\tilde{C}_x)_{x\in X} \in \prod_{x\in X} \mathcal{E}_x\Big).$$

By the choice of $(\mathcal{E}_x)_{x\in X}$, we obtain (see (3.7))

$$\Big(\prod_{x\in X} \tilde{A}_x \in \mathcal{U}\Big)\&\Big(\prod_{x\in X} \tilde{B}_x \in \mathcal{U}\Big)\&\Big(\prod_{x\in X} \tilde{C}_x \in \mathcal{U}\Big).$$

As a corollary, by (6.28), we have the following important statement:

$$\Big(\prod_{x\in X} \tilde{A}_x\Big) \cap \Big(\prod_{x\in X} \tilde{B}_x\Big) \cap \Big(\prod_{x\in X} \tilde{C}_x\Big) \neq \varnothing.$$

Then, from (6.30), we obtain $\tilde{A}_x \cap \tilde{B}_x \cap \tilde{C}_x \neq \varnothing$ for $x \in X$. In particular, $\tilde{A}_\chi \cap \tilde{B}_\chi \cap \tilde{C}_\chi \neq \varnothing$. As a corollary, $A \cap B \cap C \neq \varnothing$. Since the choice of A, B, and C was arbitrary, the following property holds:

$$\Sigma_1 \cap \Sigma_2 \cap \Sigma_3 \neq \varnothing \quad \forall \Sigma_1 \in \mathcal{E}_\chi \quad \forall \Sigma_2 \in \mathcal{E}_\chi \quad \forall \Sigma_3 \in \mathcal{E}_\chi.$$

From (6.29), we obtain $\mathcal{E}_\chi \in \mathbb{F}_0^*(\mathcal{L}_\chi)$. Since the choice of χ was arbitrary,

$$\mathcal{E}_x \in \mathbb{F}_0^*(\mathcal{L}_x) \quad \forall x \in X.$$

As a corollary, by the choice of $(\mathcal{E}_x)_{x\in X}$, we obtain

$$(\mathcal{E}_x)_{x\in X} \in \prod_{x\in X} \mathbb{F}_0^*(\mathcal{L}_x) : \; \mathcal{U} = \bigodot_{x\in X} \mathcal{E}_x.$$

□

Theorem 2. *The following equality is true*:

$$\mathbb{F}_0^*(\bigodot_{x\in X} \mathcal{L}_x) = \Big\{ \bigodot_{x\in X} \mathcal{U}_x : \; (\mathcal{U}_x)_{x\in X} \in \prod_{x\in X} \mathbb{F}_0^*(\mathcal{L}_x) \Big\}.$$

The proof immediately follows from Propositions 5 and 6. Returning to (6.21), we note that

$$\mathbf{f} \triangleq \big(\bigodot_{x\in X} \mathcal{E}_x \big)_{(\mathcal{E}_x)_{x\in X} \in \prod\limits_{x\in X} \langle \mathcal{L}_x - \mathrm{link}\rangle_0[E_x]} \in \Big\langle \bigodot_{x\in X} \mathcal{L}_x - \mathrm{link}\Big\rangle_0[\mathbb{E}]^{\prod\limits_{x\in X} \langle \mathcal{L}_x - \mathrm{link}\rangle_0[E_x]} \tag{6.31}$$

is a surjection. Moreover (see (2.11)), by [14, Proposition 4.3], for $(L_x)_{x\in X} \in \prod_{x\in X} \mathcal{L}_x$, we have

$$\mathbf{f}^{-1}\Big(\langle \bigodot_{x\in X} \mathcal{L}_x - \mathrm{link}\rangle^0[\mathbb{E}| \prod_{x\in X} L_x] \Big) = \prod_{x\in X} \langle \mathcal{L}_x - \mathrm{link}\rangle^0[E_x|L_x]. \tag{6.32}$$

Moreover, the following set-product is defined:

$$\prod_{x\in X} \hat{\mathfrak{C}}_0^*[E_x; \mathcal{L}_x] = \big\{ (\mathbb{H}_x)_{x\in X} \in \mathcal{P}(\mathcal{P}'(\mathcal{P}(\mathbf{E})))^X |\; \mathbb{H}_\chi \in \hat{\mathfrak{C}}_0^*[E_\chi; \mathcal{L}_\chi] \;\forall \chi \in X \big\}.$$

In addition (see Section 2), $\hat{\mathfrak{C}}_0^*[E_x; \mathcal{L}_x] \in \mathcal{P}'(\mathcal{P}(\langle \mathcal{L}_x - \mathrm{link}\rangle_0[E_x]))$ for $x \in X$. Then, by (3.5), we have

$$\bigodot_{x\in X} \hat{\mathfrak{C}}_0^*[E_x; \mathcal{L}_x] = \Big\{ \prod_{x\in X} \mathbb{H}_x : \; (\mathbb{H}_x)_{x\in X} \in \prod_{x\in X} \hat{\mathfrak{C}}_0^*[E_x; \mathcal{L}_x] \Big\} \in \mathcal{P}'\Big(\mathcal{P}\Big(\prod_{x\in X} \langle \mathcal{L}_x - \mathrm{link}\rangle_0[E_x] \Big)\Big);$$

thus, the box product of the families $\hat{\mathfrak{C}}_0^*[E_x; \mathcal{L}_x]$, $x \in X$, is defined. Moreover, we have the property

$$\hat{\mathfrak{C}}_0^*\Big[\mathbb{E}; \bigodot_{x\in X} \mathcal{L}_x\Big] \in (\mathrm{p} - \mathrm{BAS})\Big[\langle \bigodot_{x\in X} \mathcal{L}_x - \mathrm{link}\rangle_0[\mathbb{E}] \Big].$$

From (6.32), we obtain the following statement:

$$\mathbf{f}^{-1}(\mathbb{H}) \in \bigodot_{x\in X} \hat{\mathfrak{C}}_0^*[E_x; \mathcal{L}_x] \quad \forall \mathbb{H} \in \hat{\mathfrak{C}}_0^*\Big[\mathbb{E}; \bigodot_{x\in X} \mathcal{L}_x\Big]. \tag{6.33}$$

Now, we recall that (see (2.12)), for $x \in X$,

$$\mathbb{T}_*\langle E_x|\mathcal{L}_x\rangle \in (\mathrm{top})[\langle \mathcal{L}_x - \mathrm{link}\rangle_0[E_x]]:\ \hat{\mathfrak{C}}_0^*[E_x;\mathcal{L}_x] \subset \mathbb{T}_*\langle E_x|\mathcal{L}_x\rangle. \tag{6.34}$$

Then,

$$(\mathbb{T}_*\langle E_x|\mathcal{L}_x\rangle)_{x\in X} \in \prod_{x\in X} (\mathrm{top})[\langle \mathcal{L}_x - \mathrm{link}\rangle_0[E_x]].$$

By (3.9),

$$\bigodot_{x\in X} \mathbb{T}_*\langle E_x|\mathcal{L}_x\rangle = \Big\{ \prod_{x\in X} G_x :\ (G_x)_{x\in X} \in \prod_{x\in X} \mathbb{T}_*\langle E_x|\mathcal{L}_x\rangle \Big\} \in \pi\Big[\prod_{x\in X} \langle \mathcal{L}_x - \mathrm{link}\rangle_0[E_x]\Big]$$

is used as an open base for the corresponding box topology:

$$\mathbf{t}_{\odot}[(\mathbb{T}_*\langle E_x|\mathcal{L}_x\rangle)_{x\in X}] = \{\cup\}(\bigodot_{x\in X} \mathbb{T}_*\langle E_x|\mathcal{L}_x\rangle) \in (\mathrm{top})\Big[\prod_{x\in X} \langle \mathcal{L}_x - \mathrm{link}\rangle_0[E_x]\Big].$$

Moreover, by (6.34), we obtain

$$\bigodot_{x\in X} \hat{\mathfrak{C}}_0^*[E_x;\mathcal{L}_x] \subset \bigodot_{x\in X} \mathbb{T}_*\langle E_x|\mathcal{L}_x\rangle \subset \mathbf{t}_{\odot}[(\mathbb{T}_*\langle E_x|\mathcal{L}_x\rangle)_{x\in X}]. \tag{6.35}$$

On the other hand, by (2.12), the following inclusion holds:

$$\hat{\mathfrak{C}}_0^*[\mathbb{E};\bigodot_{x\in X} \mathcal{L}_x] \in (\mathrm{p}-\mathrm{BAS})_0\Big[\Big\langle \bigodot_{x\in X} \mathcal{L}_x - \mathrm{link}\Big\rangle_0[\mathbb{E}];\mathbb{T}_*\Big\langle\mathbb{E}|\bigodot_{x\in X} \mathcal{L}_x\Big\rangle\Big]. \tag{6.36}$$

Therefore, from (6.33) and (6.35), we find that $\mathbf{f}$ is a continuous mapping in the sense of topologies

$$\mathbf{t}_{\odot}\big[(\mathbb{T}_*\langle E_x|\mathcal{L}_x\rangle)_{x\in X}\big], \quad \mathbb{T}_*\big\langle\mathbb{E}|\bigodot_{x\in X} \mathcal{L}_x\big\rangle; \tag{6.37}$$

we use [15, Proposition 1.4.1]. So, we established the continuity of the mapping (6.31). In addition, the space-product of the families $\langle \mathcal{L}_x - \mathrm{link}\rangle_0[E_x]$, $x \in X$, is equipped with box topology. Moreover, note that $\mathbf{f}$ (6.31) is a bijection from

$$\prod_{x\in X} \langle \mathcal{L}_x - \mathrm{link}\rangle_0[E_x]$$

onto $\langle \bigodot_{x\in X} \mathcal{L}_x - \mathrm{link}\rangle_0[\mathbb{E}]$; see [14, Proposition 5.2]. As a result, $\mathbf{f}$ (6.31) is a continuous bijection, i.e., condensation in the sense of topologies (6.37). So, the TS

$$\Big(\prod_{x\in X} \langle \mathcal{L}_x - \mathrm{link}\rangle_0[E_x], \mathbf{t}_{\odot}[(\mathbb{T}_*\langle E_x|\mathcal{L}_x\rangle)_{x\in X}]\Big)$$

condenses on the following space of Stone type:

$$\Big(\Big\langle \bigodot_{x\in X} \mathcal{L}_x - \mathrm{link}\Big\rangle_0[\mathbb{E}], \mathbb{T}_*\Big\langle\mathbb{E}|\bigodot_{x\in X} \mathcal{L}_x\Big\rangle\Big). \tag{6.38}$$

In addition, by (4.7), we obtain

$$\mathbb{F}_0^*(\bigodot_{x\in X} \mathcal{L}_x) \in \mathbf{C}_{\langle \bigodot_{x\in X} \mathcal{L}_x - \mathrm{link}\rangle_0[\mathbb{E}]}\big[\mathbb{T}_*\langle\mathbb{E}|\bigodot_{x\in X} \mathcal{L}_x\rangle\big].$$

Theorem 2 reveals the structure of the set $\mathbb{F}_0^*(\bigodot_{x\in X} \mathcal{L}_x)$. By (4.5), we have

$$\mathbf{T}^*_{\bigodot_{x\in X} \mathcal{L}_x}[\mathbb{E}] = \mathbb{T}_*\langle\mathbb{E}|\bigodot_{x\in X} \mathcal{L}_x\rangle|_{\mathbb{F}_0^*(\bigodot_{x\in X} \mathcal{L}_x)};$$

thus, ultrafilters of the π-system $\bigodot_{x\in X} \mathcal{L}_x$ form a closed subspace of the space (6.38).

7. Infinite products of maximal linked systems, 2

We use the notation of the previous section: X, $\mathbf{E}$, $(E_x)_{x\in X}$, and $\mathbb{E}$. By (3.8), (6.3), and(6.5), we have

$$\bigotimes_{x\in X}\mathcal{L}_x=\Big\{\Lambda\in\mathcal{P}(\mathbb{E})|\ \exists(L_x)_{x\in X}\in\prod_{x\in X}\mathcal{L}_x: \\ \Big(\Lambda=\prod_{x\in X}L_x\Big)\&\big(\exists K\in\mathrm{Fin}(X):\ L_s=E_s\ \forall s\in X\setminus K\big)\Big\}\in\pi[\mathbb{E}]\cap\mathcal{P}\Big(\bigodot_{x\in X}\mathcal{L}_x\Big). \tag{7.1}$$

Consider a widely understood MS

$$\Big(\mathbb{E},\bigotimes_{x\in X}\mathcal{L}_x\Big):\ \bigotimes_{x\in X}\mathcal{L}_x\subset\bigodot_{x\in X}\mathcal{L}_x. \tag{7.2}$$

Note that (see (2.10)) the following inclusion is true:

$$\prod_{x\in X}\langle\mathcal{L}_x-\mathrm{link}\rangle_0[E_x]\subset\prod_{x\in X}(\mathrm{Fam})[E_x].$$

Therefore, by [13, (4.5), Proposition 4.1], we obtain

$$\bigotimes_{x\in X}\mathcal{E}_x=\Big\{H\in\mathcal{P}(\mathbb{E})|\,\exists(\Sigma_x)_{x\in X}\in\prod_{x\in X}\mathcal{E}_x: \\ \Big(H=\prod_{x\in X}\Sigma_x\Big)\&\big(\exists K\in\mathrm{Fin}(X):\ \Sigma_s=E_s\ \ \forall s\in X\setminus K\big)\Big\} \\ \in\Big\langle\bigotimes_{x\in X}\mathcal{L}_x-\mathrm{link}\Big\rangle_0[\mathbb{E}]\quad\forall(\mathcal{E}_x)_{x\in X}\in\prod_{x\in X}\langle\mathcal{L}_x-\mathrm{link}\rangle_0[E_x]. \tag{7.3}$$

We recall that (see [13, Theorem 4.1]) the following equality is true:

$$\Big\langle\bigotimes_{x\in X}\mathcal{L}_x-\mathrm{link}\Big\rangle_0[\mathbb{E}]=\Big\{\bigotimes_{x\in X}\mathcal{E}_x:\ (\mathcal{E}_x)_{x\in X}\in\prod_{x\in X}\langle\mathcal{L}_x-\mathrm{link}\rangle_0[E_x]\Big\} \tag{7.4}$$

By (6.10), (6.12), (6.13), and (7.2), we have

$$\Lambda=\prod_{x\in X}\mathbf{P}_x(\Lambda)\quad\forall\Lambda\in\big(\bigotimes_{x\in X}\mathcal{L}_x\big)\setminus\{\varnothing\}.$$

We use notation (6.15) for the image operation. Then, by [13, Propostion 4.2], we have

$$(\mathbf{P}^1_x(\mathcal{E}))_{x\in X}\in\prod_{x\in X}\langle\mathcal{L}_x-\mathrm{link}\rangle_0[E_x]\quad\forall\mathcal{E}\in\Big\langle\bigotimes_{x\in X}\mathcal{L}_x-\mathrm{link}\Big\rangle_0[\mathbb{E}] \tag{7.5}$$

(we use (5.3)). In this connection, we use the following useful property:

$$(\mathcal{E}_x)_{x\in X}=\Big(\mathbf{P}^1_\chi\big(\bigotimes_{x\in X}\mathcal{E}_x\big)\Big)_{\chi\in X}\quad\forall(\mathcal{E}_x)_{x\in X}\in\prod_{x\in X}\langle\mathcal{L}_x-\mathrm{link}\rangle_0[E_x]; \tag{7.6}$$

in (7.6), we use (7.4), (7.5), and [14, Proposition 6.1]. Now, we recall (6.22); hence (see (7.4)),

$$\bigotimes_{x\in X}\mathcal{U}_x\in\Big\langle\bigotimes_{x\in X}\mathcal{L}_x-\mathrm{link}\Big\rangle_0[\mathbb{E}]\quad\forall(\mathcal{U}_x)_{x\in X}\in\prod_{x\in X}\mathbb{F}^*_0(\mathcal{L}_x). \tag{7.7}$$

Moreover, by (7.1), the general constructions imply the following obvious inclusion:

$$\mathbb{F}_0^*(\bigotimes_{x\in X}\mathcal{L}_x) \subset \big\langle \bigotimes_{x\in X}\mathcal{L}_x - \mathrm{link}\big\rangle_0[\mathbb{E}]. \tag{7.8}$$

In what follows, we consider questions related to a representation of ultrafilters on $\bigotimes_{x\in X}\mathcal{L}_x$ as products (7.7). In this connection, we recall [4]. But, in the present constructions, we use a scheme based on (4.6).

Proposition 7. *If* $(\mathcal{U}_x)_{x\in X} \in \prod\limits_{x\in X}\mathbb{F}_0^*(\mathcal{L}_x)$, *then* $\bigotimes\limits_{x\in X}\mathcal{U}_x \in \mathbb{F}_0^*(\bigotimes\limits_{x\in X}\mathcal{L}_x)$.

P r o o f. Fix

$$(\mathcal{U}_x)_{x\in X} \in \prod_{x\in X}\mathbb{F}_0^*(\mathcal{L}_x).$$

Then, by (7.7), we have

$$\bigotimes_{x\in X}\mathcal{U}_x \in \big\langle \bigotimes_{x\in X}\mathcal{L}_x - \mathrm{link}\big\rangle_0[\mathbb{E}]. \tag{7.9}$$

The inclusion $\mathcal{U}_x \in \mathbb{F}_0^*(\mathcal{L}_x)$ holds for $x\in X$; therefore,

$$\Sigma_1\cap\Sigma_2\cap\Sigma_3 = (\Sigma_1\cap\Sigma_2)\cap\Sigma_3 \neq \varnothing \;\forall \Sigma_1\in\mathcal{U}_x \quad \forall\Sigma_2\in\mathcal{U}_x \quad \forall\Sigma_3\in\mathcal{U}_x \tag{7.10}$$

(we use the axioms of filter). Moreover, by (4.6) and (7.9), we obtain the following implication:

$$\begin{gathered}\Big(\Sigma_1\cap\Sigma_2\cap\Sigma_3\neq\varnothing \quad \forall\Sigma_1\in\bigotimes_{x\in X}\mathcal{U}_x \quad \forall\Sigma_2\in\bigotimes_{x\in X}\mathcal{U}_x \quad \forall\Sigma_3\in\bigotimes_{x\in X}\mathcal{U}_x\Big)\\ \Longrightarrow \Big(\bigotimes_{x\in X}\mathcal{U}_x\in\mathbb{F}_0^*\big(\bigotimes_{x\in X}\mathcal{L}_x\big)\Big).\end{gathered} \tag{7.11}$$

Now, we choose arbitrary sets

$$\Big(\mathbb{A}\in\bigotimes_{x\in X}\mathcal{U}_x\Big)\&\Big(\mathbb{B}\in\bigotimes_{x\in X}\mathcal{U}_x\Big)\&\Big(\mathbb{C}\in\bigotimes_{x\in X}\mathcal{U}_x\Big). \tag{7.12}$$

Using (7.3), (7.8), (7.9), and (7.12), we obtain

$$\big(\mathbb{A}\in\mathcal{P}(\mathbb{E})\big)\&\big(\mathbb{B}\in\mathcal{P}(\mathbb{E})\big)\&\big(\mathbb{C}\in\mathcal{P}(\mathbb{E})\big).$$

In addition, for some $(\tilde{\mathbb{A}}_x)_{x\in X}\in\prod_{x\in X}\mathcal{U}_x$, we have

$$\Big(\mathbb{A}=\prod_{x\in X}\tilde{\mathbb{A}}_x\Big)\&\big(\exists K\in\mathrm{Fin}(X):\ \tilde{\mathbb{A}}_s=E_s \quad \forall s\in X\setminus K\big). \tag{7.13}$$

Similarly, for some $(\tilde{\mathbb{B}}_x)_{x\in X}\in\prod_{x\in X}\mathcal{U}_x$, we have

$$\Big(\mathbb{B}=\prod_{x\in X}\tilde{\mathbb{B}}_x\Big)\&\big(\exists K\in\mathrm{Fin}(X):\ \tilde{\mathbb{B}}_s=E_s \quad \forall s\in X\setminus K\big).$$

Finally, for some $(\tilde{\mathbb{C}}_x)_{x\in X}\in\prod_{x\in X}\mathcal{U}_x$, we obtain

$$\Big(\mathbb{C}=\prod_{x\in X}\tilde{\mathbb{C}}_x\Big)\&\big(\exists K\in\mathrm{Fin}(X):\ \tilde{\mathbb{C}}_s=E_s \quad \forall s\in X\setminus K\big). \tag{7.14}$$

Then, $\tilde{\mathbb{A}}_x \in \mathcal{U}_x$, $\tilde{\mathbb{B}}_x \in \mathcal{U}_x$, and $\tilde{\mathbb{C}}_x \in \mathcal{U}_x$ for $x \in X$. Therefore, by (7.10), for $x \in X$, we have

$$\tilde{\mathbb{A}}_x \cap \tilde{\mathbb{B}}_x \cap \tilde{\mathbb{C}}_x \neq \varnothing; \tag{7.15}$$

as a corollary, $\tilde{\mathbb{A}}_x \cap \tilde{\mathbb{B}}_x \cap \tilde{\mathbb{C}}_x \in \mathcal{P}'(\mathbf{E})$. Then, by (7.15), we have

$$\prod_{x \in X} (\tilde{\mathbb{A}}_x \cap \tilde{\mathbb{B}}_x \cap \tilde{\mathbb{C}}_x) \neq \varnothing \tag{7.16}$$

(we use the axiom of choice). In addition, $(\tilde{\mathbb{A}}_x)_{x\in X} \in \mathcal{P}(\mathbf{E})^X$, $(\tilde{\mathbb{B}}_x)_{x\in X} \in \mathcal{P}(\mathbf{E})^X$, and $(\tilde{\mathbb{C}}_x)_{x\in X} \in \mathcal{P}(\mathbf{E})^X$. Then, by (6.8) and (7.13)–(7.14), we have

$$\mathbb{A} \cap \mathbb{B} \cap \mathbb{C} = \prod_{x \in X} (\tilde{\mathbb{A}}_x \cap \tilde{\mathbb{B}}_x \cap \tilde{\mathbb{C}}_x).$$

From (7.16), the property $\mathbb{A} \cap \mathbb{B} \cap \mathbb{C} \neq \varnothing$ follows. Since the choice of $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$ was arbitrary (see (7.12)), the premise of implication (7.11) is true. As a corollary, we obtain

$$\bigotimes_{x \in X} \mathcal{U}_x \in \mathbb{F}_0^*\Big(\bigotimes_{x \in X} \mathcal{L}_x\Big).$$

$\square$

Proposition 8. *If* $\mathcal{U} \in \mathbb{F}_0^*(\bigotimes_{x\in X} \mathcal{L}_x)$, *then*

$$\exists (\mathcal{U}_x)_{x\in X} \in \prod_{x \in X} \mathbb{F}_0^*(\mathcal{L}_x) : \mathcal{U} = \bigotimes_{x \in X} \mathcal{U}_x. \tag{7.17}$$

P r o o f. Fix $\mathcal{U} \in \mathbb{F}_0^*(\bigotimes_{x\in X} \mathcal{L}_x)$. Then, by (7.8), we have

$$\mathcal{U} \in \langle \bigotimes_{x \in X} \mathcal{L}_x - \mathrm{link}\rangle_0[\mathbb{E}]. \tag{7.18}$$

Therefore, from (7.4) and (7.18), we find that, for some mapping

$$(\mathcal{E}_x)_{x\in X} \in \prod_{x \in X} \langle \mathcal{L}_x - \mathrm{link}\rangle_0[E_x],$$

the following equality holds:

$$\mathcal{U} = \bigotimes_{x \in X} \mathcal{E}_x. \tag{7.19}$$

In addition, $\mathcal{E}_x \in \langle \mathcal{L}_x - \mathrm{link}\rangle_0[E_x]$ for $x \in X$. Fix $\chi \in X$; then $\mathcal{E}_\chi \in \langle \mathcal{L}_\chi - \mathrm{link}\rangle_0[E_\chi]$. By (4.6), we obtain the following implication:

$$\big(\Sigma_1 \cap \Sigma_2 \cap \Sigma_3 \neq \varnothing \quad \forall \Sigma_1 \in \mathcal{E}_\chi \quad \forall \Sigma_2 \in \mathcal{E}_\chi \quad \forall \Sigma_3 \in \mathcal{E}_\chi\big) \Longrightarrow (\mathcal{E}_\chi \in \mathbb{F}_0^*(\mathcal{L}_\chi)). \tag{7.20}$$

Choose arbitrary sets $A \in \mathcal{E}_\chi$, $B \in \mathcal{E}_\chi$, and $C \in \mathcal{E}_\chi$. Using (2.10), we introduce

$$(\tilde{A}_x)_{x\in X} \in \prod_{x \in X} \mathcal{E}_x$$

by the following rule: $\tilde{A}_\chi \triangleq A$ and $\tilde{A}_x \triangleq E_x$ for $x \in X \setminus \{\chi\}$. We obtain

$$\mathbb{A} \triangleq \prod_{x \in X} \tilde{A}_x \in \mathcal{P}(\mathbb{E}). \tag{7.21}$$

Therefore, for $\mathbb{A}$ (7.21), we find that $\exists(\Sigma_x)_{x\in X} \in \prod_{x\in X} \mathcal{E}_x$:

$$\Big(\mathbb{A} = \prod_{x\in X} \Sigma_x\Big)\&(\exists K \in \mathrm{Fin}(X):\ \Sigma_s = E_s \quad \forall s \in X \setminus K).$$

Then, by (3.6), (3.8), and (7.19), we conclude that $\mathbb{A} \in \mathcal{U}$. Introduce (see (2.10)) a mapping

$$(\tilde{B}_x)_{x\in X} \in \prod_{x\in X} \mathcal{E}_x$$

by the rule: $\tilde{B}_\chi \triangleq B$ and $\tilde{B}_x \triangleq E_x$ for $x \in X \setminus \{\chi\}$. Then

$$\mathbb{B} \triangleq \prod_{x\in X} \tilde{B}_x \in \mathcal{P}(\mathbb{E}). \tag{7.22}$$

So, $\mathbb{B}$ (7.22) is a set, for which $\exists(\Sigma_x)_{x\in X} \in \prod_{x\in X} \mathcal{E}_x$:

$$\Big(\mathbb{B} = \prod_{x\in X} \Sigma_x\Big)\&(\exists K \in \mathrm{Fin}(X):\ \Sigma_s = E_s\ \forall s \in X \setminus K).$$

As a result, we conclude that (see (3.6) and (7.19)) $\mathbb{B} \in \mathcal{U}$. Finally, we introduce (see (2.10)) a mapping

$$(\tilde{C}_x)_{x\in X} \in \prod_{x\in X} \mathcal{E}_x$$

by the following rule: $\tilde{C}_\chi \triangleq C$ and $\tilde{C}_x \triangleq E_x$ for $x \in X \setminus \{\chi\}$. Then

$$\mathbb{C} \triangleq \prod_{x\in X} \tilde{C}_x \in \mathcal{P}(\mathbb{E}) \tag{7.23}$$

is a set, for which, by (7.23), $\exists(\Sigma_x)_{x\in X} \in \prod_{x\in X} \mathcal{E}_x$:

$$(\mathbb{C} = \prod_{x\in X} \Sigma_x)\&(\exists K \in \mathrm{Fin}(X):\ \Sigma_s = E_s\ \forall s \in X \setminus K).$$

From (3.6) and (7.19), we conclude that $\mathbb{C} \in \mathcal{U}$. So, $\mathbb{A} \in \mathcal{U}$, $\mathbb{B} \in \mathcal{U}$, and $\mathbb{C} \in \mathcal{U}$. By the choice of $\mathcal{U}$, we have (see (4.6)) the property

$$\mathbb{A} \cap \mathbb{B} \cap \mathbb{C} \neq \varnothing. \tag{7.24}$$

But $(\tilde{A}_x)_{x\in X} \in \mathcal{P}(\mathbf{E})^X$, $(\tilde{B}_x)_{x\in X} \in \mathcal{P}(\mathbf{E})^X$, and $(\tilde{C}_x)_{x\in X} \in \mathcal{P}(\mathbf{E})^X$; therefore (see (6.8) and (7.21)–(7.23)),

$$\mathbb{A} \cap \mathbb{B} \cap \mathbb{C} = \prod_{x\in X} (\tilde{A}_x \cap \tilde{B}_x \cap \tilde{C}_x).$$

Then, by (7.24), we obtain $\tilde{A}_x \cap \tilde{B}_x \cap \tilde{C}_x \neq \varnothing\ \ \forall x \in X$. In particular,

$$A \cap B \cap C = A_\chi \cap B_\chi \cap C_\chi \neq \varnothing.$$

Since the choice of A, B, and C was arbitrary, we obtain

$$\Sigma_1 \cap \Sigma_2 \cap \Sigma_3 \neq \varnothing \quad \forall \Sigma_1 \in \mathcal{E}_\chi \quad \forall \Sigma_2 \in \mathcal{E}_\chi \quad \forall \Sigma_3 \in \mathcal{E}_\chi.$$

Then (see (7.20)) $\mathcal{E}_\chi \in \mathbb{F}_0^*(\mathcal{L}_\chi)$. Since the choice of χ was arbitrary, it is established that $\mathcal{E}_x \in \mathbb{F}_0^*(\mathcal{L}_x)$ $\forall x \in X$. So,

$$(\mathcal{E}_x)_{x\in X} \in \prod_{x\in X} \mathbb{F}_0^*(\mathcal{L}_x).$$

Using (7.19), we obtain the required statement (7.17). $\square$

Theorem 3. *The following equality holds*:

$$\mathbb{F}_0^*(\bigotimes_{x\in X}\mathcal{L}_x) = \Big\{\bigotimes_{x\in X}\mathcal{U}_x:\ (\mathcal{U}_x)_{x\in X}\in\prod_{x\in X}\mathbb{F}_0^*(\mathcal{L}_x)\Big\}.$$

The proof immediately follows from Propositions 7 and 8. In connection with Theorem 3, we recall constructions of [4].

Following to [14], we introduce the following natural mapping:

$$\mathbf{g}\stackrel{\triangle}{=}\big(\bigotimes_{x\in X}\mathcal{E}_x\big)_{(\mathcal{E}_x)_{x\in X}\in\prod\limits_{x\in X}\langle\mathcal{L}_x-\mathrm{link}\rangle_0[E_x]}\in\big\langle\bigotimes_{x\in X}\mathcal{L}_x-\mathrm{link}\big\rangle_0[\mathbb{E}]^{\prod\limits_{x\in X}\langle\mathcal{L}_x-\mathrm{link}\rangle_0[E_x]}. \tag{7.25}$$

So,

$$\mathbf{g}:\ \prod_{x\in X}\langle\mathcal{L}_x-\mathrm{link}\rangle_0[E_x]\longrightarrow\big\langle\bigotimes_{x\in X}\mathcal{L}_x-\mathrm{link}\big\rangle_0[\mathbb{E}];$$

in addition,

$$\mathbf{g}((\mathcal{E}_x)_{x\in X})=\bigotimes_{x\in X}\mathcal{E}_x\quad\forall(\mathcal{E}_x)_{x\in X}\in\prod_{x\in X}\big\langle\mathcal{L}_x-\mathrm{link}\big\rangle_0[E_x]. \tag{7.26}$$

The properties of $\mathbf{g}$ (see (7.25), (7.26)) were considered in [14]. Now we will restrict ourselves to listing them. Note that

$$\begin{gathered}\bigotimes_{x\in X}\hat{\mathfrak{C}}_0^*[E_x;\mathcal{L}_x]=\Big\{\mathbb{C}\in\mathcal{P}\big(\prod_{x\in X}\langle\mathcal{L}_x-\mathrm{link}\rangle_0[E_x]\big)\,|\,\exists(\mathbb{F}_x)_{x\in X}\in\prod_{x\in X}\hat{\mathfrak{C}}_0^*[E_x;\mathcal{L}_x]:\\ (\mathbb{C}=\prod_{x\in X}\mathbb{F}_x)\&(\exists K\in\mathrm{Fin}(X):\ \mathbb{F}_s=\langle\mathcal{L}_s-\mathrm{link}\rangle_0[E_s]\quad\forall s\in X\setminus K)\Big\}\end{gathered} \tag{7.27}$$

(in (7.27), we use (3.6) and take into account that, for $x\in X$,

$$\langle\mathcal{L}_x-\mathrm{link}\rangle^0[E_x|E_x]=\langle\mathcal{L}_x-\mathrm{link}\rangle_0[E_x],$$

see [7, (4.7)]). Then, by [14, Proposition 6.2], we obtain

$$\mathbf{g}^{-1}(\mathbf{H})\in\bigotimes_{x\in X}\hat{\mathfrak{C}}_0^*[E_x;\mathcal{L}_x]\quad\forall\mathbf{H}\in\hat{\mathfrak{C}}_0^*[\mathbb{E};\bigotimes_{x\in X}\mathcal{L}_x]. \tag{7.28}$$

Now, we recall (6.34). As a corollary, the following π-system is defined:

$$\begin{gathered}\bigotimes_{x\in X}\mathbb{T}_*\langle E_x|\mathcal{L}_x\rangle=\Big\{\mathbb{H}\in\mathcal{P}\Big(\prod_{x\in X}\langle\mathcal{L}_x-\mathrm{link}\rangle_0[E_x]\Big)\,|\,\exists(\mathbb{B}_x)_{x\in X}\in\prod_{x\in X}\mathbb{T}_*\langle E_x|\mathcal{L}_x\rangle:\\ \Big(\mathbb{H}=\prod_{x\in X}\mathbb{B}_x\Big)\&(\exists K\in\mathrm{Fin}(X):\ \mathbb{B}_s=\langle\mathcal{L}_s-\mathrm{link}\rangle_0[E_s]\quad\forall s\in X\setminus K)\Big\}\\ \in\pi\Big[\prod_{x\in X}\langle\mathcal{L}_x-\mathrm{link}\rangle_0[E_x]\Big];\end{gathered} \tag{7.29}$$

we use (3.6) and (3.9). By means of (2.7), (3.10), and (7.29), the topology

$$\mathbf{t}_\otimes[(\mathbb{T}_*\langle E_x|\mathcal{L}_x\rangle)_{x\in X}]=\{\cup\}\big(\bigotimes_{x\in X}\mathbb{T}_*\langle E_x|\mathcal{L}_x\rangle\big)\in(\mathrm{top})\Big[\prod_{x\in X}\langle\mathcal{L}_x-\mathrm{link}\rangle_0[E_x]\Big] \tag{7.30}$$

is defined. From (6.34) and (7.30), we obtain

$$\bigotimes_{x\in X}\hat{\mathfrak{C}}_0^*[E_x;\mathcal{L}_x]\subset\bigotimes_{x\in X}\mathbb{T}_*\langle E_x|\mathcal{L}_x\rangle\subset\mathbf{t}_\otimes[(\mathbb{T}_*\langle E_x|\mathcal{L}_x\rangle)_{x\in X}]. \tag{7.31}$$

Therefore, by (7.28) and (7.31), we have the following property:

$$\mathbf{g}^{-1}(\mathbf{H}) \in \mathbf{t}_{\otimes}[(\mathbb{T}_*\langle E_x|\mathcal{L}_x\rangle)_{x\in X}] \quad \forall \mathbf{H} \in \hat{\mathfrak{C}}_0^*[\mathbb{E}; \bigotimes_{x\in X} \mathcal{L}_x]. \tag{7.32}$$

Using (6.36) and (7.32), we obtain the following important property: $\mathbf{g}$ (7.25) is a continuous mapping in the sense of TS

$$\Big(\prod_{x\in X}\langle \mathcal{L}_x - \mathrm{link}\rangle_0[E_x], \mathbf{t}_{\otimes}[(\mathbb{T}_*\langle E_x|\mathcal{L}_x\rangle)_{x\in X}]\Big), \quad \Big(\langle\bigotimes_{x\in X}\mathcal{L}_x - \mathrm{link}\rangle_0[\mathbb{E}], \mathbb{T}_*\langle\mathbb{E}|\bigotimes_{x\in X}\mathcal{L}_x\rangle\Big); \tag{7.33}$$

we use [15, Proposition 1.4.1]. Now, we recall [14, Proposition 6.4] that $\mathbf{g}$ is a bijection from $\prod_{x\in X}\langle \mathcal{L}_x - \mathrm{link}\rangle_0[E_x]$ onto $\langle\bigotimes_{x\in X}\mathcal{L}_x - \mathrm{link}\rangle_0[\mathbb{E}]$ (in this connection, we recall (7.6)). In addition, we recall the following useful statement [14, Proposition 6.5]:

$$\mathbf{g}^1(\mathbb{H}) \in \hat{\mathfrak{C}}_0^*[\mathbb{E}; \bigotimes_{x\in X}\mathcal{L}_x] \quad \forall \mathbb{H} \in \bigotimes_{x\in X}\mathfrak{C}_0^*[E_x;\mathcal{L}_x].$$

By means of this property, the following important statement was established in [14, Proposition 7.1]: $\mathbf{g}$ is an open mapping in the sense of TS (7.33). So, we obtain the following basic statement (see [14, Theorem 7.1]).

Theorem 4. *The mapping* $\mathbf{g}$ (7.25) *is a homeomorphism from the TS*

$$\Big(\prod_{x\in X}\langle \mathcal{L}_x - \mathrm{link}\rangle_0[E_x], \mathbf{t}_{\otimes}[(\mathbb{T}_*\langle E_x|\mathcal{L}_x\rangle)_{x\in X}]\Big)$$

onto the TS

$$(\langle\bigotimes_{x\in X}\mathcal{L}_x - \mathrm{link}\rangle_0[\mathbb{E}], \mathbb{T}_*\langle\mathbb{E}|\bigotimes_{x\in X}\mathcal{L}_x\rangle).$$

From (4.7), we obtain

$$\mathbb{F}_0^*(\bigotimes_{x\in X}\mathcal{L}_x) \in \mathbf{C}_{\langle\bigotimes_{x\in X}\mathcal{L}_x - \mathrm{link}\rangle_0[\mathbb{E}]}[\mathbb{T}_*\langle\mathbb{E}|\bigotimes_{x\in X}\mathcal{L}_x\rangle].$$

Theorem 3 reveals the structure of the set $\mathbb{F}_0^*(\bigotimes_{x\in X}\mathcal{L}_x)$. By (4.5), we have

$$\mathbf{T}^*_{\bigotimes_{x\in X}\mathcal{L}_x}[\mathbb{E}] = \mathbb{T}_*\langle\mathbb{E}|\bigotimes_{x\in X}\mathcal{L}_x\rangle)|_{\mathbb{F}_0^*(\bigotimes_{x\in X}\mathcal{L}_x)}.$$

Thus, ultrafilters of our π-system $\bigotimes_{x\in X}\mathcal{L}_x$ form a closed subspace of the second TS in (7.33).

8. Some corollaries for ultrafilter spaces

In this section, we consider some statements related to products of spaces with topologies of type (4.3). But, at first, we note general properties connected with subspaces of TSs.

For every TS $(\mathbf{X},\tau)$, $\mathbf{X} \neq \varnothing$, and $(\mathbf{Y},\vartheta)$, $\mathbf{Y} \neq \varnothing$, denote by $C(\mathbf{X},\tau,\mathbf{Y},\vartheta)$ the set of all mappings from $\mathbf{Y}^{\mathbf{X}}$ continuous with respect to the topologies τ and ϑ. Similarly, for nonempty sets $\mathbf{X}$ and $\mathbf{Y}$, let

$$\mathbf{Y}^{\mathbf{X}}_{(*)} \triangleq \{f \in \mathbf{Y}^{\mathbf{X}}|\ f^1(\mathbf{X}) = \mathbf{Y}\}$$

be the set of all surjections from $\mathbf{X}$ onto $\mathbf{Y}$, let

$$(\mathrm{bi})[\mathbf{X};\mathbf{Y}] \triangleq \{f \in \mathbf{Y}^{\mathbf{X}}_{(*)} \mid \forall x_1 \in \mathbf{X}\ \forall x_2 \in \mathbf{X}\ \ (f(x_1) = f(x_2)) \Longrightarrow (x_1 = x_2)\}$$

be the set of all bijections from $\mathbf{X}$ onto $\mathbf{Y}$; finally, for $\tau_1 \in (\mathrm{top})[\mathbf{X}]$ and $\tau_2 \in (\mathrm{top})[\mathbf{Y}]$, let

$$C^0_{\mathbf{c}}(\mathbf{X},\tau_1,\mathbf{Y},\tau_2) \triangleq C(\mathbf{X},\tau_1,\mathbf{Y},\tau_2) \cap (\mathrm{bi})[\mathbf{X};\mathbf{Y}] \tag{8.1}$$

be the set of all condensations from $(\mathbf{X},\tau_1)$ onto $(\mathbf{Y},\tau_2)$. We note yet another important notion: for every TS $(\mathbf{X},\tau_1)$, $\mathbf{X} \neq \varnothing$, and $(\mathbf{Y},\tau_2)$, $\mathbf{Y} \neq \varnothing$, let

$$C_{\mathrm{op}}(\mathbf{X},\tau_1,\mathbf{Y},\tau_2) \triangleq \{f \in C(\mathbf{X},\tau_1,\mathbf{Y},\tau_2) \mid f^1(G) \in \tau_2\ \forall G \in \tau_1\}$$

be the set of all open mappings from $(\mathbf{X},\tau_1)$ in $(\mathbf{Y},\tau_2)$. Then

$$(\mathrm{Hom})[\mathbf{X};\tau_1;\mathbf{Y};\tau_2] \triangleq C_{\mathrm{op}}(\mathbf{X},\tau_1,\mathbf{Y},\tau_2) \cap (\mathrm{bi})[\mathbf{X};\mathbf{Y}] \in \mathcal{P}(C^0_{\mathbf{c}}(\mathbf{X},\tau_1,\mathbf{Y},\tau_2))$$

is the set (possibly empty) of all homeomorphisms from $(\mathbf{X},\tau_1)$ onto $(\mathbf{Y},\tau_2)$. Now, we note several simple general properties.

(1) If $(\mathbf{X},\tau_1)$, $\mathbf{X} \neq \varnothing$, and $(\mathbf{Y},\tau_2)$, $\mathbf{Y} \neq \varnothing$, are two TS, $f \in C(\mathbf{X},\tau_1,\mathbf{Y},\tau_2)$, and $A \in \mathcal{P}'(\mathbf{X})$, then $f^1(A) \in \mathcal{P}'(\mathbf{Y})$ and

$$(f|A) \in C\big(A,\tau_1|_A, f^1(A), \tau_2|_{f^1(A)}\big).$$

(2) If $\mathbf{X}$ and $\mathbf{Y}$ are nonempty sets, $f \in (\mathrm{bi})[\mathbf{X};\mathbf{Y}]$, and $A \in \mathcal{P}'(\mathbf{X})$, then $(f|A) \in (\mathrm{bi})[A; f^1(A)]$.

Immediate combination of (1) and (2) implies the following properties.

(3) If $(\mathbf{X},\tau_1)$, $\mathbf{X} \neq \varnothing$, and $(\mathbf{Y},\tau_2)$, $\mathbf{Y} \neq \varnothing$, are two TS, $f \in C^0_{\mathbf{c}}(\mathbf{X},\tau_1,\mathbf{Y},\tau_2)$, and $A \in \mathcal{P}'(\mathbf{X})$, then $(f|A) \in C^0_{\mathbf{c}}(A,\tau_1|_A, f^1(A), \tau_2|_{f^1(A)})$.

(4) If $(\mathbf{X},\tau_1)$, $\mathbf{X} \neq \varnothing$, and $(\mathbf{Y},\tau_2)$, $\mathbf{Y} \neq \varnothing$, are two TS, $f \in (\mathrm{Hom})[\mathbf{X};\tau_1;\mathbf{Y};\tau_2]$, and $A \in \mathcal{P}'(\mathbf{X})$, then

$$(f|A) \in (\mathrm{Hom})\big[A;\tau_1|_A, f^1(A), \tau_2|_{f^1(A)}\big].$$

Now, we note some statements on the structure of a subspace of the product of TSs. If $(\mathbf{X},\tau_1)$, $\mathbf{X} \neq \varnothing$, and $(\mathbf{Y},\tau_2)$, $\mathbf{Y} \neq \varnothing$, are two TS, then, similarly to Section 5, in what follows, we suppose that

$$\tau_1 \bigotimes \tau_2 \triangleq \{\cup\}(\tau_1\{\times\}\tau_2). \tag{8.2}$$

Note that (3.6) and (8.2) should be distinguished; in (8.2), we consider a topology. Then, using [15, Proposition 2.3.2], for every TS $(\mathbf{X},\tau_1)$, $\mathbf{X} \neq \varnothing$, and $(\mathbf{Y},\tau_2)$, $\mathbf{Y} \neq \varnothing$, and sets $A \in \mathcal{P}'(\mathbf{X})$ and $B \in \mathcal{P}'(\mathbf{Y})$, we obtain

$$(\tau_1 \bigotimes \tau_2)|_{A\times B} = \tau_1|_A \bigotimes \tau_2|_B. \tag{8.3}$$

Moreover, if $\mathbf{X}$ and $\mathbf{Y}$ are nonempty sets, $(Y_x)_{x\in\mathbf{X}} \in \mathcal{P}'(\mathbf{Y})^{\mathbf{X}}$, $(\tau_x)_{x\in\mathbf{X}} \in \prod_{x\in\mathbf{X}}(\mathrm{top})[Y_x]$, and $(A_x)_{x\in\mathbf{X}} \in \prod_{x\in\mathbf{X}}\mathcal{P}'(Y_x)$, then

$$\mathbf{t}_{\odot}[(\tau_x)_{x\in\mathbf{X}}]|_{\prod_{x\in\mathbf{X}} A_x} = \mathbf{t}_{\odot}\big[(\tau_x|_{A_x})_{x\in\mathbf{X}}\big]; \tag{8.4}$$

of course, we keep in mind that, in the case under consideration,

$$(A_x)_{x\in\mathbf{X}} \in \mathcal{P}'(\mathbf{Y})^X, \quad (\tau_x|_{A_x})_{x\in\mathbf{X}} \in \prod_{x\in\mathbf{X}}(\mathrm{top})[A_x], \quad \prod_{x\in\mathbf{X}} A_x \in \mathcal{P}'(\prod_{x\in\mathbf{X}} Y_x).$$

In (8.4), we have an analogy with [15, Proposition 2.3.2] (an obvious verification of (8.4) we omit). Finally, for every nonempty sets $\mathbb{X}$ and $\mathbb{Y}$, mappings $(Y_x)_{x\in\mathbb{X}} \in \mathcal{P}'(\mathbb{Y})^{\mathbb{X}}$, $(\tau_x)_{x\in\mathbb{X}} \in \prod_{x\in\mathbb{X}}(\text{top})[Y_x]$, and $(A_x)_{x\in\mathbb{X}} \in \prod_{x\in\mathbb{X}} \mathcal{P}'(Y_x)$, we have

$$\mathbf{t}_{\otimes}[(\tau_x)_{x\in\mathbb{X}}]|_{\prod_{x\in\mathbb{X}} A_x} = \mathbf{t}_{\otimes}[(\tau_x|_{A_x})_{x\in\mathbb{X}}]. \tag{8.5}$$

Now, we consider some topological properties for products of ultrafilter spaces. We begin with the simplest case.

The case of product of two ultrafilter spaces. In this subsection, we fix nonempty sets X and Y. In addition, we fix π-systems $\mathcal{X} \in \pi[X]$ and $\mathcal{Y} \in \pi[Y]$. Then

$$\mathbf{T}^*_{\mathcal{X}}[X] \in (\text{top})[\mathbb{F}^*_0(\mathcal{X})] \quad \text{and} \quad \mathbf{T}^*_{\mathcal{Y}}[Y] \in (\text{top})[\mathbb{F}^*_0(\mathcal{Y})];$$
$$\mathbb{F}^*_0(\mathcal{X}) \in \mathcal{P}'(\langle \mathcal{X} - \text{link}\rangle_0[X]) \quad \text{and} \quad \mathbb{F}^*_0(\mathcal{Y}) \in \mathcal{P}'(\langle \mathcal{Y} - \text{link}\rangle_0[Y]).$$

We recall (4.5):

$$\big(\mathbf{T}^*_{\mathcal{X}}[X] = \mathbb{T}_*\langle X|\mathcal{X}\rangle|_{\mathbb{F}^*_0(\mathcal{X})}\big)\&\big(\mathbf{T}^*_{\mathcal{Y}}[Y] = \mathbb{T}_*\langle Y|\mathcal{Y}\rangle|_{\mathbb{F}^*_0(\mathcal{Y})}\big). \tag{8.6}$$

By (8.2), the following topology is defined:

$$\mathbf{T}^*_{\mathcal{X}}[X] \bigotimes \mathbf{T}^*_{\mathcal{Y}}[Y] \in (\text{top})[\mathbb{F}^*_0(\mathcal{X}) \times \mathbb{F}^*_0(\mathcal{Y})].$$

Using (8.3) and (8.6), we obtain

$$\mathbf{T}^*_{\mathcal{X}}[X] \bigotimes \mathbf{T}^*_{\mathcal{Y}}[Y] = \big(\mathbb{T}_*\langle X|\mathcal{X}\rangle \bigotimes \mathbb{T}_*\langle Y|\mathcal{Y}\rangle\big)|_{\mathbb{F}^*_0(\mathcal{X})\times\mathbb{F}^*_0(\mathcal{Y})}, \tag{8.7}$$

where

$$\mathbb{T}_*\langle X|\mathcal{X}\rangle \bigotimes \mathbb{T}_*\langle Y|\mathcal{Y}\rangle \in (\text{top})\big[\langle \mathcal{X} - \text{link}\rangle_0[X] \times \langle \mathcal{Y} - \text{link}\rangle_0[Y]\big].$$

The mapping (5.15) is a homeomorphism. Finally, we recall (5.17). Now, we note that

$$z \longmapsto \text{pr}_1(z)\{\times\}\text{pr}_2(z): \quad \mathbb{F}^*_0(\mathcal{X}) \times \mathbb{F}^*_0(\mathcal{Y}) \longrightarrow \mathbb{F}^*_0(\mathcal{X}\{\times\}\mathcal{Y}) \tag{8.8}$$

is defined correctly (see Theorem 1). In addition, the mapping (8.8) is a restriction of (5.15) to the set $\mathbb{F}^*_0(\mathcal{X}) \times \mathbb{F}^*_0(\mathcal{Y})$. To make this and subsequent statements shorter, we introduce new notation. In this subsection, denote by $\mathbf{u}$ and $\mathbf{v}$ the mappings (5.15) and (8.8), respectively. Then,

$$\mathbf{v} = (\mathbf{u}|\mathbb{F}^*_0(\mathcal{X}) \times \mathbb{F}^*_0(\mathcal{Y})). \tag{8.9}$$

Moreover, by Theorem 1 and (5.15), we obtain

$$\mathbb{F}^*_0(\mathcal{X}\{\times\}\mathcal{Y}) = \mathbf{u}^1(\mathbb{F}^*_0(\mathcal{X}) \times \mathbb{F}^*_0(\mathcal{Y})). \tag{8.10}$$

Theorem 5. *The mapping (8.8) is a homeomorphism in the sense of topologies (8.7) and*

$$\mathbf{T}^*_{\mathcal{X}\{\times\}\mathcal{Y}}[X \times Y]: \mathbf{v} \in (\text{Hom})\big[\mathbb{F}^*_0(\mathcal{X}) \times \mathbb{F}^*_0(\mathcal{Y}); \mathbf{T}^*_{\mathcal{X}}[X] \bigotimes \mathbf{T}^*_{\mathcal{Y}}[Y]; \mathbb{F}^*_0(\mathcal{X}\{\times\}\mathcal{Y}); \mathbf{T}^*_{\mathcal{X}\{\times\}\mathcal{Y}}[X \times Y]\big].$$

P r o o f. We use (8.9) and (8.10) in constructions connected with (4). For this, we note that (see Section 5)

$$\mathbf{u} \in (\text{Hom})\big[\langle \mathcal{X} - \text{link}\rangle_0[X] \times \langle \mathcal{Y} - \text{link}\rangle_0[Y]; \mathbb{T}_*\langle X|\mathcal{X}\rangle \bigotimes \mathbb{T}_*\langle Y|\mathcal{Y}\rangle; \langle \mathcal{X}\{\times\}\mathcal{Y} - \text{link}\rangle_0[X \times Y];$$
$$\mathbb{T}_*\langle X \times Y|\mathcal{X}\{\times\}\mathcal{Y}\rangle\big].$$

Consider (4) with the following specific definitions:

$$\mathbf{X} = \langle \mathcal{X} - \mathrm{link}\rangle_0[X] \times \langle \mathcal{Y} - \mathrm{link}\rangle_0[Y], \quad \tau_1 = \mathbb{T}_*\langle X|\mathcal{X}\rangle \bigotimes \mathbb{T}_*\langle Y|\mathcal{Y}\rangle,$$
$$\mathbf{Y} = \langle \mathcal{X}\{\times\}\mathcal{Y} - \mathrm{link}\rangle_0[X \times Y], \quad \tau_2 = \mathbb{T}_*\langle X \times Y|\mathcal{X}\{\times\}\mathcal{Y}\rangle, \tag{8.11}$$
$$f = \mathbf{u}, \quad A = \mathbb{F}_0^*(\mathcal{X}) \times \mathbb{F}_0^*(\mathcal{Y}).$$

Using (4), (8.9) and (8.11), we obtain

$$\mathbf{v} \in (\mathrm{Hom})\big[\mathbb{F}_0^*(\mathcal{X}) \times \mathbb{F}_0^*(\mathcal{Y}); (\mathbb{T}_*\langle X|\mathcal{X}\rangle \bigotimes \mathbb{T}_*\langle Y|\mathcal{Y}\rangle)|_{\mathbb{F}_0^*(\mathcal{X})\times\mathbb{F}_0^*(\mathcal{Y})}, \mathbf{u}^1(\mathbb{F}_0^*(\mathcal{X}) \times \mathbb{F}_0^*(\mathcal{Y})); \mathbb{T}_*\langle X \times Y|\mathcal{X}\{\times\}\mathcal{Y}\rangle|_{\mathbf{u}^1(\mathbb{F}_0^*(\mathcal{X})\times\mathbb{F}_0^*(\mathcal{Y}))}\big]. \tag{8.12}$$

Then, we use (4.5), (8.7), (8.9), and (8.10). We have the chain of equalities

$$\mathbf{T}^*_{\mathcal{X}\{\times\}\mathcal{Y}}[X \times Y] = \mathbb{T}_*\langle X \times Y|\mathcal{X}\{\times\}\mathcal{Y}\rangle|_{\mathbb{F}_0^*(\mathcal{X}\{\times\}\mathcal{Y})} = \mathbb{T}_*\langle X \times Y|\mathcal{X}\{\times\}\mathcal{Y}\rangle|_{\mathbf{u}^1(\mathbb{F}_0^*(\mathcal{X})\times\mathbb{F}_0^*(\mathcal{Y}))}.$$

Using (8.7), (8.9), (8.10), and (8.12), we obtain the required property of $\mathbf{v}$. □

The case of box topology on the product of ultrafilter spaces. In this and subsequent subsections, we use nonempty sets X and $\mathbf{E}$ and the mapping $(E_x)_{x\in X} \in \mathcal{P}'(\mathbf{E})^X$ defined in Section 6. Moreover, we follow (6.1) for the set $\mathbb{E}$. In what follows, we fix $(\mathcal{L}_x)_{x\in X}$ (6.2). Then, by (4.3), we have

$$(\mathbf{T}^*_{\mathcal{L}_x}[E_x])_{x\in X} \in \prod_{x\in X} (\mathrm{top})[\mathbb{F}_0^*(\mathcal{L}_x)]. \tag{8.13}$$

In addition,

$$\mathbb{F}_0^*(\mathcal{L}_x) \in \mathcal{P}'(\langle \mathcal{L}_x - \mathrm{link}\rangle_0[E_x]) \quad \forall x \in X.$$

Therefore,

$$(\mathbb{F}_0^*(\mathcal{L}_x))_{x\in X} \in \prod_{x\in X} \mathcal{P}'(\langle \mathcal{L}_x - \mathrm{link}\rangle_0[E_x]).$$

Using (4.5), we obtain

$$\mathbf{T}^*_{\mathcal{L}_x}[E_x] = \mathbb{T}_*\langle E_x|\mathcal{L}_x\rangle|_{\mathbb{F}_0^*(\mathcal{L}_x)} \quad \forall x \in X. \tag{8.14}$$

From (3.10) and (8.13), the following property is extracted:

$$\mathbf{t}_\odot[(\mathbf{T}^*_{\mathcal{L}_x}[E_x])_{x\in X}] \in (\mathrm{top})[\prod_{x\in X} \mathbb{F}_0^*(\mathcal{L}_x)]. \tag{8.15}$$

We recall that, by Proposition 5, the mapping

$$(\mathcal{U}_x)_{x\in X} \longmapsto \bigodot_{x\in X} \mathcal{U}_x : \prod_{x\in X} \mathbb{F}_0^*(\mathcal{L}_x) \longrightarrow \mathbb{F}_0^*(\bigodot_{x\in X} \mathcal{L}_x) \tag{8.16}$$

is defined correctly. By (6.31), this mapping (8.16) is a restriction of (6.31) to the set $\prod_{x\in X} \mathbb{F}_0^*(\mathcal{L}_x)$. For brevity, we denote the mapping (8.16) by $\mathbf{w}$. By (6.22), we have

$$\mathbf{w} \triangleq (\bigodot_{x\in X} \mathcal{U}_x)_{(\mathcal{U}_x)_{x\in X}\in \prod\limits_{x\in X} \mathbb{F}_0^*(\mathcal{L}_x)} \in \mathbb{F}_0^*(\bigodot_{x\in X} \mathcal{L}_x)^{\prod_{x\in X} \mathbb{F}_0^*(\mathcal{L}_x)}. \tag{8.17}$$

Then

$$\prod_{x\in X} \mathbb{F}_0^*(\mathcal{L}_x) \in \mathcal{P}'(\prod_{x\in X} \langle \mathcal{L}_x - \mathrm{link}\rangle_0[E_x])$$

and

$$\mathbf{w} = (\mathbf{f}|\prod_{x\in X}\mathbb{F}_0^*(\mathcal{L}_x)). \tag{8.18}$$

Moreover, we recall that, by (6.31) and Theorem 2,

$$\mathbb{F}_0^*(\bigodot_{x\in X}\mathcal{L}_x) = \mathbf{f}^1(\prod_{x\in X}\mathbb{F}_0^*(\mathcal{L}_x)). \tag{8.19}$$

Theorem 6. *The mapping* (8.16) *is a condensation in the sense of topologies* (8.15) *and* $\mathbf{T}^*_{\mathbb{E}}[\bigodot_{x\in X}\mathcal{L}_x]$:

$$\mathbf{w} \in C^0_{\mathbf{c}}\Big(\prod_{x\in X}\mathbb{F}_0^*(\mathcal{L}_x), \mathbf{t}_{\odot}[(\mathbf{T}^*_{\mathcal{L}_x}[E_x])_{x\in X}], \mathbb{F}_0^*(\bigodot_{x\in X}\mathcal{L}_x), \mathbf{T}^*_{\bigodot_{x\in X}\mathcal{L}_x}[\mathbb{E}]\Big). \tag{8.20}$$

P r o o f. We use (8.17)–(8.19) in constructions connected with (3). For this, we recall that (see Section 6)

$$\mathbf{f} \in C^0_{\mathbf{c}}\Big(\prod_{x\in X}\langle\mathcal{L}_x - \mathrm{link}\rangle_0[E_x], \mathbf{t}_{\odot}[(\mathbb{T}_*\langle E_x|\mathcal{L}_x\rangle)_{x\in X}], \langle\bigodot_{x\in X}\mathcal{L}_x - \mathrm{link}\rangle_0[\mathbb{E}], \mathbb{T}_*\langle\mathbb{E}|\bigodot_{x\in X}\mathcal{L}_x\rangle\Big).$$

Now, we use (3) with the following specific definitions:

$$\begin{gathered}\mathbf{X} = \prod_{x\in X}\langle\mathcal{L}_x - \mathrm{link}\rangle_0[E_x], \quad \tau_1 = \mathbf{t}_{\odot}[(\mathbb{T}_*\langle E_x|\mathcal{L}_x\rangle)_{x\in X}],\\ \mathbf{Y} = \langle\bigodot_{x\in X}\mathcal{L}_x - \mathrm{link}\rangle_0[\mathbb{E}], \quad \tau_2 = \mathbb{T}_*\langle\mathbb{E}|\bigodot_{x\in X}\mathcal{L}_x\rangle,\\ f = \mathbf{f}, \quad A = \prod_{x\in X}\mathbb{F}_0^*(\mathcal{L}_x).\end{gathered} \tag{8.21}$$

Then, by (3), (8.18), and (8.21), we obtain

$$\mathbf{w} \in C^0_{\mathbf{c}}\Big(\prod_{x\in X}\mathbb{F}_0^*(\mathcal{L}_x), \mathbf{t}_{\odot}[(\mathbb{T}_*\langle E_x|\mathcal{L}_x\rangle)_{x\in X}]|_{\prod\limits_{x\in X}\mathbb{F}_0^*(\mathcal{L}_x)}, \mathbf{f}^1(\prod_{x\in X}\mathbb{F}_0^*(\mathcal{L}_x)), \mathbb{T}_*\langle\mathbb{E}|\bigodot_{x\in X}\mathcal{L}_x\rangle|_{\mathbf{f}^1(\prod\limits_{x\in X}\mathbb{F}_0^*(\mathcal{L}_x))}\Big).$$

By (8.19), the following inclusion holds:

$$\mathbf{w} \in C^0_{\mathbf{c}}\Big(\prod_{x\in X}\mathbb{F}_0^*(\mathcal{L}_x), \mathbf{t}_{\odot}[(\mathbb{T}_*\langle E_x|\mathcal{L}_x\rangle)_{x\in X}]|_{\prod\limits_{x\in X}\mathbb{F}_0^*(\mathcal{L}_x)}, \mathbb{F}_0^*(\bigodot_{x\in X}\mathcal{L}_x), \mathbb{T}_*\langle\mathbb{E}|\bigodot_{x\in X}\mathcal{L}_x\rangle|_{\mathbb{F}_0^*(\bigodot\limits_{x\in X}\mathcal{L}_x)}\Big). \tag{8.22}$$

Now, we use (8.4) with the following specific definitions:

$$\mathbf{X} = X, \quad \mathbf{Y} = \mathcal{P}'(\mathcal{P}(\mathbf{E})). \tag{8.23}$$

Using (8.23), we also suppose that

$$(Y_x)_{x\in\mathbf{X}} = (\langle\mathcal{L}_x - \mathrm{link}\rangle_0[E_x])_{x\in X}, \quad (\tau_x)_{x\in\mathbf{X}} = (\mathbb{T}_*\langle E_x|\mathcal{L}_x\rangle)_{x\in X}, \quad (A_x)_{x\in\mathbf{X}} = (\mathbb{F}_0^*(\mathcal{L}_x))_{x\in X}. \tag{8.24}$$

In this connection (see (8.23) and (8.24)), we recall that, by (2.8) and (2.9), the following chain of inclusions holds:

$$\langle\mathcal{L}_x - \mathrm{link}\rangle_0[E_x] \subset \langle\mathcal{L}_x - \mathrm{link}\rangle[E_x] \subset \mathcal{P}'(\mathcal{L}_x) \subset \mathcal{P}'(\mathcal{P}(E_x)) \subset \mathcal{P}'(\mathcal{P}(\mathbf{E})) = \mathbf{Y};$$

moreover, $\langle \mathcal{L}_x - \mathrm{link}\rangle_0[E_x] \neq \varnothing$ for $x \in X$. Therefore (see (8.23)),

$$(\langle \mathcal{L}_x - \mathrm{link}\rangle_0[E_x])_{x\in\mathbf{X}} \in \mathcal{P}'(\mathbf{Y})^{\mathbf{X}}.$$

This corresponds to the conditions for (8.4). Then, from (8.4), (8.23), and (8.24), we have

$$\mathbf{t}_{\odot}[(\mathbb{T}_*\langle E_x|\mathcal{L}_x\rangle)_{x\in X}]|_{\prod\limits_{x\in X}\mathbb{F}_0^*(\mathcal{L}_x)} = \mathbf{t}_{\odot}[(\mathbb{T}_*\langle E_x|\mathcal{L}_x\rangle|_{\mathbb{F}_0^*(\mathcal{L}_x)})_{x\in X}] = \mathbf{t}_{\odot}[(\mathbf{T}^*_{\mathcal{L}_x}[E_x])_{x\in X}]; \tag{8.25}$$

of course, in (8.25), we use (4.5). Moreover, by (4.5), we have

$$\mathbb{T}_*\langle \mathbb{E}|\bigodot_{x\in X}\mathcal{L}_x\rangle|_{\mathbb{F}_0^*(\bigodot\limits_{x\in X}\mathcal{L}_x)} = \mathbf{T}_{\bigodot\limits_{x\in X}\mathcal{L}_x}[\mathbb{E}]. \tag{8.26}$$

From (8.22), (8.25), and (8.26), we obtain (8.20). □

The case of generalized Cartesian product of ultrafilter spaces. We follow the previous subsection (see also Sections 6 and 7), using X, $\mathbf{E}$, $(E_x)_{x\in X}$, $\mathbb{E}$, and $(\mathcal{L}_x)_{x\in X}$. Of course, we use (8.13)–(8.14). Then, by (3.10) and (8.13), we have

$$\mathbf{t}_{\otimes}[(\mathbf{T}^*_{\mathcal{L}_x}[E_x])_{x\in X}] \in (\mathrm{top})[\prod_{x\in X}\mathbb{F}_0^*(\mathcal{L}_x)]. \tag{8.27}$$

From Proposition 7, we conclude that

$$(\mathcal{U}_x)_{x\in X} \longmapsto \bigotimes_{x\in X}\mathcal{U}_x : \prod_{x\in X}\mathbb{F}_0^*(\mathcal{L}_x) \longrightarrow \mathbb{F}_0^*(\bigotimes_{x\in X}\mathcal{L}_x) \tag{8.28}$$

is a restriction of the mapping $\mathbf{g}$ (7.25) to the set $\prod_{x\in X}\mathbb{F}_0^*(\mathcal{L}_x)$. We denote this mapping (8.28) by $\mathbf{r}$ for brevity; so,

$$\mathbf{r} \triangleq (\bigotimes_{x\in X}\mathcal{U}_x)_{(\mathcal{U}_x)_{x\in X}\in\prod\limits_{x\in X}\mathbb{F}_0^*(\mathcal{L}_x)} \in \mathbb{F}_0^*(\bigotimes_{x\in X}\mathcal{L}_x)^{\prod_{x\in X}\mathbb{F}_0^*(\mathcal{L}_x)}. \tag{8.29}$$

Similar to (8.18), we obtain the following equality:

$$\mathbf{r} = (\mathbf{g}|\prod_{x\in X}\mathbb{F}_0^*(\mathcal{L}_x)). \tag{8.30}$$

Moreover, note that, by (7.25) and Theorem 3, we have

$$\mathbb{F}_0^*(\bigotimes_{x\in X}\mathcal{L}_x) = \mathbf{g}^1(\prod_{x\in X}\mathbb{F}_0^*(\mathcal{L}_x)). \tag{8.31}$$

Theorem 7. *The mapping (8.28) is a homeomorphism in the sense of topologies (8.27) and* $\mathbf{T}^*_{\bigotimes\limits_{x\in X}\mathcal{L}_x}[\mathbb{E}]$ *:*

$$\mathbf{r} \in (\mathrm{Hom})\Big[\prod_{x\in X}\mathbb{F}_0^*(\mathcal{L}_x); \mathbf{t}_{\otimes}[(\mathbf{T}^*_{\mathcal{L}_x}[E_x])_{x\in X}]; \mathbb{F}_0^*(\bigotimes_{x\in X}\mathcal{L}_x); \mathbf{T}^*_{\bigotimes\limits_{x\in X}\mathcal{L}_x}[\mathbb{E}]\Big]. \tag{8.32}$$

P r o o f. We use (8.29)–(8.31) in constructions connected with (4). For this, we note that, by Theorem 4,

$$\mathbf{g} \in (\mathrm{Hom})\Big[\prod_{x\in X}\langle \mathcal{L}_x - \mathrm{link}\rangle_0[E_x]; \mathbf{t}_{\otimes}[(\mathbb{T}_*\langle E_x|\mathcal{L}_x\rangle)_{x\in X}]; \langle\bigotimes_{x\in X}\mathcal{L}_x - \mathrm{link}\rangle_0[\mathbb{E}]; \mathbb{T}_*\langle\mathbb{E}|\bigotimes_{x\in X}\mathcal{L}_x\rangle\Big].$$

Now, we use (4) with

$$\begin{gathered}\mathbf{X} = \prod_{x \in X} \langle \mathcal{L}_x - \mathrm{link}\rangle_0[E_x], \quad \tau_1 = \mathbf{t}_{\otimes}[(\mathbb{T}_*\langle E_x|\mathcal{L}_x\rangle)_{x\in X}],\\ \mathbf{Y} = \langle \bigotimes_{x\in X} \mathcal{L}_x - \mathrm{link}\rangle_0[\mathbb{E}], \quad \tau_2 = \mathbb{T}_*\langle \mathbb{E}|\bigotimes_{x\in X}\mathcal{L}_x\rangle,\\ f = \mathbf{g}, \quad A = \prod_{x\in X}\mathbb{F}_0^*(\mathcal{L}_x).\end{gathered} \tag{8.33}$$

Then, by (4), (8.30), and (8.31), we obtain (see (8.33)) the following property:

$$\begin{gathered}\mathbf{r} \in (\mathrm{Hom})\Big[\prod_{x\in X}\mathbb{F}_0^*(\mathcal{L}_x); \mathbf{t}_{\otimes}[(\mathbb{T}_*\langle E_x|\mathcal{L}_x\rangle)_{x\in X}]|_{\prod\limits_{x\in X}\mathbb{F}_0^*(\mathcal{L}_x)};\\ \mathbf{g}^1(\prod_{x\in X}\mathbb{F}_0^*(\mathcal{L}_x)); \mathbb{T}_*\langle\mathbb{E}|\bigotimes_{x\in X}\mathcal{L}_x\rangle|_{\mathbf{g}^1(\prod\limits_{x\in X}\mathbb{F}_0^*(\mathcal{L}_x))}\Big].\end{gathered}$$

Using (8.31), we get the obvious inclusion:

$$\begin{gathered}\mathbf{r} \in (\mathrm{Hom})\Big[\prod_{x\in X}\mathbb{F}_0^*(\mathcal{L}_x); \mathbf{t}_{\otimes}[(\mathbb{T}_*\langle E_x|\mathcal{L}_x\rangle)_{x\in X}]|_{\prod\limits_{x\in X}\mathbb{F}_0^*(\mathcal{L}_x)}; \mathbb{F}_0^*(\bigotimes_{x\in X}\mathcal{L}_x);\\ \mathbb{T}_*\langle\mathbb{E}|\bigotimes_{x\in X}\mathcal{L}_x\rangle|_{\mathbb{F}_0^*(\bigotimes\limits_{x\in X}\mathcal{L}_x)}\Big].\end{gathered} \tag{8.34}$$

Now, using (4.5) and (8.34), we obtain

$$\mathbf{r} \in (\mathrm{Hom})\Big[\prod_{x\in X}\mathbb{F}_0^*(\mathcal{L}_x); \mathbf{t}_{\otimes}[(\mathbb{T}_*\langle E_x|\mathcal{L}_x\rangle)_{x\in X}]|_{\prod\limits_{x\in X}\mathbb{F}_0^*(\mathcal{L}_x)}; \mathbb{F}_0^*(\bigotimes_{x\in X}\mathcal{L}_x); \mathbf{T}^*_{\bigotimes\limits_{x\in X}\mathcal{L}_x}[\mathbb{E}]\Big]. \tag{8.35}$$

In what follows, we use (8.5). In addition, we suppose that, in (8.5),

$$(\mathbb{X} = X)\&(\mathbb{Y} = \mathcal{P}'(\mathcal{P}(\mathbf{E}))). \tag{8.36}$$

Using (8.36), we suppose that, in (8.5),

$$\begin{gathered}(Y_x)_{x\in\mathbb{X}} = (\langle\mathcal{L}_x - \mathrm{link}\rangle_0[E_x])_{x\in X}, \quad (\tau_x)_{x\in\mathbb{X}} = (\mathbb{T}_*\langle E_x|\mathcal{L}_x\rangle)_{x\in X},\\ (A_x)_{x\in\mathbb{X}} = (\mathbb{F}_0^*(\mathcal{L}_x))_{x\in X}.\end{gathered}$$

Then we obtain the following chain of equalities:

$$\mathbf{t}_{\otimes}\big[(\mathbb{T}_*\langle E_x|\mathcal{L}_x\rangle)_{x\in X}\big]|_{\prod\limits_{x\in X}\mathbb{F}_0^*(\mathcal{L}_x)} = \mathbf{t}_{\otimes}\big[(\mathbb{T}_*\langle E_x|\mathcal{L}_x\rangle|_{\mathbb{F}_0^*(\mathcal{L}_x)})_{x\in X}\big] = \mathbf{t}_{\otimes}\big[(\mathbf{T}^*_{\mathcal{L}_x}[E_x])_{x\in X}\big].$$

Therefore, by (8.35), the following inclusion holds:

$$\mathbf{r} \in (\mathrm{Hom})\Big[\prod_{x\in X}\mathbb{F}_0^*(\mathcal{L}_x); \mathbf{t}_{\otimes}\big[(\mathbf{T}^*_{\mathcal{L}_x}[E_x])_{x\in X}\big], \mathbb{F}_0^*(\bigotimes_{x\in X}\mathcal{L}_x); \mathbf{T}^*_{\bigotimes\limits_{x\in X}\mathcal{L}_x}[\mathbb{E}]\Big].$$

So, the property (8.32) is established. □

9. Conclusion

In this paper, some questions related to the structure of ultrafilters and MLSs on products of widely understood MSs were considered. In this connection, two basic directions were developed: the direction connected with representations for ultrafilter and MLSs on the products of MSs (set-theoretical direction) and (topological) direction connected with topological relations between TSs of Stone type arising under consideration of topology products (in the box and Cartesian variants) and topologies on the sets of ultrafilters and MLSs for the product of the corresponding measurable structures. In the first direction, the following property is established: ultrafilters and MLSs on products of MSs are exhausted by products of ultrafilters and MLSs, respectively. In the second direction, important properties of homeomorphism and compaction were obtained. In addition, the compaction property is established for the box products of TSs. In the case of the generalized Cartesian product, the homeomorphism property holds. This comparison shows the better character of Tychonoff's product of TSs compared to box TSs.

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LINEARIZATION OF POISSON–LIE STRUCTURES ON THE $2D$ EUCLIDEAN AND $(1+1)$ POINCARÉ GROUPS

Bousselham Ganbouri

Mohammed First University,
Mohammed V Avenue, P.O. Box 524, 60000 Oujda, Morocco

b.ganbouri@ump.ac.ma

Mohamed Wadia Mansouri

Ibn Tofail University,
P.O. Box 242, 14000 Kénitra, Morocco

mansourimohammed.wadia@uit.ac.ma

Abstract: The paper deals with linearization problem of Poisson-Lie structures on the $(1+1)$ Poincaré and $2D$ Euclidean groups. We construct the explicit form of linearizing coordinates of all these Poisson-Lie structures. For this, we calculate all Poisson-Lie structures on these two groups mentioned above, through the correspondence with Lie Bialgebra structures on their Lie algebras which we first determine.

Keywords: Poisson-Lie groups, Lie bialgebras, Linearization.

1. Introduction

Poisson-Lie structure on a Lie group G is a Poisson structure $\{.,.\}$ on $C^\infty(G)$, such that the multiplication $\mu : G\times G \longrightarrow G$ is a Poisson map, namely

$$\{f\circ\mu,\, g\circ\mu\}_{C^\infty(G\times G)}(x,y)=\{f,g\}_{C^\infty(G)}(\mu(x,y)),\quad x,y\in G,\quad f,g\in C^\infty(G).$$

By Drinfel'd [5, 6], this is equivalent to giving an antisymmetric contravariant $2-$tensor π on G such that the Schouten–Nijenhuis bracket $[\pi,\pi]=0$ and satisfies the multiplicativity relation

$$\pi(xy)=l_{x*}\pi(y)+r_{y*}\pi(x),\quad \forall x,y\in G,$$

where l_{x*} and r_{y*} are the left and right translations in G by x and y, respectively.

The relation above shows that the Poisson-Lie structure π must vanishing at the identity $e\in G$, so that its derivative $d_e\pi:\mathcal{G}\to\bigwedge^2\mathcal{G}$ at e is well defined, where $\mathcal{G}$ is the Lie algebra of G. This linear homomorphism turns out to be a 1-cocycle with respect to the adjoint action, and the dual homomorphism $\bigwedge^2\mathcal{G}^*\to\mathcal{G}^*$ satisfies the Jacobi identity; i.e., the dual $\mathcal{G}^*$ of $\mathcal{G}$ becomes a Lie algebra. Satisfying these properties, the map $d_e\pi$ is said to be a Lie bialgebra structure associated to π.

Recall that the preceding construction is in some sense invertible [10]. Namely, if G is simply connected then any Lie bialgebra structure $\delta:\mathcal{G}\to\bigwedge^2\mathcal{G}$ on the Lie algebra $\mathcal{G}=\mathrm{Lie}\,(G)$ carries uniquely defined Poisson–Lie structure π on G such that

$$(d_e\pi)(S)=\delta(S),\quad \forall S\in\mathcal{G}. \tag{1.1}$$

If we choose a local coordinates $(x_1, x_2, ..., x_n)$ in a neighborhood U of the unity e, the Poisson–Lie structure π is given by

$$\pi(x) = \sum_{1\le i<j\le n} \pi_{ij}(x)\partial x_i \wedge \partial x_j, \quad \forall\, x \in U,$$

where π_{ij} are smooth functions vanishing at e and

$$\{x_i, x_j\}(x) = \pi_{ij}(x), \quad \forall\, x \in U.$$

The Taylor series of the functions π_{ij} is given by

$$\pi_{ij}(x) = \sum_{1\le k\le n} c_{ij}^k x_k + \theta_{ij}(x),$$

where *order* $(\theta_{ij}) \ge 2$ and $c_{ij}^k = \partial\pi_{ij}/\partial x_k(e)$.

In particular, the terms $c_{ij}^k x_k$ define a linear Poisson structure π_0, called the *linear part* of π, there Poisson bracket is written in terms of the local coordinates $(x_1, x_2, ..., x_n)$ as

$$\{x_i, x_j\}_0 = \sum_{1\le k\le n} c_{ij}^k x_k. \tag{1.2}$$

Further, since π satisfies the Jacobi identity, the $\{c_{ij}^k\}_{\substack{1\le i<j\le n\\ 1\le k\le n}}$ form a set of structure constants for the Lie algebra $(\mathcal{G}^*, \delta^*)$ dual of Lie algebra $(\mathcal{G}, [,])$. In other words, $\mathcal{G}^*$ is called the linearizing Lie algebra of Poisson–Lie structure π.

In this paper we are interested in the following *linearization problem:*
Are there new coordinates where the terms θ_{ij} vanish identically, so that the Poisson-Lie structure coincides with its linear part?

For a Poisson structure P vanishing at a point x_0, Weinstein [11] proved that if the linearizing Lie algebra is semisimple, then P is formally linearizable at x_0. Furthermore, Conn [3] proved that if the linearizing Lie algebra is semisimple, then P is analytically linearizable. Duffour [7] showed that semisimplicity does not imply smooth linearizability by giving a counterexample of a three-dimensional solvable Lie algebra. In the case of smooth Poisson structures, Conn [4] proved that if the linearizing Lie algebra is semisimple and of compact type then the linearization is smooth.

For a Poisson–Lie structures, Chloup–Arnould [2] gave examples of linearizable and non linearizable Poisson–Lie structures. Recently, Enriquez–Etingof–Marshal [8] constructed a Poisson isomorphism between the formal Poisson manifolds g^* and G^*, where g is a finite dimensional quasitriangular Lie bialgebra and Alekseev–Meinrenken [1] showed that for any coboundary Poisson–Lie group G, the Poisson structure on G^* is linearisable at the group unit.

The aim of this paper is the explicit construction of smooth linearizing coordinates for the Poisson-Lie structures on the $2D$ Euclidean group generated by the Lie algebra $s_3(0)$ and the $(1+1)$ Poincaré group generated by the Lie algebra $\tau_3(-1)$. We note that the notations $s_3(0)$ and $\tau_3(-1)$ are the same as in [9], where all real three-dimensional Lie algebras are classified. We adopt the same notification throughout this paper.

In this work we present a Lie bialgebra structures on the Lie algebras $s_3(0)$ and $\tau_3(-1)$ and we adopt the classification given in [9]. Then, we give the corresponding Poisson–Lie structures on $2D$ Euclidean and $(1+1)$ Poincaré groups and present their Casimir functions, which describe a symplectic leaves for all Poisson–Lie structures. Finally, we show that all these Poisson–Lie structures are linearizable near the unity by constructing the explicit forme of linearizing coordinates.

The paper is organized as follows. In Section 2 we treat the $2D$ Euclidean group and explain the technical methods, in Section 3 we investigate the $(1+1)$ Poincaré group for which we list in a schematic way our results in the same order and with the same notations.

2. Poisson–Lie structures on $2D$ Euclidean group

2.1. $2D$ Euclidean Lie algebra and group

The $2D$ Euclidean Lie algebra $s_3(0)$ is defined by the Lie brackets:

$$[e_3, e_1] = e_2, \quad [e_3, e_2] = -e_1, \quad [e_1, e_2] = 0.$$

The relation above defines a solvable three-dimensional real Lie algebra where its adjoint representation ρ is as follows:

$$\rho(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho(e_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho(e_3) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The generic Lie group element M with a local coordinates (x, y, z) "near $\{e\}$" is as follows

$$M = \exp(x\rho(e_1)) \exp(y\rho(e_2)) \exp(z\rho(e_3)) = \begin{pmatrix} \cos(z) & -\sin(z) & y \\ \sin(z) & \cos(z) & -x \\ 0 & 0 & 1 \end{pmatrix}.$$

If M' is another generic Lie group element with "local coordinates" (x', y', z'), then the multiplication of two group elements would be

$$M.M' = \begin{pmatrix} \cos(z+z') & -\sin(z+z') & y + y'\cos(z) + x'\sin(z) \\ \sin(z+z') & \cos(z+z') & -x - x'\cos(z) + y'\sin(z) \\ 0 & 0 & 1 \end{pmatrix}.$$

Therewith, the $2D$ Euclidean group can be identified by $\mathbb{R}^3$ associated with the group multiplication law:

$$(x, y, z).(x', y', z') = (x + x'\cos(z) - y'\sin(z), y + y'\cos(z) + x'\sin(z), z + z')$$

with the unity $e = (0, 0, 0)$.

The left invariant fields (E_1, E_2, E_3) associated to the basis (e_1, e_2, e_3) have this local expression

$$E_1 = \cos(z)\partial_x + \sin(z)\partial_y, \quad E_2 = -\sin(z)\partial_x + \cos(z)\partial_y, \quad E_3 = \partial_z.$$

2.2. Bialgebra and Poisson-Lie structures on $2D$ Euclidean group

Let δ be a bialgebra structure on the Lie algebra $s_3(0)$. In the basis (e_1, e_2, e_3) of $s_3(0)$ we write

$$\begin{aligned} \delta(e_1) &= a_1 e_2 \wedge e_3 + b_1 e_3 \wedge e_1 + c_1 e_1 \wedge e_2, \\ \delta(e_2) &= a_2 e_2 \wedge e_3 + b_2 e_3 \wedge e_1 + c_2 e_1 \wedge e_2, \\ \delta(e_3) &= a_3 e_2 \wedge e_3 + b_3 e_3 \wedge e_1 + c_3 e_1 \wedge e_2, \end{aligned}$$

this is equivalent to

$$\begin{pmatrix} \delta(e_1) \\ \delta(e_2) \\ \delta(e_3) \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} e_2 \wedge e_3 \\ e_3 \wedge e_1 \\ e_1 \wedge e_2 \end{pmatrix} = U \begin{pmatrix} e_2 \wedge e_3 \\ e_3 \wedge e_1 \\ e_1 \wedge e_2 \end{pmatrix}.$$

If $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ is the dual basis of (e_1, e_2, e_3), then the Lie bracket on $s_3^*(0)$ given by δ^* can be written:

$$\begin{aligned}\delta^*(\varepsilon_2 \wedge \varepsilon_3) &= a_1\varepsilon_1 + a_2\varepsilon_2 + a_3\varepsilon_3,\\ \delta^*(\varepsilon_3 \wedge \varepsilon_1) &= b_1\varepsilon_1 + b_2\varepsilon_2 + b_3\varepsilon_3,\\ \delta^*(\varepsilon_1 \wedge \varepsilon_2) &= c_1\varepsilon_1 + c_2\varepsilon_2 + c_3\varepsilon_3.\end{aligned}$$

By a straightforward computation, we show that in order to ensure that δ is a 1-cocycle, the system below must to be verified

$$\begin{pmatrix} a_2+b_1 & b_2-a_1 & b_3+c_2 \\ a_1-b_2 & a_2+b_1 & a_3+c_2 \\ 0 & 0 & a_1+b_2 \end{pmatrix} \begin{pmatrix} e_2\wedge e_3 \\ e_3\wedge e_1 \\ e_1 \wedge e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence, the matrix U has the form

$$U = \begin{pmatrix} 0 & b_1 & c_1 \\ -b_1 & 0 & c_2 \\ -c_1 & -c_2 & c_3 \end{pmatrix}, \tag{2.1}$$

where the Jacobi identity fulfilled by δ^* is $b_1c_3 = 0$.
Therefore, we get

Proposition 1. *The Lie bialgebra structures δ on 2D Euclidean Lie algebra are written in terms of the basis (e_1, e_2, e_3) as follows:*

$$\begin{aligned}\delta(e_1) &= b_1e_3\wedge e_1 + c_1e_1\wedge e_2,\\ \delta(e_2) &= -b_1e_2\wedge e_3 + c_2e_1\wedge e_2,\\ \delta(e_3) &= -c_1e_2\wedge e_3 - c_2e_3\wedge e_1 + c_3e_1\wedge e_2,\end{aligned}$$

where b_1, c_1, c_2 and c_3 are reals such that $b_1c_3 = 0$.

Now, let π be the Poisson-Lie structures corresponding to the bialgebra structures δ. We set:

$$\pi = \pi_{23}E_2\wedge E_3 + \pi_{31}E_3\wedge E_1 + \pi_{12}E_1\wedge E_2,$$

where $(E_2\wedge E_3, E_3\wedge E_1, E_1\wedge E_2)$ is the basis of the bivector fields on the 2D Euclidean group.

For any element E_k of the basis (E_1, E_2, E_3), the Lie derivative of π in the direction of E_k is written as

$$L_{E_k}\pi = \sum_{1\le i<j\le 3} E_k(\pi_{ij})E_i\wedge E_j + \pi_{ij}\left([E_k, E_i]\wedge E_j - [E_k, E_j]\wedge E_i\right), \quad k = 1, 2, 3.$$

By a technical and explicit computation using the above relation, we show that the equation (1.1) which describes the correspondence between π and δ can be transformed into the following system

$$\begin{cases} E_1(\pi_{23}) = 0, \\ E_2(\pi_{23}) = -b_1, \\ E_3(\pi_{23}) - \pi_2 = -c_1, \end{cases} \quad \begin{cases} E_1(\pi_{31}) = b_1, \\ E_2(\pi_{31}) = 0, \\ E_3(\pi_{31}) + \pi_{23} = -c_2, \end{cases} \quad \begin{cases} E_1(\pi_{12}) + \pi_{31} = c_1, \\ E_2(\pi_{12}) - \pi_{23} = c_2, \\ E_3(\pi_{12}) = c_3. \end{cases} \tag{2.2}$$

The system (2.2) has for solutions:

$$\begin{aligned}\pi_{23}(x,y,z) &= (b_1x - c_1)\sin(z) - (b_1y - c_2)\cos(z) - c_2,\\ \pi_{31}(x,y,z) &= (b_1x - c_1)\cos(z) + (b_1y - c_2)\sin(z) + c_1,\\ \pi_{12}(x,y,z) &= -\frac{b_1}{2}x^2 - \frac{b_1}{2}y^2 + c_1x + c_2y + c_3z.\end{aligned}$$

Since

$$
\begin{aligned}
E_2 \wedge E_3 &= \cos(z)\partial_y \wedge \partial_z + \sin(z)\partial_z \wedge \partial_x, \\
E_3 \wedge E_1 &= -\sin(z)\partial_y \wedge \partial_z + \cos(z)\partial_z \wedge \partial_x, \\
E_1 \wedge E_2 &= \partial_x \wedge \partial_y,
\end{aligned}
$$

we have:

Proposition 2. *In the local coordinates (x, y, z), the Poisson–Lie bracket $\{.,.\}$ on $2D$ Euclidean group is*:

$$
\begin{aligned}
\{y, z\} &= -b_1 y - c_1 \sin(z) - c_2(\cos(z) - 1), \\
\{z, x\} &= b_1 x - c_2 \sin(z) + c_1(\cos(z) - 1), \\
\{x, y\} &= -\frac{b_1}{2}x^2 - \frac{b_1}{2}y^2 + c_1 x + c_2 y + c_3 z.
\end{aligned}
$$

We will call this four-parametric Poisson–Lie brackets as $\mathcal{PL}(b_1, c_1, c_2, c_3)$.

The linear part π_0 of π is straightforwardly obtained as

$$
\begin{aligned}
\{y, z\}_0 &= -b_1 y - c_1 z, \\
\{z, x\}_0 &= b_1 x - c_2 z, \\
\{x, y\}_0 &= c_1 x + c_2 y + c_3 z.
\end{aligned}
$$

2.3. Classification of Poisson–Lie structures on $2D$ Euclidean group

The Poisson–Lie structures on a Lie group G are in one-to-one correspondence with the bialgebra structures on its Lie algebra $\mathcal{G}$. Thus, we obtain the complete classes of the Poisson-Lie structures on $2D$ Euclidean group by using the classification of Lie bialgebra structures on $s_3(0)$, which was given by Gomez in [9].

In [9], we find four nonequivalents (under Lie algebra automorphisms) classes of Lie bialgebra structures on $s_3(0)$. By taking into account the change of basis:

$$
e_1 = \mathfrak{e}_1, \quad e_2 = \mathfrak{e}_2, \quad e_3 = -\mathfrak{e}_0,
$$

we get a correspondence between each one of those classes and our presented cocommutator δ given in Proposition 1. This correspondence is specified by a fixed values of the parameters (b_1, c_1, c_2, c_3) of the matrix (2.1), as presented in the table below

Table 1. Correspondence with the classification [9] of Lie bialgebra structures on $s_3(0)$.

Lie bialgebra in [6]	b_1	c_1	c_2	c_3
(9)	$-\lambda$	0	0	0
15'	0	0	0	$-\omega$
11'	0	1	0	0
(14')	0	α	0	$-\lambda$

In Table 1, the first column describe the number that identifies the type of Lie bialgebra (last column of table III in [9]). The remaining of columns describe the particular values of the parameters (b_1, c_1, c_2, c_3) for which the cocommutator given in Proposition 1 coincides with the Lie bialgebra parameters from [9]. Note, the parameters λ and ω are nonzero reals.

Thus, we have four nonequivalents (under group automorphisms) classes of Poisson–Lie structures on the $2D$ Euclidean group, that would be explicitly obtained by substituting the values of the parameters (b_1, c_1, c_2, c_3) into the full Poisson–Lie bracket expressions $\mathcal{PL}(b_1, c_1, c_2, c_3)$ given in Proposition 2 as shown in table below

Table 2. Classification of Poisson-Lie structures on the $2D$ Euclidean group corresponding to the Lie bialgebra structures given in Table 1.

$\{,\}$	$\{y,z\}$	$\{z,x\}$	$\{x,y\}$
$\mathcal{PL}(-\lambda,0,0,0)$	λy	$-\lambda x$	$\lambda/2\cdot(x^2+y^2)$
$\mathcal{PL}(0,0,0,-\omega)$	0	0	$-\omega z$
$\mathcal{PL}(0,1,0,0)$	$-\sin(z)$	$\cos(z)-1$	x
$\mathcal{PL}(0,\alpha,0,-\lambda)$	$-\alpha\sin(z)$	$\alpha(\cos(z)-1)$	$\alpha x-\lambda z$

Now, recall that a local Casimir function on a Poisson–Lie group G is a function C such that $\{C, f\} = 0$ for any function f on G. Note that the local Casimir functions on a Poisson–Lie group (G, π) are constant in symplectic leaves of G.

Let $\mathcal{C}_{\mathcal{PL}(b_1,c_1,c_2,c_3)}$ be the Casimir functions for the Poisson–Lie structures $\mathcal{PL}(b_1, c_1, c_2, c_3)$. For the classes of Poisson–Lie structures given in Table 2, we get

$$\begin{aligned}
\mathcal{C}_{\mathcal{PL}(-\lambda,0,0,0)} &= 2\arctan\left(\frac{x}{y}\right) + z,\\
\mathcal{C}_{\mathcal{PL}(0,0,0,-\omega)} &= f(z),\\
\mathcal{C}_{\mathcal{PL}(0,1,0,0)} &= \frac{x\sin(z)}{\cos(z)-1} - y,\\
\mathcal{C}_{\mathcal{PL}(0,\alpha,0,-\lambda)} &= -\alpha y + \frac{(\alpha x-\lambda z)\sin(z)}{\cos(z)-1} - \lambda\ln(1-\cos(z)),
\end{aligned}$$

where f is a $C^\infty-$function that depends only on z.

2.4. Linearization of Poisson–Lie structures on $2D$ Euclidean group

Now, we consider the formula (1.2), than the linear part π_0 of π can be written as

$$\pi_0(x) = \sum_{1\le i<j\le n}\left(\sum_{1\le k\le n} c_{ij}^k x_k\right)\partial_{x_i}\wedge\partial_{x_j}.$$

Note, the Lie bialgebra structure δ associated to π defines a linear Poisson–Lie structure on the additive group $\mathcal{G}$ ($\mathcal{G} \simeq \mathbb{R}^n$), that can be expressed as

$$\delta(a) = \sum_{1\le i<j\le n}\left(\sum_{1\le k\le n} c_{ij}^k a_k\right)\partial_i\wedge\partial_j, \quad a = (a_1, ..., a_n) \in \mathbb{R}^n, \tag{2.3}$$

where $(\partial_1, ..., \partial_n)$ is the canonical basis of $\mathbb{R}^n$.

The expression (2.3) coincides with the linear part π_0, hence the linearization problem becomes as follows:
Is there a local Poisson diffeomorphism $\varphi : G \longrightarrow \mathcal{G}$ of a neighborhood of e in G into a neighborhood of 0 in $\mathcal{G}$ such that $\varphi(e) = 0$?

A such diffeomorphism preserves necessarily the subgroup of singular points: $\{x \in G : \pi(x) = 0\}$ and the symplectics leaves.

If $(\varphi_1, ..., \varphi_n)$ are the components of φ, then φ is solution of the system of equations

$$\{\varphi_i, \varphi_j\} = \sum_{1\leq k\leq n} c_{ij}^k \varphi_k, \qquad 1 \leq i < j \leq n. \tag{2.4}$$

Method. We calculate the equations which determine the symplectics leaves for the four classes of Poisson–Lie structures given in Table 2, using the Casimir functions (each symplectic leaf is the common level manifold of Casimir functions) and we determine their subgroup of singular points.

The identification of the subgroup of the singular points and the symplectics leaves of the $2D$ Euclidean group with those of its Lie algebra $s_3(0)$ allows us to solve the system of equations (2.4) for each class of Poisson–Lie structures given in Table 2. Consequently, our main result is the following

Theorem 1. *All Poisson–Lie structures on $2D$ Euclidean group which are given in Table 2 are linearizable near the unity. The linearizing coordinates of each class are given in Table 3:*

Table 3. Components of linearizing diffeomorphisms φ corresponding to the Poisson–Lie structures given in Table 2.

$\varphi_i(x,y,z)$	$\varphi_1(x,y,z)$	$\varphi_2(x,y,z)$	$\varphi_3(x,y,z)$
$\mathcal{PL}(-\lambda,0,0,0)$	$x\cos(\frac{z}{2}) + y\sin(\frac{z}{2})$	$-x\sin(\frac{z}{2}) + y\cos(\frac{z}{2})$	z
$\mathcal{PL}(0,0,0,-\omega)$	x	y	z
$\mathcal{PL}(0,1,0,0)$	$x + y\tan(\frac{z}{2})$	y	$\tan(\frac{z}{2})$
$\mathcal{PL}(0,\alpha,0,-\lambda)$	$x - \frac{\lambda}{\alpha}z +$ $+\big(y - \frac{\lambda}{\alpha}\ln(1+\tan^2(\frac{z}{2}))\big)\tan(\frac{z}{2})$	y	$2\tan(\frac{z}{2})$

Remark 1. The class $\mathcal{PL}(0,0,0,-\omega)$ is linear in the local coordinates (x,y,z) (trivial case).

3. Poisson–Lie structures on $(1+1)$ Poicaré group

3.1. $(1+1)$ Poincaré Lie algebra and group

The $(1+1)$ Poincaré Lie algebra $\tau_3(-1)$ (presented in null coordinates) is defined by the Lie brackets

$$[e_3, e_1] = -e_1, \quad [e_3, e_2] = e_2, \quad [e_1, e_2] = 0.$$

1. Adjoint representation

$$\rho(e_1) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho(e_3) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

2. Matrix group element

$$M = \exp(x\rho(e_1))\exp(y\rho(e_2))\exp(z\rho(e_3)) = \begin{pmatrix} \exp(-z) & 0 & x \\ 0 & \exp(z) & -y \\ 0 & 0 & 1 \end{pmatrix}.$$

3. Group multiplication law

The $(1+1)$ Poincaré group can be identified by $\mathbb{R}^3$ associated with the group multiplication law:

$$(x, y, z).(x', y', z') = (x + x' \exp(-z), y + y' \exp(z), z + z'),$$

with the unity $e = (0, 0, 0)$.

4. Basis of left invariant fields

$$E_1 = \exp(-z)\partial_x, \quad E_2 = \exp(z)\partial_y, \quad E_3 = \partial_z.$$

3.2. Lie bialgebra and Poisson–Lie structures on $(1+1)$ Poincaré group

1. Lie bialgebra structures on $\tau_3(-1)$

Proposition 3. *The Lie bialgebra structures δ on $(1+1)$ Poincaré Lie algebra are written in terms of the basis (e_1, e_2, e_3) as follows*:

$$\delta(e_1) = b_1 e_3 \wedge e_1 + c_1 e_1 \wedge e_2,$$
$$\delta(e_2) = -b_1 e_2 \wedge e_3 + c_2 e_1 \wedge e_2,$$
$$\delta(e_3) = -c_1 e_2 \wedge e_3 - c_2 e_3 \wedge e_1 + c_3 e_1 \wedge e_2,$$

with a_1, b_1, c_1, c_2 are real such that $b_1 c_3 = 0$.

P r o o f. Similar to the proof of Proposition 1. □

2. Poisson–Lie structures on (1+1) Poincaré group

Proposition 4. *In the local coordinates (x, y, z), the Poisson–Lie bracket $\{.,.\}$ on $(1+1)$ Poincaré group is written as*

$$\{y, z\} = -b_1 y + c_1(1 - \exp(z)),$$
$$\{z, x\} = b_1 x + c_2(\exp(-z) - 1),$$
$$\{x, y\} = c_1 x + c_2 y + c_3 z - b_1 xy.$$

We will call this six-parametric Poisson–Lie brackets as $\mathcal{PL}(b_1, c_1, c_2, c_3)$.

P r o o f. Similar to the proof of Proposition 1. □

3. The linear part is as follows

$$\{y, z\}_0 = -b_1 y - c_1 z,$$
$$\{z, x\}_0 = b_1 x - c_2 z,$$
$$\{x, y\}_0 = c_1 x + c_2 y + c_3 z.$$

3.3. Classification of Lie bialgebra and Poisson–Lie structures on $(1+1)$ Poincaré group

1. Isomorphic to the Lie algebra $\tau_3(-1)$ in [9] through the change of variables

$$e_1 = \mathfrak{e}_1, \quad e_2 = \mathfrak{e}_2, \quad e_3 = -\mathfrak{e}_0$$

2. Correspondence with the classification of Lie bialgebras on $\tau_3(-1)$

Table 4. Correspondence with the classification [9] of Lie bialgebra structures on $\tau_3(-1)$.

Lie bialgebra in [9]	b_1	c_1	c_2	c_3
6 $(\rho = -1, \chi = e_0 \wedge e_1)$	0	0	1	0
7 $(\rho = -1)$	$-\lambda$	0	0	0
(11)	0	$\alpha\beta$	α	0
5’	0	0	0	-1
8	0	$-\alpha$	0	-1
(14)	0	$\alpha\lambda$	α	-1

In Table 4, the first column describes the number that identifies the type of Lie bialgebra (last column of table III in [9]. Note, the parameters λ, α and β are nonzero reals.

3. Classification of Poisson Lie structures on (1+1) Poincaré group

Table 5. Correspondence with the Lie bialgebra structures given in Table 4 of Poisson–Lie structures on the $(1+1)$ Poincaré group.

$\{,\}$	$\{y,z\}$	$\{z,x\}$	$\{x,y\}$
$\mathcal{PL}(0,0,1,0)$	0	$\exp(-z)-1$	y
$\mathcal{PL}(-\lambda,0,0,0)$	λy	$-\lambda x$	λxy
$\mathcal{PL}(0,\alpha\beta,\alpha,0)$	$\alpha\beta(1-\exp(z))$	$\alpha(\exp(-z)-1)$	$\alpha\beta x+\alpha y$
$\mathcal{PL}(0,0,0,-1)$	0	0	$-z$
$\mathcal{PL}(0,-\alpha,0,-1)$	$\alpha(\exp(z)-1)$	0	$-\alpha x - z$
$\mathcal{PL}(0,\alpha\lambda,\alpha,-1)$	$\alpha\lambda(1-\exp(z))$	$\alpha(\exp(-z)-1))$	$\alpha\lambda x+\alpha y - z$

4. Casimir functions

$$\mathcal{C}_{\mathcal{PL}(0,0,1,0)} = \frac{y}{\exp(z)-1}, \quad \mathcal{C}_{\mathcal{PL}(-\lambda,0,0,0)} = -\frac{x\exp(z)}{y}, \quad \mathcal{C}_{\mathcal{PL}(0,\alpha\beta,\alpha,0)} = \frac{\beta x\exp(z)+y}{\exp(z)-1},$$
$$\mathcal{C}_{\mathcal{PL}(0,0,0,-1)} = f(z), \quad \mathcal{C}_{\mathcal{PL}(0,-\alpha,0,-1)} = \frac{\alpha x\exp(z)+z}{\exp(z)-1} - \ln(\exp(-z)-1),$$
$$\mathcal{C}_{\mathcal{PL}(0,\alpha\lambda,\alpha,-1)} = \frac{\alpha\lambda x\exp(z)+\alpha y - z}{1-\exp(z)} - \ln(\exp(-z)-1),$$

where f is a $C^\infty-$function of the only variable z.

3.4. Linearization of Poisson-Lie structures on (1+1) Poincaré group

Theorem 2. *All Poisson-Lie structures on* $(1+1)$ *Poincaré group which are given in Table 5 are linearizable near the unity. The linearizing coordinates of each class are given below*:

Table 6. Components of linearizing diffeomorphisms φ corresponding to the Poisson–Lie structures given in Table 5.

$\varphi_i(x,y,z)$	$\varphi_1(x,y,z)$	$\varphi_2(x,y,z)$	$\varphi_3(x,y,z)$
$\mathcal{PL}(0,0,1,0)$	x	$-y$	$\exp(z)-1$
$\mathcal{PL}(-\lambda,0,0,0)$	$x\exp(z)$	$-y$	z
$\mathcal{PL}(0,\alpha\beta,\alpha,0)$	$x+(\frac{1}{\beta}y+1)(\exp(-z)-1)$	y	$\exp(-z)-1$
$\mathcal{PL}(0,0,0,-1)$	x	y	z
$\mathcal{PL}(0,-\alpha,0,1)$	$-x-\frac{1}{\alpha}z\exp(-z)$	y	$\exp(-z)-1$
$\mathcal{PL}(0,-\alpha,0,1)$	$x+\frac{1}{\lambda}y(\exp(-z)-1)-\frac{1}{\alpha\lambda}z\exp(-z)$	y	$1-\exp(-z)$

P r o o f. We use the same method as in Theorem 1. □

Remark 2. The class $\mathcal{PL}(0,0,0,-1)$ is linear in the local coordinates (x,y,z) (trivial case).

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UNIT AND UNITARY CAYLEY GRAPHS FOR THE RING OF EISENSTEIN INTEGERS MODULO n

Reza Jahani-Nezhad[†], **Ali Bahrami**[††]

Department of Pure Mathematics, Faculty of Mathematical Sciences,
University of Kashan,
Kashan 87317-53153, I. R. Iran

[†]jahanian@kashanu.ac.ir, [††]alibahrami1972@gmail.com

Abstract: Let E_n be the ring of Eisenstein integers modulo n. We denote by $G(E_n)$ and G_{E_n}, the unit graph and the unitary Cayley graph of E_n, respectively. In this paper, we obtain the value of the diameter, the girth, the clique number and the chromatic number of these graphs. We also prove that for each $n > 1$, the graphs $G(E_n)$ and G_{E_n} are Hamiltonian.

Keywords: Unit graph, Unitary Cayley graph, Eisenstein integers, Hamiltonian graph.

1. Introduction

Associating a graph with an algebraic object is an active research subject in algebraic graph theory, an area of mathematics in which methods of abstract algebra are employed in studying various graph invariants and tools in graph theory are used in studying various properties of the associated algebraic structure. The Cayley graph of a finite group was first considered in 1878 by Arthur Cayley [12]. Research on graphs associated with rings was started in 1988 by I. Beck [10]. In the literature, there are some other graphs associated with rings, such as the Cayley graph of a commutative ring [1], the unitary Cayley graph of a ring [3], the total graph of a ring [5], the zero divisor graph of a ring [6], the unit graph of a ring [7] and the comaximal graph of a ring [16].

Let R be a commutative ring with non-zero identity. We denote by $U(R)$, $J(R)$ and $Z(R)$ the group of units of R, the Jacobson radical of R and the set of zero divisors of R, respectively. The *unitary Cayley graph* of a ring R, denoted by G_R, is the graph whose vertex set is R, and in which $\{a, b\}$ is an edge if and only if $a - b \in U(R)$. In 1995 this graph was initially introduced by Dejter and Giudici [14] for $\mathbb{Z}_n$, the ring of integers modulo n. In 2009, Akhtar et al. [3] generalized the unitary Cayley graph $G_{\mathbb{Z}_n}$ to G_R for a finite ring R. The *unit graph* of a ring R, denoted by $G(R)$, is a graph whose vertices are elements of R and two distinct vertices a and b are adjacent if and only if $a + b$ in $U(R)$. In 1990, the unit graph was first investigated by Chung [13] and Grimaldi [16] for $\mathbb{Z}_n$. In 2010, Ashrafi, et al. [7] generalized the unit graph $G(\mathbb{Z}_n)$ to $G(R)$ for an arbitrary ring R. Numerous results about unit and unitary Cayley graphs were obtained, see for examples [3, 7, 18, 19, 21, 22].

The following facts are well known, see for example [4] and [17]. Let ω be a primitive third root of unity. Then the set of all complex numbers $a+b\omega$, where a and b are integers, forms an Euclidean domain with the usual complex number operations and Euclidean norm $N(a + b\omega) = a^2 + b^2 - ab$. This domain will be denoted by E and will be called the *ring of Eisenstein integers*. The units of E are ± 1, $\pm\omega$ and $\pm\overline{\omega}$. The primes of E (up to a unit multiple) are the usual prime integers that are congruent to 2 modulo 3 and Eisenstein integers whose norm is a usual prime integer. Let n be a natural number and let $\langle n\rangle$ be the principal ideal generated by n in E. Then the factor ring

$E/\langle n\rangle$ is isomorphic to the ring ${}_n = \{a + b\omega | a, b \in \mathbb{Z}_n\}$, where $\mathbb{Z}_n$ is the ring of integers modulo n. Thus E_n is a principal ideal ring. This ring is called the *ring of Eisenstein integers modulo n*. In [4] this ring is studied and its properties are investigated, its units are characterized and counted. It is easy to see that $a + b\omega$ is a unit in E_n if and only if $N(a + b\omega)$ is a unit in $\mathbb{Z}_n$. Recall that a ring is *local* if it has a unique maximal ideal. It is shown that

(1) if p is a prime integer, then the ring E_{p^k} is local if and only if $p = 3$ or $p \equiv 2 \pmod 3$;

(2) let $\varphi(R)$ denote the number of units in a ring R, then $\varphi(E_{3^k}) = 2 \times 3^{2k-1}$ and

$$\varphi(E_{p^k}) = \begin{cases} p^{2k-2}(p^2 - 1) & \text{if} \quad p \equiv 2 \pmod 3, \\ (p^k - p^{k-1})^2 & \text{if} \quad p \equiv 1 \pmod 3. \end{cases}$$

In this article, some properties of the graphs $G(E_n)$ and G_{E_n} are studied. The diameter, the girth, chromatic number and clique number, in terms of n, are found. Also, we prove that for each $n > 1$, the graphs $G(E_n)$ and G_{E_n} are *Hamiltonian* and the independence number of G_{E_n} is calculated. An earlier study was carried out for the unit and unitary graphs for the ring of Gaussian integers modulo n, see [9].

Throughout the article, by a *graph* G we mean a fnite undirected graph without loops or multiple edges. If the degree of each vertex in G is equal to k, where k is a positive integer, then G is called *k-regular graph*. For a graph G and for any two vertices a and b of G, we recall that a *walk* between a and b is an alternating sequence $a = v_0$, e_1, v_1, e_2, ..., e_k, $v_k = b$ of vertices and edges of G, denoted by

$$a = v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} \dots \xrightarrow{e_k} v_k = b,$$

such that for every i with $1 \le i \le k$, the edge e_i has endpoints v_{i-1} and v_i. Also, a *path* between a and b is a walk between a and b without repeated vertices. A *cycle* of a graph is a path such that the start and end vertices are the same. The number of edges (counting repeats) in a walk, path or a cycle, is called its *length*. A *Hamiltonian path (cycle)* in G is a path (cycle) in G that visits every vertex of G exactly once. A graph is called *Hamitonian* if it contains a Hamiltonian cycle. For vertices a and b of G, we define $d(a, b)$ to be the length of a shortest path from a to b ($d(a, a) = 0$ and $d(a, b) = \infty$ if there is no such path). The *diameter* of G is

$$\operatorname{diam}(G) = \sup\{d(a, b) \mid \ a, b \in V(G)\}.$$

The *girth* of G, denoted by $gr(G)$ is the length of a shortest cycle in G, ($gr(G) = \infty$ if G contains no cycle). For a positive integer r, a graph is called *r-partite* if the vertex set admits a partition into r classes such that vertices in the same partition class are not adjacent. A r-partite graph is called *complete* if every two vertices in different parts are adjacent. The complete 2-partite graph (also called the *complete bipartite* graph) with exactly two partitions of size n and m, is denoted by $K_{n,m}$. A complete graph on the n vertices, denoted by K_n, is a graph such that every two of distinct vertices are adjacent. A *clique* in G is a set of pairwise adjacent vertices of G. A clique of the maximum size is called a *maximum clique*. The *clique number* of G, denoted by $\omega(G)$, is the number of vertices of a maximum clique in G. We color the vertices of G so that no two joined vertices have the same color. If we color the vertices, we call it a coloring of G. The *chromatic number* $\chi(G)$ of the graph G is the minimum number of colors of colorings of G. The *tensor product* or *Kronecker product* $G \otimes H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, in which (a, b) is adjacent to (c, d) if and only if a is adjacent to c in G and b is adjacent to d in H. For other notions not mentioned in this introduction, one can refer to [11, 15].

Throughout this article, the integers p and p_i are used implicitly to denote primes congruent to 2 modulo 3, while q and q_j likewise denote prime integers congruent to 1 modulo 3. For classical theorems and notations in commutative algebra, the interested reader is referred to [8].

2. The Unit and Unitary Cayley Graphs of E_n

In this section, we determine diameter and girth of the unit and unitary Cayley graphs of E_n. The case when n is a power of a prime is considered first. Then the general case is considered.

2.1. The Unit and Unitary Cayley Graphs of E_{t^n}

For the sake of compeleteness, we mention here two important results that will be kept throghout the paper.

Proposition 1 [3, Proposition 2.2].

(a) *Let R be a ring. Then G_R is a regular graph of degree $|U(R)|$.*

(b) *Let R be a local ring with the maximal ideal $\underline{m}$. Then G_R is a complete mutipartite graph whose partite sets are the cosets of $\underline{m}$ in R. In paticular, G_R is a compelete graph if and only if R is a field.*

Theorem 1 [2, Theorem 3.1]. *Let R be a ring. Then $G(R)$ is a complete r-partite graph if and only if R is a local ring with the maximal ideal $\underline{m}$ and $r = |R/\underline{m}| = 2^n$, for some $n \in \mathbb{N}$ or R is a finite field.*

Theorem 2. *Let n be a positive integer. Then the following statements hold*

(1) $\operatorname{diam}(G(E_{3^n})) = \operatorname{diam}(G_{E_{3^n}}) = 2$;

(2) $\operatorname{gr}(G(E_{3^n})) = \operatorname{gr}(G_{E_{3^n}}) = 3$.

P r o o f. For each positive integer n, E_{3^n} is a local ring with the maximal ideal $\langle 2+\omega\rangle$, see [4]. Since $\varphi(E_{3^n}) = 2 \times 3^{2n-1}$, we have

$$\left|\frac{E_{3^n}}{\langle 2+\omega\rangle}\right| = 3.$$

Therefore, by Proposition 1, $G_{E_{3^n}}$ is a complete 3-partite graph and hence $\operatorname{diam}(G_{E_{3^n}}) = 2$ and $\operatorname{gr}(G_{E_{3^n}}) = 3$. Also, by Theorem 1, $G(E_{3^n}/\langle 2+\omega\rangle)$ is a complete bipartite graph. Thus $\operatorname{diam}(G(E_{3^n})) = 2$ and $\operatorname{gr}(G(E_{3^n})) = 3$. □

Theorem 3. *Let n be a positive integer and q be a prime integer congruent to 1 modulo 3. Then the following statements hold,*

(1) $\operatorname{diam}(G(E_{q^n})) = \operatorname{diam}(G_{E_{q^n}}) = 2$;

(2) $\operatorname{gr}(G(E_{q^n})) = \operatorname{gr}(G_{E_{q^n}}) = 3$.

P r o o f. Since q is a prime integer congruent to 1 modulo 3, the ring E_{q^n} is the product of the two local rings $E/\langle (a+b\omega)^n\rangle$ and $E/\langle (a+b\overline{\omega})^n\rangle$ that have the same number of elements. The ideals $\langle a+b\omega\rangle$ and $\langle a+b\overline{\omega}\rangle$ are the only maximal ideals of E_{q^n}, see [4]. Therefore, by [18, Theorem 3.5], we have

$$\operatorname{diam}(G(E_{q^n})) = \operatorname{diam}(G_{E_{q^n}}) = 2.$$

On the other hand, in view of the proof of [7, Proposition 5.10] and [3, Theorem 3.2], we obtain

$$\operatorname{gr}(G(E_{q^n})) = \operatorname{gr}(G_{E_{3^n}}) = 3.$$

□

Lemma 1 [19, Lemma 4.1]. *Let R be a finite ring and $a \in R$. The following statements are equivalent*:

(1) $a \in J(R)$;

(2) $a + u \in U(R)$ *for any* $u \in U(R)$.

Theorem 4 [9, Theorem 2.6]. *Let $R \cong R_1 \times R_2 \times .... \times R_n$ be a finite ring, where $(R_i, \underline{m}_i)$ is a local ring, for each $i = 1, \cdots, n$. Then the following statements are equivalent*:

(1) $2 \in J(R)$;

(2) $G_R = G(R)$;

(3) *for every* $i = 1, ..., n$, $|R_i|$ *is even.*

Theorem 5. *Let n be a positive integer and p be a prime integer congruent to 2 modulo 3. Then the following statements hold*

(1) $\operatorname{diam}(G_{E_{p^n}}) = \begin{cases} 1 & \textit{if} \quad n = 1, \\ 2 & \textit{if} \quad n > 1; \end{cases}$

(2) $\operatorname{diam}(G(E_{p^n})) = 2$;

(3) $\operatorname{gr}(G(E_{p^n})) = \operatorname{gr}(G_{E_{p^n}}) = 3$.

P r o o f. Since p is a prime integer congruent to 2 modulo 3, p is an Eisenstein prime integer. Hence E_p is a field. If $n > 1$, then the ring E_{p^n} is a local ring with the maximal ideal $\langle p \rangle$. Since

$$\varphi(E_{p^n}) = p^{2k-2}(p^2 - 1),$$

we obtain that (see [4]):

$$\left| \frac{E_{p^n}}{\langle p \rangle} \right| = p^2.$$

If $p = 2$, then it follows from Theorem 4 that $G(E_{p^n}) = G_{E_{p^n}}$. In this case, by [3, Theorem 3.1] and [3, Theorem 3.2], we obtain $\operatorname{gr}(G(E_{p^n})) = \operatorname{gr}(G_{E_{p^n}}) = 3$ and

$$\operatorname{diam}(G(E_{p^n})) = \operatorname{diam}(G_{E_{p^n}}) = \begin{cases} 1 & \text{if} \quad n = 1, \\ 2 & \text{if} \quad n > 1. \end{cases}$$

We now assume that $p \neq 2$. Then $G_{E_{p^n}}$ is a complete p^2-partite graph. Therefore, $\operatorname{diam}(G_{E_{p^n}}) = 2$ and $\operatorname{gr}(G_{E_{p^n}}) = 3$. Since $G(E_{p^n}/\langle p \rangle)$ is a complete $(p^2+1)/2$-partite graph, we obtain that

$$\operatorname{diam}(G(E_{p^n})) = 2, \quad \operatorname{gr}(G(E_{p^n})) = 3.$$

□

2.2. Diameter and Girth for the Graphs G_{E_n} and $G(E_n)$

The general case is now investigated. It is well known [8, Theorem 8.7] that every finite commutative ring can be expressed as a direct product of finite local rings, and this decomposition is unique up to permutations of such local rings. Throughout this section we assume that $R \cong R_1 \times R_2 \times .... \times R_t$ is a finite commutative ring, where each R_i is a finite commutative local ring with the maximal ideal $\underline{m}_i$. Since $(u_1, ..., u_t)$ is a unit of R if and only if each u_i is a unit in R_i, we see immediately that $G_R \cong G_{R_1} \otimes G_{R_2} ... \otimes G_{R_t}$ and $G(R) \cong G(R_1) \otimes G(R_2) ... \otimes G(R_t)$. We denote by K_i the (finite) residue field $R_i/\underline{m}_i$ and $f_i = |K_i|$. We also assume (after appropriate permutation of factors) that $f_1 \leq f_2 \leq ... \leq f_t$.

Theorem 6. *Let $n > 1$ be an integer with at least two distinct prime factors. Then*

$$\operatorname{diam}(G_{E_n}) = \operatorname{diam}(G(E_n)) = 2.$$

P r o o f. Let

$$n = 3^k \times \prod_{i=1}^{m} p_i^{\alpha_i} \times \prod_{j=1}^{l} q_j^{\beta_j},$$

where p_i and q_j are prime integers such that $p_i \equiv 2 \pmod 3$ and $q_j \equiv 1 \pmod 3$, then

$$E_n \cong E_{3^k} \times \prod_{i=1}^{m} E_{p_i^{\alpha_i}} \times \prod_{j=1}^{l} E_{q_j^{\beta_j}},$$

see [4]. This shows that, E_n is isomorphic to a direct product of finite local rings $(R_i, \underline{m}_i)$, such that for every i, $|R_i/\underline{m}_i| = 3$ or p_i^2 or q_j. By [3, Theorem 3.5 (b)], we conclude that

$$\operatorname{diam}(G_{E_n}) = \operatorname{diam}(G(E_n) = 2.$$

□

Theorem 7. *Let $n > 1$ be an integer with at least two distinct prime factors. Then*

$$\operatorname{gr}(G_{E_n}) = \operatorname{gr}(G(E_n)) = 3.$$

P r o o f. By the argument similar to that above, we conclude that

$$E_n \cong E_{3^k} \times \prod_{i=1}^{m} E_{p_i^{\alpha_i}} \times \prod_{j=1}^{l} E_{q_j^{\beta_j}}.$$

Thus, by [3, Theorem 3.2], we obtain $\operatorname{gr}(G_{E_n}) = 3$. On the other hand, in view of the proof of [7, Theorem 5.10], we have $gr(G(E_n)) \in \{3, 4\}$.

Since n is an integer with at least two distinct prime factors, we can assume that $n = ab$ with $\gcd(a, b) = 1$. It is clear that

$$\gcd(a^2 + b^2 - ab, n) = 1.$$

Thus, $N(a + b\omega)$ is a unit in $\mathbb{Z}_n$, and so $a + b\omega$ is a unit in E_n. This showes that $x = a$ and $y = b\omega$ are adjacent. Now, by taking $z = b + a\omega$, we have $x + z = (a + b) + a\omega$ and $y + z = b + (a + b)\omega$. Clearly, $N(x + z) = N(y + z) = a^2 + b^2 + ab$ is a unit in $\mathbb{Z}_n$ which implies that $x + z$ and $y + z$ are unit elements of E_n. Therefore, we obtain the cycle

$$x \longrightarrow y \longrightarrow z \longrightarrow x.$$

This implies that $\operatorname{gr}(G(E_n)) = 3$. □

2.3. Some Graph Invariants of Graphs G_{E_n} and $G(E_n)$

In the sequel, we obtain the clique number and the chromatic number for the graphs G_{E_n} and $G(E_n)$.

Theorem 8. *Let $n > 1$ be an integer and*

$$n = 3^k \times \prod_{i=1}^{m} p_i^{\alpha_i} \times \prod_{j=1}^{l} q_j^{\beta_j}.$$

Then the following statements hold

(1) *if $3 \mid n$, then $\chi(G_{E_n}) = \omega(G_{E_n}) = 3$ and $\alpha(G_{E_n}) = n^2/3$;*

(2) *if $3 \nmid n$, then*

$$\chi(G_{E_n}) = \omega(G_{E_n}) = \min\left\{p_i^2, q_j \mid 1 \le i \le m,\ 1 \le j \le l,\ p_i \mid n,\ q_j \mid n\right\}$$

and

$$\alpha(G_{E_n}) = \frac{n^2}{\min\left\{p_i^2,\ q_j \mid 1 \le i \le m,\ 1 \le j \le l,\ p_i \mid n,\ q_j \mid n\right\}}.$$

P r o o f. 1. Let k be the biggest positive integer such that $3^k \mid n$. Then

$$E_n = E_{3^k} \times \prod_{i=1}^{m} E_{p_i^{\alpha_i}} \times \prod_{j=1}^{l} E_{q_j^{\beta_j}}.$$

Since E_{3^k} is a local ring with the maximal ideal $\langle 2+\omega \rangle$ and

$$\left| \frac{E_{3^k}}{\langle 2+\omega \rangle} \right| = 3,$$

it follows from [3, Proposition 6.1] that $\chi(G_{E_n}) = \omega(G_{E_n}) = 3$ and $\alpha(G_{E_n}) = n^2/3$.

2. If $3 \nmid n$, then it yields that E_n is isomorphic to a direct product of finite local rings $(R_i, \underline{m}_i)$, such that for every i, $|R_i/\underline{m}_i| = p_i^2$ *or* q_j. Thus by [3, Proposition 6.1], we have

$$\chi(G_{E_n}) = \omega(G_{E_n}) = k = \min\left\{\ p_i^2, q_j \ \mid\ 1 \le i \le m,\ 1 \le j \le l\right\}$$

and $\alpha(G_{E_n}) = n^2/k$. □

Theorem 9. *Let $n > 1$ be an integer and*

$$n = 3^k \times \prod_{i=1}^{m} p_i^{\alpha_i} \times \prod_{j=1}^{l} q_j^{\beta_j}.$$

Then the following statements hold

(1) *if $2 \mid n$, then $\chi(G(E_n)) = \omega(G(E_n)) = 4$;*

(2) *if $2 \nmid n$, then*

$$\chi(G(E_n)) = \omega(G(E_n)) = \frac{1}{2^{1+m+l}} \times \prod_{i=1}^{m}(p_i^{2\alpha_i} - p_i^{2\alpha_i-2}) \times \prod_{j=1}^{l}(q_j^{\beta_j} - q_j^{\beta_j-1})^2 + m + 2l + 1.$$

P r o o f. Since $n = 3^k \times \prod_{i=1}^{m} p_i^{\alpha_i} \times \prod_{j=1}^{l} q_j^{\beta_j}$, we have

$$E_n = E_{3^k} \times \prod_{i=1}^{m} E_{p_i^{\alpha_i}} \times \prod_{j=1}^{l} E_{q_j^{\beta_j}}.$$

1. If $2 \mid n$, then $2 \notin U(E_n)$. Hence, in view of the proof of [21, Theorem 2.2], we have

$$\chi(G(E_n)) = \omega(G(E_n)) = 4.$$

2. If $2 \nmid n$, Then $2 \in U(E_n)$. By an argument similar to that above, we conclude that

$$\chi(G(E_n)) = \omega(G(E_n)) = \frac{1}{2^{1+m+l}} \times \prod_{i=1}^{m} (p_i^{2\alpha_i} - p_i^{2\alpha_i-2}) \times \prod_{j=1}^{l} (q_j^{\beta_j} - q_j^{\beta_j-1})^2 + m + 2l + 1.$$

□

We now state our final result.

Theorem 10. *For each integer $n > 1$, the graphs $G(E_n)$ and G_{E_n} are Hamitonian.*

P r o o f. Let $n > 1$ be an integer. By Theorem 2, Theorem 3, Theorem 5 and Theorem 6, the graphs $G(E_n)$ and G_{E_n} are connected. Thus $G(E_n)$ is Hamiltonian graph, by [22, Theorem 2.1]. Also, it follows from [20, Lemma 4] that G_{E_n} is Hamiltonian graph. □

3. Concluding Remarks

In this article, the diameter, the girth, the chromatic number and the clique number of $G(E_n)$ and G_{E_n} are studied. We also prove that for each $n > 1$, the graphs $G(E_n)$ and G_{E_n} are *Hamiltonian* and the independence number of G_{E_n} is calculated. We end our paper with the following two open questions:

Question 1. Is there any closed formula for $\alpha(G(E_n))$?

Question 2. When are $G(E_n)$ and G_{E_n} Eulerian?

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SHILLA GRAPHS WITH $b = 5$ AND $b = 6$[1]

Alexander A. Makhnev[†], **Ivan N. Belousov**[††]

Krasovskii Institute of Mathematics and Mechanics,
Ural Branch of the Russian Academy of Sciences,
16 S. Kovalevskaya Str., Ekaterinburg, 620990, Russia

Ural Federal University,
19 Mira str., Ekaterinburg, 620002, Russia

[†]makhnev@imm.uran.ru, [††]i_belousov@mail.ru

Abstract: A Q-polynomial Shilla graph with $b = 5$ has intersection arrays $\{105t, 4(21t + 1), 16(t + 1); 1, 4(t + 1), 84t\}$, $t \in \{3, 4, 19\}$. The paper proves that distance-regular graphs with these intersection arrays do not exist. Moreover, feasible intersection arrays of Q-polynomial Shilla graphs with $b = 6$ are found.

Keywords: Shilla graph, Distance-regular graph, Q-polynomial graph.

1. Introduction

We consider undirected graphs without loops or multiple edges. For a vertex a of a graph Γ, denote by $\Gamma_i(a)$ the ith neighborhood of a, i.e., the subgraph induced by Γ on the set of all vertices at distance i from a. Define $[a] = \Gamma_1(a)$ and $a^\perp = \{a\} \cup [a]$.

Let Γ be a graph, and let $a, b \in \Gamma$. Denote by $\mu(a, b)$ (by $\lambda(a, b)$) the number of vertices in $[a] \cap [b]$ if a and b are at distance 2 (are adjacent) in Γ. Further, the induced $[a] \cap [b]$ subgraph is called *μ-subgraph* (*λ-subgraph*).

If vertices u and w are at distance i in Γ, then we denote by $b_i(u, w)$ (by $c_i(u, w)$) the number of vertices in the intersection of $\Gamma_{i+1}(u)$ (of $\Gamma_{i-1}(u)$, respectively) with $[w]$. A graph Γ of diameter d is called *distance-regular with intersection array* $\{b_0, b_1, \ldots, b_{d-1}; c_1, \ldots, c_d\}$ if, for each $i = 0, \ldots, d$, the values $b_i(u, w)$ and $c_i(u, w)$ are independent of the choice of vertices u and w at distance i in Γ. Define $a_i = k - b_i - c_i$. Note that, for a distance regular graph, b_0 is the degree of the graph and a_1 is the degree of the local subgraph (the neighborhood of the vertex). Further, for vertices x and y at distance l in the graph Γ, denote by $p^l_{ij}(x, y)$ the number of vertices in the subgraph $\Gamma_i(x) \cap \Gamma_j(y)$. The numbers $p^l_{ij}(x, y)$ are called the intersection numbers of Γ (see [2]). In a distance-regular graph, they are independent of the choice of x and y.

A Shilla graph is a distance-regular graph Γ of diameter 3 with second eigenvalue θ_1 equal to $a = a_3$. In this case, a divides k and b is defined by $b = b(\Gamma) = k/a$. Morover, $a_1 = a - b$ and Γ has intersection array $\{ab, (a+1)(b-1), b_2; 1, c_2, a(b-1)\}$. Feasible intersection arrays of Shilla graphs are found in [6] for $b \in \{2, 3\}$.

Feasible intersection arrays of Shilla graphs are found in [1] for $b = 4$ (50 arrays) and for $b = 5$ (82 arrays). At present, a list of feasible intersection arrays of Shilla graphs for $b = 6$ is unknown. Moreover, the existence of Q-polynomial Shilla graphs with $b = 5$ also is unknown.

In this paper, we find feasible intersection arrays of Q-polynomial Shilla graphs with $b = 6$ and prove that Q-polynomial Shilla graphs with $b = 5$ do not exist.

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Theorem 1. *A Q-polynomial Shilla graph with $b = 6$ has intersection array*

(1) $\{42t, 5(7t+1), 3(t+3); 1, 3(t+3), 35t\}$, *where* $t \in \{7, 12, 17, 27, 57\}$;

(2) $\{372, 315, 75; 1, 15, 310\}$, $\{744, 625, 125; 1, 25, 620\}$ *or* $\{930, 780, 150; 1, 30, 775\}$;

(3) $\{312, 265, 48; 1, 24, 260\}$, $\{624, 525, 80; 1, 40, 520\}$, $\{1794, 1500, 200; 1, 100, 1495\}$ *or* $\{5694, 4750, 600; 1, 300, 4745\}$.

In view of Theorem 2 from [1], a Q-polynomial Shilla graph with $b = 5$ has intersection array $\{105t, 4(21t+1), 16(t+1); 1, 4(t+1), 84t\}$, $t \in \{3, 4, 19\}$.

Theorem 2. *Distance-regular graphs with intersection arrays* $\{315, 256, 64; 1, 16, 252\}$ *and* $\{1995, 1600, 320; 1, 80, 1596\}$ *do not exist.*

Theorem 3. *Distance-regular graphs with intersection array* $\{420, 340, 80; 1, 20, 336\}$ *do not exist.*

2. Proof of Theorem 1

In this section, Γ is a Q-polynomial Shilla graph with $b = 6$. Then $(a_2 - 5a - 6)^2 - 4(5b_2 - a_2)$ is the square of an integer. By [6, Lemma 8], we have

$$2a \leq c_2 b(b+1) + b^2 - b - 2;$$

therefore, $a \leq 21c_2 + 14$. It follows from the proof of Theorem 9 in [6] that either $k < b^3 - b = 6 \cdot 35$ or $v < k(2b^3 - b + 1) = 428k$. By [6, Corollary 17 and Theorem 20], the number $b_2 + c_2$ divides $b(b-1)b_2$ and

$$-34 = -b^2 + 2 \leq \theta_3 \leq -b^2(b+3)/(3b+1) \leq -18.$$

Theorem 2 from [7] implies the following lemma.

Lemma 1. *If* $b_2 = c_2$, *then* Γ *has an intersection arrays* $\{42t, 5(7t+1), 3(t+3); 1, 3(t+3), 35t\}$ *and* $t \in \{7, 12, 17, 27, 57\}$.

To the end of this section, assume that $b_2 \neq c_2$ and $k > \theta_1 > \theta_2 > \theta_3$ are eigenvalues of the graph Γ. Then

$$6(6b_2 + c_2)/(b_2 + c_2) = -\theta_3.$$

On the other hand, according to [6, Lemma 10], the number c_2 divides $(a+6)b_2$, $30a(a+1)$ and $(a+6)b_2 \geq (a+1)c_2$.

Lemma 2. *If* $-34 \leq \theta_3 \leq -18$, *then one of the following statements holds*:

(1) $\theta_3 = -31$ *and* Γ *has one of the intersection arrays* $\{372, 315, 75; 1, 15, 310\}$, $\{744, 625, 125; 1, 25, 620\}$, *and* $\{930, 780, 150; 1, 30, 775\}$;

(2) $\theta_3 = -26$ *and* Γ *has one of the intersection arrays* $\{312, 265, 48; 1, 24, 260\}$, $\{624, 525, 80; 1, 40, 520\}$, $\{1794, 1500, 200; 1, 100, 1495\}$, *and* $\{5694, 4750, 600; 1, 300, 4745\}$;

(3) $\theta_3 = -21$ *and* Γ *has one of the intersection arrays* $\{42t, 5(7t+1), 3(t+3); 1, 3(t+3), 35t\}$ *for* $t \in \{7, 12, 17, 27, 57\}$.

P r o o f. By [6, Lemma 10], c_2 divides $b(b-1)b_2 = 30b_2$ and, by [6, Corollary 17], the smallest nonprinciple eigenvalue θ_3 is equal to $b(bb_2 + c_2)/(b_2 + c_2)$. Therefore, $30(\theta_3 + 6)/(\theta_3 + 36)$ is an integer and $\theta_3 \in \{-34, -33, -32, -31, -30, -27, -26, -24, -21, -18\}$.

Let $\theta_3 = -34$. Then $3(6b_2 + c_2) = 17(b_2 + c_2)$ and $b_2 = 14c_2$. Further, θ_3 is a root of the equation $x^2 - (a_1 + a_2 - k)x + (b-1)b_2 - a_2 = 0$; therefore, $a = 425/28 \cdot c_2 - 34$. In this case, the multiplicity of the first nonprincipal eigenvalue is $m_1 = 6/5 \cdot (2545c_2 - 5544)/c_2$, a contradiction with the fact that 5 does not divide $6 \cdot 5544$.

Let $\theta_3 = -33$. Then $2(6b_2 + c_2) = 11(b_2 + c_2)$ and $b_2 = 9c_2$. Further, $a = 275/27 \cdot c_2 - 33$ and the multiplicity of the first nonprincipal eigenvalue is equal to $m_1 = 6/5 \cdot (1645c_2 - 5184)/c_2$, a contradiction as above.

Let $\theta_3 = -32$. Then $3(6b_2 + c_2) = 16(b_2 + c_2)$ and $2b_2 = 13c_2$. Further, $a = 100/13 \cdot c_2 - 32$ and the multiplicity of the first nonprincipal eigenvalue is $m_1 = 6/5 \cdot (1195c_2 - 4836)/c_2$, a contradiction as above.

Let $\theta_3 = -31$. Then $6(6b_2 + c_2) = 31(b_2 + c_2)$ and $b_2 = 5c_2$. Further, $a = 31/5 \cdot c_2 - 31$ and the multiplicity of the first nonprincipal eigenvalue is $m_1 = 30(37c_2 - 180)/c_2 = 1110 - 5400/c_2$. The number of vertices in the graph is $31/5 \cdot (222c_2^2 - 2005c_2 + 4500)/c_2$; hence, c_2 divides 900 and is a multiple of 5. By computer enumeration, we find that, only for $c_2 = 15, 25$ and 30, we have admissible intersection arrays $\{372, 315, 75; 1, 15, 310\}$, $\{744, 625, 125; 1, 25, 620\}$ and $\{930, 780, 150; 1, 30, 775\}$.

Let $\theta_3 = -30$. Then $(6b_2 + c_2) = 5(b_2 + c_2)$ and $b_2 = 4c_2$. Further, $a = 125/24 \cdot c_2 - 30$ and the multiplicity of the first nonprincipal eigenvalue is $m_1 = 6/5 \cdot (745c_2 - 4176)/c_2$, a contradiction as above.

Let $\theta_3 = -27$. Then $2(6b_2 + c_2) = 9(b_2 + c_2)$ and $3b_2 = 7c_2$. Further, $a = 25/7 \cdot c_2 - 25$ and the multiplicity of the first nonprincipal eigenvalue is $m_1 = 6/5 \cdot (445c_2 - 3276)/c_2$, a contradiction as above.

Let $\theta_3 = -26$. Then $3(6b_2 + c_2) = 13(b_2 + c_2)$ and $b_2 = 2c_2$. Further, $a = 13/4 \cdot c_2 - 26$ and the multiplicity of the first nonprincipal eigenvalue is $m_1 = 6(77c_2 - 600)/c_2 = 462 - 3600/c_2$. The number of vertices in the graph is $13/8 \cdot (231c_2^2 - 3340c_2 + 12000)/c_2$; hence, c_2 divides 1200 and is a multiple of 4. By computer enumeration, we find that only for $c_2 = 24, 40, 100$, and 300 we have admissible intersection arrays $\{312, 265, 48; 1, 24, 260\}$, $\{624, 525, 80; 1, 40, 520\}$, $\{1794, 1500, 200; 1, 100, 1495\}$, and $\{5694, 4750, 600; 1, 300, 4745\}$.

Let $\theta_3 = -21$. Then $2(6b_2 + c_2) = 7(b_2 + c_2)$ and $b_2 = c_2$. Further, $a = 7/3 \cdot c_2 - 21$ and the multiplicity of the first nonprincipal eigenvalue is $m_1 = 6(41c_2 - 360)/c_2 = 246 - 2160/c_2$. The number of vertices in the graph is $7/3 \cdot (82c_2^2 - 1335c_2 + 5400)/c_2$; hence, c_2 divides 1080 and is a multiple of 3. By computer enumeration, we find that, only for $c_2 = 18, 30, 45, 60, 90$, and 180, we have admissible intersection arrays $\{42t, 5(7t+1), 3(t+3); 1, 3(t+3), 35t\}$ for $t \in \{3, 7, 12, 17, 27, 57\}$. A graph with the array obtained for $t = 3$ does not exist by [5].

Let $\theta_3 = -18$. Then $6(6b_2 + c_2) = 19(b_2 + c_2)$, so $3b_2 = 2c_2$. Further, $a = 2512 \cdot c_2 - 18$ and the multiplicity of the first nonprincipal eigenvalue is $m_1 = 6/5 \cdot (145c_2 - 1224)/c_2$, a contradiction. The lemma is proved. $\square$

Theorem 1 follows from Lemmas 1–2.

3. Triple intersection numbers

In the proof of Theorem 3, the triple intersection numbers [3] are used.

Let Γ be a distance-regular graph of diameter d. If u_1, u_2, u_3 are vertices of the graph Γ, then r_1, r_2, r_3 are non-negative integers not greater than d. Denote by $\left\{ {u_1u_2u_3 \atop r_1r_2r_3} \right\}$ the set of vertices $w \in \Gamma$ such that $d(w, u_i) = r_i$ and by $\left[{u_1u_2u_3 \atop r_1r_2r_3} \right]$ the number of vertices in $\left\{ {u_1u_2u_3 \atop r_1r_2r_3} \right\}$. The numbers $\left[{u_1u_2u_3 \atop r_1r_2r_3} \right]$ are called the triple intersection numbers. For a fixed triple of vertices u_1, u_2, u_3, instead of $\left[{u_1u_2u_3 \atop r_1r_2r_3} \right]$, we will write $[r_1r_2r_3]$. Unfortunately, there are no general formulas for the numbers $[r_1r_2r_3]$. However, [3] outlines a method for calculating some numbers $[r_1r_2r_3]$.

Let u, v, w be vertices of the graph Γ, $W = d(u, v)$, $U = d(v, w)$, and let $V = d(u, w)$. Since there is exactly one vertex $x = u$ such that $d(x, u) = 0$, then the number $[0jh]$ is 0 or 1. Hence $[0jh] = \delta_{jW}\delta_{hV}$. Similarly, $[i0h] = \delta_{iW}\delta_{hU}$ and $[ij0] = \delta_{iU}\delta_{jV}$.

Another set of equations can be obtained by fixing the distance between two vertices from $\{u, v, w\}$ and counting the number of vertices located at all possible distances from the third:

$$\begin{cases} \sum\limits_{l}^{d} [ljh] = p^U_{jh} - [0jh] \\ \sum\limits_{l}^{d} [ilh] = p^V_{ih} - [i0h] \\ \sum\limits_{l}^{d} [ijl] = p^W_{ij} - [ij0] \end{cases} \tag{3.1}$$

However, some triplets disappear. For $|i - j| > W$ or $i + j < W$, we have $p^W_{ij} = 0$; therefore, $[ijh] = 0$ for all $h \in \{0, ..., d\}$.

We set

$$S_{ijh}(u, v, w) = \sum_{r,s,t=0}^{d} Q_{ri}Q_{sj}Q_{th} \begin{bmatrix} uvw \\ rst \end{bmatrix}.$$

If the Krein parameter $q^h_{ij} = 0$, then $S_{ijh}(u, v, w) = 0$.

We fix vertices u, v, w of a distance-regular graph Γ of diameter 3 and set

$$\{ijh\} = \begin{Bmatrix} uvw \\ ijh \end{Bmatrix}, \quad [ijh] = \begin{bmatrix} uvw \\ ijh \end{bmatrix}, \quad [ijh]' = \begin{bmatrix} uwv \\ ihj \end{bmatrix}, \quad [ijh]^* = \begin{bmatrix} vuw \\ jih \end{bmatrix}, \quad [ijh]^{\sim} = \begin{bmatrix} wvu \\ hji \end{bmatrix}.$$

Calculating the numbers

$$[ijh]' = \begin{bmatrix} uwv \\ ihj \end{bmatrix}, \quad [ijh]^* = \begin{bmatrix} vuw \\ jih \end{bmatrix}, \quad [ijh]^{\sim} = \begin{bmatrix} wvu \\ hji \end{bmatrix}$$

(symmetrization of the triple intersection numbers) can give new relations that make it possible to prove the nonexistence of a graph.

4. Graphs with intersection arrays $\{315, 256, 64; 1, 16, 252\}$ and $\{1995, 1600, 320; 1, 80, 1596\}$

Let Γ be a distance-regular graph with intersection array $\{315, 256, 64; 1, 16, 252\}$. By [2, Theorem 4.4.3], the eigenvalues of the local subgraph of the graph Γ are contained in the interval $[-5, 59/5)$. Since the Terwilliger polynomial (see [4]) is $-4(5x - 59)(x + 5)(x + 1)(x - 43)$, then these eigenvalues lie in $[-5, -1] \cup (59/5.43]$. Hence, all nonprinciple eigenvalues are negative and the local subgraph is a union of isolated $(a_1 + 1)$-cliques, a contradiction with the fact that $a_1 + 1 = 49$ does not divide $k = 315$.

Thus, a distance-regular graph with intersection array $\{315, 256, 64; 1, 16, 252\}$ does not exist.

Let Γ be a distance-regular graph with intersection array $\{1995, 1600, 320; 1, 80, 1596\}$. Then Γ has $1 + 1995 + 39900 + 8000 = 49896$ vertices, spectrum $1995^1, 399^{495}, 15^{23275}, -21^{26125}$, and the dual matrix of eigenvalues

$$Q = \begin{pmatrix} 1 & 495 & 23275 & 26125 \\ 1 & 99 & 175 & -275 \\ 1 & 0 & -56 & 55 \\ 1 & -99/4 & 931/4 & -209 \end{pmatrix}.$$

The Terwilliger polynomial of the graph Γ is $-20(x + 5)(x + 1)(x - 79)(x - 299)$; hence, the eigenvalues of the local subgraph are contained in $[-5, -1] \cup \{79\} \cup \{394\}$.

Note that the multiplicity $m_1 = 495$ of the eigenvalue $\theta_1 = 399$ is less than k. By the corollary to Theorem 4.4.4 from [2] for $b = b_1/(\theta_1 + 1) = 4$, the graph $\Sigma = [u]$ has an eigenvalue $-1 - b = -5$ of multiplicity at least $k - m_1 = 1500$.

Let the number of eigenvalues 79 of the graph Σ be equal to y. Then the sum of eigenvalues of the graph Σ is at most $-7500 - (494 - y) + 79y + 394$; therefore, $y \geq 95$. Now twice the number of edges in Σ is equal to

$$786030 = 1995 \cdot 394 = \sum_i m_i \theta_i^2$$

but not less than

$$25 \cdot 1500 + 399 + 95 \cdot 79^2 + 394^2 = 786030.$$

Hence, Σ has spectrum $394^1.79^{95}, -1^{399}, -5^{1500}$.

Now the number $t = k_\Sigma \lambda_\Sigma/2$ of triangles in Σ containing this vertex is equal to $\sum_i m_i \theta_i^3/(2v)$. Therefore,

$$t = \sum_i m_i \theta_i^3/(2v) = (394^3 + 79^3 \cdot 95 - 399 - 125 \cdot 1500)/3990 = 27021$$

and $\lambda_\Sigma = 54042/394$ is approximately equal to 137.16, a contradiction.

Thus, a distance-regular graph with intersection array $\{1995, 1600, 320; 1, 80, 1596\}$ does not exist.

Theorem 2 is proved.

5. Graph with array $\{420, 340, 80; 1, 20, 336\}$

Let Γ be a distance-regular graph with intersection array $\{420, 340, 80; 1, 20, 336\}$. Then Γ is a formally self-dual graph having $1 + 420 + 7140 + 1700 = 9261$ vertices, spectrum $420^1, 84^{420}, 0^{7140}, -21^{1700}$, and the dual matrix of eigenvalues

$$Q = \begin{pmatrix} 1 & 420 & 7140 & 1700 \\ 1 & 84 & 0 & -85 \\ 1 & 0 & -21 & 20 \\ 1 & -21 & 84 & -64 \end{pmatrix}.$$

The Terwilliger polynomial of the graph Γ is $-20(x+5)(x+1)(x-16)(x-59)$ and the eigenvalues of the local subgraph are contained in $[-5, -1] \cup \{16\} \cup \{79\}$. If the nonprinciple eigenvalues of a local subgraph are negative, then this subgraph is a union of isolated (a_1+1)-cliques, a contradiction with the fact that $a_1+1 = 80$ does not divide $k = 420$. Hence, the local subgraph has eigenvalue 6.

Lemma 3. *Intersection numbers of a graph Γ satisfy the equalities*

(1) $p^1_{11} = 79$, $p^1_{21} = 340$, $p^1_{32} = 1360$, $p^1_{22} = 5440$, $p^1_{33} = 340$,

(2) $p^2_{11} = 20$, $p^2_{12} = 320$, $p^2_{13} = 80$, $p^2_{22} = 5519$, $p^2_{23} = 1300$, $p^2_{33} = 320$;

(3) $p^3_{12} = 336$, $p^3_{13} = 84$, $p^3_{22} = 5460$, $p^3_{23} = 1344$, $p^3_{33} = 271$.

P r o o f. Direct calculations. $\square$

Let u, v, and w be vertices of a graph Γ, $[rst] = \left[\begin{smallmatrix} uvw \\ rst \end{smallmatrix}\right]$, $\Omega = \Gamma_3(u)$, and let $\Sigma = \Omega_2$. Then Σ is a regular graph of degree 1344 on 1700 vertices.

Lemma 4. *Let $d(u, v) = d(u, w) = 3$ and $d(v, w) = 1$. Then the following equalities hold*:

(1) $[122] = 2r_6/5 - 136$, $[123] = [132] = -2r_6/5 + 472$, $[133] = 2r_6/5 - 388$;

(2) $[211] = r_6/10 - 38$, $[212] = [221] = -r_6/10 + 374$, $[222] = -14r_6/10 + 5576$, $[223] = [232] = 3r_6/2 - 490$, $[233] = -3r_6/2 + 1834$;

(3) $[311] = -r_6/10 + 117$, $[312]=[321]=r_6/10 - 34$, $[322] = r_6$, $[323] = [332] = -11r_6/10 + 1378$, $[333] = 11r_6/10 - 1107$,

where $r_6 \in \{1010, 1020, \ldots, 1170\}$.

P r o o f. A simplification of formulas (3.1) taking into account the equalities $S_{113}(u, v, w) = S_{131}(u, v, w) = S_{311}(u, v, w) = 0$. $\square$

By Lemma 4, we have $1010 \leq [322] = r_6 \leq 1170$.

Lemma 5. *Let $d(u, v) = d(u, w) = d(v, w) = 3$. Then the following equalities hold*:

(1) $[122] = -r_{17} + 336$, $[123] = [132] = r_{17}$, $[133] = -r_{17} + 84$;

(2) $[213] = [231] = r_{17}$, $[212] = [221] = -r_{17} + 336$, $[222] = 39r_{17}/4 + 3444$, $[223] = [232] = -35r_{17}/4 + 1680$, $[233] = 31r_{17}/4 - 336$;

(3) $[313] = [331] = -r_{17} + 84$, $[312] = [321] = r_{17}$, $[322] = -35r_{17}/4 + 1680$, $[323] = [332] = 31r_{17}/4 - 336$, $[333] = -27r_{17}/4 + 522$,

where $r_{17} \in \{44, 48, \ldots, 76\}$.

P r o o f. A simplification of formulas (3.1) taking into account the equalities $S_{113}(u, v, w) = S_{131}(u, v, w) = S_{311}(u, v, w) = 0$. $\square$

By Lemma 5, we have $1015 \leq [322] = -35r_{17}/4 + 1680 \leq 1295$.

The number d of edges between $\Sigma(w)$ and $\Sigma - (\{w\} \cup \Lambda(w))$ satisfies the inequalities

$$359905 = 84 \cdot 1010 + 271 \cdot 1015 \leq d \leq 84 \cdot 1170 + 271 \cdot 1295 = 449225,$$
$$267.786 \leq 1343 - \lambda \leq 334.245,$$
$$1008.755 \leq \lambda \leq 1075.214,$$

where λ is the mean value of the parameter $\lambda(\Sigma)$.

Lemma 6. *Let $d(u, v) = d(u, w) = 3$ and $d(v, w) = 2$. Then the following equalities hold*:

(1) $[122] = (-64r_{15} + 4r_{16} + 7364)/27$, $[123] = [132] = (64r_{15} - 4r_{16} + 1708)/27$, $[133] = (-64r_{15} + 4r_{16} + 560)/27$;

(2) $[211] = -r_{15}+20$, $[212] = [221] = (71r_{15}+4r_{16}+6392)/27$, $[222] = (-17r_{15}-13r_{16}+38311)/9$, $[223] = [232] = (-20r_{15} + 35r_{16} + 26095)/27$, $[233] = (64r_{15} - 31r_{16} + 8053)/27$;

(3) $[311] = r_{15}$, $[312] = [321] = (-71r_{15} - 4r_{16} + 2248)/27$, $[313] = (44r_{15} + 4r_{16} + 20)/27$, $[322] = (115r_{15}+35r_{16}+26716)/27$, $[323] = [332] = (-44r_{15}-31r_{16}+7297)/27$, $[333] = r_{16}$,

where $-10r_{15} + 4r_{16} + 20$ is a multiple of 27, $r_{15} \in \{0, 1, \ldots, 20\}$, and $r_{16} \in \{0, 1, \ldots, 235\}$.

P r o o f. A simplification of formulas (3.1) taking into account the equalities $S_{113}(u, v, w) = S_{131}(u, v, w) = S_{311}(u, v, w) = 0$. □

By Lemma 6, we have

$$998 \leq [322] = (115r_{15} + 35r_{16} + 26716)/27 \leq 1294.$$

Let us count the number h of pairs of vertices y and z at distance 3 in the graph Ω, where

$$y \in \left\{ \begin{matrix} uv \\ 31 \end{matrix} \right\}, \quad z \in \left\{ \begin{matrix} uv \\ 32 \end{matrix} \right\}.$$

On the one hand, by Lemma 4, we have $[323] = -11r_6/10+1378$, where $r_6 \in \{1010, 1020, ..., 1170}$, therefore

$$7644 = 8491 \leq h \leq 84267 = 22428.$$

On the other hand, by Lemma 6, we have $[313] = (44r_{15} + 4r_{16} + 20)/27$, where $r_{15} \in \{0, 1, ..., 20\}$, $r_{16} \in \{0, 1, ..., 235\}$, therefore

$$7644 \leq \sum_i (44r_{15}^i + 4r_{16}^i) + 995.55 \leq 22428,$$

$$6648.44 \leq \sum_i (44r_{15}^i + 4r_{16}^i) \leq 21432.45,$$

$$4.946 \leq \sum_i (11r_{15}^i + r_{16}^i)/1344 \leq 15.947.$$

If $r_{15} = 0$, then $r_{16} + 5$ is a multiple of 27 and $r_{16} = 22.49, \ldots$.
If $r_{15} = 1$, then $2r_{16} + 5$ is a multiple of 27 and $r_{16} = 11.38, \ldots$.
In any case,

$$\sum_i (11r_{15}^i + r_{16}^i)/1344 \geq 22,$$

a contradiction.

Theorem 3 is proved. □

Conclusion

The following are the main steps in creating a theory of Shilla graphs:
(1) finding a list of feasible intersection arrays of Shilla graphs with $b = 6$;
(2) classification of Q-polynomial Shilla graphs with $b_2 = c_2$.

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A ROBUST ITERATIVE APPROACH FOR SOLVING NONLINEAR VOLTERRA DELAY INTEGRO–DIFFERENTIAL EQUATIONS

Austine Efut Ofem

Department of Mathematics, University of Uyo,
Uyo, Nigeria

ofemaustine@gmail.com

Unwana Effiong Udofia

Department of Mathematics and Statistics, Akwa Ibom State University,
Ikot Akpaden, Mkpatenin, Nigeria

unwanaudofia.aksu@yahoo.com

Donatus Ikechi Igbokwe

Department of Mathematics, Michael Okpara University of Agriculture,
Umudike, Nigeria

igbokwedi@yahoo.com

Abstract: This paper presents a new iterative algorithm for approximating the fixed points of multivalued generalized α-nonexpansive mappings. We study the stability result of our new iterative algorithm for a larger concept of stability known as weak w^2–stability. Weak and strong convergence results of the proposed iterative algorithm are also established. Furthermore, we show numerically that our new iterative algorithm outperforms several known iterative algorithms for multivalued generalized α-nonexpansive mappings. Again, as an application, we use our proposed iterative algorithm to find the solution of nonlinear Volterra delay integro-differential equations. Finally, we provide an illustrative example to validate the mild conditions used in the result of the application part of this study. Our results improve, generalize and unify several results in the existing literature.

Keywords: Banach space, Uniformly convex Banach space, Multivalued generalized α-nonexpansive mapping, Convergence, Nonlinear Volterra delay integro-differential equations.

1. Introduction

A mapping g on a nonempty subset $\mathcal{K}$ of a Banach $\mathscr{G}$ is called nonexpansive if

$$\|g(x) - g(y)\| \leq \|x - y\|, \quad \text{for all} \quad x, y \in \mathcal{K}.$$

A point x in $\mathcal{K}$ is said to be a fixed point of g if $g(x) = x$. We denote the set of all fixed points of g by

$$\mathcal{F}(g) = \{x \in \mathcal{K} : x = g(x)\}.$$

Let $\mathbb{R}$ denote the set of all real numbers and $\mathbb{N}$ be the set of all natural numbers.

In 1965, Browder [7], Göhde [10] and Kirk [17] independently studied the existence of fixed points of nonexpansive mappings in Banach spaces. The authors showed that every nonexpansive mapping defined on a bounded closed convex subset of a uniformly convex Banach space always has a fixed point. Recently, many authors have introduced and studied some new classes of mappings which are considered to be larger than the single-valued nonexpansive mappings.

One of the first extensions and generalizations of single-valued nonexpansive mapping which has fascinated many authors was introduced by Suzuki [33] in 2008. Such mappings are generally known as mappings satisfying condition (C). The author proved some existence and convergence results for such mapping.

Definition 1. *A mapping $g : \mathcal{K} \to \mathcal{K}$ is said to be Suzuki generalized nonexpansive mapping or mapping satisfying condition (C) if for all $x, y \in \mathcal{K}$ we have*

$$\frac{1}{2}\|x - gx\| \leq \|x - y\| \quad \textit{implies} \quad \|gx - gy\| \leq \|x - y\|.$$

In 2011, Aoyama and Kohshaka [5] introduced a new class of single-valued mappings known as α-nonexpansive mappings and obtained some fixed point theorems for such mappings.

Definition 2. *A mapping $g : \mathcal{K} \to \mathcal{K}$ is said to be an α-nonexpansive with $\alpha \in [0, 1)$ if*

$$\|gx - gy\|^2 \leq \alpha\|gx - y\|^2 + \alpha\|gy - x\|^2 + (1 - 2\alpha)\|x - y\|^2,$$

for all $x, y \in \mathcal{K}$.

Obviously, every nonexpansive mapping is an α-nonexpansive with $\alpha = 0$ (i.e., 0-nonexpansive mapping).

In 2017, Pant and Shukla [29] introduced a new type of single-valued nonexpansive mappings known as generalized α-nonexpansive mappings and obtained some existence and convergence theorems.

Definition 3. *A mapping $g : \mathcal{K} \to \mathcal{K}$ is said to be generalized α-nonexpansive with $\alpha \in [0, 1)$ if*

$$\frac{1}{2}\|x - gx\| \leq \|x - y\| \quad \textit{implies}$$
$$\|gx - gy\| \leq \alpha\|gx - y\| + \alpha\|gy - x\| + (1 - 2\alpha)\|x - y\|,$$

for all $x, y \in \mathcal{K}$.

This class of mappings properly includes nonexpansive and Suzuki generalized nonexpansive mappings [29].

Fixed point theory for multivalued mappings has useful applications in control theory, convex optimization, differential equations and economics. The fixed points of multivalued mappings were first studied by Markin [20] and Nadler [21].

A set $\mathcal{K}$ is said to be proximinal if for each $x \in \mathscr{G}$, there exists an element $y \in \mathcal{K}$ such that $\|x - y\| = d(x, \mathcal{K})$, where

$$d(x, \mathcal{K}) = \inf\{\|x - \ell\| : \ell \in \mathcal{K}\}.$$

We denote by $CB(\mathcal{K})$, $C(\mathcal{K})$ and $\mathcal{P}(\mathcal{K})$ the families of nonempty closed and bounded subsets, nonempty compact subsets and nonempty proximinal subsets of $\mathcal{K}$, respectively. Let $\mathscr{H}$ be the Hausdorff metric induced by d of $\mathscr{G}$ which is defined as:

$$\mathscr{H}(\mathscr{U}, \mathscr{V}) = \max\left\{\sup_{x\in\mathscr{U}}(x, \mathscr{V}), \sup_{y\in\mathscr{V}}(y, \mathscr{U})\right\}, \quad \text{for all} \quad \mathscr{U}, \mathscr{V} \in \mathrm{CB}(\mathcal{K}).$$

An element $x \in \mathcal{K}$ is said to be a fixed point of a multivalued mapping $\mathcal{T} : \mathcal{K} \to \mathcal{P}(\mathcal{K})$ if $x \in \mathcal{T}x$. Let $F(\mathcal{T}) = \{x \in \mathcal{K} : x \in \mathcal{T}x\}$ denote the set of all fixed points of $\mathcal{T}$.

A multivalued mapping $\mathcal{T} : \mathcal{K} \to \mathcal{P}(\mathcal{K})$ is said to be a contraction if there exists a constant $\delta \in [0, 1)$ such that for all $x, y \in \mathcal{K}$,

$$\mathscr{H}(\mathcal{T}x, \mathcal{T}y) \leq \delta\|x - y\|, \tag{1.1}$$

and nonexpansive if

$$\mathscr{H}(\mathcal{T}x, \mathcal{T}y) \leq \|x - y\|,$$

for all $x, y \in \mathcal{K}$. The study of fixed points for multivalued contraction and nonexpansive mappings using the Hausdorff metric was initiated by Markin [20].

In 2011, Abkar and Eslamian [2] gave the multivalued version of Suzuki generalized nonexpansive mappings.

Definition 4. *A multivalued mapping $\mathcal{T} : \mathcal{K} \to CB(\mathcal{K})$ is said to be Suzuki generalized nonexpansive mappings or said to satisfy condition (C) if for all $x, y \in \mathcal{K}$, we have*

$$\frac{1}{2}d(x, \mathcal{T}x) \leq \|x - y\| \quad \textit{implies} \quad \mathscr{H}(\mathcal{T}x, \mathcal{T}y) \leq \|x - y\|.$$

Recently, Iqbal et al. [15] introduced a multivalued generalized α-nonexpansive mapping and obtained some fixed points results in uniformly convex Banach spaces.

Definition 5. *A mapping $\mathcal{T} : \mathcal{K} \to CB(\mathcal{K})$ is said to be a multivalued generalized α-nonexpansive if there exists $\alpha \in [0, 1)$ such that*

$$\frac{1}{2}d(x, \mathcal{T}x) \leq \|x - y\| \quad \textit{implies}$$
$$\mathscr{H}(\mathcal{T}x, \mathcal{T}y) \leq \alpha d(x, \mathcal{T}y) + \alpha d(y, \mathcal{T}x) + (1 - 2\alpha)\|x - y\|,$$

for all $x, y \in \mathcal{K}$.

It is not hard to see that every multivalued mapping satisfying condition (C) is multivalued generalized α-nonexpansive mapping with $\alpha = 0$ and also, every multivalued generalized α-nonexpansive mapping with a nonempty fixed point set is multivalued quasi-nonexpansive.

The fixed point theory of the classes of multivalued nonexpansive mappings is more cumbersome than the corresponding theory for the classes of single valued nonexpansive mappings. But the numerous applications of the former have caused several researchers to study not only the existence and uniqueness of fixed points of different classes of multivalued nonexpansive mappings, but also approximated the fixed points of different classes of multivalued nonexpansive mappings.

In the course of approximating the fixed points of the classes of nonexpansive mappings, several iterative algorithms have be introduced and studied. Some of the well known iterative algorithms in existing literature are given in Mann [19], Ishikawa [16], Noor [22], S [3], Abbas and Nazir [1], Tharkur [34] and many more.

In 2009, Shahzad and Zegeye [11] studied convergence of the Mann and Ishikawa iterative algorithms for multivalued nonexpansive mappings in a nonempty closed convex subset of a uniformly convex Banach space. The authors defined

$$\mathcal{P}_{\mathcal{T}}(x) = \{y \in \mathcal{T}x : \|x - y\| = d(x, \mathcal{T}x)\}$$

for a multivalued mapping $\mathcal{T} : \mathcal{K} \to \mathcal{P}(\mathcal{K})$ to make it well defined.

The famous Mann iterative algorithms is defined as:

$$\begin{cases} x_0 = x \in \mathcal{K}, \\ x_{n+1} = (1 - u_n)x_n + u_n\ell_n, \end{cases} \qquad \forall n \geq 1, \tag{1.2}$$

where $\{u_n\}$ is a sequence in (0,1) and $\ell_n \in \mathcal{P}_{\mathcal{T}}(x_n)$.

The Ishikawa iterative algorithms is defined as:

$$\begin{cases} x_0 = x \in \mathcal{K}, \\ y_n = (1-v_n)x_n + v_n\ell_n, \\ x_{n+1} = (1-u_n)x_n + u_n\zeta_n, \end{cases} \quad \forall n \geq 1, \tag{1.3}$$

where $\{u_n\}$ and $\{v_n\}$ are sequences in (0,1), $\zeta_n \in \mathcal{P}_{\mathcal{T}}(y_n)$ and $\ell_n \in \mathcal{P}_{\mathcal{T}}(x_n)$.

The following iterative algorithm which was introduced by Argawal et al. [3], is known as S-iterative algorithm:

$$\begin{cases} x_0 = x \in \mathcal{K}, \\ y_n = (1-v_n)x_n + v_n\ell_n, \\ x_{n+1} = (1-u_n)\ell_n + u_n\zeta_n, \end{cases} \quad \forall n \geq 1, \tag{1.4}$$

where $\{u_n\}$ and $\{v_n\}$ are sequences in (0,1), $\zeta_n \in \mathcal{P}_{\mathcal{T}}(y_n)$ and $\ell_n \in \mathcal{P}_{\mathcal{T}}(x_n)$. The authors proved that (1.4) converges at the same rate of the Picard iteration and faster than the Ishikawa iteration for contractions mappings.

In 2018, Gunduz et al. [11] introduced the multivalued version of Thakur iteration process as follows:

$$\begin{cases} x_0 = x \in \mathcal{K}, \\ z_n = (1-t_n)x_n + t_n\ell_n, \\ y_n = (1-v_n)x_n + v_n\omega_n, \\ x_{n+1} = (1-u_n)\omega_n + u_n\zeta_n, \end{cases} \quad \forall n \geq 1, \tag{1.5}$$

where $\{u_n\}$, $\{t_n\}$ and $\{v_n\}$ are sequences in (0,1), $\zeta_n \in \mathcal{P}_{\mathcal{T}}(y_n)$, $\omega_n \in \mathcal{P}_{\mathcal{T}}(z_n)$ and $\ell_n \in \mathcal{P}_{\mathcal{T}}(x_n)$. The authors proved numerically that (1.5) convergence is faster than each of Mann, Ishikawa, Noor, S, Abass iteration processes.

Recently, Okeke et al. [28] introduced the multivalued version of Picard-Ishikawa hybrid iterative algorithm which was considered in [26] as follows:

$$\begin{cases} x_0 = x \in \mathcal{K}, \\ z_n = (1-v_n)x_n + v_n\ell_n, \\ y_n = (1-u_n)x_n + u_n\omega_n, \\ x_{n+1} = \zeta_n, \end{cases} \quad \forall n \geq 1, \tag{1.6}$$

where $\{u_n\}$ and $\{v_n\}$ are sequences in (0,1), $\zeta_n \in \mathcal{P}_{\mathcal{T}}(y_n)$, $\omega_n \in \mathcal{P}_{\mathcal{T}}(z_n)$ and $\ell_n \in \mathcal{P}_{\mathcal{T}}(x_n)$. The authors proved analytically and numerally that (1.6) converges faster than a number of existing iterative algorithms for quasi-nonexpansive mapping.

On the other hand, a fixed point iteration procedure is said to be stable numerically if small errors or modifications in the data or procedure has small control on the computed value of the fixed point.

The concept of stability of fixed point iteration process was rigorously studied by Harder in her Ph.D thesis which was published in [13, 14].

Definition 6 [13, 14]. *Let $\mathcal{T} : \mathcal{K} \to \mathcal{P}(\mathcal{K})$. Define a fixed point iteration algorithm by $x_{n+1} = f(\mathcal{T}, x_n)$ such that $\{x_n\}$ converges to a fixed point $q \in \mathcal{T}$. Let $\{t_n\}$ be an arbitrary sequence in $\mathscr{G}$. Define*

$$\epsilon_n = \|t_n - f(\mathcal{T}, t_n)\|, \quad \forall n \geq 1.$$

A fixed point iterative algorithm is said to be $\mathcal{T}$-stable if the following condition is fulfilled:

$$\lim_{n\to\infty} \epsilon_n = 0 \quad \textit{if and only if} \quad \lim_{n\to\infty} t_n = q.$$

The notion of stability in Definition 6 has been studied by several authors for both single and multivalued mappings (see [15, 24, 28] and the references in them).

In [6], Berinde showed that the concept of stability in Definition 6 is not precise because of the sequence $\{t_n\}$ which is arbitrary taken. To overcome this limitation, Berinde [6] observed that it would be more natural that $\{t_n\}$ be an approximate sequence of $\{x_n\}$. Therefore, any iteration algorithm which is stable will also be weakly stable but the converse is generally not true.

Definition 7 [6]. *Let $\{x_n\} \subset \mathscr{G}$ be a given sequence. Then a sequence $\{t_n\} \subset \mathscr{G}$ is an approximate sequence of $\{x_n\}$ if, for any $k \in \mathbb{N}$, there exists $\eta = \eta(k)$ such that*

$$\|x_n - t_n\| \leq \eta, \quad \forall\, n \geq k.$$

Definition 8 [6]. *Let $\mathcal{T} : \mathcal{K} \to \mathcal{P}(\mathcal{K})$. Let $\{x_n\}$ be a sequence defined by an iterative algorithm with $x_0 \in \mathscr{G}$ and*

$$x_{n+1} = f(\mathcal{T}, x_n), \quad n \geq 0. \tag{1.7}$$

Let $\{x_n\}$ converge to a fixed point q of $\mathcal{T}$. Suppose for any approximate sequence $\{t_n\} \subset \mathscr{G}$ of $\{x_n\}$

$$\lim_{n\to\infty} \epsilon_n = \lim_{n\to\infty} \|t_{n+1} - f(\mathcal{T}, t_n)\| = 0$$

implies

$$\lim_{n\to\infty} t_n = q,$$

then we say that (1.7) is weakly $\mathcal{T}$-stable or weakly stable with respect to $\mathcal{T}$.

In 2010, Timis [35] studied a new concept of weak stability which is obtained from Definition 8 by replacing of the approximate sequence with the notion of the equivalent sequence which is more general.

Definition 9 [8]. *Let $\{x_n\}$ and $\{t_n\}$ be two sequences. We say that these sequences are equivalent if*

$$\lim_{n\to\infty} \|x_n - t_n\| = 0.$$

Definition 10 [35]. *Let $\mathcal{T} : \mathcal{K} \to \mathcal{P}(\mathcal{K})$. Let $\{x_n\}$ be an iterative algorithm defined for $x_0 \in \mathscr{G}$ and*

$$x_{n+1} = f(\mathcal{T}, x_n), \quad n \geq 0. \tag{1.8}$$

Let $\{x_n\}$ converge to a fixed point q of $\mathcal{T}$. Suppose for any equivalent sequence $\{t_n\} \subset \mathscr{G}$ of $\{x_n\}$

$$\lim_{n\to\infty} \epsilon_n = \lim_{n\to\infty} \|t_{n+1} - f(\mathcal{T}, t_n)\| = 0$$

implies

$$\lim_{n\to\infty} t_n = q,$$

then we shall say that (1.8) is weak w^2–stable with respect to $\mathcal{T}$.

Interestingly, the concept of w^2–stability has not been consumed by many authors for multivalued mappings.

Motivated by the above results, firstly, we construct a new four step iterative algorithm for approximating the fixed points of multivalued generalized α-nonexpansive mappings as follows:

$$\begin{cases} x_0 = x \in \mathcal{K}, \\ s_n = (1-v_n)x_n + v_n\ell_n, \\ z_n = (1-u_n)\ell_n + u_n h_n, \qquad \forall n \geq 1, \\ y_n \in \mathcal{P}_{\mathcal{T}}(z_n), \\ x_{n+1} = \zeta_n, \end{cases} \tag{1.9}$$

where $\{u_n\}$ and $\{v_n\}$ are sequences in (0,1), $\zeta_n \in \mathcal{P}_{\mathcal{T}}(y_n)$, $h_n \in \mathcal{P}_{\mathcal{T}}(s_n)$ and $\ell_n \in \mathcal{P}_{\mathcal{T}}(x_n)$.

Secondly, we will show that our new iterative algorithm (1.9) is w^2–stable with respect to $\mathcal{T}$. The stability results are supported with some illustrative examples.

Thirdly, we prove the weak and strong convergence results of the iterative algorithms (1.9) for multivalued generalized α-nonexpansive mappings in Banach spaces. Furthermore, a numerical experiment is performed to show that the iterative algorithm (1.9) enjoys a better speed of convergence than all of the iterative processes (1.2)–(1.6) for multivalued generalized α-nonexpansive mappings.

Finally, as an application, we will utilize the new iterative method (1.9) to find the solutions of nonlinear Volterra delay integro-differential equations in Banach spaces. An example is also provided to show that our results are applicable.

2. Preliminaries

The following definitions, propositions and lemmas will be useful in proving our main results.

Definition 11. *A Banach space $\mathcal{G}$ is said to be uniformly convex if for each $\epsilon \in (0,2]$, there exists $\delta > 0$ such that for $x, y \in \mathcal{G}$ satisfying $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x-y\| > \epsilon$, we have*

$$\left\| \frac{x+y}{2} \right\| < 1 - \delta.$$

Definition 12. *A Banach space $\mathcal{G}$ is said to satisfy Opial's condition if for any sequence $\{x_n\}$ in $\mathcal{G}$ which converges weakly to $x \in \mathcal{G}$ implies*

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \quad \forall\, y \in \mathcal{G} \quad \textit{with} \quad y \neq x.$$

Definition 13. *Let $\mathcal{G}$ be a Banach space and $\mathcal{K}$ a nonempty closed convex subset of $\mathcal{G}$. Let $\{x_n\}$ be a bounded sequence in $\mathcal{G}$. For $x \in \mathcal{G}$, we put*

$$r(x, \{x_n\}) = \limsup_{n\to\infty} \|x_n - x\|.$$

The asymptotic radius of $r(\{x_n\})$ relative to $\{x_n\}$ is defined by

$$r(\mathcal{K}, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in \mathcal{K}\}.$$

The asymptotic center of $A(\{x_n\})$ relative to $\{x_n\}$ is given as:

$$A(\mathcal{K}, \{x_n\}) = \big\{x \in \mathcal{K} : r(x, \{x_n\}) = r(\mathcal{K}, \{x_n\})\big\}.$$

In a uniformly convex Banach space, it is well known that $A(\mathcal{K}, \{x_n\})$ consists of exactly one point.

Definition 14. *A multivalued mapping $T : \mathcal{K} \to \mathcal{P}(\mathcal{K})$ is said to be demiclosed at $y \in \mathcal{K}$ if for any sequence $\{x_n\} \in \mathcal{K}$ weakly convergent to x and $y_n \in \mathcal{T}x_n$ strongly convergent to y, we have $y \in \mathcal{T}x$.*

Definition 15 [31]. *A multivalued mapping $\mathcal{T} : \mathcal{K} \to CB(\mathcal{K})$ is said to satisfy condition (I) if a nondecreasing function $f : [0, \infty) \to [0, \infty)$ exists with $f(0) = 0$ and for all $r > 0$ then $f(r) > 0$ such that $d(x, \mathcal{T}x) \geq f(d(x, F(\mathcal{T}))))$, for all $x \in \mathcal{K}$, where*

$$d(x, F(\mathcal{T})) = \inf_{z \in F(\mathcal{T})} \|x - z\|.$$

Lemma 1 [15]. *Let $\mathcal{K}$ be a nonempty subset of a Banach space $\mathcal{G}$ and $\mathcal{T} : \mathcal{K} \to CB(\mathcal{K})$ be a multivalued mapping. If $\mathcal{T}$ is a generalized α-nonexpansive mapping, then the following inequality holds*:

$$d(x, \mathcal{T}y) \leq \left(\frac{3 + \alpha}{1 - \alpha}\right) d(x, \mathcal{T}x) + \|x - y\|, \quad \forall\, x, y \in \mathcal{K}.$$

Lemma 2 [37]. *Let $\{\theta_n\}$ be a nonnegative real sequence satisfying the following inequality*:

$$\theta_{n+1} \leq (1 - \sigma_n)\theta_n,$$

where $\sigma_n \in (0, 1)$ for all $n \in \mathbb{N}$ and

$$\sum_{n=0}^{\infty} \sigma_n = \infty,$$

then

$$\lim_{n \to \infty} \theta_n = 0.$$

Lemma 3 [30]. *Suppose $\mathcal{G}$ is a uniformly convex Banach space and $\{\iota_n\}$ is any sequence satisfying $0 < p \leq \iota_n \leq q < 1$ for all $n \geq 1$. Suppose $\{x_n\}$ and $\{y_n\}$ are any sequences of $\mathcal{G}$ such that*

$$\limsup_{n \to \infty} \|x_n\| \leq b, \quad \limsup_{n \to \infty} \|y_n\| \leq b$$

and

$$\limsup_{n \to \infty} \|\iota_n x_n + (1 - \iota_n) y_n\| = b$$

hold for some $b \geq 0$. Then $\lim\limits_{n \to \infty} \|x_n - y_n\| = 0$.

Lemma 4 [32]. *Let $\mathcal{T} : \mathcal{K} \to \mathcal{P}(\mathcal{K})$ and*

$$\mathcal{P}_{\mathcal{T}}(x) = \big\{y \in \mathcal{T}x : \|x - y\| = d(x, \mathcal{T}x)\big\}.$$

Then the following are equivalent

(*a*) $x \in F(\mathcal{T})$;

(*b*) $\mathcal{P}_{\mathcal{T}}(x) = \{x\}$;

(*c*) $x \in F(\mathcal{P}_{\mathcal{T}})$.

Moreover, $F(\mathcal{T}) = F(\mathcal{P}_{\mathcal{T}})$.

3. Stability result

In this section, we will show that the iterative algorithm (1.9) is w^2–stable with respect to $\mathcal{T}$.

Theorem 1. *Let $\mathcal{K}$ be a nonempty closed convex subset of a Banach space $\mathscr{G}$. Let $\mathcal{T} : \mathcal{K} \to \mathcal{P}(\mathcal{K})$ be a multivalued mapping and $\mathcal{P}_{\mathcal{T}}$ is a multivalued contraction mapping with $[0,1)$. Let $\{x_n\}$ be the iterative algorithm defined by (1.9), then $\{x_n\}$ converges to a fixed point of $\mathcal{T}$.*

P r o o f. In [21], the existence of the fixed point of $\mathcal{P}_{\mathcal{T}}$ is guaranteed. Now, we show that $\{x_n\}$ converges to some fixed point q. Using (1.9), we have

$$\begin{aligned}
\|s_n - q\| &= \|(1-v_n)x_n + v_n\ell_n - q\| \\
&\le (1-v_n)\|x_n - q\| + v_n\|\ell_n - q\| \\
&\le (1-v_n)\|x_n - q\| + v_n d(\ell_n, \mathcal{P}_{\mathcal{T}}(q)) \\
&\le (1-v_n)\|x_n - q\| + v_n \mathscr{H}(\mathcal{P}_{\mathcal{T}}(x_n), \mathcal{P}_{\mathcal{T}}(q)) \\
&\le (1-v_n)\|x_n - q\| + v_n\delta\|x_n - q\| \\
&= (1 - v_n(1-\delta))\|x_n - q\|,
\end{aligned}$$

$$\begin{aligned}
\|z_n - q\| &= \|(1-u_n)\ell_n + u_n h_n - q\| \\
&\le (1-u_n)\|\ell_n - q\| + u_n\|h_n - q\| \\
&\le (1-u_n)d(\ell_n, \mathcal{P}_{\mathcal{T}}(q)) + u_n d(h_n, \mathcal{P}_{\mathcal{T}}(q)) \\
&\le (1-u_n)\mathscr{H}(\mathcal{P}_{\mathcal{T}}(x_n), \mathcal{P}_{\mathcal{T}}(q)) + u_n\mathscr{H}(\mathcal{P}_{\mathcal{T}}(s_n), \mathcal{P}_{\mathcal{T}}(q)) \\
&\le (1-u_n)\delta\|x_n - q\| + u_n\delta\|s_n - q\| \\
&\le \delta(1 - u_n v_n(1-\delta))\|x_n - q\|,
\end{aligned}$$

$$\begin{aligned}
\|y_n - q\| &\le \mathscr{H}(\mathcal{P}_{\mathcal{T}}(z_n), \mathcal{P}_{\mathcal{T}}(q)) \\
&\le \delta\|z_n - q\| \\
&\le \delta^2(1 - u_n v_n(1-\delta))\|x_n - q\|,
\end{aligned}$$

$$\begin{aligned}
\|x_{n+1} - q\| &= \|\zeta_n - q\| \\
&\le \mathscr{H}(\mathcal{P}_{\mathcal{T}}(y_n), \mathcal{P}_{\mathcal{T}}(q)) \\
&\le \delta\|y_n - q\| \\
&\le \delta^3(1 - u_n v_n(1-\delta))\|x_n - q\|. \qquad (3.1)
\end{aligned}$$

Since $\{u_n\}$, $\{v_n\} \in (0,1)$ and $\delta \in [0,1)$, it implies that

$$(1 - u_n v_n(1-\delta)) < 1.$$

Thus, (3.1) yields

$$\begin{aligned}
\|x_{n+1} - q\| &\le \delta^3\|x_n - q\| \\
&\ \vdots \\
&\le \gamma^{3n}\|x_1 - q\|. \qquad (3.2)
\end{aligned}$$

Taking limit on both sides of the above inequality (3.2), we get $\lim_{n\to\infty}\|x_n - q\| = 0$. Indeed, $\delta \in [0,1)$ and so $\lim_{n\to\infty}\gamma^{3n} = 0$. □

We provide the following example to support our analytical proof in Theorem 1.

Example 1. Let $\mathcal{K} = [0,1] \subset \mathscr{G} = \mathbb{R}$ be endowed with the usual norm. Define an operator $\mathcal{T} : \mathcal{K} \to \mathcal{P}(\mathcal{K})$ by

$$\mathcal{T}x = \left[0, \frac{x}{4}\right]. \tag{3.3}$$

Clearly, $q = 0 \in \mathcal{T}x$. Next we show that $\mathcal{P}_{\mathcal{T}}$ is a multivalued contraction mapping with $\delta = 1/4$. Now

$$\begin{aligned}\mathcal{P}_{\mathcal{T}} &= \left\{y \in \mathcal{T}x : |x-y| = d\left(x, \left[0, \frac{x}{4}\right]\right)\right\}\\ &= \left\{y \in \mathcal{T}x : |x-y| = \left|x - \frac{x}{4}\right|\right\}\\ &= \left\{y \in \mathcal{T}x : x - y = x - \frac{x}{4}\right\}\\ &= \left\{y \in \mathcal{T}x : y = \frac{x}{4}\right\},\end{aligned}$$

so that

$$\mathscr{H}(\mathcal{T}x, \mathcal{T}y) \leq \frac{1}{4}\|x-y\|,$$

for all $x, y \in \mathcal{T}$.

The iteration algorithm (1.9) associated with the mapping in (3.3) is as follows:

$$\begin{cases} x_1 \in \mathcal{K},\\ s_n = (1-v_n)x_n + v_n\dfrac{x_n}{4},\\ z_n = (1-u_n)\dfrac{x_n}{4} + u_n\dfrac{s_n}{4}, \qquad \forall n \geq 1.\\ y_n = \dfrac{s_n}{4},\\ x_{n+1} = \dfrac{y_n}{4}, \end{cases} \tag{3.4}$$

The following Table 1 and Fig. 1 show that $\lim\limits_{n\to\infty} = 0 = q \in \mathcal{T}x$ for different choices of real sequences $\{u_n\}$ and $\{v_n\}$ in $(0,1)$.

Table 1. Convergence behavior of iteration algorithm (3.3) for different choices of real sequences$\{u_n\}$ and $\{v_n\}$ in (0,1).

Step	(3.3) with a	(3.3) with b	(3.3) with c
1	0.9000000000	0.9000000000	0.9000000000
2	0.0114257812	0.0128906250	0.0123046875
3	0.0001450539	0.0001846313	0.0001682281
4	0.0000018415	0.0000026445	0.0000023000
5	0.0000000234	0.0000000379	0.0000000314
6	0.0000000003	0.0000000005	0.0000000004
7	0.0000000000	0.0000000000	0.0000000000

where a, b and c stand for the cases

$$u_n = v_n = \frac{1}{n+1}, \quad u_n = v_n = \frac{1}{2n+1}, \quad u_n = \frac{n+1}{2n+1}, \quad v_n = \frac{n}{n^2+1},$$

respectively.

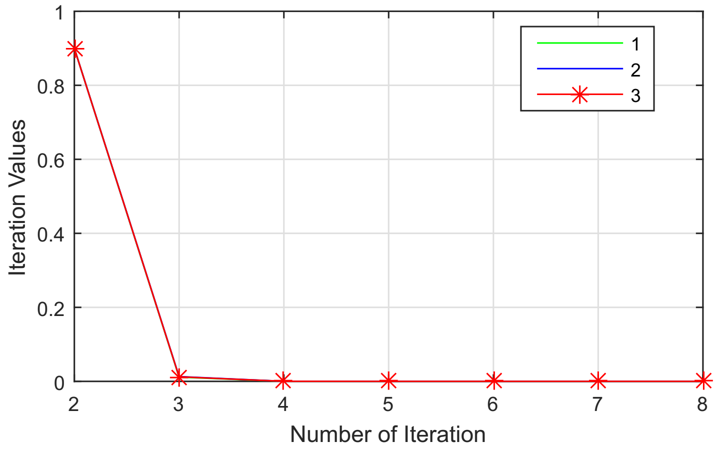

Figure 1. Graph corresponding to Table 1.

Theorem 2. *Suppose that all the conditions of Theorem 1 are satisfied. Then the iteration algorithm* (1.9) *is w^2-stable with respect to $\mathcal{T}$.*

P r o o f. Let $\{t_n\} \in \mathcal{K}$ be an equivalent sequence of $\{x_n\}$. Define a sequence $\{\epsilon_n\}$ in $\mathbb{R}^+$ by

$$\begin{cases} \epsilon_n = \|t_{n+1} - m_n\|, \\ k_n \in \mathcal{P}_{\mathcal{T}}(g_n), \\ g_n = (1-u_n)r_n + u_n d_n, \\ c_n = (1-v_n)t_n + v_n r_n, \end{cases} \quad \forall n \in \mathbb{N}, \tag{3.5}$$

where $\{u_n\}$ and $\{v_n\}$ are sequences in (0,1), $m_n \in \mathcal{P}_{\mathcal{T}}(k_n)$, $k_n \in \mathcal{P}_{\mathcal{T}}(g_n)$, $r_n \in \mathcal{P}_{\mathcal{T}}(x_n)$ and $d_n \in \mathcal{P}_{\mathcal{T}}(c_n)$. Let $\lim\limits_{n\to\infty} \epsilon_n = 0$, then from (1.1), (1.9) and (3.5), we have

$$\begin{aligned} \|t_{n+1} - q\| &\leq \|t_{n+1} - x_{n+1}\| + \|x_{n+1} - q\| \\ &\leq \|t_{n+1} - m_n\| + \|m_n - x_{n+1}\| + \|x_{n+1} - q\| \\ &= \epsilon_n + \|m_n - \zeta_n\| + \|x_{n+1} - q\| \\ &\leq \epsilon_n + \mathscr{H}(\mathcal{P}_{\mathcal{T}}(k_n), \mathcal{P}_{\mathcal{T}}(y_n)) + \|x_{n+1} - q\| \\ &\leq \epsilon_n + \delta\|k_n - y_n\| + \|x_{n+1} - q\| \\ &\leq \epsilon_n + \delta\mathscr{H}(\mathcal{P}_{\mathcal{T}}(g_n), \mathcal{P}_{\mathcal{T}}(z_n)) + \|x_{n+1} - q\| \\ &\leq \epsilon_n + \delta^2\|g_n - z_n\| + \|x_{n+1} - q\|, \end{aligned} \tag{3.6}$$

$$\begin{aligned} \|g_n - z_n\| &\leq (1-u_n)\|r_n - \ell_n\| + u_n\|d_n - h_n\| \\ &\leq (1-u_n)\mathscr{H}(\mathcal{P}_{\mathcal{T}}(t_n), \mathcal{P}_{\mathcal{T}}(x_n)) + u_n\mathscr{H}(\mathcal{P}_{\mathcal{T}}(c_n), \mathcal{P}_{\mathcal{T}}(s_n)) \\ &\leq (1-u_n)\delta\|t_n - x_n\| + u_n\delta\|c_n - s_n\|, \end{aligned} \tag{3.7}$$

$$\begin{aligned} \|c_n - s_n\| &\leq (1-v_n)\|t_n - x_n\| + u_n\|r_n - l_n\| \\ &\leq (1-v_n)\|t_n - x_n\| + u_n\mathscr{H}(\mathcal{P}_{\mathcal{T}}(t_n), \mathcal{P}_{\mathcal{T}}(x_n)) \\ &\leq (1-v_n)\|t_n - x_n\| + v_n\delta\|t_n - x_n\| \\ &= (1 - v_n(1-\delta))\|t_n - x_n\|. \end{aligned} \tag{3.8}$$

Using (3.6), (3.7) and (3.8), we get

$$\|t_{n+1}-q\| \leq \epsilon_n + \delta^3(1-u_nv_n(1-\delta))\|t_n-x_n\| + \|x_{n+1}-q\|. \tag{3.9}$$

Since $\{t_n\} \in \mathcal{K}$ and its equivalence to $\{x_n\}$ yields $\lim_{n\to\infty}\|x_n-t_n\| = 0$. We have shown in Theorem 1 that $\lim_{n\to\infty}\|x_n-q\| = 0$, consequently $\lim_{n\to\infty}\|x_{n+1}-q\| = 0$.

Thus, taking the limit on both sides of (3.9), we get

$$\lim_{n\to\infty}\|t_n-q\| = 0.$$

Hence, $\{x_n\}$ is w^2–stable with respect to $\mathcal{T}$. □

We again support the analytical proof in Theorem 2 with the following example.

Example 2. Let $\mathcal{K}$, $\mathscr{G} = \mathbb{R}$ and $\mathcal{T}$ be same as in Example 1. Let $\{x_n^{(1)}\}_{n=0}^{\infty}$, $\{x_n^{(2)}\}_{n=0}^{\infty}$ and $\{x_n^{(3)}\}_{n=0}^{\infty}$ be iterative algorithms corresponding to (3.4) with control parameters

$$\left(u_n = v_n = \frac{1}{n+1}\right), \quad \left(u_n = v_n = \frac{1}{2n+1}\right), \quad \left(u_n = \frac{n+1}{2n+1}, \quad v_n = \frac{n}{n^2+1}\right)$$

for all $n \in \mathbb{N}$, respectively.

It is shown in Example 1 that $\{x_n^{(i)}\}_{n=0}^{\infty}$ converges to $q = 0 \in \mathcal{T}x$ for each $i \in \{1,2,3\}$. Clearly,

$$\lim_{n\to\infty}\|x^{(i)}\| = \|\lim_{n\to\infty} x^{(i)}\| = 0$$

for each $i \in \{1,2,3\}$. Taking the sequence $\{t_n\}_{n=0}^{\infty}$ to be $t_n = 1/(n+4)$ for all $n \in \mathbb{N}$, then we get

$$0 \leq \lim_{n\to\infty}\|x^{(i)}-t_n\| \leq \lim_{n\to\infty}\|x^{(i)}\| + \lim_{n\to\infty}\|t_n\| = 0, \quad \text{for each} \quad i \in \{1,2,3\},$$

which shows that $\lim_{n\to\infty}\|x^{(i)}-t_n\| = 0$ for each $i \in \{1,2,3\}$, in other words, each of $\{x_n^{(i)}\}_{n=0}^{\infty}$, $i \in \{1,2,3\}$ and

$$\{t_n\}_{n=0}^{\infty} = \left\{\frac{1}{n+4}\right\}_{n=0}^{\infty}$$

are equivalent sequences.

Let $\epsilon_n^{(1)}$, $\epsilon_n^{(2)}$ and $\epsilon_n^{(3)}$ be corresponding sequences to the iterative algorithms $\{x_n^{(1)}\}_{n=0}^{\infty}$, $\{x_n^{(2)}\}_{n=0}^{\infty}$ and $\{x_n^{(3)}\}_{n=0}^{\infty}$, respectively. Then we have

$$\epsilon_n^{(1)} = \left|\frac{1}{n+5} - \frac{1}{4}\left(\frac{1}{4}\left(\frac{n}{n+1}\cdot\frac{1}{4}\cdot\frac{1}{n+4} + \frac{1}{n+1}\cdot\frac{1}{4}\left(\frac{n}{n+1}\cdot\frac{1}{n+4} + \frac{1}{n+1}\cdot\frac{1}{4}\cdot\frac{1}{n+4}\right)\right)\right)\right|,$$

$$\epsilon_n^{(2)} = \left|\frac{1}{n+5} - \frac{1}{4}\left(\frac{1}{4}\left(\frac{2n}{2n+1}\cdot\frac{1}{4}\cdot\frac{1}{n+4} + \frac{1}{2n+1}\cdot\frac{1}{4}\left(\frac{2n}{2n+1}\cdot\frac{1}{n+4} + \frac{1}{2n+1}\cdot\frac{1}{4}\cdot\frac{1}{n+4}\right)\right)\right)\right|,$$

and

$$\epsilon_n^{(3)} = \left|\frac{1}{n+5} - \frac{1}{4}\left(\frac{1}{4}\left(\frac{n^2-n+1}{n^2+1}\cdot\frac{1}{4}\cdot\frac{1}{n+4} + \frac{n}{n^2+1}\cdot\frac{1}{4}\left(\frac{n}{2n+1}\cdot\frac{1}{n+4} + \frac{n+1}{2n+1}\cdot\frac{1}{4}\cdot\frac{1}{n+4}\right)\right)\right)\right|.$$

Obviously, $\lim_{n\to\infty}\epsilon_n^{(i)} = 0$ for each $i \in \{1,2,3\}$. Hence, all the iterative algorithms $\{x_n^{(i)}\}_{n=0}^{\infty}$, $i \in \{1,2,3\}$ are w^2-stable with respect to $\mathcal{T}$.

Remark 1. Since the notion of w^2–stability is more general the concept of simple stability considered in [15, 24, 28], hence, our result improves and generalizes the corresponding in [15, 24, 28] and several others.

4. Convergence results

In this section, we will prove the weak and strong convergence results of our new iterative algorithm (1.9) for multivalued generalized α-nonexpansive mappings in uniformly convex Banach spaces.

Lemma 5. *Let $\mathcal{K}$ be a nonempty closed convex subset of a real Banach space $\mathscr{G}$. Let $\mathcal{T}:\mathcal{K}\to\mathcal{P}(\mathcal{K})$ be a multivalued mapping such that $F(\mathcal{T})\neq\emptyset$ and $\mathcal{P}_{\mathcal{T}}$ is a generalized α-nonexpansive mapping. Let $\{x_n\}$ be the iterative algorithm defined by (1.9), then $\lim_{n\to\infty}\|x_n-q\|$ exists for all $q\in F(\mathcal{T})$.*

P r o o f. Taking $q\in F(\mathcal{T})$, then from Lemma 4, we have $\mathcal{P}_{\mathcal{T}}(q)=\{q\}$ and $\mathcal{P}(\mathcal{T})=F(\mathcal{P}_{\mathcal{T}})$. Since $\mathcal{P}_{\mathcal{T}}$ is a generalized α-nonexpansive mapping, we get

$$\frac{1}{2}d(q,\mathcal{P}_{\mathcal{T}}(q))=0=\|x_n-q\|.$$

On the other hand,

$$\begin{aligned}\mathscr{H}(\mathcal{P}_{\mathcal{T}}(x_n),\mathcal{P}_{\mathcal{T}}(q))&\leq \alpha d(x_n,\mathcal{P}_{\mathcal{T}}(q))+\alpha d(q,\mathcal{P}_{\mathcal{T}}(x_n))+(1-2\alpha)\|x_n-q\|\\&\leq \alpha\|x_n-q\|+\alpha\mathscr{H}(\mathcal{P}_{\mathcal{T}}(q),\mathcal{P}_{\mathcal{T}}(x_n))+(1-2\alpha)\|x_n-q\|\\&\leq\|x_n-q\|.\end{aligned}$$

Similarly, for any $q\in F(\mathcal{T})$, we obtain

$$\begin{cases}\mathscr{H}(\mathcal{P}_{\mathcal{T}}(y_n),\mathcal{P}_{\mathcal{T}}(q))\leq\|y_n-q\|,\\ \mathscr{H}(\mathcal{P}_{\mathcal{T}}(z_n),\mathcal{P}_{\mathcal{T}}(q))\leq\|z_n-q\|,\\ \mathscr{H}(\mathcal{P}_{\mathcal{T}}(s_n),\mathcal{P}_{\mathcal{T}}(q))\leq\|s_n-q\|,\\ \mathscr{H}(\mathcal{P}_{\mathcal{T}}(\zeta_n),\mathcal{P}_{\mathcal{T}}(q))\leq\|\zeta_n-q\|.\end{cases}$$

Now from (1.9), we have

$$\begin{aligned}\|s_n-q\|&=\|(1-v_n)x_n+v_n\ell_n-q\|\\&\leq(1-v_n)\|x_n-q\|+v_n\|\ell_n-q\|\\&\leq(1-v_n)\|x_n-q\|+v_nd(\ell_n,\mathcal{P}_{\mathcal{T}}(q))\\&\leq(1-v_n)\|x_n-q\|+v_n\mathscr{H}(\mathcal{P}_{\mathcal{T}}(x_n),\mathcal{P}_{\mathcal{T}}(q))\\&\leq(1-v_n)\|x_n-q\|+v_n\|x_n-q\|\\&=\|x_n-q\|.\end{aligned}\tag{4.1}$$

Also,

$$\begin{aligned}\|z_n-q\|&=\|(1-u_n)\ell_n+u_nh_n-q\|\\&\leq(1-u_n)\|\ell_n-q\|+u_n\|h_n-q\|\\&\leq(1-u_n)d(\ell_n,\mathcal{P}_{\mathcal{T}}(q))+u_nd(h_n,\mathcal{P}_{\mathcal{T}}(q))\\&\leq(1-u_n)\mathscr{H}(\mathcal{P}_{\mathcal{T}}(x_n),\mathcal{P}_{\mathcal{T}}(q))+u_n\mathscr{H}(\mathcal{P}_{\mathcal{T}}(s_n),\mathcal{P}_{\mathcal{T}}(q))\\&\leq(1-u_n)\|x_n-q\|+u_n\|s_n-q\|\\&\leq\|x_n-q\|.\end{aligned}\tag{4.2}$$

Again,

$$\begin{aligned}\|y_n-q\|&\leq\mathscr{H}(\mathcal{P}_{\mathcal{T}}(z_n),\mathcal{P}_{\mathcal{T}}(q))\\&\leq\|z_n-q\|\\&\leq\|x_n-q\|.\end{aligned}\tag{4.3}$$

Finally,

$$\begin{aligned}\|x_{n+1}-q\| &= \|\zeta_n - q\| \\ &\le \mathscr{H}(\mathcal{P}_{\mathcal{T}}(y_n), \mathcal{P}_{\mathcal{T}}(q)) \\ &\le \|y_n - q\| \\ &\le \|x_n - q\|.\end{aligned}$$

Thus, $\{\|x_n - q\|\}$ is bounded and non-increasing, which implies that $\lim_{n\to\infty} \|x_n - q\|$ exists for all $q \in F(\mathcal{T})$. ☐

Lemma 6. *Let $\mathcal{K}$ be a nonempty subset of a uniformly convex Banach space $\mathscr{G}$. Let $\mathcal{T} : \mathcal{K} \to \mathcal{P}(\mathcal{K})$ be a multivalued mapping such that $F(\mathcal{T}) \neq \emptyset$ and $\mathcal{P}_{\mathcal{T}}$ is a generalized α-nonexpansive mapping. Let $\{x_n\}$ be the iterative algorithm defined by (1.9), then $\lim_{n\to\infty} d(x_n, \mathcal{T}x_n) = 0$.*

P r o o f. From Lemma 5, we have that $\lim_{n\to\infty} \|x_n - q\|$ exists for all $q \in F(\mathcal{T})$. We suppose that

$$\lim_{n\to\infty} \|x_n - q\| = b \quad \text{for some} \quad b \ge 0. \tag{4.4}$$

Now from (4.1), (4.2), (4.3) and (4.4), we have

$$\limsup_{n\to\infty} \|s_n - q\| \le b, \tag{4.5}$$

$$\limsup_{n\to\infty} \|z_n - q\| \le b, \tag{4.6}$$

$$\limsup_{n\to\infty} \|y_n - q\| \le b \tag{4.7}$$

and

$$\lim_{n\to\infty} \|\zeta_n - q\| \le b. \tag{4.8}$$

Now, we have the following inequalities

$$\begin{aligned}\limsup_{n\to\infty} \|\ell_n - q\| &\le \limsup_{n\to\infty} \mathscr{H}(\mathcal{P}_{\mathcal{T}}(x_n), \mathcal{P}_{\mathcal{T}}(q)) \\ &\le \limsup_{n\to\infty} \|x_n - q\| = b, \\ \limsup_{n\to\infty} \|h_n - q\| &\le \limsup_{n\to\infty} \mathscr{H}(\mathcal{P}_{\mathcal{T}}(s_n), \mathcal{P}_{\mathcal{T}}(q)) \\ &\le \limsup_{n\to\infty} \|s_n - q\| \le b\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}\limsup_{n\to\infty} \|\zeta_n - q\| &\le \limsup_{n\to\infty} \mathscr{H}(\mathcal{P}_{\mathcal{T}}(y_n), \mathcal{P}_{\mathcal{T}}(q)) \\ &\le \limsup_{n\to\infty} \|y_n - q\| \le b.\end{aligned}$$

Using (1.9) and (4.4), we have

$$\begin{aligned}b = \lim_{n\to\infty} \|x_{n+1} - q\| &= \lim_{n\to\infty} \|\zeta_n - q\| \\ &\le \lim_{n\to\infty} \mathscr{H}(\mathcal{P}_{\mathcal{T}}(y_n), \mathcal{P}_{\mathcal{T}}(q)) \\ &\le \lim_{n\to\infty} \|y_n - q\| \\ &\le \lim_{n\to\infty} \mathscr{H}(\mathcal{P}_{\mathcal{T}}(z_n), \mathcal{P}_{\mathcal{T}}(q)) \\ &\le \lim_{n\to\infty} \|z_n - q\| \\ &= \lim_{n\to\infty} \|(1-u_n)\ell_n + u_n h_n - q\|.\end{aligned}$$

By Lemma 3, we have

$$\lim_{n\to\infty} \|\ell_n - h_n\| = 0. \tag{4.10}$$

Again, from (1.9) we have

$$\begin{aligned}\|x_{n+1} - q\| &= \|\zeta_n - q\| \\ &\leq \mathscr{H}(\mathcal{P}_{\mathcal{T}}(y_n), \mathcal{P}_{\mathcal{T}}(q)) \\ &\leq \|y_n - q\|,\end{aligned}$$

which gives

$$b \leq \liminf_{n\to\infty} \|y_n - q\|. \tag{4.11}$$

From (4.7) and (4.11), we obtain

$$\lim_{n\to\infty} \|y_n - q\| = b.$$

Again from (1.9), we have

$$\begin{aligned}\|y_n - q\| &\leq \mathscr{H}(\mathcal{P}_{\mathcal{T}}(z_n), \mathcal{P}_{\mathcal{T}}(q)) \\ &\leq \|z_n - q\|,\end{aligned}$$

which yields

$$b \leq \liminf_{n\to\infty} \|z_n - q\|. \tag{4.12}$$

From (4.6) and (4.12), we have

$$\lim_{n\to\infty} \|z_n - q\| = b.$$

By (1.9) and (4.10), we get

$$\begin{aligned}\|z_n - q\| &= \|(1 - u_n)\ell_n + u_n h_n - q\| \\ &\leq \|\ell_n - q\| + u_n\|h_n - \ell_n\|,\end{aligned}$$

which gives

$$b \leq \liminf_{n\to\infty} \|\ell_n - q\|. \tag{4.13}$$

Using (4.9) and (4.13), we have

$$\lim_{n\to\infty} \|\ell_n - q\| = b.$$

Also,

$$\begin{aligned}\|\ell_n - q\| &\leq \|\ell_n - h_n\| + \|h_n - q\| \\ &\leq \|\ell_n - h_n\| + \mathscr{H}(\mathcal{P}_{\mathcal{T}}(s_n), \mathcal{P}_{\mathcal{T}}(q)) \\ &\leq \|\ell_n - h_n\| + \|h_n - q\|\end{aligned}$$

gives

$$b \leq \liminf_{n\to\infty} \|s_n - q\|. \tag{4.14}$$

From (4.5) and (4.14), we obtain

$$\lim_{n\to\infty} \|s_n - q\| = b.$$

Finally, from (1.9), we have

$$\lim_{n\to\infty} \|s_n - q\| = \lim_{n\to\infty} \|(1 - \upsilon_n)(x_n - q) + \upsilon_n(\ell_n - q)\| = b. \tag{4.15}$$

Now, due to (4.4), (4.9), (4.15) and Lemma 3 we obtain

$$\lim_{n\to\infty} \|x_n - \ell_n\| = 0. \tag{4.16}$$

Since $d(x_n, \mathcal{T}x_n) \le \|x_n - \ell_n\|$, we get

$$\lim_{n\to\infty} d(x_n, \mathcal{T}x_n) = 0.$$

□

Next we prove weak convergence of the iterative algorithm (1.9) to the fixed point of multivalued generalized α-nonexpansive mapping.

Theorem 3. *Let $\mathcal{K}$ be a nonempty subset of a uniformly convex Banach space $\mathcal{G}$ which satisfies Opial's condition. Let $\mathcal{T} : \mathcal{K} \to \mathcal{P}(\mathcal{K})$ be a multivalued mapping such that $F(\mathcal{T}) \neq \emptyset$ and $\mathcal{P}_{\mathcal{T}}$ is a generalized α-nonexpansive mapping. Let $I - \mathcal{P}_{\mathcal{T}}$ be demiclosed with respect to zero and $\{x_n\}$ be the iterative algorithm defined by (1.9), then $\{x_n\}$ converges weekly to a fixed point of $\mathcal{T}$.*

P r o o f. Let $q \in F(\mathcal{T}) = F(\mathcal{P}_{\mathcal{T}})$. From Lemma 5 we have that $\lim_{n\to\infty} \|x_n - q\|$ exists. Now we show that the sequence $\{x_n\}$ has a unique weak sequential limit in $F(\mathcal{T})$. To prove this, let p_1 and p_2 be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$, respectively. From (4.16), there exists $\ell_n \in \mathcal{T}x_n$ such that $\lim_{n\to\infty} \|x_n - \ell_n\| = 0$. Therefore, from the demiclosedness of $I - \mathcal{P}_{\mathcal{T}}$ with respect to zero, we have $p_1 \in F(\mathcal{T}) = F(\mathcal{P}_{\mathcal{T}})$. Following the same method of proof, we can show that $p_2 \in F(\mathcal{T})$. Next, we prove uniqueness. To show this, suppose that $p_1 \neq p_2$, then from Opial's condition we obtain

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - p_1\| &= \lim_{n_i\to\infty} \|x_{n_i} - p_1\| \\
&< \lim_{n_i\to\infty} \|x_{n_i} - p_2\| \\
&= \lim_{n\to\infty} \|x_n - p_2\| \\
&= \lim_{n_k\to\infty} \|x_{n_k} - p_2\| \\
&< \lim_{n_k\to\infty} \|x_{n_k} - p_1\| \\
&= \lim_{n\to\infty} \|x_n - p_1\|,
\end{aligned}$$

which is a contradiction, so $p_1 = p_2$. Hence, $\{x_n\}$ converges weakly to a fixed point of $\mathcal{T}$. □

Furthermore, we state and prove strong convergence theorems of the new iterative algorithm (1.9) for multivalued generalized α-nonexpansive mappings.

Theorem 4. *Let $\mathcal{K}$ be a nonempty closed convex subset of a real Banach space $\mathcal{G}$. Let $\mathcal{T} : \mathcal{K} \to \mathcal{P}(\mathcal{K})$ be a multivalued mapping such that $F(\mathcal{T}) \neq \emptyset$ and $\mathcal{P}_{\mathcal{T}}$ is a generalized α-nonexpansive mapping. If $\{x_n\}$ be the iterative algorithm defined by (1.9), then $\{x_n\}$ converges strongly to a fixed point of $\mathcal{T}$ if and only if $\liminf_{n\to\infty} d(x_n, F(\mathcal{T})) = 0$.*

P r o o f. The necessity is obvious. Conversely, assume that $\liminf_{n\to\infty} d(x_n, F(\mathcal{T})) = 0$. By Lemma 5, it is proved that

$$\|x_{n+1} - q\| \leq \|x_n - q\|.$$

This yields

$$d(x_{n+1}, F(\mathcal{T})) \leq d(x_n, F(\mathcal{T})).$$

Thus $\liminf_{n\to\infty} d(x_n, F(\mathcal{T}))$ exists. By hypothesis,

$$\liminf_{n\to\infty} d(x_n, F(\mathcal{T})) = 0$$

so we must have

$$\lim_{n\to\infty} d(x_n, F(\mathcal{T})) = 0.$$

Next, we prove that the sequence $\{x_n\}$ is Cauchy in $\mathcal{K}$. We choose arbitrary $\epsilon > 0$. Since $\lim_{n\to\infty} d(x_n, F(\mathcal{T})) = 0$, then there exists n_0 such that for all $n \geq n_0$.

$$d(x_n, F(\mathcal{T})) < \frac{\epsilon}{4}.$$

Particularly,

$$\inf\{\|x_{n_0} - q\| : q \in F(\mathcal{T})\} < \frac{\epsilon}{4},$$

so an element $p \in F(\mathcal{T})$ must exist such that

$$\|x_{n_0} - q\| < \frac{\epsilon}{2}.$$

Now for $n, s \geq n_0$, we have

$$\|x_{n+s} - x_n\| \leq \|x_{n+s} - p\| + \|x_n - p\| \leq 2\|x_{n_0} - p\| < 2\left(\frac{\epsilon}{2}\right) = \epsilon.$$

Hence, $\{x_n\}$ is the Cauchy sequence in the closed subset $\mathcal{K}$ of the Banach space $\mathscr{G}$. It follows that $\{x_n\}$ must converge in $\mathcal{K}$. Now let $\lim_{n\to\infty} x_n = p^*$, then from Lemma 1 we obtain

$$\begin{aligned} d(p^*, \mathcal{P}_{\mathcal{T}}(p^*)) &\leq \|x_n - p^*\| + d(x_n, \mathcal{P}_{\mathcal{T}}(p^*)) \\ &\leq \|x_n - p^*\| + \left(\frac{3+\alpha}{1-\alpha}\right) d(x_n, \mathcal{P}_{\mathcal{T}}(x_n)) + \|x_n - p^*\| \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

This implies that $p^* \in \mathcal{P}_{\mathcal{T}}(p^*)$ and $p^* \in F(\mathcal{P}_{\mathcal{T}})$. From Lemma 4, we have $p^* \in F(\mathcal{P}_{\mathcal{T}})$. Hence, $\{x_n\}$ converges strongly to a fixed point of $\mathcal{T}$. □

Theorem 5. *Let $\mathcal{K}$ be a nonempty compact convex subset of a uniformly convex Banach space $\mathscr{G}$. Let $\mathcal{T} : \mathcal{K} \to \mathcal{P}(\mathcal{K})$ be a multivalued mapping such that $F(\mathcal{T}) \neq \emptyset$ and $\mathcal{P}_{\mathcal{T}}$ is a generalized α-nonexpansive mapping. Suppose $\{x_n\}$ is the iterative algorithm defined by (1.9), then $\{x_n\}$ converges strongly to a fixed point of $\mathcal{T}$.*

P r o o f. By Lemma 5, we know that $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, \mathcal{T}x_n) = 0$. Since $\mathcal{K}$ is compact, it follows that a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ exists such that x_{n_i} converges to some $y \in \mathcal{K}$. Since $\mathcal{P}_{\mathcal{T}}$ is a multivalued generalized α nonexpansive mappings, then from Lemma 1 we obtain

$$d(x_{n_i}, \mathcal{P}_{\mathcal{T}}(y)) \leq \left(\frac{3+\alpha}{1-\alpha}\right) d(x_{n_i}, \mathcal{T}(x_{n_i}) + \|x_{n_i} - y\|.$$

Again, since $F(\mathcal{T}) = F(\mathcal{P}_{\mathcal{T}})$, by taking the limit as $i \to \infty$, we have that $y \in \mathcal{T}y$. Hence, $\{x_n\}$ converges strongly to $y \in F(\mathcal{T})$. □

Theorem 6. *Let $\mathcal{K}$ be a nonempty closed convex subset of a real Banach space $\mathscr{G}$. Let $\mathcal{T} : \mathcal{K} \to \mathcal{P}(\mathcal{K})$ be a multivalued mapping satisfying condition (I) such that $F(\mathcal{T}) \neq \emptyset$ and $\mathcal{P}_{\mathcal{T}}$ is a generalized α-nonexpansive mapping. If $\{x_n\}$ be the iterative algorithm defined by (1.9), then $\{x_n\}$ converges strongly to a fixed point of $\mathcal{T}$.*

P r o o f. From Lemma 5, $\lim_{n\to\infty} \|x_n - q\|$ exits for all $q \in F(\mathcal{T})$ and therefore $\{x_n\}$ is bounded. Let $\lim_{n\to\infty} \|x_n - q\| = b$ for some $b \geq 0$. If $b = 0$, then the result follows trivially. Suppose $b > 0$, then by Lemma 5 we have

$$\|x_{n+1} - q\| \leq \|x_n - q\|$$

which gives

$$\inf_{q\in F(\mathcal{T})} \|x_{n+1} - q\| \leq \inf_{q\in F(\mathcal{T})} \|x_n - q\|.$$

It follows that

$$d(x_{n+1}, F(\mathcal{T})) \leq d(x_n, F(\mathcal{T})),$$

so $\lim_{n\to\infty} d(x_n, F(\mathcal{T}))$ exists. From condition (I) and Lemma 6, we get

$$\lim_{n\to\infty} f(d(x_n, F(\mathcal{T}))) \leq \lim_{n\to\infty} d(x_n, \mathcal{T}x_n) = 0.$$

Since f is a nondecreasing function and $f(0) = 0$, it follows that $\lim_{n\to\infty} d(x_n, F(\mathcal{T})) = 0$. Conclusion of the result follows from Theorem 4. $\square$

5. Numerical Experiment

In this section, we give an example of a multivalued generalized α-nonexpansive mapping which does not satisfy condition (C). We will also compare the convergence of our new iterative algorithm with the iterative algorithms (1.2)–(1.6) using the provided example.

Example 3. Let $(\mathbb{R}, \|\cdot\|)$ be a normed space with the usual norm and $\mathcal{K} = [2, 4]$. Define $\mathcal{T} : \mathcal{K} \to \mathcal{P}(\mathcal{K})$ as:

$$\mathcal{T}x = \begin{cases} \left[2, \dfrac{x+2}{2}\right], & \text{if} \quad x \in [2,3], \\ 2, & \text{if} \quad x \in (3,4]. \end{cases}$$

Then $\mathcal{T}$ is a multivalued generalized α-nonexpansive mapping, but $\mathcal{T}$ does not satisfy condition (C).

First, we show that $\mathcal{T}$ is a multivalued generalized 1/3-nonexpansive mapping. For this, we consider the following possible cases:

Case (a): If $x, y \in [2, 3]$, then

$$\begin{aligned} \alpha d(\mathcal{T}x, y) + \alpha d(\mathcal{T}y, x) + (1 - 2\alpha)\|x - y\| &= \frac{1}{3}\left|\frac{x+2}{2} - y\right| + \frac{1}{3}\left|\frac{y+2}{2} - x\right| + \frac{1}{3}|x - y| \\ &\geq \frac{1}{3}\left|\frac{3x}{2} - \frac{3x}{2}\right| + \frac{1}{3}|x - y| \\ &= \frac{1}{2}|x - y| + \frac{1}{3}|x - y| \\ &\geq \frac{1}{2}|x - y| \\ &= \mathscr{H}(\mathcal{T}x, \mathcal{T}y). \end{aligned}$$

Case (b): If $x \in [2,3]$ and $y \in (3,4]$, we obtain

$$\begin{aligned}\alpha d(\mathcal{T}x,y)+\alpha d(\mathcal{T}y,x)+(1-2\alpha)\|x-y\| &= \frac{1}{3}\left|\frac{x+2}{2}-y\right|+\frac{1}{3}|x-2|+\frac{1}{3}|x-y| \\ &\geq \frac{1}{3}\left|\frac{x}{2}+y-\frac{6}{2}\right|+\frac{1}{3}|x-y| \\ &\geq \frac{1}{3}\left|\frac{3x}{2}-\frac{6}{2}\right|+\frac{1}{3}|x-y| \\ &= \frac{1}{2}|x-2| \\ &= \mathscr{H}(\mathcal{T}x,\mathcal{T}y).\end{aligned}$$

Case (c): If $x, y \in (3,4]$, then we have

$$d(\mathcal{T}y,x)+\alpha d(\mathcal{T}y,x)+(1-2\alpha)\|x-y\| \geq 0 = \mathscr{H}(\mathcal{T}x,\mathcal{T}y).$$

Hence, $\mathcal{T}$ is a multivalued generalized 1/3-nonexpansive mapping.

Next we show that $\mathcal{T}$ does not satisfy condition (C). Now, take $x = 29/10$ and $y = 19/6$, then we obtain

$$\frac{1}{2}d(x,\mathcal{T}x) = \left(\frac{29}{10},\mathcal{T}\frac{29}{10}\right) = \frac{9}{40} < \frac{16}{60} = |x-y|.$$

But,

$$\mathscr{H}(\mathcal{T}x,\mathcal{T}y) = \frac{9}{20} > \frac{16}{60} = |x-y|.$$

Hence, $\mathcal{T}$ does not satisfy condition (C).

Finally, we will now show that $\mathcal{P}_{\mathcal{T}}$ is a multivalued generalized α-nonexpansive mapping. Note that $q = 2 \in \mathcal{T}x$. We consider the following cases:

Case (I): If $x \in [2,3]$, then

$$\begin{aligned}\mathcal{P}_{\mathcal{T}} &= \left\{y \in \mathcal{T}x : |y-x| = d\Big(x,\Big[1,\frac{x+2}{2}\Big]\Big)\right\} \\ &= \left\{y \in \mathcal{T}x : |y-x| = \left|x-\frac{x+2}{2}\right|\right\} \\ &= \left\{y \in \mathcal{T}x : |y-x| = \left|\frac{x-2}{2}\right|\right\} \\ &= \left\{y \in \mathcal{T}x : x-y = \left|\frac{x-2}{2}\right|\right\} \\ &= \left\{y \in \mathcal{T}x : y = \frac{x+2}{2}\right\}.\end{aligned}$$

Case (II): If $x \in (3,4]$, then we get

$$\begin{aligned}\mathcal{P}_{\mathcal{T}} &= \{y \in \mathcal{T}x : |y-x| = d(x,\{2\}\} \\ &= \{y \in \mathcal{T}x : |y-x| = |x-2|\} \\ &= \{y \in \mathcal{T}x : x-y = x-2\} \\ &= \{y \in \mathcal{T}x : y = 2\}.\end{aligned}$$

Following the same argument as those in Example 3, one can easily show that $\mathcal{P}_{\mathcal{T}}$ is a multivalued generalized α-nonexpansive mapping.

With the aid of MATLAB (R2015a), we will use the above example to show that our new iterative algorithm (1.9) converges faster than the iterative algorithms (1.2)–(1.6) and the comparison Table 2 and Fig. 2 are obtained for various iterative algorithms with control sequences $u_n = v_n = t_n = 3/4$ and initial guess $x_0 = 4$.

Table 2. Comparison of speed of convergence of our new iterative algorithm (1.9) with Mann, Ishikawa, S, Thakur, Picard-Ishikawa iterative schemes.

Step	Mann	Ishikawa	S	Thakur	Picard-Ishikawa	New
1	4.0000000	4.0000000	4.0000000	4.0000000	4.0000000	4.0000000
2	3.2500000	2.9687500	2.7187500	2.6054688	2.4843750	2.1796875
3	2.7812500	2.4692383	2.2583008	2.1832962	2.1173096	2.0161438
4	2.4882813	2.2272873	2.0928268	2.0554901	2.0284109	2.0014504
5	2.3051758	2.1100923	2.0333596	2.0167987	2.0068808	2.0001303
6	2.1907349	2.0533259	2.0119886	2.0050856	2.0016664	2.0000117
7	2.1192093	2.0258298	2.0043084	2.0015396	2.0004036	2.0000011
8	2.0745058	2.0125113	2.0015483	2.0004661	2.0000977	2.0000001
9	2.0465661	2.0060602	2.0005564	2.0001411	2.0000237	2.0000000
10	2.0291038	2.0029354	2.0002000	2.0000427	2.0000057	2.0000000
11	2.0181899	2.0014218	2.0000719	2.0000129	2.0000014	2.0000000
12	2.0113687	2.0006887	2.0000258	2.0000039	2.0000003	2.0000000
13	2.0071054	2.0003336	2.0000093	2.0000012	2.0000001	2.0000000
14	2.0044409	2.0001616	2.0000033	2.0000004	2.0000000	2.0000000
15	2.0027756	2.0000783	2.0000012	2.0000001	2.0000000	2.0000000
16	2.0017347	2.0000379	2.0000004	2.0000000	2.0000000	2.0000000
17	2.0010842	2.0000184	2.0000002	2.0000000	2.0000000	2.0000000
18	2.0006776	2.0000089	2.0000001	2.0000000	2.0000000	2.0000000
19	2.0004235	2.0000043	2.0000000	2.0000000	2.0000000	2.0000000

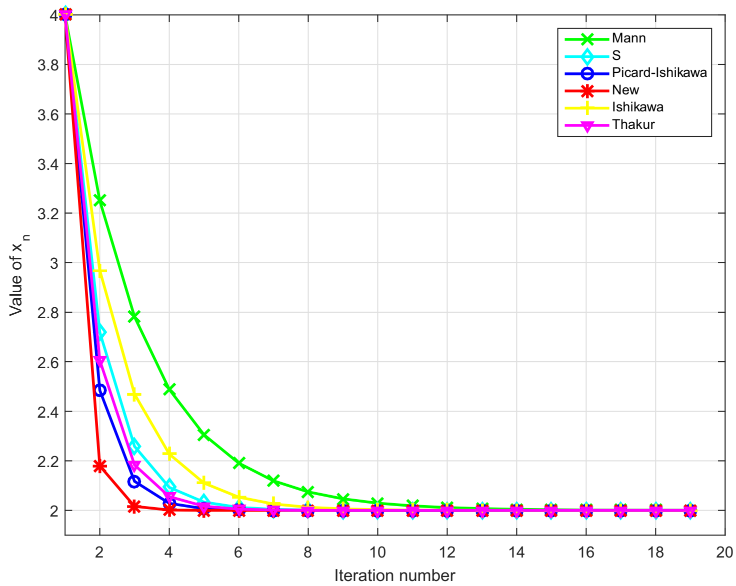

Figure 2. Graph corresponding to Table 2.

6. Application

Existence theorem for fixed points of an operator is concerned with establishing sufficient conditions in which the operator will have solution, but does not necessarily show how to find it. On the other hand, the iteration method of fixed points is concerned with approximation or computation of sequences which converge to the solution of such operator.

In this section, we will approximate the solution of nonlinear Volterra delay integro-differential equations by utilizing the following iterative algorithm recently introduced by Ofem et al. [24]:

$$\begin{cases} x_0 \in \mathcal{K}, \\ s_n = (1-v_n)x_n + v_n T x_n, \\ z_n = (1-u_n)Tx_n + u_n T s_n, \qquad \forall n \geq 1, \\ y_n = T z_n, \\ x_{n+1} = T y_n, \end{cases} \tag{6.1}$$

where $\{u_n\}$ and $\{v_n\}$ are sequences in $(0,1)$.

Remark 2. We remark that the iterative algorithm (1.9) is the multivalued conversion of the iterative algorithm (6.1). It is shown in [24] that the iterative algorithm (6.1) has a better speed of convergence than S [3], Picard-S [12], Thakur [34] and M [36] iteration processes for single-valued generalized α-nonexpansive mappings.

In particular, we will be interested in the following nonlinear Volterra delay integro–differential equation (VDIE):

$$x'(t) = f\Big(t, x(t), x(\hbar(t)), \int_0^t \wp\big(t, s, x(s), x(\hbar(s))\big) ds\Big), \quad t \in I, \tag{6.2}$$

$$x(t) = \psi(t), \quad t \in [-r, 0], \tag{6.3}$$

where $I = [0,k]$, $k > 0$ and $\psi \in C([-r,0], \Re)$.

A function $x \in C([r,k], \Re) \cap C'([0,k], \Re)$ that satisfies the equations (6.2)–(6.3) is called a solution of the initial value problem (6.2)–(6.3).

Suppose that the following conditions are performed:

(M_1) Let $f \in C([0,k] \times \Re^3, \Re)$, $\wp \in C([0,k] \times [0,k] \times \Re^2, \Re)$ and $\hbar \in C([0,k], [-r,k])$ be such that $\hbar(t) \leq t$.

(M_2) There exists constants $L_f, L_\wp > 0$ such that

$$|f(t, \sharp_1, \sharp_2, \sharp_3) - f(t, \flat_1, \flat_2, \flat_3)| \leq L_f(|\sharp_1 - \flat_1| + |\sharp_2 - \flat_2| + |\sharp_3 - \flat_3|);$$
$$|\wp(t, s, \sharp_1, \sharp_2) - \wp(t, s, \flat_1, \flat_2)| \leq L_f(|\sharp_1 - \flat_1| + |\sharp_2 - \flat_2|)$$

for all $t, s \in I$, $\sharp_i, \flat_i \in \Re$ $(i = 1, 2, 3)$.

(M_3) $kL_f[2 + L_f k] < 1$.

(M_4) The function $\phi : [-r,k] \to \Re_+$ is positive, nondecreasing and continuous and there exists $\daleth > 0$ such that

$$\int_0^t \phi(s) ds \leq \daleth \phi(t), \quad t \in [0,k].$$

Clearly, from assumption (M_1), the initial value problem (6.2)–(6.3) is equivalent to the following integral equations:

$$x(t) = \psi(0) + \int_0^t f\Big(s, x(s), x(\hbar(s)), \int_0^s \wp\big(s, \tau, x(\tau), x(\hbar(\tau))\big)\,d\tau\Big)ds, \quad t \in I,$$
$$x(t) = \psi(t), \quad t \in [-r, 0].$$

The following existence result for initial value problem is due to Kucche and Shikhare [18].

Theorem 7. *If the assumptions (M_1)–(M_4) hold, then the problem (6.2)–(6.3) has a unique solution and the equation (6.2) is generalized Ulam–Hyers–Rassias stable with respect to the function ϕ.*

We now present our main result in section.

Theorem 8. *Let $\{x_n\}$ be the iterative procedure (6.1) with $u_n, v_n \in (0,1)$ such that $\sum_{n=0}^{\infty} u_n v_n = \infty$. Suppose that the conditions (M_1)–(M_3) are fulfilled. Then the initial value problem (6.2)–(6.3) has a unique solution, say, q in $C([r,k],\Re) \cap C'([0,k],\Re)$ and $\{x_n\}$ converges to q.*

P r o o f. Consider the Banach space $\mathscr{G} = C([-r,k],\Re)$ with Chebyshev norm $\|\cdot\|_C$. Let $\{x_n\}$ be an iterative sequence generated by the iteration process (6.1) for the operator $T : \mathscr{G} \to \mathscr{G}$ define by

$$Tx(t) = \psi(0) + \int_0^t f\Big(s, x(s), x(\hbar(s)), \int_0^s \wp\big(s, \tau, x(\tau), x(\hbar(\tau))\big)\,d\tau\Big)ds, \quad t \in I,$$
$$Tx(t) = \psi(t), \quad t \in [-r, 0].$$

Let q stand for the fixed point of T. We now prove that $x_n \to q$ as $n \to \infty$. It obvious that for $t \in [-r, 0]$, $x_n \to q$ as $n \to \infty$. Next for $t \in I$, we get

$$\begin{aligned}
\|s_n - q\| &= \|(1 - v_n)x_n + v_n T x_n - Tq\| \\
&\le (1 - v_n)\|x_n - q\| + v_n\|Tx_n - Tq\| \\
&= (1 - v_n)|x_n(t) - q(t)| + v_n|T(x_n)(t) - T(q)(t)| \\
&= (1 - v_n)|x_n(t) - q(t)| \\
&\quad + v_n\Big|\psi(0) + \int_0^t f\Big(s, x_n(s), x_n(\hbar(s)), \int_0^s \wp\big(s, \tau, x_n(\tau), x_n(\hbar(\tau))\big)\,d\tau\Big) \\
&\quad - \psi(0) - \int_0^t f\Big(s, q(s), q(\hbar(s)), \int_0^s \wp\big(s, \tau, q(\tau), q(\hbar(\tau))\big)\,d\tau\Big)\Big| \\
&\le (1 - v_n)|x_n(t) - q(t)| \\
&\quad + v_n \int_0^t L_f\Big\{\max_{0\le d_1\le s}|x_n(d_1) - q(d_1)| + \max_{0\le d_1\le s}|x_n(\hbar(d_1)) - q(\hbar(d_1))| \\
&\quad + \int_0^s L_\wp\big[\max_{0\le d_2\le \tau}|x_n(d_2) - q(d_2)| + \max_{0\le d_1\le \tau}|x_n(\hbar(d_2)) - q(\hbar(d_2))|\big]\,d\tau\Big\}ds \\
&\le (1 - v_n)|x_n(t) - q(t)| \\
&\quad + v_n \int_0^t L_f\Big\{\max_{-r\le d_1\le k}|x_n(d_1) - q(d_1)| + \max_{-r\le \tau_1\le k}|x_n(\tau_1) - q(\tau_1)| \\
&\quad + \int_0^s L_\wp\big[\max_{-r\le d_2\le k}|x_n(d_2) - q(d_2)| + \max_{-r\le \tau_2\le k}|x_n(\tau_2) - q(\tau_2)|\big]\,d\tau\Big\}ds
\end{aligned}$$

$$\begin{aligned}
&\leq (1-v_n)\|x_n-q\|_C + v_n\int_0^t L_f\Big\{2\|x_n-q\|_C + 2\int_0^s L_\wp\|x_n-q\|_C d\tau\Big\}ds \\
&\leq (1-v_n)\|x_n-q\|_C + v_n k L_f(2+L_\wp k)\|x_n-q\|_C \\
&= \big[1-v_n(1-kL_f(2+L_\wp k))\big]\|x_n-q\|_C; \qquad (6.4)
\end{aligned}$$

$$\begin{aligned}
\|z_n-q\| &\leq (1-u_n)\|Tx_n-Tq\| + u_n\|Ts_n-Tq\| \\
&= (1-u_n)\Big|\int_0^t f\Big(s, x_n(s), x_n(\hbar(s)), \int_0^s \wp\big(s,\tau,x_n(\tau),x_n(\hbar(\tau))\big)d\tau\Big) \\
&- \int_0^t f\Big(s, q(s), q(\hbar(s)), \int_0^s \wp(s,\tau,q(\tau),q(\hbar(\tau)))d\tau\Big)\Big| \\
&+ u_n\Big|\int_0^t f\Big(s, s_n(s), s_n(\hbar(s)), \int_0^s \wp\big(s,\tau,s_n(\tau),s_n(\hbar(\tau))\big)d\tau\Big) \\
&- \int_0^t f\Big(s, q(s), q(\hbar(s)), \int_0^s \wp(s,\tau,q(\tau),q(\hbar(\tau)))d\tau\Big)\Big| \\
&\leq (1-u_n)\int_0^t L_f\Big\{\max_{0\leq d_1\leq s}|x_n(d_1)-q(d_1)| + \max_{0\leq d_1\leq s}|x_n(\hbar(d_1))-q(\hbar(d_1))| \\
&+ \int_0^s L_\wp\big[\max_{0\leq d_2\leq \tau}|x_n(d_2)-q(d_2)| + \max_{0\leq d_1\leq \tau}|x_n(\hbar(d_2))-q(\hbar(d_2))|\big]d\tau\Big\}ds \\
&+ u_n\int_0^t L_f\Big\{\max_{0\leq d_1\leq s}|s_n(d_1)-q(d_1)| + \max_{0\leq d_1\leq s}|s_n(\hbar(d_1))-q(\hbar(d_1))| \\
&+ \int_0^s L_\wp\big[\max_{0\leq d_2\leq \tau}|s_n(d_2)-q(d_2)| + \max_{0\leq d_1\leq \tau}|s_n(\hbar(d_2))-q(\hbar(d_2))|\big]d\tau\Big\}ds \\
&\leq (1-u_n)\int_0^t L_f\Big\{\max_{-r\leq d_1\leq k}|x_n(d_1)-q(d_1)| + \max_{-r\leq \tau_1\leq k}|x_n(\tau_1)-q(\tau_1)| \\
&+ \int_0^s L_\wp\big[\max_{-r\leq d_2\leq k}|x_n(d_2)-q(d_2)| + \max_{-r\leq \tau_2\leq k}|x_n(\tau_2)-q(\tau_2)|\big]d\tau\Big\}ds \\
&+ u_n\int_0^t L_f\Big\{\max_{-r\leq d_1\leq k}|s_n(d_1)-q(d_1)| + \max_{-r\leq \tau_1\leq k}|s_n(\tau_1)-q(\tau_1)| \\
&+ \int_0^s L_\wp\big[\max_{-r\leq d_2\leq k}|s_n(d_2)-q(d_2)| + \max_{-r\leq \tau_2\leq k}|s_n(\tau_2)-q(\tau_2)|\big]d\tau\Big\}ds \\
&\leq (1-u_n)\int_0^t L_f\Big\{2\|x_n-q\|_C + 2\int_0^s L_\wp\|x_n-q\|_C d\tau\Big\}ds \\
&+ u_n\int_0^t L_f\Big\{2\|s_n-q\|_C + 2\int_0^s L_\wp\|s_n-q\|_C d\tau\Big\}ds \\
&\leq (1-u_n)kL_f(2+L_\wp k)\|x_n-q\|_C + u_n kL_f(2+L_\wp k)\|s_n-q\|_C \\
&= kL_f(2+L_\wp k)\big[(1-u_n)\|x_n-q\|_C + u_n\|s_n-q\|_C\big]; \qquad (6.5)
\end{aligned}$$

$$\begin{aligned}
\|z_n-q\| &= \|Tz_n-Tq\| \\
&= \Big|\int_0^t f\Big(s, z_n(s), z_n(\hbar(s)), \int_0^s \wp\big(s,\tau,z_n(\tau),z_n(\hbar(\tau))\big)d\tau\Big) \\
&- \int_0^t f\Big(s, q(s), q(\hbar(s)), \int_0^s \wp\big(s,\tau,q(\tau),q(\hbar(\tau))\big)d\tau\Big)\Big| \\
&\leq \int_0^t L_f\Big\{\max_{0\leq d_1\leq s}|z_n(d_1)-q(d_1)| + \max_{0\leq d_1\leq s}|z_n(\hbar(d_1))-q(\hbar(d_1))|
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^s L_\wp\big[\max_{0\le d_2\le\tau}|z_n(d_2)-q(d_2)|+\max_{0\le d_1\le\tau}|z_n(\hbar(d_2))-q(\hbar(d_2))|\big]d\tau\Big\}ds\\
&\le\int_0^t L_f\Big\{\max_{-r\le d_1\le k}|z_n(d_1)-q(d_1)|+\max_{-r\le\tau_1\le k}|z_n(\tau_1)-q(\tau_1)|\\
&+\int_0^s L_\wp\big[\max_{-r\le d_2\le k}|z_n(d_2)-q(d_2)|+\max_{-r\le\tau_2\le k}|z_n(\tau_2)-q(\tau_2)|\big]d\tau\Big\}ds\\
&\le\int_0^t L_f\Big\{2\|z_n-q\|_C+2\int_0^s L_\wp\|z_n-q\|_C d\tau\Big\}ds\\
&\le kL_f(2+L_\wp k)\|z_n-q\|_C. \qquad (6.6)
\end{aligned}
$$

$$
\begin{aligned}
\|x_{n+1}-q\|&=\|Ty_n-Tq\|\\
&=\Big|\int_0^t f\Big(s,y_n(s),y_n(\hbar(s)),\int_0^s\wp\big(s,\tau,y_n(\tau),y_n(\hbar(\tau))\big)d\tau\Big)\\
&-\int_0^t f\Big(s,q(s),q(\hbar(s)),\int_0^s\wp\big(s,\tau,q(\tau),q(\hbar(\tau))\big)d\tau\Big)\Big|\\
&\le\int_0^t L_f\Big\{\max_{0\le d_1\le s}|y_n(d_1)-q(d_1)|+\max_{0\le d_1\le s}|y_n(\hbar(d_1))-q(\hbar(d_1))|\\
&+\int_0^s L_\wp\big[\max_{0\le d_2\le\tau}|y_n(d_2)-q(d_2)|+\max_{0\le d_1\le\tau}|y_n(\hbar(d_2))-q(\hbar(d_2))|\big]d\tau\Big\}ds\\
&\le\int_0^t L_f\Big\{\max_{-r\le d_1\le k}|y_n(d_1)-q(d_1)|+\max_{-r\le\tau_1\le k}|y_n(\tau_1)-q(\tau_1)|\\
&+\int_0^s L_\wp\big[\max_{-r\le d_2\le k}|y_n(d_2)-q(d_2)|+\max_{-r\le\tau_2\le k}|y_n(\tau_2)-q(\tau_2)|\big]d\tau\Big\}ds\\
&\le\int_0^t L_f\Big\{2\|y_n-q\|_C+2\int_0^s L_\wp\|y_n-q\|_C d\tau\Big\}ds\\
&\le kL_f(2+L_\wp k)\|y_n-q\|_C. \qquad (6.7)
\end{aligned}
$$

Using (6.4), (6.5), (6.6) and (6.7), we get

$$\|x_{n+1}-q\|\le[kL_f(2+L_\wp k)]^3[1-u_nv_n(1-kL_f(2+L_\wp k))]\|x_n-q\|_C.$$

Using assumption (M_3), we obtain

$$\|x_{n+1}-q\|\le[1-u_nv_n(1-kL_f(2+L_\wp k))]\|x_n-q\|_C. \qquad (6.8)$$

Now define

$$\sigma_n=u_nv_n(1-kL_f(2+L_\wp k))<1,$$

then $\sigma_n\in(0,1)$ such that $\sum_0^\infty\sigma_n=\infty$ and set $\theta_n=\|x_n-q\|_C$. Then (6.8) can be rewritten as

$$\theta_{n+1}=(1-\sigma_n)\theta.$$

Therefore, all the conditions of Lemma 2 are satisfied. Hence, $\lim_{n\to\infty}\|x_n-q\|=0$. □

Now, we furnish the following example in support of the above claims in Theorem 8.

Example 4. Consider the following nonlinear delay Volterra integro-differential equations:

$$
\begin{aligned}
x'(t)=1+\frac{tx(t)}{25}-\frac{7tx(\hbar(t))}{50}+\frac{1}{5}\int_0^t\frac{1}{10}[x(s)-x(\hbar(s))]ds,&\quad t\in[0,3],\\
x(t)=0,&\quad t\in[-1,0],
\end{aligned}
$$

where $\hbar(t)=t/3$, $t\in[0,3]$. Obviously, we have that $\hbar(t)=t/3\le t$, $t\in[0,3]$.

(i) Define $\wp : [0,3] \times [0,3] \times \Re \times \Re \to \Re$ by

$$\wp(t,s,x(s),x(\hbar(s))) = \frac{1}{5}[x(s) - x(\hbar(s))], \quad t,s \in [0,3].$$

Then for $t,s \in [0,3]$ and $\sharp_i, \flat_i \in \Re$ $(i = 1,2)$, we have

$$|\wp(t,s,\sharp_1,\sharp_2) - \wp(t,s,\flat_1,\flat_2)| \le \frac{1}{5}(|\sharp_1 - \flat_1| + |\sharp_2 - \flat_2|).$$

(ii) Define $f : [0,3] \times \Re \times \Re \times \Re \to \Re$ by

$$\begin{aligned} & f\Big(t, x(t), x(\hbar(t)), \int_0^s \wp(t,s,x(s),x(\hbar(s)))ds\Big) \\ &= 1 + \frac{tx(t)}{25} - \frac{7tx(\hbar(t))}{50} + \frac{1}{10}\int_0^t \frac{1}{5}[x(s) - x(\hbar(s))]ds \\ &= 1 + \frac{tx(t)}{25} - \frac{7tx(\hbar(t))}{50} + \frac{1}{10}\int_0^t \wp(t,s,x(s),x(\hbar(s)))ds, \quad t \in [0,3]. \end{aligned}$$

Then, for any $t,s \in [0,3]$ and $\sharp_i, \flat_i \in \Re$ $(i = 1,2,3)$, we have

$$\begin{aligned} |f(t,\sharp_1,\sharp_2,\sharp_3) - f(t,\flat_1,\flat_2,\flat_3)| &\le \frac{1}{25}|\sharp_1 - \flat_2| + \frac{7}{50}|\sharp_2 - \flat_2| + \frac{1}{10}|\sharp_3 - \flat_3| \\ &\le \frac{7}{50}(|\sharp_1 - \flat_2| + |\sharp_2 - \flat_2| + |\sharp_3 - \flat_3|). \end{aligned}$$

Thus the above defined functions f and $\wp$ satisfy the assumptions (M_1) and (M_2) with $L_f = 7/50$, $L_\wp = 1/10$. Further, we see that

$$kL_f(2 + kL_h) = 3 \cdot \frac{7}{50}(2 + 3 \cdot \frac{1}{10}) = \frac{483}{500} < 1.$$

Thus condition (M_3) holds. Now, if we take $u_n = n/(n+1)$ and $v_n = 1/n$, it follows that

$$\sum_{n=0}^{\infty} u_n v_n = \infty.$$

In addition, we notice that the exact solution of the problem (6.2)–(6.3) is the function

$$x(t) = \begin{cases} t, & \text{if} \quad t \in [0,3], \\ 0, & \text{if} \quad t \in [-1,0]. \end{cases}$$

Indeed, for $x(t) = t$, $t \in [0,3]$ and $\hbar(t) = t/3$, $t \in [0,3]$, we get

$$1+\frac{tx(t)}{25}-\frac{7tx(\hbar(t))}{50}+\frac{1}{5}\int_0^t \frac{1}{10}[x(s) - x(\hbar(s))]ds=1+\frac{t^2}{25}-\frac{7t^2}{150} + \frac{1}{5}\int_0^t \frac{1}{10}\left[s - \frac{s}{3}\right] ds = 1 = x'(t).$$

Thus, all the conditions of Theorem 8 are fulfilled. Hence, Theorem 8 is applicable.

Remark 3. For any fixed $r > 0$, define

$$\hbar_1(t) = t - r, \quad t \in [0,k].$$

Then we get the following special case of the Problem (6.2)–(6.3) as follows:

$$x'(t) = f_1\Big(t, x(t), x(t-r), \int_0^t \wp_1(t, s, x(s), x(t-r))ds\Big), \quad t \in [0, k], \tag{6.9}$$

$$x(t) = \psi(t), \quad t \in [-r, 0], \tag{6.10}$$

which is the initial value problem for a nonlinear Volterra integro–differential equation.

The approximation of solution the problem (6.9)–(6.10) has been studied by several authors for $\wp_1(t, s, x(s), x(t-r)) = 0$ (see for example [4, 9, 12, 23, 25–27] and the references there in). Hence, our result in Theorem 8 generalizes the corresponding results in [4, 9, 12, 23, 25–27] and several others.

7. Conclusion

In this paper, we have studied the stability result of our newly introduced iterative algorithm (1.9) for a wider concept of stability known as w^2–stability instead of the simple notion of stability considered in [15, 24, 28]. A numerical example is also used to support the analytical proof of our stability theorem. We have also proved the weak and strong convergence theorems of our new iterative algorithm (1.9) for fixed points of multivalued generalized α–nonexpansive mappings. In addition, a numerical experiment was also carried out to illustrate the advantage of our iterative method over some well known iterative methods in the literature. Further, as application of our new iterative algorithm (1.9), we approximated the solution of nonlinear Volterra delay integro-differential equations (6.2)–(6.3). A nontrivial example of a nonlinear Volterra delay integro-differential equation which satisfies all the mild conditions used in obtaining our result has been provided. We have also seen that the class of delay differential equation studied in [4, 9, 12, 23, 25–27] is a special case of the class nonlinear Volterra delay integro-differential equation considered in this article. Hence, our results generalize, improve and unify the corresponding results in [4, 9, 12, 15, 23–28] and several others in the existing literature.

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NOTE ON SUPER $(a, 1)$–P_3–ANTIMAGIC TOTAL LABELING OF STAR S_n

S. Rajkumar[†], **M. Nalliah**[††] **and Madhu Venkataraman**[†††]

Department of Mathematics, School of Advanced Sciences,
Vellore Institute of Technology
Vellore-632 014, India

[†]raj26101993@gmail.com, [††]nalliahklu@gmail.com, [†††]madhu.v@vit.ac.in

Abstract: Let $G = (V, E)$ be a simple graph and H be a subgraph of G. Then G admits an H-covering, if every edge in $E(G)$ belongs to at least one subgraph of G that is isomorphic to H. An $(a, d) - H$-antimagic total labeling of G is bijection $f : V(G) \cup E(G) \to \{1, 2, 3, \ldots, |V(G)| + |E(G)|\}$ such that for all subgraphs H' of G isomorphic to H, the H' weights $w(H') = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e)$ constitute an arithmetic progression $\{a, a + d, a + 2d, \ldots, a + (n - 1)d\}$, where a and d are positive integers and n is the number of subgraphs of G isomorphic to H. The labeling f is called a super $(a, d) - H$-antimagic total labeling if $f(V(G)) = \{1, 2, 3, \ldots, |V(G)|\}$. In [5], David Laurence and Kathiresan posed a problem that characterizes the super $(a, 1) - P_3$-antimagic total labeling of Star S_n, where $n = 6, 7, 8, 9$. In this paper, we completely solved this problem.

Keywords: H-covering, Super $(a, d) - H$-antimagic, Star.

1. Introduction

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be simple and finite graphs. Let $|V(G)| = v_G$, $|E(G)| = e_G$, $|V(H)| = v_H$ and $|E(H)| = e_H$. An edge covering of G is a family of different subgraphs $H_1, H_2, H_3, \ldots, H_k$ such that any edge of $E(G)$ belongs to at least one of the subgraphs H_j, $1 \leq j \leq k$. If the $H_j^{'}$s are isomorphic to a given graph H, then G admits an H-covering. Gutienrez and Lladó [2] defined $H-$magic labeling, which is a generalization of Kotzig and Rosa's edge magic total labeling [4]. A bijection $f : V(G) \cup E(G) \to \{1, 2, 3, ..., v_G + e_G\}$ is called an H-magic labeling of G if there exists a positive integer k such that each subgraph H' of G isomorphic to H satisfies

$$w(H') = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e) = k.$$

In this case, they say that G is H-magic. When $f(V(G)) = \{1, 2, 3, \ldots, v_G\}$, we say that G is $H-$super magic. On the other hand, Inayah et al. [3] introduced $(a, d) - H$-antimagic total labeling of G which is defined as a bijection $f : V(G) \cup E(G) \to \{1, 2, 3, \ldots, v_G + e_G\}$ such that for all subgraphs H' of G isomorphic to H, the set of H'-weights

$$w(H') = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e)$$

constitutes an arithmetic progression $a, a + d, a + 2d, \ldots, a + (n - 1)d$, where a and d are some positive integers and n is the number of subgraphs isomorphic to H. In this case, they say that G is $(a, d) - H-$antimagic. If $f(V(G)) = \{1, 2, 3, \ldots, v_G\}$, they say that f is a super $(a, d) - H$-antimagic total labeling and G is super $(a, d) - H$-antimagic. This labeling is a more general case of super (a, d)-edge-antimagic total labelings. If $H \cong K_2$, then we say that super $(a, d) - H$-antimagic labelings, which is also called super (a, d)-edge-antimagic total labelings and have been introduced in [6]. They studied some basic properties of such labeling and also proved the following theorem.

Theorem 1 [3]. *If G has a super $(a, d) - H$-antimagic total labeling and t is the number of subgraphs of G isomorphic to H, then G has a super $(a', d) - H$-antimagic total labeling, where $a' = [(v_G + 1)v_H + (2v_G + e_G + 1)e_H] - a - (t - 1)d$.*

Several authors are studied antimagic type labeling of graphs see [1]. In 2015, Laurence and Kathiresan [5] obtained an upper bound of d for any graph G, and they investigated the existence of super $(a, d) - P_3$-antimagic total labeling of star graph S_n. First, they observed that S_n admits a P_h−covering for $h = 2, 3$, and the star S_n contains

$$t = \binom{n}{h-1}$$

subgraphs P_h, $h = 2, 3$, which is denoted by P_h^j, $1 \le j \le h$. In 2005, Sugeng et al. [7] investigated the case $h = 2$ using super (a, d)-edge-antimagic total labeling. In 2015, the case of $h = 3$ was investigated by Laurence and Kathiresan [5]. Here they observed that if the star $S_n, n \ge 3$ admits a super $(a, d) - P_3$-antimagic total labeling then $d \in \{0, 1, 2\}$. Now, they proved the star S_n, $n \ge 3$ has super $(4n + 7, 0) - P_3$-antimagic total labeling and S_n, $n \ge 3$ admits a super $(a, 2) - P_3$-antimagic total labeling if and only if $n = 3$. Also, they proved the following theorems and posed a problem.

Theorem 2 [5]. *If the star S_n, $n \ge 3$ has super $(a, 1)$-P_3-antimagic total labeling, then $3 \le n \le 9$. Moreover, the star S_n admits a super $(a, 1)$-P_3-antimagic total labeling, where $a = 19$, for $n = 3$ and $a = 21$, for $n = 4$.*

Theorem 3 [5]. *For $n = 5$, the star S_n has no super $(a, 1)$-P_3-antimagic total labeling.*

Problem 1. [5] *For each $n, 6 \le n \le 9$ characterize the super $(a, 1) - P_3$-antimagic total labeling for the star S_n.*

In this paper, we present the complete solution to the above problem.

2. Main Results

Let $S_n \cong K_{1,n}, n \ge 1$ be the star graph and let v_0 be the central vertex and let $v_i, 1 \le i \le n$ be its adjacent vertices. Thus S_n has $n + 1$ vertices and n edges.

Theorem 4. *The star S_6 has no super $(a, 1) - P_3$-antimagic total labeling.*

P r o o f. Let $V(S_6) = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E(S_6) = \{v_0v_1, v_0v_2, v_0v_3, v_0v_4, v_0v_5, v_0v_6\}$ be the vertex set and the edge set of Star S_6. Suppose there exists a super $(a, 1) - P_3$-antimagic total labeling $f : V \cup E \to \{1, 2, 3, \ldots, 13\}$ for S_6 and let v_0 be the central vertex of S_6. In the computation of P_3 — weights the label of the central vertex v_0,$f(v_0)$ is used 15 times and label of other vertices and edges say i are used 5 times each. Therefore,

$$10f(v_0) + 5\sum_{i=1}^{13}(i) = \frac{15}{2}[2a + 14],$$

which implies $a = (70 + 2f(v_0))/3$. Since $1 \le f(v_0) \le 7$, it follows that $a = 24$ if $f(v_0) = 1$, $a = 26$ if $f(v_0) = 4$ and $a = 28$ if $f(v_0) = 7$.

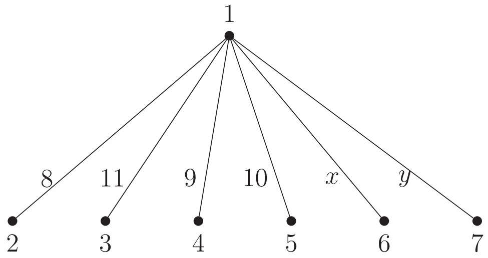

Figure 1. There is no possible to obtain P_3-weight 27.

Case (i): $f(v_0) = 1$. Then $a = 24$ and the P_3 — weights of S_6 are given by $W = \{24, 25, \ldots, 38\}$. Now, the P_3 — weight 24 is getting exactly two possible 5 elements sum $(1, 2, 4, 8, 9)$ or $(1, 2, 3, 8, 10)$ and hence the label of edges $e_1 = v_0v_1$ and $e_2 = v_0v_3$ or v_0v_2 is $f(e_1) = 8$ and $f(e_2) = 9$ or 10.

Subcase (i): $f(e_2 = v_0v_3) = 9$. Then $a = 24$ and hence the label of the vertices and edges are $f(v_0) = 1, f(v_1) = 2, f(v_3) = 4, f(e_1 = v_0v_1) = 8$ and $f(e_2 = v_0v_3) = 9$. Now, the P_3 — weight 25 is getting exactly one possible 5 elements sum $(1, 2, 3, 8, 11)$ and hence the label of an edge $e_3 = v_0v_2$ is $f(e_3) = 11$. Also,the P_3 — weight 26 is getting exactly one possible 5 elements sum $(1, 2, 5, 8, 10)$ and hence the label of an edge $e_4 = v_0v_4$ is $f(e_4) = 10$.

Let $x = v_0v_5$ and $y = v_0v_6$ be two edges of S_6 (see Fig. 1). Clearly, the label of the edges x and y is $f(x) = 12$ or 13 and $f(y) = 13$ or 12. If $f(x) = 12$ then $f(y) = 13$ and hence there is no P_3 — weight 27. Also, if $f(x) = 13$ then $f(y) = 12$ and hence there is no P_3 — weight 27, which is a contradiction.

A similar contradiction arises, if the edges $e_1 = v_0v_1$ and $e_2 = v_0v_2$ with $f(e_1 = 9)$ and $f(e_2) = 8$ for the P_3 — weight 24 is used to getting the P_3 — weight 27.

Subcase (ii): $f(e_2 = v_0v_2) = 10$. Then $a = 24$ and hence the label of the vertices and edges of P_3 — weight 24 is $f(v_0) = 1, f(v_1) = 2, f(v_2) = 3, f(e_1 = v_0v_1) = 8$ and $f(e_2 = v_0v_2) = 10$. Now, the P_3 — weight 25 is getting exactly one possible 5 elements sum $(1, 2, 5, 8, 9)$ and hence the label of an edge $e_3 = v_0v_4$ is $f(e_3) = 9$. Also, the P_3 — weight 26 is getting exactly one possible 5 elements sum $(1, 2, 4, 8, 11)$ and hence the label of an edge $e_4 = v_0v_3$ is $f(e_4) = 11$. Let $x = v_0v_5$ and $y = v_0v_6$ be two edges of S_6 (see Fig. 2). Clearly, the label of the edges x and y is $f(x) = 12$ or 13 and $f(y) = 13$ or 12. If $f(x) = 12$ then $f(y) = 13$ and hence there is no P_3 — weight 27. Also, If $f(x) = 13$ then $f(y) = 12$ and hence there is no P_3 — weight 27, which is a contradiction.

A similar contradiction arises, if the edges $e_1 = v_0v_1$ and $e_2 = v_0v_2$ with $f(e_1) = 10$ and $f(e_2) = 8$ for the P_3 — weight 24 is used to getting the P_3 – weight 27.

Case (ii): $f(v_0) = 7$. Then $a = 28$. Now, if f is a super $(28, 1) - P_3$-antimagic total labeling of S_6, then by Theorem 1 [3], $\bar{f}$ is a super $(24, 1) - P_3$-antimagic total labeling, which does not exist by Case (i).

Case (iii): $f(v_0) = 4$. Then $a = 26$ and hence the P_3 — weights of S_6 are given by $W = \{26, 27, \ldots, 40\}$. Now, the P_3 — weight 26 is getting exactly four possibles 5 elements sum such as $(4, 1, 2, 8, 11)$, $(4, 1, 2, 9, 10)$, $(4, 2, 3, 8, 9)$ and $(4, 1, 3, 8, 10)$ and hence the edges $e_1 = v_0v_1$ or v_0v_2 and $e_2 = v_0v_2$ or v_0v_3 with $f(e_1) = 8$ or 9 and $f(e_2) = 9$ or 10 or 11.

Subcase (i): $f(e_1 = v_0v_1) = 8$ and $f(e_2 = v_0v_2) = 11$. Then $a = 26$ and hence the label of the vertices and edges of P_3 — weight 26 is $f(v_0) = 4, f(v_1) = 1, f(v_2) = 2, f(e_1 = v_0v_1) = 8$ and

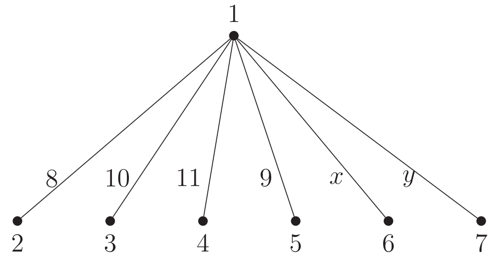

Figure 2. The possible edge labels x and y are obtain P_3-weight 27.

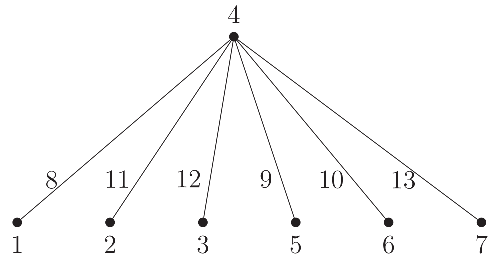

Figure 3. There is no possible to obtain P_3-weight 30.

$f(e_2 = v_0v_2) = 11$. Now, the P_3 — weight 27,28 and 29 are getting exactly one possible 5 elements sum $(4, 1, 5, 8, 9)$,$(4, 1, 3, 8, 12)$ and $(4, 1, 6, 8, 10)$. Hence the label of the edges $e_3 = v_0v_3$, $e_4 = v_0v_4$, $e_5 = v_0v_5$ and $e_6 = v_0v_6$ is $f(e_3) = 12, f(e_4) = 9, f(e_5) = 10$ and $f(e_6) = 13$. From Fig. 3, there is no P_3 — weight is 30, which is a contradiction.

A similar contradiction arises, if the edges e_1 and e_2 with $f(e_1 = v_0v_1) = 11$ and $f(e_2 = v_0v_2) = 8$ for P_3 — weight 26 are used to getting the P_3 — weight 33, for more details see Fig. 4.

Subcase (ii): $f(e_1 = v_0v_1) = 9$ and $f(e_2 = v_0v_2) = 10$. Then $a = 26$ and hence the label of the vertices and edges of P_3 — weight 26 is $f(v_0) = 4, f(v_1) = 1, f(v_2) = 2, f(e_1 = v_0v_1) = 9$ and $f(e_2 = v_0v_2) = 10$. Now, the P_3 — weight 27 is getting exactly two possibles 5 elements sum such as $(4, 2, 3, 10, 8)$, $(4, 1, 5, 9, 8)$ and hence the label of the edges $e_3 = v_0v_3$ or v_0v_4 is $f(e_3) = 8$. If an edge $e_3 = v_0v_3$ with $f(e_3) = 8$ then we get the P_3 — weight as sum of 5 elements $(4, 1, 3, 9, 8)$ is 25, which is a contradiction. If an edge $e_3 = v_0v_4$ with $f(e_3) = 8$ then we get the P_3 — weights from 28 to 32 are getting exactly one possible 5 elements sum such as $(4, 1, 3, 9, 11), (4, 2, 5, 10, 8), (4, 2, 3, 10, 11), (4, 3, 5, 11, 8)$ and $(4, 1, 6, 9, 12)$. From Fig. 5, there is no P_3 — weight 33, which is a contradiction.

A similar contradiction arises, if the edges $e_1 = v_0v_1$ and $e_2 = v_0v_2$ with $f(e_1 = v_0v_1) = 10$ and $f(e_2 = v_0v_2) = 9$ for the P_3 — weight 26 is used to getting the P_3 — weight 27, which is a contradiction.

Subcase (iii): $f(e_1 = v_0v_2) = 8$ and $f(e_2 = v_0v_3) = 9$. Then $a = 26$ and hence the label of the vertices and edges of P_3 — weight 26 is $f(v_0) = 4, f(v_2) = 2, f(v_3) = 3, f(e_1 = v_0v_2) = 8$ and $f(e_2 = v_0v_3) = 9$. Now, the P_3 — weight 27 is getting exactly one possible 5 elements sum $(4, 1, 3, 9, 10)$ and hence the label of an edge $e_3 = v_0v_1$ is $f(e_3) = 10$. Thus, we get a P_3 — weight

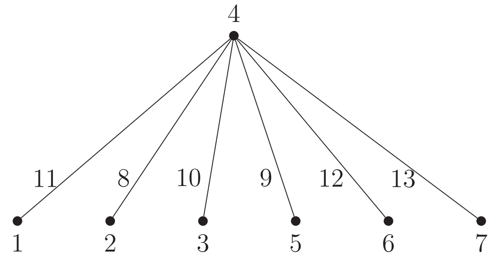

Figure 4. The possible edge label is obtain to P_3-weight 33.

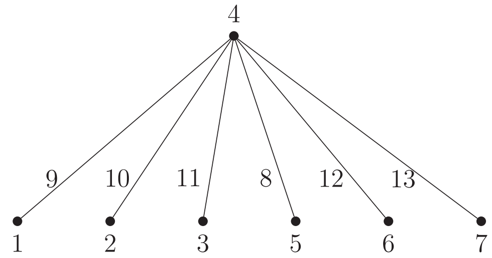

Figure 5. There is no possible to obtain P_3-weight 33.

as sum of 5 elements $(4, 1, 2, 10, 8)$ is 25, which is a contradiction.

A similar contradiction arises, if the edges $e_1 = v_0v_2$ and $e_2 = v_0v_3$ with $f(e_1 = v_0v_2) = 9$ and $f(e_2 = v_0v_3) = 8$ for the P_3 — weight 26. The P_3 — weight 27 is getting exactly one possible 5 elements sum $(4, 1, 2, 11, 9)$ and hence the label of an edge $f(e_3 = v_0v_1) = 11$. Thus, we get the $P_3 = (v_0, v_1, v_3, e_3 = v_0v_1, e_2 = v_0v_3)$ with weight $(4+1+3+11+8)$ is 27, which is a contradiction.

Subcase (iv): $f(e_1 = v_0v_1) = 8$ and $f(e_2 = v_0v_3) = 10$. Then $a = 26$ and hence the label of the vertices and edges of P_3 — weight 26 is $f(v_0) = 4, f(v_1) = 1, f(v_3) = 3, f(e_1 = v_0v_1) = 8$ and $f(e_2 = v_0v_3) = 10$. Now, the P_3 — weight 27 is getting exactly two possibles 5 elements sum such as $(4, 1, 2, 8, 12)$, $(4, 1, 5, 8, 9)$ and hence the label of the edges $e_3 = v_0v_2$ or v_0v_4 is $f(e_3) = 12$ or 9. If an edge $e_3 = v_0v_2$ with $f(e_3) = 12$ then the P_3 — weights 28 and 29 are getting exactly one possible 5 elements sum $(4, 1, 6, 8, 9)$ and $(4, 1, 5, 8, 11)$. From Fig. 6, there is no P_3 — weight 30, which is a contradiction. If an edge $e_4 = v_0v_4$ with $f(e_4) = 9$ then the P_3 — weight 28 is getting exactly one possible 5 elements sum $(4, 1, 2, 8, 13)$ and hence the label of an edge $e_5 = v_0v_2$ is $f(e_5) = 13$. From Fig. 7, there is no P_3 — weight 29 when $x = 11$ or 12 and $y = 12$ or 11, which is a contradiction.

A similar contradiction arises, if the edges $e_1 = v_0v_1$ and $e_2 = v_0v_3$ with $f(e_1 = v_0v_1) = 10$ and $f(e_2 = v_0v_3) = 8$ for the P_3 — weight 26 are used to getting the P_3 — weight 27, which is a contradiction. □

Theorem 5. *The star S_7 has no super $(a, 1) - P_3$-antimagic total labeling.*

P r o o f. Let $V(S_7) = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and $E(S_7) = \{v_0v_1, v_0v_2, v_0v_3, v_0v_4, v_0v_5, v_0v_6, v_0v_7\}$ be the vertex and edge set of star S_7. Suppose there exists a super $(a, 1) - P_3$-antimagic total labeling $f : V \cup E \to \{1, 2, 3, \ldots, 15\}$ for S_7 and let v_0 be the central vertex of S_7. In the

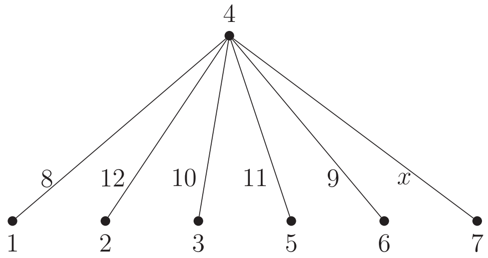

Figure 6. There is no possible to obtain P_3-weight 30.

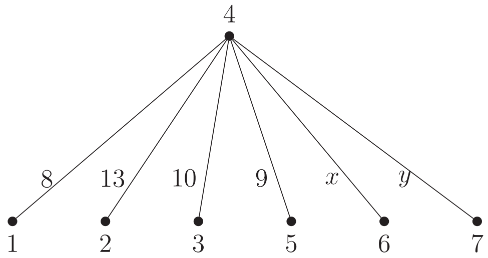

Figure 7. There is no possible to obtain P_3-weight 29.

computation of P_3 — weights the label of the central vertex v_0, $f(v_0)$ is used 21 times and label of other vertices and edges say i are used 6 times each. Therefore,

$$15f(v_0)+6\sum_{i=1}^{15}(i)=\frac{21}{2}[2a+20],$$

which implies that we get

$$a=\frac{15f(v_0)+510}{21}.$$

Since $1\leq f(v_0)\leq 8$, we have only two values a such as $a=25$ if $f(v_0)=1$ and $a=30$ if $f(v_0)=8$.

Case (i): $f(v_0)=1$. Then $a=25$ and the P_3 — weights of S_7 is given by $W=\{25,26,\ldots,45\}$. Now, the P_3 — weight 25 is getting exactly one possible 5 elements sum $(1,2,3,9,10)$ and hence the label of edges $e_1=v_0v_1$ and $e_2=v_0v_2$ is $f(e_1)=9$ and $f(e_2)=10$. Since the minimum possible sum of vertices labels for P_3 — weight is 7, it follows that there is no P_3 — weight 26, which is a contradiction. A similar contradiction arises, if the edges $e_1=v_0v_1$ and $e_2=v_0v_2$ with $f(e_1)=10$ and $f(e_2)=9$ for the P_3 — weight 25 is used to getting the P_3 — weight 27.

Case (ii): $f(v_0)=8$. Then $a=30$. Now, if f is a super $(30,1)-P_3$-antimagic total labeling of S_6, then by Theorem 1 [3], $\bar{f}$ is a super $(25,1)-P_3$-antimagic total labeling, which does not exist by Case (i). □

Theorem 6. *The star S_8 has no super $(a,1)-P_3$-antimagic total labeling.*

P r o o f. Let $V(S_8)=\{v_0,v_1,v_2,v_3,v_4,v_5,v_6,v_7,v_8\}$ and $E(S_8)=\{v_0v_1,v_0v_2,v_0v_3,v_0v_4,v_0v_5,v_0v_6,v_0v_7,v_0v_8\}$ be the vertex and edge set of star S_8. Suppose there exists a super $(a,1)-P_3$-antimagic total labeling $f:V\cup E\to\{1,2,3,\ldots,17\}$ for S_8 and let v_0 be the central vertex of S_8.

In the computation of P_3 — weights the label of the central vertex v_0, $f(v_0)$ is used 28 times and label of other vertices and edges say i are used 7 times each. Therefore,

$$21f(v_0) + 7\sum_{i=1}^{17}(i) = \frac{28}{2}[2a + 27],$$

which implies that we get

$$a = \frac{21f(v_0) + 693}{28}.$$

Since $1 \leq f(v_0) \leq 9$, we have only two values a such as $a = 27$, if $f(v_0) = 3$ and $a = 30$, if $f(v_0) = 7$.

Case (i): $f(v_0) = 3$. Then $a = 27$ and the P_3 — weights of S_8 is given by $W = \{27, 28, \ldots, 54\}$. Now, the P_3 — weight 27 is getting exactly one possible 5 elements sum $(3, 1, 2, 10, 11)$ and hence the label of edges $e_1 = v_0v_1$ and $e_2 = v_0v_2$ is $f(e_1) = 10$ and $f(e_2) = 11$. Since the minimum possible sum of vertices labels for P_3 — weight is 8, it follows that there is no P_3 — weight 29, which is a contradiction. A similar contradiction arises, if the edges $e_1 = v_0v_1$ and $e_2 = v_0v_2$ with $f(e_1) = 11$ and $f(e_2) = 10$ for the P_3 — weight 27 is used to getting the P_3 — weight 29.

Case (ii) $f(v_0) = 7$ Then $a = 30$. Now, if f is a super $(30, 1) - P_3$-antimagic total labeling of S_6, then by Theorem 1 [3], $\bar{f}$ is a super $(27, 1) - P_3$-antimagic total labeling, which does not exist by Case (i). □

Theorem 7. *The star S_9 has no super $(a, 1) - P_3$-antimagic total labeling.*

P r o o f. Let $V(S_9) = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$ be the vertex set of star S_9. Suppose there exists a super $(a, 1) - P_3$-antimagic total labeling $f : V \cup E \to \{1, 2, 3, \ldots, 19\}$ for S_9 and let v_0 be the central vertex of S_9. In the computation of P_3 — weights the label of the central vertex v_0, $f(v_0)$ is used 36 times and label of other vertices and edges say i are used 8 times each. Therefore,

$$28f(v_0) + 8\sum_{i=1}^{19}(i) = \frac{36}{2}[2a + 35],$$

which implies that we get

$$a = \frac{14f(v_0) + 445}{18}.$$

Since $1 \leq f(v_0) \leq 10$, we have that a is not an integer, which is a contradiction. □

From Theorem 2–3 [5], Theorem 4–7, we get the following result.

Theorem 8. *The star $S_n, n \geq 3$ admits a super $(a, 1) - P_3$-antimagic total labeling if and only if $n = 3$ and 4.*

3. Conclusion and Scope

In [5], they investigated the existence of super (a, d)-P_3-antimagic total labeling of star S_n and posed the Problem 1 [5]. This paper proved the star S_n has no super $(a, 1)$-P_3-antimagic total labeling, where $n = 6, 7, 8, 9$. Therefore, we have entirely solved Problem 1 [5].

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DIFFERENTIAL GAME WITH A LIFELINE FOR THE INERTIAL MOVEMENTS OF PLAYERS

Bahrom T. Samatov[†], **Ulmasjon B. Soyibboev**[††]

Namangan State University,
316 Uychi Str., Namangan, 116019, Uzbekistan

[†]samatov57@inbox.ru, [††]ulmasjonsoyibboev@gmail.com

Abstract: In this paper, we study the well-known problem of Isaacs called the "Lifeline" game when movements of players occur by acceleration vectors, that is, by inertia in Euclidean space. To solve this problem, we investigate the dynamics of the attainability domain of an evader through finding solvability conditions of the pursuit–evasion problems in favor of a pursuer or an evader. Here a pursuit problem is solved by a parallel pursuit strategy. To solve an evasion problem, we propose a strategy for the evader and show that the evasion is possible from given initial positions of players. Note that this work develops and continues studies of Isaacs, Petrosjan, Pshenichnii, Azamov, and others performed for the case of players' movements without inertia.

Keywords: Differential game, Pursuit, Evasion, Acceleration, Strategy, Attainability domain, Lifeline.

1. Introduction

Differential game theory deals with conflict problems in systems expressed by differential equations. As a result of developing Pontryagin's maximum principle, it became apparent that there was a link between optimal control theory and differential games. Actually, problems of differential games describe a generalization of optimal control problems in cases where more than one player is involved.

The study of differential games was initiated by American mathematician R. Isaacs. His research was published in the form of a monograph [20] in 1965, in which a great number of examples were considered, and theoretical questions were only affected. Differential games have been one of the basic research fields since 1960, and their fundamental results were gained by Pontryagin [29], Krasovskii [23], Bercovitz [5], Dar'in and Kurzhanskii [9], Elliot and Kalton [10], Isaacs [20], Fleming [11], Friedman [12], Hajek [14], Ho, Bryson and Baron [15], Petrosjan [28], Pshenichnii [30, 31], Subbotin [40, 41], Ushakov [42], Chikrii [8], and others.

The monograph of Isaacs [20] includes certain game problems that were discussed in detail and put forward for further study. One of these problems is called the "Lifeline" problem, which was initially formulated and solved for certain special cases [20, Problem 9.5.1]. A simplified analytical solution to this problem in the half-plane was proposed by Isaacs in [20]. For the case when controls of both players are subjected to geometric constraints, this game was rather thoroughly considered by Petrosjan [28] based on approximating measurable controls with the most efficient piecewise constant controls that realize the parallel convergence strategy. Later, this approach to control in differential pursuit games was termed the Π-strategy. The strategy proposed in [28], for a simple pursuit game with geometric constraints, became the starting point for the development of the pursuit method in games with multiple pursuers [6, 13, 39]. In 1986, a simplified analytical solution of the "Lifeline" problem was suggested by A. Azamov [3]. Later on, for the cases when controls of both players are subjected to integral, Grönwall or mixed constraints, this game was investigated in the works of Samatov [4, 32–37].

Based on the fundamental approaches in the theory of differential games developed by Pontryagin [29] and Krasovskii [23], a differential game is considered as a control problem from the point of view of either pursuer or evader. According to this view, the game reduces to either pursuit (convergence) problem or evasion (escape) problem. The main technique for solving the pursuit and evasion problems is constructing optimal strategies of players and defining the value of the game. The works [16–19, 21] are devoted to studying differential games of simple motion and by optimal strategies of players, it was proved that the value of the game exists.

In [24, 25], the classic time-optimal differential games with a lifeline were investigated. The first player seeks to lead the system to a given closed terminal set with a smooth boundary, and the second player strives to guide the same system to another given set whose boundary is also smooth and is also called a lifeline. To solve this problem, the authors adhered to the formalization of positional differential games proposed by N.N. Krasovskii and A.I. Subbotin.

In the present paper, we discuss the pursuit–evasion problems and the "Lifeline" game for the inertial movements of players. We impose geometric constraints on controls of the players. In order to solve a pursuit problem, we suggest the Π-strategy for the pursuer and prove that this is an optimal strategy. After that, necessary and sufficient condition of pursuit is originated and optimal pursuit time (guaranteed pursuit time) is determined. To solve an evasion problem, we propose a strategy for the evader and show that the evasion is possible from given initial positions of the players. Here, any closed set given in the space is considered as a lifeline. In this case, the first player (a pursuer) aims to coincide with the second player (an evader) as quickly as possible and, by this occurrence, a trajectory of the evader shouldn't intersect the lifeline. The aim of the evader is to reach the lifeline by the time of the coincidence or is not to encounter the pursuer during the game. To solve the "Lifeline" problem, conditions of monotone embedding in respect to time for an attainability domain are given. In this paper, the statement and solution method of a differential game with a lifeline differ significantly from those from the works [24, 25]. Results of the paper rely on the works [1, 7, 22, 26–29, 42, 43] and adjoin the works [2–4, 20, 38, 41].

2. Statement of problem

Assume that in the space $\mathbb{R}^n$ a controlled object P, called a pursuer, chases another object E, called an evader. Denote by x a state of the pursuer and by y that of the evader in $\mathbb{R}^n$.

Let the motion dynamics of the players be generated by the following differential equations and initial conditions respectively:

$$P: \ \ddot{x} = u, \quad x(0) = x_0, \quad \dot{x}(0) = x_1, \tag{2.1}$$

$$E: \ \ddot{y} = v, \quad y(0) = y_0, \quad \dot{y}(0) = y_1, \tag{2.2}$$

where $x, y, x_0, y_0, x_1, y_1, u, v \in \mathbb{R}^n$, $n \geq 2$; x_0 and y_0 are initial states of the players, and x_1 and y_1 are their initial velocity vectors, respectively. We suppose that $x_0 \neq y_0$ and $x_1 = y_1$.

Here the temporal variation of u must be a measurable function $u(\cdot) : [0, \infty) \to \mathbb{R}^n$, and we impose a geometrical constraint on this vector-function (briefly, G-constraint) in the form

$$|u(t)| \leq \alpha \quad \text{almost everywhere} \quad t \geq 0, \tag{2.3}$$

where α is a given positive parametric number, which means the maximal acceleration value of the pursuer.

In a similar way, the temporal variation of v must be a measurable function $v(\cdot) : [0, \infty) \to \mathbb{R}^n$, and we impose on this vector-function a G-constraint in the form

$$|v(t)| \leq \beta, \quad \text{almost everywhere} \quad t \geq 0, \tag{2.4}$$

where β is a given non-negative parametric number, which means the maximal acceleration value of the evader.

The control parameters of the players are the acceleration vectors u and v, which depend on time $t \geq 0$. We denote by U and V the set of all control parameters u and v satisfying conditions (2.3) and (2.4), respectively.

Definition 1. *We call the measurable function $u(\cdot)$ $(v(\cdot))$ that satisfies the condition (2.3) (the condition (2.4)) an admissible control of the pursuer (the evader) of the class U (the class V), where the pair of classes of admissible controls introduced (U, V) defines a differential game.*

By means of the equations (2.1) and (2.2), each triplet $(x_0, x_1, u(\cdot))$ and $(y_0, y_1, v(\cdot))$ generates the trajectories of motion

$$x(t) = x_0 + x_1 t + \int_0^t (t-s)u(s)ds, \tag{2.5}$$

$$y(t) = y_0 + y_1 t + \int_0^t (t-s)v(s)ds \tag{2.6}$$

of the pursuer and evader respectively.

Suppose that a closed subset M called a lifeline is given in the space $\mathbb{R}^n$. The main goal of a pursuer P is to catch an evader E, that is, to achieve the equality $x(t^*) = y(t^*)$ at some time t^*, $t^* > 0$, while the evader remains in the zone $\mathbb{R}^n \setminus M$. The goal of the evader is to reach the zone M before being caught by the pursuer or to maintain the relation $x(t) \neq y(t)$ for all $t,\ t \geq 0$. We should note that M does not restrict the motion of the pursuer. Further, it is assumed that the initial states x_0 and y_0 are given under the conditions $x_0 \neq y_0$ and $y_0 \notin M$.

It is known that control functions depending only on the time parameter t, $t \geq 0$,, are insufficient for the pursuer to solve the pursuit problem, and suitable types of control must be strategies. There are several methods for defining such a concept. Below we will give some concepts to define.

First, we introduce the following notation:

$$z(t) = x(t) - y(t), \quad z_0 = x_0 - y_0, \quad \dot{z}(0) = x_1 - y_1.$$

Definition 2. *A function $\boldsymbol{u} : \mathbb{R}^n \times S_\beta \to S_\alpha$ is called a strategy of the pursuer if the following conditions are valid.*

- (*a*) *$\boldsymbol{u}(z_0, v)$ is a Borel measurable function in v, $v \in V$.*
- (*b*) *Admissibility: The inclusion $\boldsymbol{u}(z_0, v(\cdot)) \in U$ holds for each $v(\cdot) \in V$ on some time interval $[0, t]$. In this case, the function $\boldsymbol{u}(z_0, v(\cdot))$ is called a realization of the strategy $\boldsymbol{u}(z_0, v)$.*
- (*c*) *Volterra property: If $v_1(s) = v_2(s)$ almost everywhere on $[0, t]$ for every $v_1(\cdot)$, $v_2(\cdot) \in V$, and $t,\ t \geq 0$, then $u_1(s) = u_2(s)$ almost everywhere on $[0, t]$, where $u_i(\cdot) = \boldsymbol{u}(z_0, v_i(\cdot))$, $i = 1,\ 2$; S_α and S_β are the balls with radii α and β, respectively, centered at the origin of the space $\mathbb{R}^n$.*

Definition 3. *A strategy $\boldsymbol{u} = \boldsymbol{u}(z_0, v)$ is called a parallel pursuit strategy, or Π-strategy if, for any $v(\cdot) \in V$, a solution $z(t)$ of the Cauchy problem*

$$\ddot{z} = \boldsymbol{u}(z_0, v(t)) - v(t), \quad z(0) = z_0, \quad \dot{z}(0) = 0 \tag{2.7}$$

can be expressed as

$$z(t) = z_0\Lambda(t, v(\cdot)), \quad \Lambda(0, v(\cdot)) = 1,$$

where $\Lambda(t, v(\cdot))$ is a scalar function of t, $t \geq 0$. Usually, this function is called a convergence function in the pursuit problem.

Definition 4. *In the pursuit problem, it is said that a Π-strategy guarantees catching an evader on the time interval $[0, t_g]$ if, for every $v(\cdot) \in V$,*

(*a*) *there exists some time t^*, $t^* \in [0, t_g]$ that generates the equality $z(t^*) = 0$;*

(*b*) *the inclusion $\boldsymbol{u}(z_0, v(\cdot)) \in U$ is satisfied on the time interval $[0, t^*]$.*

Here, the number t_g is called a guaranteed pursuit (or capture) time.

Definition 5. *We call the function $\boldsymbol{v}^* : \mathbb{R}_+ \to \mathbb{R}^n$ a strategy of the evader if $\boldsymbol{v}^*(t)$ is a Lebesgue measurable function in t.*

Now we will consider the game (U, V) from the standpoint of an evader.

Definition 6. *In the evasion problem, it is said that a control $\boldsymbol{v}^*(\cdot) \in V$ guarantees escaping if, for any $u(\cdot) \in U$, a solution $z(t)$ of the Cauchy problem*

$$\ddot{z} = u(t) - \boldsymbol{v}^*(t), \quad z(0) = z_0, \quad \dot{z}(0) = 0 \tag{2.8}$$

is nonzero, that is, $z(t) \neq 0$ for all $t \geq 0$.

This paper is dedicated to solving the following problems when the controls of the players are subject to constraints (2.3) and (2.4), respectively.

Problem 1. Pursuit problem: Construct a Π-strategy of the pursuer and find the guaranteed capture time in the game (U, V).

Problem 2. Evasion problem: Construct a strategy of the evader and evaluate how to vary distance between the players.

Problem 3. Solve the “Lifeline” game.

3. A solution of the pursuit problem

In a great number of mathematical problems with parameters, an interesting property of the final analytic results is their explicit dependence on these parameters, which are regarded as constants in the solution. However, these parameters can help to determine feasibility conditions for these problems. In this section, we are going to present necessary and sufficient feasibility conditions for the pursuit problem in the game (U, V).

If the pursuer and evader choose admissible controls $u(\cdot) \in U$ and $v(\cdot) \in V$, respectively, then, depending on equation (2.7), we obtain the solution

$$z(t) = z_0 + \int_0^t (t - s)(u(s) - v(s))ds. \tag{3.1}$$

In the new notation introduced, the goal of the pursuer is now to fulfill the equality $z(t^*) = 0$ at some time t^*, $t^* > 0$. As for evader’s goal, it is to maintain the relation $z(t) \neq 0$ for all $t \geq 0$.

For the pursuer, it is not enough to achieve his goal only by program strategies, i.e., admissible controls depending only on time t. Therefore, similarly to [4], in the case under consideration, the strategy of the pursuer can also be determined depending only on the current states of the acceleration function $v(t)$, $t \geq 0$, and given constants z_0 and α.

For solving the pursuit problem, suppose that at the current time t, the pursuer is aware of the initial parameters x_0, y_0, x_1, y_1, and the constants α, β, the current time t, and the value of evader's control $v(t)$.

Definition 7. *Assume that $\alpha \geq \beta$. Then, in the game (U, V), we call the function*

$$\boldsymbol{u}(z_0, v) = v - \lambda(z_0, v)\xi_0, \tag{3.2}$$

a parallel pursuit strategy (briefly, Π-strategy) of the pursuer, where

$$\lambda(z_0, v) = \langle v, \xi_0 \rangle + \sqrt{\langle v, \xi_0 \rangle^2 + \alpha^2 - |v|^2}, \quad \xi_0 = z_0/|z_0|, \tag{3.3}$$

and $\langle v, \xi_0 \rangle$ is the scalar product of the vectors v and ξ_0 in $\mathbb{R}^n$. The function $\lambda(z_0, v)$ is usually called a resolving function.

Now we will indicate some important features for the strategy (3.2) and the resolving function (3.3).

Lemma 1. *The strategy (3.2) is defined and continuous for all $v \in S_\beta$, and the equality $|\boldsymbol{u}(z_0, v)| = \alpha$ holds during the pursuit game.*

Lemma 2. *The resolving function (3.3) is defined, continuous, and non-negative for all $v \in S_\beta$, and this function is bounded as follows:*

$$\alpha - \beta \leq \lambda(z_0, v) \leq \alpha + \beta.$$

Definition 8. *If $\alpha > \beta$, then the scalar function*

$$\Lambda(t, v(\cdot)) = 1 - \frac{1}{|z_0|} \int_0^t (t - s)\lambda(z_0, v(s))ds \tag{3.4}$$

is called a convergence function of the players in the game (U, V).

Lemma 3. *Let $\alpha > \beta$. Then*

(*a*) *for all $v(\cdot) \in V$, the function (3.4) is monotone decreasing with respect to t, $t \geq 0$;*

(*b*) *the function (3.4) is bounded for all $t \in [0, t_g]$ as follows:*

$$\Lambda_1(t) \leq \Lambda(t, v(\cdot)) \leq \Lambda_2(t), \tag{3.5}$$

where

$$\Lambda_1(t) = 1 - \frac{t^2}{2|z_0|}(\alpha + \beta), \quad \Lambda_2(t) = 1 - \frac{t^2}{2|z_0|}(\alpha - \beta).$$

P r o o f. (a) According to Lemma 2, it follows that

$$\frac{d\Lambda(t, v(\cdot))}{dt} = -\frac{1}{|z_0|}\int_0^t \lambda(z_0, v(s))ds \le -\frac{t}{|z_0|}(\alpha - \beta) < 0.$$

(b) Relying on the minimum lemma in the elementary optimal control problem [1, p. 360], we get the following estimation:

$$\Lambda(t, v(\cdot)) \le 1 - \frac{1}{|z_0|}\min_{v(\cdot)\in V}\int_0^t (t-s)\lambda(z_0, v(s))ds \le 1 - \frac{t^2}{2|z_0|}\min_{|v|\le\beta}\lambda(z_0, v) = \Lambda_2(t).$$

On the other hand, by Lemma 2, we have

$$\Lambda_1(t) = 1 - \frac{t^2}{2|z_0|}\max_{|v|\le\beta}\lambda(z_0, v) = 1 - \frac{1}{|z_0|}\max_{v(\cdot)\in V}\int_0^t (t-s)\lambda(z_0, v(s))ds \le \Lambda(t, v(\cdot)).$$

This completes the proof. □

Theorem 1. *Let $\alpha > \beta$ in the game (U, V). Then the Π-strategy (3.2) guarantees catching the evader on the time interval $[0, t_g]$, where $t_g = \sqrt{2|z_0|/(\alpha-\beta)}$.*

P r o o f. Assume that the evader chooses some control $v(\cdot) \in V$ while the pursuer implements the Π-strategy (3.2). Then, by (3.1), we obtain the function

$$z(t) = z_0 + \int_0^t (t-s)\lambda(z_0, v(s))\xi_0 ds,$$

which can be written as follows:

$$z(t) = z_0\Lambda(t, v(\cdot)), \tag{3.6}$$

where $\Lambda(t, v(\cdot))$ is the players convergence function of the form (3.4). Taking into account the right-hand side of (3.5), we conclude that the function $\Lambda_2(t)$ is equal to zero at $t = t_g$. Therefore, there exists some $t^* \in [0, t_g]$ such that $\Lambda(t^*, v(\cdot)) = 0$, and this (see (3.6)) results in $z(t^*) = 0$. Theorem 1 is proved. □

4. A solution of the evasion problem

In this section, we will suggest an admissible strategy for the evader, which guarantees escaping in the evasion problem. Using this strategy, we will prove that the strategy (3.2) is an optimal pursuit strategy and t_g is an optimal pursuit time.

Definition 9. *We call the control function*

$$\boldsymbol{v}^*(t) = -\beta\xi_0 \tag{4.1}$$

a strategy of the evader in the game (U, V).

Definition 10. *It is said that the strategy $\boldsymbol{v}^*(t)$ guarantees escaping on the time interval $[0, t_g)$ if for any control function of the pursuer $u(\cdot) \in U$, the relation $z(t) \ne 0$ is valid for all $t \in [0, t_g)$, where $z(t)$ is the solution of the Cauchy problem (2.8).*

Theorem 2. *(a) Let $\alpha > \beta$. Then the strategy (4.1) guarantees escaping on the time interval $[0, t_g)$ in the game (U, V), where t_g is the guaranteed pursuit time (see Theorem 1).*

(b) Let $\alpha \le \beta$. Then the strategy (4.1) guarantees escaping on the time interval $[0, +\infty)$ in the game (U, V), and the distance between the players is estimated as follows:

$$|z(t)| \ge \begin{cases} |z_0|, & \text{if} \quad \alpha = \beta, \\ |z_0| - \dfrac{(\alpha-\beta)t^2}{2}, & \text{if} \quad \alpha < \beta. \end{cases}$$

P r o o f. (*a*) Assume that $\alpha > \beta$ and the pursuer picks a control $u(\cdot) \in U$ while the evader applies the strategy (4.1). In accordance with (3.1), it follows the function

$$z(t) = z_0 + \int_0^t (t-s)u(s)ds + \int_0^t (t-s)\beta\xi_0 ds. \tag{4.2}$$

Estimate the absolute value of (4.2) as follows:

$$|z(t)| \ge \left| z_0 + \int_0^t (t-s)\beta\xi_0 ds \right| - \left| \int_0^t (t-s)u(s)ds \right| \ge$$

$$\ge |z_0|\Big(1 - \frac{\beta t^2}{2|z_0|}\Big) - \int_0^t (t-s)\alpha ds = |z_0| - \frac{t^2}{2}(\alpha - \beta).$$

Relying on Theorem 1, we can write the estimation

$$|z(t)| \ge |z_0| - \frac{(\alpha-\beta)t^2}{2} > 0$$

for all $t,\ 0 \le t < t_g$.

(*b*) Suppose that $\alpha \le \beta$. In this case, a proof is similar to the proof of item (a), i. e.,

$$|z(t)| \ge |z_0| - \frac{(\alpha-\beta)t^2}{2} > |z_0| > 0$$

for all $t \in [0, +\infty)$. Theorem 2 is proved. □

5. An attainability domain of the pursuer

In accordance with Theorem 1, if $\alpha > \beta$ then, by the Π-strategy (3.2), the evader is captured at some point in the space $\mathbb{R}^n$. In the considered game, we will construct a set of meeting points of the players for the case $\alpha > \beta$.

Let a triplet $(y_0, y_1, v(\cdot))$, $v(\cdot) \in V$, generates a trajectory of an evader E in the form (2.6) while a triplet $(x_0, x_1, \mathbf{u}(z_0, v(\cdot)))$, $\mathbf{u}(z_0, v(\cdot)) \in U$, generates a trajectory of a pursuer P in the form

$$x(t) = x_0 + x_1 t + \int_0^t (t-s)\mathbf{u}(z_0, v(s))ds, \tag{5.1}$$

where $t \in [0, t^*]$, $0 < t^* \le t_g$, and t^* is the encounter time of the players, that is, the equality $x(t^*) = y(t^*)$ holds. Thus, for each pair $(x(t), y(t))$, we form the set

$$W(t) = W(x(t), y(t)) = \big\{\omega : \ |\omega - x(t)| \ge (\alpha/\beta)|\omega - y(t)|\big\}, \tag{5.2}$$

on $[0, t^*]$. Note that

$$W(0) = W(x_0, y_0) = \{\omega : \ |\omega - x_0| \geq (\alpha/\beta)|\omega - y_0|\}.$$

Since $|z(t)| \geq 0$ on $[0, t^*]$, it is obvious that the inclusion $y(t) \in W(t)$ is valid on this time interval.

Remark 1. Note that the trajectories $x(t)$ and $y(t)$ of the players and the multi-valued mapping $W(t)$ directly depend on the choice of a control $v(\cdot) \in V$. This dependence is omitted for brevity.

Lemma 4. *The multi-valued mapping $W(t)$ can be expressed as*

$$W(t) = x(t) + \Lambda(t, v(\cdot))[W(0) - x_0], \tag{5.3}$$

where $\Lambda(t, v(\cdot))$ is the convergence function of the form (3.4) and

$$W(0) = x_0 - c(z_0) + R(z_0)S, \quad c(z_0) = \frac{\alpha^2 z_0}{\alpha^2 - \beta^2}, \quad R(z_0) = \frac{\alpha\beta|z_0|}{\alpha^2 - \beta^2}, \tag{5.4}$$

and S is the unit ball centered at the origin of the space $\mathbb{R}^n$.

P r o o f. First, write the set (5.2) as follows:

$$W(t) = x(t) + \{\omega : \ |\omega| \geq (\alpha/\beta)|\omega + z(t)|\} = x(t) - c(z(t)) + R(z(t))S,$$

where

$$c(z(t)) = \frac{\alpha^2 z(t)}{\alpha^2 - \beta^2}, \quad R(z(t)) = \frac{\alpha\beta|z(t)|}{\alpha^2 - \beta^2}.$$

Now, by (3.6) and (5.4), the functions $c(z(t))$ and $R(z(t))$ can be written in the form

$$c(z(t)) = c(z_0)\Lambda(t, v(\cdot)), \ R(z(t)) = R(z_0)\Lambda(t, v(\cdot)).$$

Hence, we derive the validity of formula (5.3). □

Corollary 1. *Lemma 4 implies that, for each $t \in [0, t^*]$, the multi-valued mapping $W(t)$ is a ball of radius $R(z_0)\Lambda(t, v(\cdot))$ centered at the point $x(t) - c(z_0)\Lambda(t, v(\cdot))$ and the set $W(0)$ is a ball of the radius $R(z_0)$ centered at the point $x(0) - c(z_0)$.*

Lemma 5 (The main lemma). *The multi-valued mapping $W(t) - tx_1$ is monotone decreasing in $t \in [0, t^*]$ with respect to embedding, i.e., if $t_1, t_2 \in [0, t^*]$ and $t_1 < t_2$, then*

$$W(t_2) - t_2 x_1 \subset W(t_1) - t_1 x_1.$$

P r o o f. From the statement of the problem, we have geometric constraint of the form (2.4) on values of the evader's acceleration vector. As a consequence, we find that

$$|v(t)|^2 \leq \frac{\beta^2}{\alpha^2 - \beta^2}\left(\alpha^2 - |v(t)|^2\right). \tag{5.5}$$

From the form of the resolving function (3.3), it is easy to ensure in the validity of the equality

$$\alpha^2 - |v(t)|^2 = \lambda(z_0, v(t))\left(\lambda(z_0, v(t)) - 2\langle v(t), \xi_0\rangle\right).$$

Due to this, inequality (5.5) takes the form

$$|v(t)|^2 + \frac{2\beta^2}{\alpha^2-\beta^2}\langle v(t),\xi_0\rangle\lambda(z_0,v(t)) \le \frac{\beta^2}{\alpha^2-\beta^2}\lambda^2(z_0,v(t)).$$

Completing the square in this inequality, we obtain the following inequality:

$$\left|v(t)+\frac{\beta^2}{\alpha^2-\beta^2}\lambda(z_0,v(t))\xi_0\right| \le \frac{\alpha\beta}{\alpha^2-\beta^2}\lambda(z_0,v(t)). \tag{5.6}$$

However, for all $\psi\in\mathbb{R}^n$, $|\psi|=1$, the relation

$$\left\langle v(t)+\frac{\beta^2}{\alpha^2-\beta^2}\lambda(z_0,v(t))\xi_0,\ \psi\right\rangle \le \left|v(t)+\frac{\beta^2}{\alpha^2-\beta^2}\lambda(z_0,v(t))\xi_0\right|$$

holds. Then, using the inequality (5.6), we find that

$$\left\langle v(t)+\frac{\beta^2}{\alpha^2-\beta^2}\lambda(z_0,v(t))\xi_0,\ \psi\right\rangle \le \frac{\alpha\beta}{\alpha^2-\beta^2}\lambda(z_0,v(t)).$$

Integrate both sides of this inequality over the interval $[0,t]$:

$$\int_0^t \left\langle v(s)+\frac{\beta^2}{\alpha^2-\beta^2}\lambda(z_0,v(s))\xi_0,\ \psi\right\rangle ds \le \frac{\alpha\beta}{\alpha^2-\beta^2}\int_0^t \lambda(z_0,v(s))ds. \tag{5.7}$$

For the left-hand side of (5.7), we can write the following equalities:

$$\int_0^t \left\langle v(s)+\frac{\beta^2}{\alpha^2-\beta^2}\lambda(z_0,v(s))\xi_0,\ \psi\right\rangle ds = \int_0^t \left\langle v(s)+(\frac{\alpha^2}{\alpha^2-\beta^2}-1)\lambda(z_0,v(s))\xi_0,\ \psi\right\rangle ds =$$

$$= \int_0^t \left\langle v(s)-\lambda(z_0,v(s))\xi_0,\ \psi\right\rangle ds + \left\langle\frac{\alpha^2}{\alpha^2-\beta^2}\xi_0,\ \psi\right\rangle\int_0^t \lambda(z_0,v(s))ds.$$

By the definition of Π-strategy (3.2) and in view of the form of the vector $c(z_0)$ in (5.4), the latter equality takes the form

$$\begin{aligned}\int_0^t \left\langle v(s)+\frac{\beta^2}{\alpha^2-\beta^2}\lambda(z_0,v(s))\xi_0,\ \psi\right\rangle ds &= \left\langle\int_0^t \mathbf{u}(z_0,v(s))ds,\ \psi\right\rangle+\\ &+\frac{\langle c(z_0),\ \psi\rangle}{|z_0|}\int_0^t \lambda(z_0,v(s))ds.\end{aligned} \tag{5.8}$$

Taking into account the form of $R(z_0)$ in (5.4), for the right-hand side of inequality (5.7), we obtain

$$\frac{\alpha\beta}{\alpha^2-\beta^2}\int_0^t \lambda(z_0,v(s))ds = \frac{R(z_0)}{|z_0|}\int_0^t \lambda(z_0,v(s))ds. \tag{5.9}$$

Thus, by (5.7), (5.8), and (5.9), we have the relation

$$\left\langle\int_0^t \mathbf{u}(z_0,v(s))ds,\ \psi\right\rangle + \frac{\langle c(z_0),\ \psi\rangle - R(z_0)}{|z_0|}\int_0^t \lambda(z_0,v(s))ds \le 0. \tag{5.10}$$

Applying the properties of the support function [7]

$$F(W, \psi) = \sup_{\omega \in W} \langle \omega, \psi \rangle$$

in $\psi \in \mathbb{R}^n$, $|\psi| = 1$, and using formulas (3.4), (5.1), (5.3), and (5.4) we can find the derivative of $W(t)$ in t as follows:

$$\frac{d}{dt}F(W(t), \psi) = \frac{d}{dt}F\Big(x_0 + x_1 t + \int_0^t (t-s)\mathbf{u}(z_0, v(s))ds + \Lambda(t, v(\cdot))\,[R(z_0)S - c(z_0)]\,,\ \psi\Big) =$$

$$= \langle x_1, \psi\rangle + \Big\langle \int_0^t \mathbf{u}(z_0, v(s))ds,\ \psi\Big\rangle + \frac{\langle c(z_0),\ \psi\rangle - R(z_0)}{|z_0|}\int_0^t \lambda(z_0, v(s))ds.$$

From this and inequality (5.10), we get $\frac{d}{dt}F(W(t) - tx_1, \psi) \le 0$ for all $\psi \in \mathbb{R}^n$, $|\psi| = 1$. This completes the proof of Lemma 5. □

Corollary 2. *Lemma 5 implies the inclusion $W(t) \subset W(0) + tx_1$ for all $t \in [0, t^*]$.*

By Lemma 5, we obtain an attainability domain of the evader.

Lemma 6. *The inclusion*

$$y(t) \in W(0) + tx_1 \tag{5.11}$$

holds for all $t \in [0, t^*]$.

P r o o f. The inclusion (5.11) easily follows from the form of the multi-valued mapping (5.2) and Corollary 2. □

Corollary 3. *Lemma 6 implies that, if the initial velocities of the players are equal to zero, i.e., $x_1 = y_1 = 0$, then the boundary of the attainability set of the evader is the Apollonius sphere of the form (5.4) in $\mathbb{R}^n$.*

To solve the "Lifeline" game in favor of the pursuer, using (5.11), one can define the set

$$W_P = \bigcup_{t=0}^{t_g} \{W(0) + tx_1\},$$

which is obviously convex and closed. Here

$$t_g = \sqrt{2|z_0|/(\alpha - \beta)}$$

(see Theorem 1). We call the set W_P the attainability domain of the pursuer.

6. A solution of the "Lifeline" game

In this section, the "Lifeline" game will be considered only in the case $\alpha > \beta$ and $x_1 = y_1$. Hereinafter, we will omit these conditions in statements for brevity.

Definition 11. *We say that the Π-strategy (3.2) provides winning for the pursuer in the "Lifeline" game on the time interval $[0, t_g]$ if there exists a time $t^* \in [0, t_g]$ such that the following conditions are valid:*

(*a*) $x(t^*) = y(t^*)$;

(*b*) $y(s) \notin M$ *for all* $s \in [0, t^*]$.

Theorem 3. *If the attainability domain* W_P *of the pursuer does not intersect the set* M*, i.e.,* $W_P \cap M = \emptyset$ *in the "Lifeline" game, then the* Π*-strategy* (3.2) *provides winning for the pursuer* P*.*

P r o o f follows immediately from Theorem 1, Lemma 5, and Lemma 6. □

Now let us consider a solution of the "Lifeline" game in favor of the evader E.

Definition 12. *It is said that a control* $v^*(\cdot) \in V$ *provides winning for the evader in the "Lifeline" game if, for any control of the pursuer* $u(\cdot) \in U$*, at least one of the following conditions holds*:

a) *there exists a finite time* τ *satisfying the inclusion* $y(\tau) \in M$ *and the relation* $x(t) \neq y(t)$ *for all* $t \in [0, \tau)$*;*

b) $x(t) \neq y(t)$ *for all* $t \geq 0$*.*

Consider the set

$$W_E = \Big\{\omega^* : \omega^* = \sqrt{2|\omega - y_0|/\beta}\, x_1 + \omega,\ \omega \in W(0)\Big\}.$$

We call the set W_E the attainability domain of the evader.

Theorem 4. *If the attainability domain* W_E *of the evader intersects the set* M*, i.e.,* $W_P \cap M \neq \emptyset$ *in the "Lifeline" game, then there exists a control* $v^*(\cdot) \in V$ *that provides winning for the evader* E*.*

P r o o f. Depending on the theorem conditions, there exists a point $\omega^* \in W_E \cap M$ such that the following equality holds:

$$\omega^* = \sqrt{2|\omega - y_0|/\beta}\, x_1 + \omega, \quad \omega \in W(0).$$

Then we prescribe the constant control of the form

$$v^* = \beta(\omega - y_0)/|\omega - y_0| \tag{6.1}$$

for the evader E.

First of all, let us show that, by means of the control (6.1), the evader E reaches the chosen point w^* at the time $\tau = \sqrt{2|\omega - y_0|/\beta}$. To this end, substituting (6.1) into (2.6), we form

$$y(\tau) = y_0 + y_1\tau + \int\limits_0^\tau (\tau - s)v^* ds = y_0 + y_1\tau + \frac{\tau^2}{2}v^*. \tag{6.2}$$

From (6.1) and (6.2) and taking into account the equality $x_1 = y_1$, we obtain

$$y(\tau) = y_0 + x_1\sqrt{2|\omega - y_0|/\beta} + \omega - y_0 = \omega^*.$$

Now let us prove that condition (*a*) of Definition 12 holds, that is, the evader remains uncaught. Suppose the opposite, i.e., that there is some control $u^*(\cdot) \in U$ of the pursuer, the implementation of which satisfies the equality $x(t) = y(t)$ at time $\tilde{t}$ less than τ, i.e., $\tilde{t} < \tau$. Then, by equation (3.1), we can write

$$z(\tilde{t}) = z_0 + \int_0^{\tilde{t}} (\tilde{t} - s)(u^*(s) - v^*)ds = 0.$$

Thus, for the right-hand side of $z(\tilde{t})$, we can write the absolute estimations:

$$\left| z_0 - \int_0^{\tilde{t}} (\tilde{t} - s)v^* ds \right| \leq \int_0^{\tilde{t}} (\tilde{t} - s)|u^*(s)|ds \leq \alpha \frac{\tilde{t}^2}{2}$$

or

$$\left| z_0 - \frac{\tilde{t}^2}{2} v^* \right| \leq \alpha \frac{\tilde{t}^2}{2}. \tag{6.3}$$

Introducing the notation $\eta = \tilde{t}^2/2$ in (6.3) and taking into account that $|v^*| = \beta$, we obtain the quadratic inequality in terms of η in the form

$$(\alpha^2 - \beta^2)\eta^2 + 2\eta\langle z_0, v^*\rangle - |z_0|^2 \geq 0.$$

It follows that

$$\eta = \frac{\tilde{t}^2}{2} \geq \frac{1}{\alpha^2 - \beta^2} \left(\sqrt{\langle z_0, v^*\rangle^2 + (\alpha^2 - \beta^2)|z_0|^2} - \langle z_0, v^*\rangle \right). \tag{6.4}$$

By the assumption

$$\frac{\tau^2}{2} = |\omega - y_0|/\beta > \frac{\tilde{t}^2}{2}$$

and (6.4), we determine the relation

$$\frac{|w - y_0|}{\beta} > \frac{1}{\alpha^2 - \beta^2} \left(\sqrt{\langle z_0, v^*\rangle^2 + (\alpha^2 - \beta^2)|z_0|^2} - \langle z_0, v^*\rangle \right). \tag{6.5}$$

According to the control of the evader (6.1) and by inequality (6.5), we get

$$\frac{\alpha}{\beta}|w - y_0| > |w - x_0|,$$

i.e., the inclusion $\omega \in W(0)$ (see (5.2)) does not hold, which contradicts our assumption. Theorem 4 is proved. $\square$

Remark 2. Using the definitions of the attainability domain W_P of the pursuer and the attainability domain W_E of the evader, it is not difficult to ensure that the inclusion $W_E \subset W_P$ is valid in the "Lifeline" game.

7. Examples

Example 1. (Problem for the case with a lifeline). Let the game (2.1)–(2.4) be given as follows:

$$\ddot{x} = u, \quad x_0 = (0, 0), \quad x_1 = (0, 1), \quad |u(t)| \leq \sqrt{2}, \quad t \geq 0, \tag{7.1}$$

$$\ddot{y} = v, \quad y_0 = (0, -1), \quad y_1 = (0, 1), \quad |v(t)| \leq 1, \quad t \geq 0. \tag{7.2}$$

Then, according to Theorem 1, we have $t_g = \sqrt{2\sqrt{2} + 2}$. By Lemma 4, we can write the set

$$W(0) = \{\omega : |\omega - c| \leq R, \ c = (0, -2), R = \sqrt{2}\}.$$

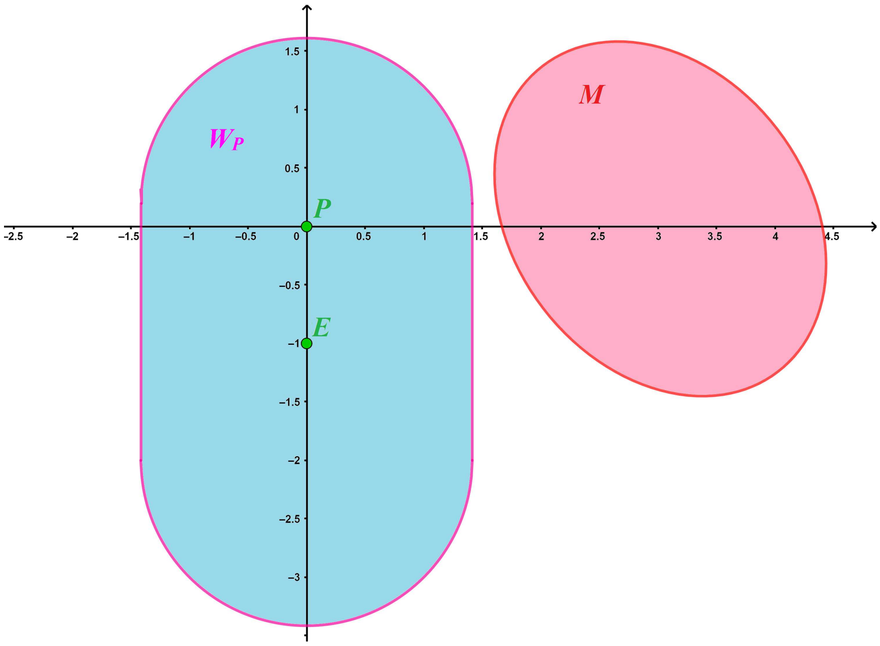

Figure 1. The case where the pursuer P wins in the "Lifeline" game (7.1)–(7.2).

The boundary of $W(0)$ is

$$\partial W(0) = \{\hat{\omega} = (\hat{\omega}_1, \hat{\omega}_2) : \hat{\omega}_1^2 + (\hat{\omega}_2 + 2)^2 = 2\}, \tag{7.3}$$

and the following equality is valid for these points:

$$|\hat{\omega} - y_0| = \sqrt{\hat{\omega}_1^2 + (\hat{\omega}_2 + 1)^2}. \tag{7.4}$$

By (7.3) and (7.4), we obtain the set

$$W_E = \left\{ \omega^* = (\tilde{\omega}_1, \tilde{\omega}_2) : \tilde{\omega}_2 = \sqrt{2\sqrt{3 \pm 2\sqrt{2 - \tilde{\omega}_1^2}}} \pm \sqrt{2 - \tilde{\omega}_1^2} - 2 \right\}.$$

Fig. 1 and 2 show the shapes of the sets W_P and W_E in the "Lifeline" game (7.1)–(7.2).

Example 2. (Attainability domain in the case of many pursuers and one evader). Consider the following game example:

$$\ddot{x}_i = u_i, \quad x_i(0) = x_{i0}, \quad \dot{x}_i(0) = \eta, \quad |u_i(t)| \le \mu_i, \quad t \ge 0, \tag{7.5}$$

$$\ddot{y} = v, \quad y(0) = y_0, \quad \dot{y}(0) = \eta, \quad |v(t)| \le 1, \quad t \ge 0, \tag{7.6}$$

where $\mu_i > 1$ and $x_{i0} \ne y_0$, $i = \overline{1, m}$.

By Lemma 6, we have

$$y(t) \in \bigcap_{i=1}^{m} W_{i0} + t\eta,$$

where

$$W_{i0} = x_{i0} - c_{i0} + R_{i0} S, \quad c_{i0} = \frac{\mu_i^2}{\mu_i^2 - 1} z_{i0}, \quad R_{i0} = \frac{\mu_i}{\mu_i^2 - 1} |z_{i0}|, \quad z_{i0} = x_{i0} - y_0.$$

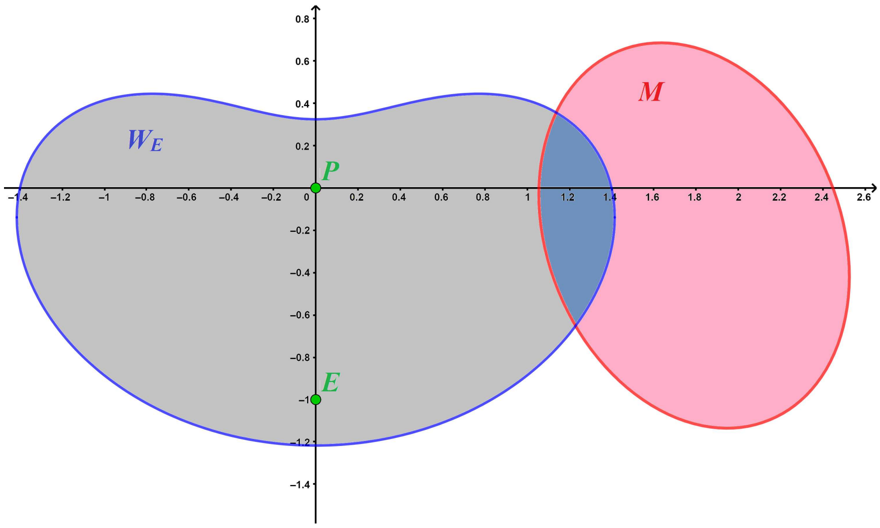

Figure 2. The case where the evader E wins in the "Lifeline" game (7.1)–(7.2).

The attainability domain of the pursuers in the game (7.5)–(7.6) has the form

$$W_P = \bigcup_{t=0}^{T^*} \Big[\bigcap_{i=1}^{m} W_{i0} + t\eta \Big],$$

where

$$T^* = \min_{i=\overline{1,m}} \sqrt{\frac{2|z_{i0}|}{\mu_i - 1}}.$$

8. Conclusion

In this paper, we have considered the pursuit–evasion problems and the "Lifeline" game of one pursuer and one evader for the inertial movements when the initial velocity vectors of the players are the same. We have imposed geometric constraints on the controls of the players. The Π-strategy was suggested for the pursuer and given optimal pursuit time in the pursuit problem. We have proposed a specific strategy for the evader. By this strategy, it was proved that the Π-strategy is optimal and the evasion is possible from the given initial states. As the main result, we have obtained the main lemma (Lemma 5) and applied this lemma to solve the "Lifeline" game.

A "Lifeline" game of many players, when geometric constraints are imposed on controls of players, can be studied in further research.

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CARLEMAN'S FORMULA OF A SOLUTIONS OF THE POISSON EQUATION IN BOUNDED DOMAIN

Ermamat N. Sattorov[†], **Zuxro E. Ermamatova**[††]

Samarkand State University,
Samarkand boulevard 15, Samarkand, Uzbekistan
[†]Sattorov-e@rambler.ru, [††]zuxroermamatova@rambler.ru

Abstract: We suggest an explicit continuation formula for a solution to the Cauchy problem for the Poisson equation in a domain from its values and values of its normal derivative on a part of the boundary. We construct the continuation formula of this problem based on the Carleman–Yarmuhamedov function method.

Keywords: Poisson equations, Ill-posed problem, Regular solution, Carleman–Yarmuhamedov function, Green's formula, Carleman formula, Mittag–Leffler entire function.

1. Introduction

In this paper, we continue the research provided in [12]. We propose an explicit formula for the reconstruction of a solution of the Poisson equation in a bounded domain from its values and the values of its normal derivative on a part of the boundary, i.e., we give an explicit continuation formula for a solution to the Cauchy problem for the Poisson equation.

Let us introduce the following notation: $\mathbb{R}^3$ is a three-dimensional real Euclidean space,

$$x=(x_1,x_2,x_3),\quad y=(y_1,y_2,y_3)\in\mathbb{R}^3,$$
$$x'=(x_1,x_2),\quad y'=(y_1,y_2)\in\mathbb{R}^2,$$
$$s=\alpha^2=|y'-x'|^2=(y_1-x_1)^2+(y_2-x_2)^2,$$
$$r^2=s+(y_3-x_3)^2=|y-x|^2,\quad \tau=tg\frac{\pi}{2\rho},\quad \rho>1,$$
$$G_\rho=\{y:|y'|<\tau y_3,\ y_3>0\},\quad \partial G_\rho=\{y:|y'|=\tau y_3,\ y_3>0\},\quad \overline{G}_\rho=G_\rho\cup\partial G_\rho,$$

ε, ε_1, and ε_2 are sufficiently small positive constants,

$$G_\rho^\varepsilon=\{y:|y'|<\tau(y_3-\varepsilon)\},\quad \partial G_\rho^\varepsilon=\{y:|y'|=\tau(y_3-\varepsilon)\},\quad \overline{G}_\rho^\varepsilon=G_\rho^\varepsilon\cup\partial G_\rho^\varepsilon,$$

and Ω_ρ is a bounded simply connected domain whose boundary $\partial\Omega_\rho$ in R^3 consists of a part of the conic surface $T\equiv\partial G_\rho$ and a smooth surface S lying inside the cone $\overline{G}_\rho$. The case $\rho=1$ is the limit case. In this case, G_1 is the half-space $y_3>0, \partial G_1$ is the hyperplane $y_3=0$, and Ω_1 is a bounded simply connected domain whose boundary consists of a compact connected part of the hyperplane $y_3=0$ and a smooth surface S in the half-space $y_3\geq 0$, $\overline{\Omega}_\rho=\Omega_\rho\cup\partial\Omega_\rho$, and S_0 is the interior of S.

The Poisson equation or potential equation [15]

$$-\triangle U(x)\equiv-\sum_{i=1}^{3}\frac{\partial^2 U}{\partial x_i^2}=f(x)\tag{1.1}$$

is a classical example of second-order elliptic partial differential equations and a mathematical model for some important physical phenomena. Let $H^\lambda(\Omega_\rho)$ be the set of real functions of the class

$C^{2,\lambda}(\Omega_\rho) \cap C^1(\overline{\Omega}_\rho)$ satisfying the Poisson equation. Let a function f be Hölder continuous with exponent $\lambda \in (0,1)$, i.e., $f \in C^{s,\lambda}(\overline{\Omega}_\rho)$ and $s \in Z_+$.

Problem 1. *Assume that we know the Cauchy data for a solution to equation (1.1) on the surface S:*

$$U(y) = f_1(y), \quad \frac{\partial U(y)}{\partial n} = f_2(y), \quad y \in S \tag{1.2}$$

where $n = (n_1, n_2, n_3)$ is the outward unit normal to the surface $\partial\Omega_\rho$ at a point y, and f_1 and f_2 are continuous functions. Given $f_1(y)$ and $f_2(y)$ on S, find $U(x)$, $x \in \Omega_\rho$.

Problem 2. *Let f_1 and f_2 be given on S. Find conditions on f_1 and f_2 that are necessary and sufficient for the existence of a solution to system (1.1) satisfying (1.2) and from the class $H(\Omega_\rho)$.*

It is well-known that the Cauchy problem (1.2) for the Poisson equation (1.1) is ill-posed [3, 5]. Hadamard [17] noted that a solution to Problem 1 is not stable. The possibility of introducing a positive parameter σ, depending on the accuracy of the initial data, was noticed by M.M. Lavrentev [23]. The uniqueness of the solution follows from the general theorem by Holmgren [6]. It has applications in many different areas such as plasma physic, electrocardiography, and corrosion non-destructive evaluation (e.g., [7, 9, 10, 13, 19]). Traditionally, regularization techniques, such as Tikhonov regularization [44] and the quasi-reversibility approach [22], were used to provide robust numerical schemes [18].

We suppose that a solution to the problem exists (in this event, it is unique) and is continuously differentiable in the closed domain, and the Cauchy data are given exactly. In this case, we establish an explicit continuation formula. This formula enables us to state a simple and convenient criterion for the solvability of the Cauchy problem.

The result established here is a multidimensional analog of theorems and Carleman-type formulas [4] by G.M. Goluzin, V.I. Krylov, V.A. Fok, and F.M. Kuni in the theory of holomorphic functions of one variable [14, 16].

The method for obtaining these results is based on an explicit form of the fundamental solution of the Poisson equation which depends on a positive parameter that vanishes together with its derivatives on a fixed cone and outside it, as the parameter tends to infinity, while the pole of the fundamental solution lies inside the cone. Following to M.M. Lavrent'ev, a fundamental solution with these properties is called a *Carleman function* for the cone [8, 23]. Having constructed a Carleman function explicitly, we write a continuation formula. The existence of a Carleman function follows from S.N. Mergelyan's approximation theorem [28]. However, this theorem shows no way for writing the Carleman function explicitly.

The Carleman function of the Cauchy problem for the Laplace equation and some close problems, in the case when $\partial\Omega_\rho \setminus S$ is a part of a conic surface, was constructed in [45]. Mergelyan [28] suggested a method to construct the Carleman function of the Cauchy problem for the Laplace equation in the case when S is a part, with a smooth boundary, of the boundary of a simply-connected domain. Based on [28] and approximative theorems, the Carleman matrix for elliptic systems was constructed in [41].

In [1], some theorems of existence of the Carleman matrix and a solvability criterion for a wider class of boundary value problems for elliptic systems were established. It was proved earlier in [1, 41] that, for every Cauchy problem for elliptic systems, the Carleman matrix exists if the Cauchy data are given on a boundary set of positive measure.

Following Tikhonov [21, 43], we call the family of functions $U_{\sigma\delta}(x)$ the regularized solution to the Cauchy problem for equation (1.1). The regularized solution determines the stability of the approximate method.

In the paper, based on results from [23, 45–48] on the Cauchy problem for the Laplace and Helmholtz equations, we construct the Carleman–Yarmuhamedov function in an explicit form. We use it to prove the Carleman formulas and a criterion for the solvability of the Cauchy problem.

In recent decades, interest in the classical ill-posed problems of mathematical physics has been preserved. This direction of investigation of the properties of solutions to the Cauchy problem for the Laplace equation was started in [2, 20, 23, 24, 42] and was further developed in [25–27, 30–40].

2. Construction of a Carleman–Yarmukhamedov function

According to [45], we define the Carleman–Yarmukhamedov function $\Phi(y,x)$ by the equality

$$-2\pi^2 K(0)\Phi(y,x) = \int\limits_0^\infty \mathrm{Im}\left[\frac{K(w)}{w}\right]\frac{du}{\sqrt{s+u^2}}, \qquad w = i\sqrt{s+u^2} + y_3 - x_3. \tag{2.1}$$

Here, $K(w)$ is an entire function of complex variable that takes real values for real w ($w = a+ib$, a and b are real numbers) such that $K(a) \neq \infty$, $|a| < \infty$, $K(0) \neq 0$, $\forall R > 0$, $\exists C_R > 0$

$$\sup_{|\mathrm{Re}\,w|<R,\ \mathrm{Im}\,w \le -C_R} \big(|K(w)| + |\mathrm{Im}\,w||K'(w)| + |\mathrm{Im}\,w|^2|K''(w)|\big) < \infty.$$

For real w, since $K(w)$ is real, we have $\overline{K(\overline{w})} = K(w)$. Then (2.1) implies that $\forall R > 0$

$$\sup_{|\mathrm{Re}\,w|<R} \{|K(w)| + (1+|\mathrm{Im}\,w|)|K'(w)| + (1+|\mathrm{Im}\,w|^2)|K''(w)|\} < \infty. \tag{2.2}$$

Now we write (2.1) in the form

$$-2\pi^2 K(0)\Phi(y,x) = \int\limits_0^\infty \left\{\frac{(y_3-x_3)\mathrm{Im}\,K(w)}{\sqrt{s+u^2}} - \mathrm{Re}\,K(w)\right\}\frac{du}{r^2+u^2}, \tag{2.3}$$

where

$$\begin{aligned}\mathrm{Im}\left(\frac{K(w)}{w}\right) &= \frac{1}{2i}\left\{\frac{K(w)}{w} - \frac{K(\overline{w})}{\overline{w}}\right\} = \frac{\overline{w}K(w) - wK(\overline{w})}{2i(r^2+u^2)}\\ &= \frac{(y_3-x_3)\,\mathrm{Im}\,K(w) - \sqrt{s+u^2}\,\mathrm{Re}\,K(w)}{r^2+u^2}.\end{aligned} \tag{2.4}$$

From (2.2) and (2.3), it follows that, for $y \neq x$, the integral in (2.1) converges absolutely.

If $K(w) \equiv 1$, then the function $\Phi(y,x)$ is the classical fundamental solution to the Laplace equation, i.e.,

$$\Phi(y,x) \equiv \Phi_0(r) = 1/(4\pi r).$$

Theorem 1 [45]. *The function $\Phi(y,x)$ defined by (2.1) or (2.3)–(2.4) is representable in the form*

$$\Phi(y,x) = \Phi_0(r) + G(y,x), \tag{2.6}$$

where $\Phi_0(r) = 1/(4\pi r)$ and the function $G(y,x)$ is harmonic in the variable y in $\mathbb{R}^3$, including $y = x$.

From Theorem 1 it follows that the function $\Phi(y,x)$ of the variable y is a fundamental solution of the Poisson equation. Therefore, for the function $U(y) \in H(\Omega_\rho)$ and for every point $x \in \Omega_\rho$, the Green's formula is valid [15]:

$$U(x) = \int\limits_{\Omega_\rho} \Phi(y,x)f(y)dy - \int\limits_{\partial\Omega_\rho} \left[U(y)\frac{\partial\Phi(y,x)}{\partial n} - \Phi(y,x)\frac{\partial U(y)}{\partial n}\right] dS_y, \tag{2.5}$$

where $f(x) \in C^\lambda(\Omega_\rho)$, $\lambda \in (0,1)$, is bounded, i.e., the former integral on the right-hand side of (2.5) satisfies equation (1.1) in the domain.

3. The Mittag-Leffler entire function

The continuation formulas below are expressed explicitly in terms of the Mittag-Leffler entire function; therefore, we now present its basic properties without proof. These properties as well as detailed proofs can be found in [11, Chapter 3, §2], [47].

The Mittag-Leffler entire function is defined by the series

$$E_\rho(w) = \sum_{n=0}^{\infty} \frac{w^n}{\Gamma(1+n/\rho)}, \quad \rho > 0, \quad w \in C, \quad E_1(w) = e^w,$$

where Γ is the Euler gamma-function. Hereinafter, we suppose that $\rho > 1$. Let

$$\gamma = \gamma(1,\beta), \quad 0 < \beta < \frac{\pi}{\rho}, \quad \rho > 1,$$

be the contour in the complex w-plane that consists of the ray $\arg w = -\beta$, $|w| \geq 1$, the arc $-\beta \leq \arg w \leq \beta$ of the circle $|w| = 1$, and the ray $\arg w = \beta$, $|w| \geq 1$, which is passed so that $\arg w$ does not decrease. The contour γ splits the complex domain C into the two simply connected infinite domains Ω^- and Ω^+ lying to the left and to the right of γ, respectively. We suppose that

$$\frac{\pi}{2\rho} < \beta < \frac{\pi}{\rho}, \quad \rho > 1.$$

Under these conditions, the following integral representations are valid:

$$E_\rho(w) = \rho e^{w^\rho} + \psi_\rho(w), \quad w \in \Omega^+,$$
$$E_\rho(w) = \psi_\rho(w), \quad E'_\rho(w) = \psi'_\rho(w), \quad w \in \Omega^-,$$

where

$$\psi_\rho(w) = \frac{\rho}{2\pi i}\int\limits_\gamma \frac{e^{\zeta^\rho}}{\zeta - w}d\zeta, \quad \psi'_\rho(w) = \frac{\rho}{2\pi i}\int\limits_\gamma \frac{e^{\zeta^\rho}}{(\zeta - w)^2}d\zeta. \tag{3.1}$$

Since $E_\rho(w)$ takes real vales for real w, we obtain

$$\operatorname{Re}\psi_\rho(w) = \frac{\psi_\rho(w) + \psi_\rho(\overline{w})}{2} = \frac{\rho}{2\pi i}\int\limits_\gamma \frac{e^{\zeta^\rho}(\zeta - \operatorname{Re} w)}{(\zeta - w)(\zeta - \overline{w})}d\zeta,$$

$$\operatorname{Im}\psi_\rho(w) = \frac{\psi_\rho(w) - \psi_\rho(\overline{w})}{2i} = \frac{\rho \operatorname{Im} w}{2\pi i}\int\limits_\gamma \frac{e^{\zeta^\rho}}{(\zeta - w)(\zeta - \overline{w})}d\zeta,$$

$$\operatorname{Im}\frac{\psi'_\rho(w)}{\operatorname{Im} w} = \frac{\rho}{2\pi i}\int\limits_\gamma \frac{2e^{\zeta^\rho}(\zeta - \operatorname{Re} w)}{(\zeta - w)^2(\zeta - \overline{w})^2}d\zeta.$$

Hereinafter, we take
$$\beta = \frac{\pi}{2\rho} + \frac{\varepsilon_2}{2}, \quad \rho > 1,$$
in the definition of the contour $\gamma(1, \beta)$. It is clear that, if
$$\frac{\pi}{2\rho} + \varepsilon_2 \le |\arg w| \le \pi, \tag{3.2}$$
then $w \in \Omega_\rho^-$ and $E_\rho(w) = \psi_\rho(w)$.

Define
$$T_{k,p}(w) = \frac{\rho}{2\pi i} \int\limits_\gamma \frac{\zeta^p e^{\zeta^p}}{(\zeta - w)^k (\zeta - \overline{w})^k} d\zeta, \quad k = 1, 2, \dots, \quad p = 0, 1, \dots.$$

The following inequalities are valid for $\pi/(2\rho) + \varepsilon_2 \le |\arg w| \le \pi$:
$$|E_\rho(w)| \le \frac{C_1}{1 + |w|}, \quad |E'_\rho(w)| \le \frac{C_2}{1 + |w|^2}, \tag{3.3}$$
$$|T_{k,p}(w)| \le \frac{C_3}{1 + |w|^{2k}}, \quad k = 1, 2, \cdots, \tag{3.4}$$
where C_1, C_2, and C_3 are constants independent of w. Take in (2.1)
$$\beta = \frac{\pi}{2\rho} + \frac{\varepsilon_2}{2} < \frac{\pi}{\rho}, \quad \rho > 1.$$
Then $E_\rho(w) = \psi_\rho(w)$, where $\psi_\rho(w)$ is defined by (3.1). Moreover, note that $\cos \rho\beta < 0$ and the integral converges:
$$\int\limits_\gamma |\zeta|^p e^{\cos \rho\beta |\zeta|^\rho} |d\zeta| < \infty, \quad p = 0, 1, \dots . \tag{3.5}$$

4. Carleman formulas

Let the Mittag-Leffler entire function be the function $K(w)$ in (2.1):
$$K(w) = e^{aw^2} E_\rho(\sigma w),$$
where
$$\rho > 1, \quad w = i\sqrt{s + u^2} + y_3 - x_3, \quad K(0) = E_\rho(0) = 1, \quad a > 0 \quad \sigma \ge 0.$$
Denote by $\Phi_\sigma(y, x)$ the corresponding fundamental solution and by $\Phi_\sigma(y - x)$ its derivative with respect to the variable σ:
$$\Psi_\sigma(y - x) \equiv \frac{d\Phi_\sigma}{d\sigma}(y - x).$$
It follows from Theorem 1 that $\Psi_\sigma(y - x)$ satisfies the Poisson equation in $\mathbb{R}^3$. Then
$$\begin{aligned} -2\pi^2 \Phi_\sigma(y - x) &= \int\limits_0^\infty \operatorname{Im} \left[\frac{e^{aw^2} E_\rho(\sigma w)}{w} \right] \frac{du}{\sqrt{s + u^2}} \\ &= e^{a(y_3 - x_3)^2} \int\limits_0^\infty \varphi_\sigma(y, x, u) \frac{e^{-as - au^2}}{u^2 + r^2} du, \end{aligned} \tag{4.1}$$

where

$$\varphi_\sigma(y-x,u)=\left[\frac{(y_3-x_3)}{\sqrt{u^2+s}}\operatorname{Im}E_\rho(\sigma w)-\operatorname{Re}E_\rho(\sigma w)\right]\cos(\nu\sqrt{s+u^2})$$
$$+\left[\operatorname{Im}E_\rho(\sigma w)+\frac{(y_3-x_3)}{\sqrt{s+u^2}}\operatorname{Re}E_\rho(\sigma w)\right]\sin(\nu\sqrt{s+u^2}),\quad \nu=2a(y_3-x_3),$$
$$\Psi_\sigma(y-x)\equiv\frac{d\Phi_\sigma}{d\sigma}(y-x)=\int\limits_0^\infty \operatorname{Im}\left[e^{aw^2}E'_\rho(\sigma w)\right]\frac{du}{\sqrt{s+u^2}}. \tag{4.2}$$

Lemma 1 [47]. *Let M be a compact set in G_ρ, and let δ be the distance from M to ∂G_ρ. Then, for $\sigma\ge 0$, the following inequalities are valid for $x\in M$ and $y\in R^3\backslash G_\rho$ $(|y'|\ge\tau y_3)$:*

$$|\Phi_\sigma(y-x)|+\left|\frac{\partial}{\partial y_k}\Phi_\sigma(y-x)\right|+\left|\frac{\partial^i}{\partial x_j^i}\frac{\partial}{\partial y_k}\Phi_\sigma(y-x)\right|\le\frac{C_4(\rho,\delta)r}{1+\sigma\delta}, \tag{4.3}$$
$$r\ge\delta>0,\quad i,=0,1,\quad k,j=1,2,3.$$
$$|\Psi_\sigma(y-x)|+\left|\frac{\partial}{\partial y_k}\Psi_\sigma(y-x)\right|+\left|\frac{\partial^i}{\partial x_j^i}\frac{\partial}{\partial y_k}\Psi_\sigma(y-x)\right|\le\frac{C_5(\rho,\delta)r}{1+\sigma\delta}, \tag{4.4}$$
$$r\ge\delta>0,\quad i=0,1,\quad k,j=1,2,3,$$

where the constants C_4 and C_5 are independent of x,y, and σ.

Theorem 2. *Let f be bounded and locally Hölder continuous in Ω_ρ, $U(y)\in H^\lambda(\Omega_\rho)$, and*

$$U(y)=f_1(y),\quad \frac{\partial U}{\partial n}(y)=f_2(y),\quad y\in S,$$

where $f_1(y)$ and $f_2(y)$ are given functions of the class $C(S)$. Then the Carleman formulas

$$\frac{\partial^i U(x)}{\partial x_j^i}=\lim_{\sigma\to\infty}\frac{\partial^i U_\sigma(x)}{\partial x_j^i}$$
$$=\lim_{\sigma\to\infty}\left[\int\limits_{\Omega_\rho}f(y)\frac{\partial^i\Phi_\sigma(y-x)}{\partial x_j^i}dy-\int\limits_S\left\{f_1(y)\frac{\partial^i}{\partial x_j^i}\frac{\partial\Phi_\sigma(y-x)}{\partial n}-f_2(y)\frac{\partial^i\Phi_\sigma(y-x)}{\partial x_j^i}\right\}dS_y\right] \tag{4.5}$$

are valid for every $x\in\Omega_\rho$, where $i=0,1$, $j=1,2,3$,

$$\frac{\partial^0 U_\sigma}{\partial x_j^0}=U_\sigma,\quad \frac{\partial^0\Phi_\sigma}{\partial x_j^0}=\Phi_\sigma,$$

and the convergence in (4.5) is uniform on compact sets in Ω_ρ.

P r o o f. From Green's formula (2.5), for every $x\in\Omega_\rho$, we obtain

$$\frac{\partial^i U(x)}{\partial x_j^i}=\int\limits_{\Omega_\rho}f(y)\frac{\partial^i\Phi_\sigma(y-x)}{\partial x_j^i}dy-\int\limits_{\partial\Omega_\rho}\left[f_1(y)\frac{\partial^i}{\partial x_j^i}\frac{\partial\Phi_\sigma(y-x)}{\partial n}-f_2(y)\frac{\partial^i}{\partial x_j^i}\Phi_\sigma(y-x)\right]dS_y, \tag{4.6}$$

$\partial\Omega_\rho=S\cup(\partial\Omega_\rho\setminus S)$. According to [47], let us estimate

$$\Phi_\sigma,\quad \frac{\partial\Phi_\sigma}{\partial y_j},\quad \frac{\partial^i}{\partial x_j^i}\frac{\partial\Phi_\sigma}{\partial y_j}.$$

Lemma 1 yields the assertion of Theorem 2. Indeed, if M is a compact set in Ω_ρ then $M \subset G_\rho$. Therefore, the inequalities in Lemma 1 for $\Phi_\sigma(y-x)$ and its derivatives remain also valid in the case where $x \in M \subset \Omega_\rho$ and $y \in \partial\Omega_\rho \backslash S \subset \partial G_\rho$ (in this case, δ is the distance from the compact set $M \subset \Omega_\rho$ to $\partial\Omega_\rho$). Now, let σ tend to infinity. The proof of Theorem 2 is complete. □

We can write (4.5) in the following equivalent form:

$$\begin{gathered}\frac{\partial^i U(x)}{\partial x_j^i} = \int_0^\infty \frac{\partial^i}{\partial x_j^i} J(\sigma,x) + \int_{\Omega_\rho} f(y) \frac{\partial^i \Phi_0(r)}{\partial x_j^i} dy \\ - \int_S \left[f_1(y) \frac{\partial^i}{\partial x_j^i} \frac{\partial \Phi_0(r)}{\partial n} - f_2(y) \frac{\partial^i \Phi_0(r)}{\partial x_j^i} \right] dS_y, \quad x \in \Omega_\rho,\end{gathered} \tag{4.7}$$

where

$$\begin{gathered}\frac{\partial^i}{\partial x_j^i} J(\sigma,x) = \int_{\Omega_\rho} f(y) \frac{\partial^i \Psi_\sigma(y-x)}{\partial x_j^i} dy - \int_S \left[f_1(y) \frac{\partial^i}{\partial x_j^i} \frac{\partial \Psi_\sigma(y-x)}{\partial n} - f_2(y) \frac{\partial^i \Psi_\sigma(y-x)}{\partial x_j^i} \right] dS_y, \\ x \in \Omega_\rho, \quad i=0,1, \quad j=1,2,3, \quad \frac{\partial^0 U}{\partial x_j^0} = U, \quad \frac{\partial^0 \Phi_0}{\partial x_j^0} = \Phi_0, \quad \frac{\partial^0 \Psi_\sigma}{\partial x_j^0} = \Psi_\sigma, \quad \frac{\partial^0 J}{\partial x_j^0} = J.\end{gathered} \tag{4.8}$$

The functions $\Psi_\sigma(y-x)$ and $\Phi_0(r)$ are defined by equalities (4.2) and (4.1), respectively. The proof of (4.7) follows from the formulas

$$\lim_{\sigma \longrightarrow \infty} \frac{\partial^i}{\partial x_j^i} P(\sigma,x) = \int_0^\infty \frac{\partial^i}{\partial x_j^i} \frac{\partial P(\sigma,x)}{\partial \sigma} + \frac{\partial^i}{\partial x_j^i} P(x)$$

and

$$\begin{gathered}\frac{\partial^i}{\partial x_j^i} \frac{\partial P(\sigma,x)}{\partial \sigma} = \int_{\Omega_\rho} f(y) \frac{\partial^i \Psi_\sigma(y-x)}{\partial x_j^i} dy - \int_S \left[f_1(y) \frac{\partial^i}{\partial x_j^i} \frac{\partial \Psi_\sigma}{\partial n}(y-x) - f_2(y) \frac{\partial^i \Psi_\sigma}{\partial x_j^i}(y-x) \right] dS_y, \\ x \in \Omega_\rho, \quad i=0,1, \quad j=1,2,3;\end{gathered}$$

moreover, the differentiation under the integral sign is legal and

$$\frac{\partial^i}{\partial x_j^i} \frac{\partial P(\sigma,x)}{\partial \sigma} = \frac{\partial^i}{\partial x_j^i} J(\sigma,x).$$

Theorem 3. *Let $S \subset C^2$, $f_1(y) \in C^1(S_0) \cap L(S)$, $f_2(y) \in C(S_0) \cap L(S)$, and let f be bounded and locally Hölder continuous in Ω_ρ. Then for the existence of a function $U(y) \in H^\lambda(\Omega_\rho) \cap C(S_0)$ such that*

$$U(y) = f_1(y), \quad \frac{\partial U}{\partial n}(y) = f_2(y), \quad y \in S_0, \tag{4.9}$$

it is necessary and sufficient that the following improper integral converge (uniformly on compact sets in G_ρ) for each $x \in G_\rho$:

$$\left| \int_1^\infty J(\sigma,x) d\sigma \right| < \infty, \tag{4.10}$$

where $J(\sigma,x)$ is defined by (4.8). If (4.10) is satisfied, then harmonic continuation is performed by equivalent formulas (4.5) and (4.7).

P r o o f. *Necessity*: Let

$$U(y) \in H(\Omega_\rho) \cap C^1(\Omega_\rho \cup S_0) \cap L(S)$$

satisfy (4.10). Let M be a compact set in G_ρ, and let $\varepsilon > 0$ be such that $M \subset \overline{G}_\rho^{2\varepsilon} \subset \overline{G}_\rho^{\varepsilon} \subset G_\rho$. It is clear that the distance from M to $\partial G_\rho^\varepsilon$ is at least $\varepsilon\tau_1$ and the distance from $\partial G_\rho^{2\varepsilon}$ to $\partial G_\rho^\varepsilon$ is $\varepsilon\tau_1$. Now, let $y \in R^3 \setminus G_\rho^\varepsilon$ ($|y'| \le \tau(y_3 - \varepsilon)$ and $y_3 > \varepsilon$) and $x \in M$ ($|x'| \le \tau(x_3 - 2\varepsilon)$ and $x_3 > 2\varepsilon$). Then $\arg w = \arg(\sigma w) = \arg(i\tau\sqrt{u^2+s} + \tau y_3 - \tau x_3)$ and

$$\tau w = i\tau\sqrt{u^2+s} + \tau y_3 - \tau x_3 = \sqrt{u^2+s}\Big(i\mathrm{tg}\frac{\pi}{2\rho} + \frac{\tau y_3 - \tau x_3}{\sqrt{u^2+s}}\Big), \quad u \ge 0, \quad \rho > 1,$$

$$\frac{\tau y_3 - \tau x_3}{\sqrt{u^2+s}} \le \frac{|y'| - |x'| - \varepsilon\tau}{|y' - x'|} \le 1 - \varepsilon_1, \quad y' \ne x', \quad \Big|\arg(a \pm \mathrm{tg}\frac{\pi}{2\rho})\Big| \ge \frac{\pi}{2\rho}; \quad a \le 1.$$

Therefore, (2.5) is valid for $\arg w$; moreover, if $y' = x'$, then $\mathrm{Re}\, w < 0$, and this inequality also holds. Consequently, $\Phi_\sigma(y-x)$ and $\Psi_\sigma(y-x)$ satisfy estimates (3.2)–(3.5) from Lemma 1, where $\delta \ge \varepsilon\tau_1$. Define $S_\varepsilon = \overline{G}_\rho^\varepsilon \cap S$; in this case, the part $S_\varepsilon \subset S$ together with the part T_ε of the cone surface $\partial G_\rho^\varepsilon$ form a closed piecewise smooth surface $S_\varepsilon \cup T_\varepsilon$ (with the consistent direction of the outer normals) which is the boundary of a simply connected bounded domain. Represent the integral on the right-hand side of (4.8) as the sum of two integrals according to the representation $S = S_\varepsilon \cup (S \setminus S_\varepsilon)$. Since $\Psi_\sigma(y-x)$ is a regular solution of the Poisson equation, by Green's formula, the integral over the part S_ε is equal to the integral over T_ε; moreover, $\Psi_\sigma(y-x)$ satisfies inequalities (4.7) and (4.9) for $y \in T_\varepsilon$ and $x \in M$, and the extended function $U(y)$ together with its gradient is bounded by a constant depending on ε. Therefore, the modulus of the integral over the part S_ε does not exceed the quantity

$$\frac{\mathrm{const}}{1 + \delta^2\sigma^2}, \quad \sigma \ge 0,$$

with a constant depending on $\rho, \varepsilon, \delta$, and the diameter of the domain Ω_ρ. Since $|y| \ge \tau(y_3 - \varepsilon)$, $y_3 \ge \varepsilon$, when $y \in S \setminus S_\varepsilon$ and $x \in K$ and $f_1(y), f_2(y) \in C(S_0) \cap L(S)$, these inequalities remain valid for the modulus of the integral over $S \setminus S_\varepsilon$ (of course, with other constants). Hence, we have (4.10).

Sufficiency: Under the assumptions of the theorem, define functions $U(x)$, $x \in G_\rho \setminus S_0$, by the right-hand side of (4.7). Consider the first term on the right-hand side of (4.7). Since $\Psi_\sigma(y)$ satisfies the Poisson equation in G_ρ for $\sigma \ge 0$, the function $J(\sigma, x)$ satisfies the Poisson equation with respect to x in G_ρ for $\sigma \ge 0$. Therefore, we conclude from (4.10) that the first term on the right-hand side of (4.7) satisfies the Poisson equation in G_ρ as the limit of the uniformly converging sequence of the solutions of the Poisson equations

$$U_n(x) = \int_0^n J(\sigma, x)d\sigma, \quad n = 1, 2, \ldots.$$

The second and third terms are the potential difference of the volume, single, and double layers and represent one solution of the Poisson equation in Ω_ρ and another in $\Omega_\rho' = G_\rho \setminus \overline{\Omega}_\rho$. Therefore, the right-hand side of (4.7) defines two different solutions of the Poisson equations $U^+(x)$ and $U^-(x)$ in Ω_ρ and Ω_ρ'. If x^1 and x^2 are two points on the normal at $x \in S_0$ symmetric with respect to x, then

$$\lim_{x^1 \to x}\big[U^+(x^1) - U^-(x^2)\big] = f_1(x), \quad \lim_{x^1 \to x}\Big[\frac{\partial U^+}{\partial n}(x^1) - \frac{\partial U^-}{\partial n}(x^2)\Big] = f_2(x), \quad x \in S_0;$$

moreover, the limit relations hold uniformly in x on each compact part S_0. If $\max y_3 < x_3$, where $y \in S$ and $x \in G_\rho$, then $\operatorname{Re} w = y_3 - x_3 < 0$ and $\Phi_\sigma(y-x)$ and its derivatives satisfy inequalities (4.6) and (4.3). Now, from formula (4.5), which is equivalent to (4.7), we see that $U^-(x) = 0$ and $U^-(x) \equiv 0$, $x \in \Omega_\rho$, by the uniqueness theorem. It is clear that $U^-(x)$ extends smoothly to $\Omega'_\rho \cup S_0$. Then $U^+(x)$ extends smoothly as a function of the class $C^1(\Omega_\rho \cup S_0)$ (see [29]). Consequently,

$$U^+(x) = f_1(x), \quad \frac{\partial U^+}{\partial n}(x) = f_2(x), \quad x \in S_0.$$

Now, we set $U(x) = U^+(x),\ x \in \Omega_\rho \cup S_0$. Theorem 3 is proved. □

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ON HOP DOMINATION NUMBER OF SOME GENERALIZED GRAPH STRUCTURES

S. Shanmugavelan[†], **C. Natarajan**[††]

Srinivasa Ramanujan Centre,
SASTRA Deemed to be University,
Kumbakonam–612001, India

[†]shanmugavelan@src.sastra.edu, [††]natarajan_c@maths.sastra.edu

Abstract: A subset $H \subseteq V(G)$ of a graph G is a hop dominating set (HDS) if for every $v \in (V \setminus H)$ there is at least one vertex $u \in H$ such that $d(u,v)=2$. The minimum cardinality of a hop dominating set of G is called the hop domination number of G and is denoted by $\gamma_h(G)$. In this paper, we compute the hop domination number for triangular and quadrilateral snakes. Also, we analyse the hop domination number of graph families such as generalized thorn path, generalized ciliates graphs, glued path graphs and generalized theta graphs.

Keywords: Hop domination number, Snake graphs, Theta graphs, Generalized thorn path.

1. Introduction

Domination in graphs is fascinating topic in the field of graph theory. It is one of the most effective mathematical models for a variety of real world problems. A simple undirected finite graph G holds a vertex set $V(G)$ with vertices and an edge set $E(G)$ whose members are unordered pair of vertices called lines or edges of G. The *degree* of a vertex v, denoted by $d(v)$, is the number of edges that are incident with v and the *distance* $d(u,v)$ between any two distinct vertices u and v is the length of the shortest path connecting u and v in G. We use the symbol $[n]=\{1,2,\dots,n\}$. For any other graph theory terminology not defined here, we follow [3].

In a graph G, a subset $D \subseteq V(G)$ is said to be a dominating set if every vertex not in D is adjacent to at least one vertex in D. The minimum cardinality of a minimal dominating set of G is the domination number $\gamma(G)$. In the last three decades, several domination parameters have been established and they have been intensively investigated with applications in communication networks, facility location problems, game theory, mathematical chemistry, and so on. For a detailed study on domination concepts, one may refer [8–10].

Ayyasamy et al. [1] defined a new distance-based domination parameter called the hop domination number of a graph G. A subset $H \subseteq V(G)$ of a graph G is a hop dominating set (HDS) if for every vertex v not in H, there exists at least one vertex $u \in H$ such that $d(u,v)=2$. The minimum cardinality of a hop dominating set of G is called the hop domination number of G and is denoted by $\gamma_h(G)$. The hop degree of a vertex v in a graph denoted by $d_h(v)$ is the number of vertices at distance= 2 from v. The hop graph $H(G)$ of a graph G is the graph having same vertex set and two vertices u,v are adjacent in $H(G)$ iff $d_G(u,v)=2$. Also, Ayyasamy et al. [2] obtained some bounds on hop domination number for trees and characterized trees attaining those bounds. Natarajan et al. [13] found characterization results for hop domination number equals other domination parameters like total domination number, connected domination number for several families of graphs. Many scholars have explored this parameter in the years thereafter, leading to novel versions such as connected hop domination, total perfect hop domination, Roman hop domination, Global hop domination, etc., [11, 12, 15–18, 20]. In 2018, Natarajan et al. [14] discussed hop domination number for some special families of graph like central graph, middle graph and total graph. Recently, Packiavathi et al. [6] obtained the hop domination number of a caterpillar graph $P_n(l_1, l_2, \ldots, l_n)$ (a caterpillar is a graph obtained from the path by attaching leaves l_i to i^{th} vertex of the path P_n) and the domination number for some special families of snake graphs which occur as hop graph of $P_n(1, 1, \ldots, 1)$ and $P_n(2, 2, \ldots, 2)$. We refine their result on caterpillar graph and present an elegant result.

2. Main results

In this section, we study the hop domination number of snake graph families like triangular, alternate triangular, quadrilateral and alternate quadrilateral snakes. In addition, the hop domination number of some generalized structures like generalized theta graphs, generalized thorn paths and generalized ciliates graphs $GC(p, q, t)$ for $p = 3$ and $p = 4$ are determined.

Definition 1 [7]. *Let $l_1, l_2, \ldots l_n$ be n positive integers. Then the thorn graph $G^t = G^t(l_1, l_2 \ldots l_n)$ is obtained from a graph G by attaching l_i pendant vertices (thorns) to each vertex v_i of G, $i \in [n]$.*

In 2020, Getchial Pon Packiavathi et al. [6] obtained the following result on caterpillar graphs.

Theorem 1 [6]. $\gamma_h(P_n(1, 1, \ldots 1)) = \gamma_h(P_n(2, 2, \ldots 2)) = \begin{cases} 2r, & \text{if} \quad n = 2r; \\ 2r + 3, & \text{if} \quad n = 2r + 1. \end{cases}$

First, we observe that the result given in Theorem 1 is wrong. For example, $\gamma_h(P_4(1, 1, 1, 1)) = 2$ whereas from their computations it is 4. So, we refine the result by taking the more generalized version of caterpillar called thorn path P_n^t.

Theorem 2. *For $n > 1$,*

$$\gamma_h(P_n^t) = \begin{cases} \left\lfloor \dfrac{n-1}{2} \right\rfloor + 1, & \text{if} \quad n \equiv 0, 1, 3 \pmod 4; \\ \left\lceil \dfrac{n}{2} \right\rceil + 1, & \text{if} \quad n \equiv 2 \pmod 4. \end{cases}$$

P r o o f. Let $v_1, v_2, \cdots, v_n$ be the vertices of the central path P_n in P_n^t (see Fig. 1).

Case 1: $n \equiv 2 \pmod 4$. In this case, any γ_h-set is of the form

$$\{v_i \mid i \equiv 1 \pmod 4,\ 1 \le i \le (n-2)\} \cup \{v_j \mid j \equiv 2 \pmod 4,\ 2 \le j \le (n-1)\} \cup \{v_{n-1}\}.$$

Thus,

$$\gamma_h(P_n^t) \le \left\lceil \frac{n}{2} \right\rceil + 1$$

and it is easily seen that

$$\gamma_h(P_n^t) \ge \left\lceil \frac{n}{2} \right\rceil + 1.$$

Therefore,

$$\gamma_h(P_n^t) = \left\lceil \frac{n}{2} \right\rceil + 1.$$

Case 2: $n \equiv 0, 1, 3 \pmod 4$. In this case, any γ_h-set is of the form

$$\{v_i \mid i \equiv 2 \pmod 4,\ 2 \le i \le (n-2)\} \cup \{v_j \mid j \equiv 3 \pmod 4,\ 3 \le i \le (n-1)\} \cup \{v_{n-1}\}.$$

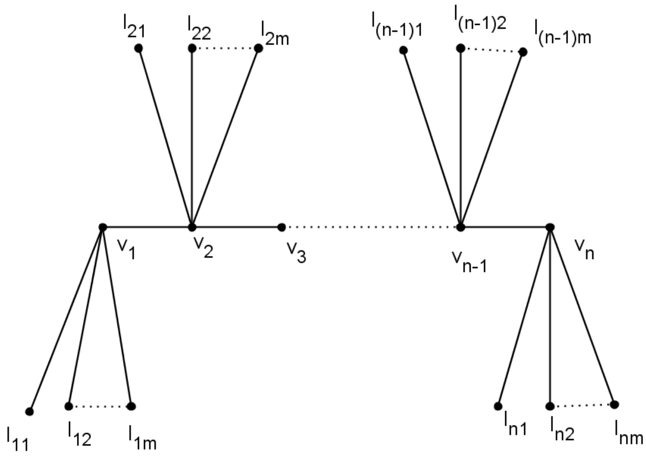

Figure 1. Thorny path P_n^t.

Thus,

$$\gamma_h(P_n^t) \leq \left\lceil \frac{n}{2} \right\rceil$$

and it is easily seen that

$$\gamma_h(P_n^t) \geq \left\lceil \frac{n}{2} \right\rceil.$$

Therefore,

$$\gamma_h(P_n^t) = \left\lceil \frac{n}{2} \right\rceil.$$

□

In [4], Derya Dogan et al. obtained some results for weak and strong domination in thorn graphs. Inspired by their results, we study our parameter namely, hop domination number for thorn rod given in [4], as well as for other generalized graph structures.

Lemma 1 [1]. $\gamma_h(P_n) = \begin{cases} 2r, & \textit{if} \quad n = 6r; \\ 2r+1, & \textit{if} \quad n = 6r+1; \\ 2r+2, & \textit{if} \quad n = 6r+s, \quad 2 \leq s \leq 5. \end{cases}$

Rewriting Lemma 1 in terms of congruence, we have

$$\gamma_h(P_n) = \begin{cases} \left\lfloor \dfrac{n}{3} \right\rfloor, & \text{if} \quad n \equiv 0 \pmod 6; \\ \left\lfloor \dfrac{n}{3} \right\rfloor + 1, & \text{if} \quad n \equiv 1, 3, 4, 5 \pmod 6; \\ \left\lfloor \dfrac{n}{3} \right\rfloor + 2, & \text{if} \quad n \equiv 2 \pmod 6. \end{cases}$$

Definition 2 [4]. *A thorn rod is a graph $P_{n,t}$ which is obtained by taking a path on $n \geq 2$ vertices and attaching $(t-1)$ leaves, known as thorns, at each of the end of P_n.*

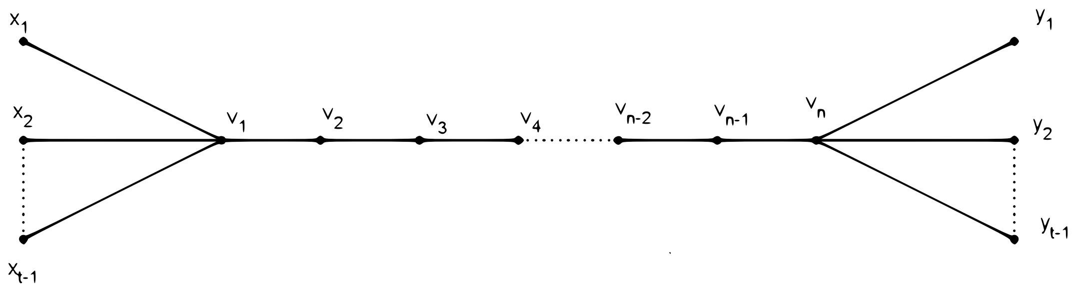

Figure 2. Thorn rod $P_{n,t}$.

Note that $P_{1,t}$ is a star graph $K_{1,t-1}$.

Theorem 3. $\gamma_h(P_{n,t}) = \begin{cases} \left\lfloor \dfrac{n-10}{3} \right\rfloor + 6, & \textit{if} \quad n \equiv 0 \pmod 6; \\ \left\lfloor \dfrac{n-10}{3} \right\rfloor + 5, & \textit{if} \quad n \equiv 1,2,3,5 \pmod 6; \\ \left\lfloor \dfrac{n-10}{3} \right\rfloor + 4, & \textit{if} \quad n \equiv 4 \pmod 6. \end{cases}$

P r o o f. Let us label the vertices of central path P_n as $v_1, v_2 \dots v_n$. Let the leaves or thorns at the vertex v_1 be $x_1, x_2, \dots x_{t-1}$ and the thorns at the vertex v_n be $y_1, y_2, \dots y_{t-1}$.

From Fig. 2, it is clear that to hop dominate $2(t-1)$ leaves and their support vertices, any γ_h-set must include the vertices $v_2, v_3, v_{n-1}, v_{n-2}$.

Now, the subgraph induced by $P_n - \{v_1, v_2, v_3, v_4, v_5, v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}, v_n\}$, is clearly a path on $(n-10)$ vertices.

By Lemma 1,

$$\gamma_h(P_{n-10}) = \begin{cases} \left\lfloor \dfrac{n-10}{3} \right\rfloor, & \text{if} \quad n \equiv 4 \pmod 6; \\ \left\lfloor \dfrac{n-10}{3} \right\rfloor + 1, & \text{if} \quad n \equiv 0,1,3,5 \pmod 6; \\ \left\lfloor \dfrac{n-10}{3} \right\rfloor + 2, & \text{if} \quad n \equiv 2 \pmod 6. \end{cases}$$

and hence $\gamma_h(G) = 4 + \gamma_h(P_{n-10})$. Thus, the result follows. □

Definition 3. *A glued path $GP(n,t)$ is a graph obtained by gluing t copies of a path $P_n (n \geq 2)$ at a common vertex v such that v is the initial vertex in each copy of P_n.*

Theorem 4. $\gamma_h(GP(n,t)) = \begin{cases} 2\left\lceil \dfrac{n}{6} \right\rceil t, & \textit{if} \quad n \equiv 0,\ 5 \pmod 6; \\ 2\left\lfloor \dfrac{n}{6} \right\rfloor t + 1, & \textit{if} \quad n \equiv 1 \pmod 6; \\ 2\left\lceil \left\lceil \dfrac{n}{6} \right\rceil t - 1 \right\rceil + 1, & \textit{if} \quad n \equiv 4 \pmod 6; \\ 2\left\lfloor \dfrac{n}{6} \right\rfloor t + 2, & \textit{if} \quad n \equiv 2,\ 3 \pmod 6. \end{cases}$

P r o o f. Let us arrange the vertices of $GP(n,t)$ row-wise subject to the following conditions:

(i) Place the common vertex in the 1^{st} row R_1.

(ii) First vertex of each copy of the path P_n be placed in the 2^{nd} row R_2.

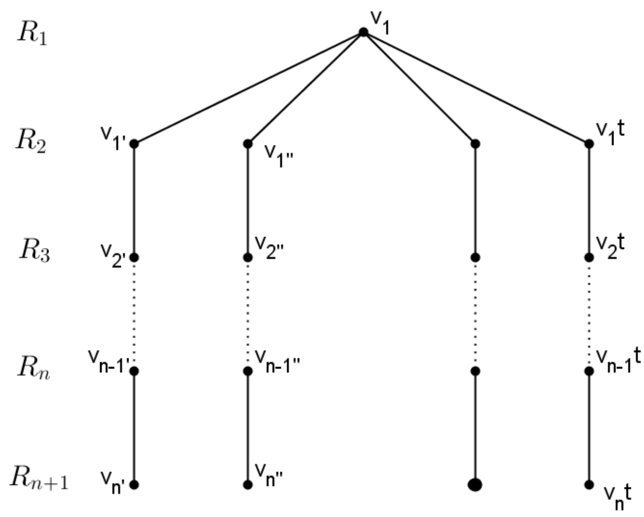

Figure 3. Glued path $GP(n,t)$.

(iii) In general, n^{th} vertex of each copy in the $(n+1)^{th}$ row R_{n+1}.

From Fig. 3, it is clear that, each row R_i has t vertices except the first row. That is, $V(R_i) = \{v_i, v_i^{''}, \dots, v_i^{(t)}\}$, $2 \leq i \leq (n+1)$. To hop dominate all those leaves and support vertices, all the vertices in R_{n-1} and R_{n-2} must be selected from a γ_h-set of $GP(n,t)$. This choice will also hop dominate all of the vertices of R_{n-3} and R_{n-4}.

Case 1: $n \equiv 0 \pmod 6$. In this case, any γ_h-set contains $2\lceil n/6 \rceil t$ vertices from the following rows $\mathcal{R} = \{R_{(n-1)}, R_{(n-2)}, R_{(n-7)}, R_{(n-8)}, \dots, R_4, R_3\}$. Thus,

$$\gamma_h(GP(n,t)) \leq 2\left\lceil \frac{n}{6} \right\rceil t.$$

It is easy to observe that any hop dominating set of $GP(n,t)$ contains at least $2\lceil n/6 \rceil t$ vertices. Therefore,

$$\gamma_h(GP(n,t)) = 2\left\lceil \frac{n}{6} \right\rceil t.$$

Case 2: $n \equiv 5 \pmod 6$. In this case, any γ_h-set includes $2\lceil n/6 \rceil t$ vertices from the rows $(\mathcal{R} \setminus R_3) \cup \{v_1\}$. Thus,

$$\gamma_h(GP(n,t)) \leq 2\left\lceil \frac{n}{6} \right\rceil t.$$

Also,

$$\gamma_h(GP(n,t)) \geq 2\left\lceil \frac{n}{6} \right\rceil t.$$

Therefore,

$$\gamma_h(GP(n,t)) = 2\left\lceil \frac{n}{6} \right\rceil t.$$

Case 3: $n \equiv 1 \pmod 6$. In this case, any γ_h-set includes vertices from the following rows $\mathcal{R}' = \{R_{(n-1)}, R_{(n-2)}, R_{(n-7)}, R_{(n-8)}, \dots, R_5, R_4\} \cup \{v_1\}$. Thus,

$$\gamma_h(GP(n,t)) \leq 2\left\lfloor \frac{n}{6} \right\rfloor t + 1.$$

One can observe that
$$\gamma_h(GP(n,t)) \geq 2\left\lfloor\frac{n}{6}\right\rfloor t+1.$$
Therefore,
$$\gamma_h(GP(n,t)) = 2\left\lfloor\frac{n}{6}\right\rfloor t+1.$$
Similarly, the proof follows for other cases. □

The generalized thorn path can be defined as follows,

Definition 4. *The graph obtained by taking a path P_n and attaching t copies of P_r to every vertex of P_n is said to be a generalized thorn path and denoted by $G(n,r,t), n>1$.*

Theorem 5. $\gamma_h[G(n,r,t)]=\begin{cases} \gamma_h(P_n)+nt\left\lfloor\frac{r}{3}\right\rfloor, & \textit{if} \quad r\equiv 0 \pmod 6; \\ n+nt\left\lfloor\frac{r}{3}\right\rfloor, & \textit{if} \quad r\equiv 1,2,3 \pmod 6; \\ nt\left(\left\lfloor\frac{r}{3}\right\rfloor+1\right), & \textit{if} \quad r\equiv 4,5 \pmod 6. \end{cases}$

P r o o f. Let $S=\{v'_{ij} : 1\leq i\leq r,\ 1\leq j\leq t\}$ denote the vertices of the i^{th} copy of P_r as shown in Fig. 4.

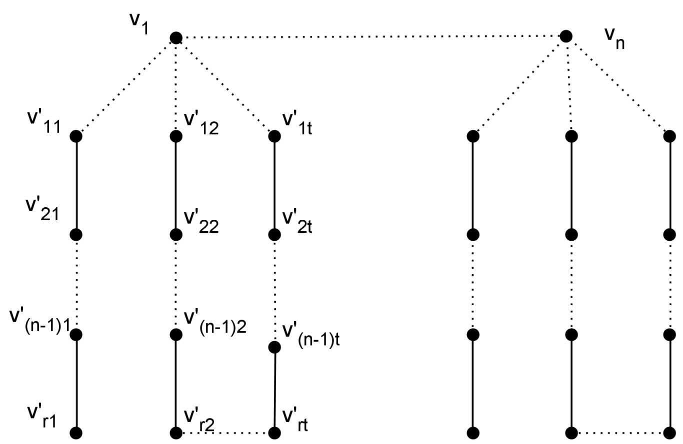

Figure 4. Generalized thorn path $G(n,r,t)$.

Case 1: $r\equiv 0 \pmod 6$. In this case, the set
$$H'=\{v'_{(n-2)j}, v'_{(n-1)j}, v'_{(n-8)j}, v'_{(n-9)j}\dots v'_{4j}, v'_{3j},\ 1\leq j\leq t\}$$
hop dominates all vertices in each copy of P_r. In order to hop dominate the vertices of the path P_n, any γ_h-set of $G(n,r,t)$ should contains $\gamma_h(P_n)$ vertices. As a result,
$$\gamma_h[G(n,r,t)]\leq \gamma_h(P_n)+|H'|=\gamma_h(P_n)+nt\left\lfloor\frac{r}{3}\right\rfloor.$$
It is easily seen that
$$\gamma_h[G(n,r,t)]\geq \gamma_h(P_n)+|H'|=\gamma_h(P_n)+nt\left\lfloor\frac{r}{3}\right\rfloor.$$

Therefore,

$$\gamma_h[G(n,r,t)] = \gamma_h(P_n) + |H'| = \gamma_h(P_n) + nt\Big\lfloor \frac{r}{3} \Big\rfloor.$$

Case 2: $r \equiv 1,\, 2,\, 3 \pmod 6$. Here, any γ_h- set contains the set

$$H' = \{v'_{(n-2)j}, v'_{(n-3)j}, v'_{(n-8)j}, v'_{(n-9)j} \dots v'_{2j}, v'_{1j},\ 1 \le j \le t\}$$

and so $H' \cup V(P_n)$ forms a γ_h-set of cardinality $n + |H'|$.

Case 3: $r \equiv 4,\, 5 \pmod 6$. In this case, vertices in attached t copies are sufficient for a γ_h-set of $G(n,r,t)$. Therefore,

$$\gamma_h[G(n,r,t)] = nt\gamma_h(P_r) \le nt\Big(\Big\lfloor \frac{r}{3} \Big\rfloor + 1\Big).$$

Also, any minimal HDS of $G(n,r,t)$ requires at least

$$\gamma_h[G(n,r,t)] = nt\gamma_h(P_r) \ge nt\Big(\Big\lfloor \frac{r}{3} \Big\rfloor + 1\Big)$$

vertices. Hence,

$$\gamma_h[G(n,r,t)] = nt\gamma_h(P_r) = nt\Big(\Big\lfloor \frac{r}{3} \Big\rfloor + 1\Big).$$

□

Definition 5 [19]. *A generalized theta graph $\theta[nP(m)]$ is a graph obtained from n-internally disjoint paths, in which each path $P(m)$ contains m internal vertices and these paths share common end vertices u and v (see Fig. 5).*

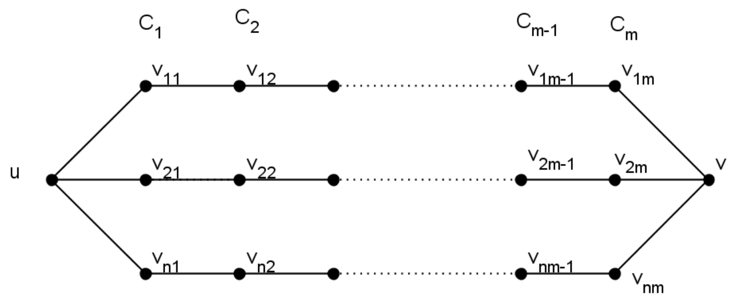

Figure 5. Generalized theta graph $\theta[nP(m)]$.

Theorem 6. $$\gamma_h(\theta[nP(m)]) = \begin{cases} 4 + \Big\lfloor \dfrac{m-6}{3} \Big\rfloor, & \text{if} \quad m \equiv 0 \pmod 6; \\ 4 + n\Big[\Big\lfloor \dfrac{m-6}{3} \Big\rfloor + 1\Big], & \text{if} \quad m \equiv 1,3,4,5 \pmod 6; \\ 4 + n\Big[\Big\lfloor \dfrac{m-6}{3} \Big\rfloor + 2\Big], & \text{if} \quad m \equiv 2 \pmod 6). \end{cases}$$

P r o o f. Let us denote $\theta[nP(m)]$ by G for convenience. Clearly, $\{u,v\}$ should be included in any γ_h-set and any one vertex from column C_1 and C_m is enough to hop dominate the vertices in C_1 and C_m. The induced subgraph $\langle G - \{u,v\} \cup C_1 \cup C_2 \cup C_{m-1} \cup C_m \rangle$ is a collection of n-distinct paths P_{m-4}. As a consequence of Lemma 1, the result follows. □

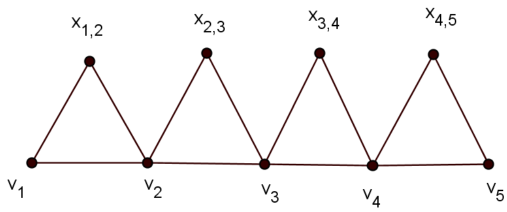

Figure 6. Triangular snake T_4.

Definition 6 [5]. *A triangular snake graph T_n is a graph obtained from the path P_n by replacing each edge by a cycle of length 3. For example, a triangular snake T_4 is shown in Fig. 6*

A double triangular snake DT_n consists of two triangular snakes that have a common path.

That is, a double triangular snake is obtained from a path $v_1, v_2, \ldots, v_n$ by joining v_i and v_{i+1} to a new vertex x_i for $i = 1, 2, \ldots, n-1$ and to a new vertex y_i for $i = 1, 2, \ldots, n-1$. For example, a double triangular snake DT_6 is illustrated in Fig. 7.

A triple triangular snake TT_n is a graph in which three triangular snakes have a common path.

Similarly, a four triangular snake FT_n is a graph in which four triangles share a common path.

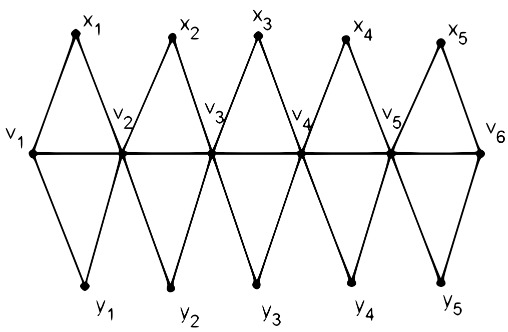

Figure 7. Double triangular snake DT_6.

Remark 1. $\gamma_h(T_n) \geq 3$.

Theorem 7.

$$\gamma_h(T_n) = \gamma_h(DT_n) = \gamma_h(TT_n) = \gamma_h(FT_n) = \begin{cases} 2 + \left\lfloor \dfrac{n-3}{2} \right\rfloor, & \textit{if} \quad n \textit{ is odd } (n \neq 3), \\ 2 + \left\lfloor \dfrac{n-2}{2} \right\rfloor, & \textit{if} \quad n \textit{ is even } (n \neq 6). \end{cases}$$

P r o o f. First, we observe that any γ_h-set of T_n will also be a γ_h-set for DT_n, TT_n and FT_n because they share a common path.

For $n = 3$ and 6, $\gamma_h(T_3) = \gamma_h(T_6) = 3$.

Let us label the vertices of the common path as $\{v_1, v_2 \ldots v_n\}$ and the remaining vertices of T_n be $S = \{x_{i,i+1}\}$, $1 \leq i \leq (n-1)$ as shown in Fig. 8.

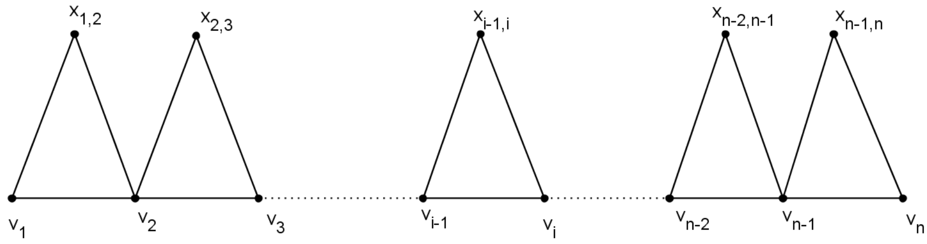

Figure 8. Triangular snake T_n.

Case 1: n is odd and $n \geq 5$. While finding any γ_h-set of T_n, the vertices v_2 and v_{n-2} are taken and the remaining vertices from the subset $S_k \subseteq S$ where

$$S_k = \{x_{i,i+1} |\, i \equiv 0 \pmod 2, \;\; 2 \leq i \leq (n-3)\}.$$

Clearly, $|S_k| = (n-3)/2$. Thus,

$$\gamma_h(T_n) \leq 2 + \left\lfloor \frac{n-3}{2} \right\rfloor.$$

It is easily seen that

$$\gamma_h(T_n) \geq 2 + \left\lfloor \frac{n-3}{2} \right\rfloor.$$

Therefore,

$$\gamma_h(T_n) = 2 + \left\lfloor \frac{n-3}{2} \right\rfloor.$$

Case 2: n is even and $n \neq 6$. Note that any common vertex say $x_{i,i+1}$ hop dominates the vertices v_{i-2} and v_i of P_n, $x_{i-2,i-1}$, $x_{i+1,i+2}$. Equivalently, the hop degree of any vertex is at most 4. Hence by choosing vertices from the set

$$S_k' = \{x_{i,i+1} |\, i \equiv 0 \pmod 2, \;\; 2 \leq i \leq (n-2)\} \subseteq S,$$

any γ_h-set can be obtained which includes the non-hop dominated vertices v_2 and v_{n-1} too. Thus,

$$\gamma_h(T_n) \leq 2 + \left\lfloor \frac{n-2}{2} \right\rfloor.$$

It is observed that

$$\gamma_h(T_n) \geq 2 + \left\lfloor \frac{n-2}{2} \right\rfloor.$$

Therefore,

$$\gamma_h(T_n) = 2 + \left\lfloor \frac{n-2}{2} \right\rfloor.$$

□

Definition 7 [5]. *An alternate triangular snake AT_n is a graph obtained from the path P_n, in which every alternate edge of a path is replaced by a cycle C_3. For example, an alternate double triangular snake is shown in Fig. 9.*

An alternate double triangular snake $AD(T_n)$ is obtained from two alternate triangular snakes that share a common path. For example, an alternate double triangular snake is illustrated in Fig. 10.

An alternate triple (four) triangular snakes $AT(T_n)(AF(T_n))$ consists of three (four) alternate triangular snakes that share a common path.

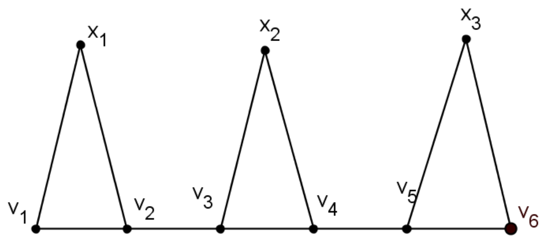

Figure 9. Alternate Triangular snake AT_6.

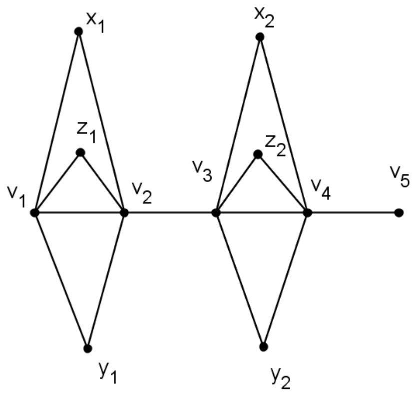

Figure 10. Alternate Triangular snake ADT_5.

Theorem 8.

$$\gamma_h(AT_n) = \gamma_h(AD(T_n)) = \gamma_h(AT(T_n)) = \gamma_h(AF(T_n)) = \begin{cases} \dfrac{n}{2}, & \textit{if} \quad n \equiv 0,2 \pmod 4; \\ \dfrac{n+1}{2}, & \textit{if} \quad n \equiv 3 \pmod 4; \\ \dfrac{n-1}{2}, & \textit{if} \quad n \equiv 1 \pmod 4. \end{cases}$$

P r o o f. Let us follow the labeling of vertices as described in Theorem 7. Here, $d(v_i) = 3$, $i \neq 1, n$ and any vertex of path P_n hop dominates at most 3 vertices. In any γ_h-set, it is clear that central vertices of P_n alone appear consecutively (see Fig. 11–12).

Case 1: n is even.

Case 1.1: $n \equiv 0 \pmod 4$. Here, any γ_h-set is of the form

$$S = \{v_i | \ i \equiv 2 \pmod 4, \ \ 2 \leq i \leq (n-2)\} \cup \{v_j | \ j \equiv 3 \pmod 4, \ \ 3 \leq j \leq (n-1)\}.$$

Thus, $\gamma_h(AT_n) \leq n/2$. It is easily seen that, $\gamma_h(AT_n) \geq n/2$. Therefore, $\gamma_h(AT_n) = n/2$.

Case 1.2: $n \equiv 2 \pmod 4$ In this case, v_{n-2} must be chosen in any γ_h-set and the remaining vertices are chosen from $\{v_2, v_3, v_6, v_7, \dots v_{n-4}, v_{n-3}\}$. Thus, $\gamma_h(AT_n) \leq (n-2)/2 + 1 = n/2$. It is easily seen that $\gamma_h(AT_n) \geq n/2$. Therefore, $\gamma_h(AT_n) = n/2$.

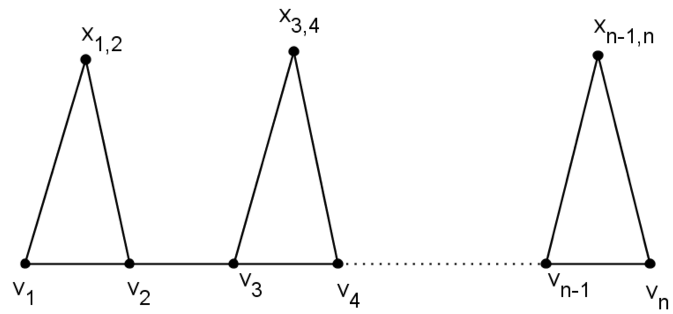

Figure 11. AT_n, when n is even.

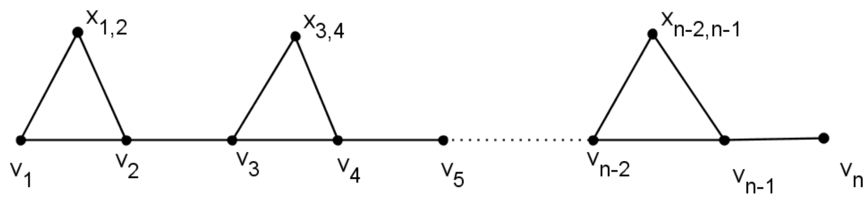

Figure 12. AT_n, when n is odd.

Case 2: n is odd.

Case 2.1: $n \equiv 1 \pmod 4$. In this case, any γ_h-set are chosen from $\{v_2, v_3, v_6, v_7, \ldots v_{n-3}, v_{n-2}\}$, with cardinality $(n-1)/2$. Thus, $\gamma_h(AT_n) \leq (n-1)/2$ and it is easy to verify that $\gamma_h(AT_n) \geq (n-1)/2$. Therefore, $\gamma_h(AT_n) = (n-1)/2$.

Similarly, the case for $n \equiv 3 \pmod 4$ follows. □

Definition 8. *A Quadrilateral snake Q_n is a graph obtained by replacing each edge of a path P_n by a cycle of length 4.*

An alternate quadrilateral snake AQ_n is obtained from the path P_n by replacing its alternate edges with C_4.

Proposition 1. (i) $\gamma_h(Q_n) = \begin{cases} \dfrac{n+2}{2}, & \text{if} \quad n \equiv 0,\, 2 \pmod 4; \\ \dfrac{n+3}{2}, & \text{if} \quad n \equiv 1 \pmod 4; \\ \dfrac{n+1}{2}, & \text{if} \quad n \equiv 3 \pmod 4. \end{cases}$

(ii) $\gamma_h(AQ_n) = \begin{cases} \dfrac{n+1}{2}, & \text{if} \quad n \equiv 1,\, 3 \pmod 4; \\ \dfrac{n+2}{2}, & \text{if} \quad n \equiv 0,\, 2 \pmod 4. \end{cases}$

Definition 9. *Ciliate is a graph $C(p, s)$ obtained from p disjoint copies of the path P_s by linking one end point of each such copy in the cycle C_p. For example, a Ciliate C(3, 3) is shown in Fig. 13.*

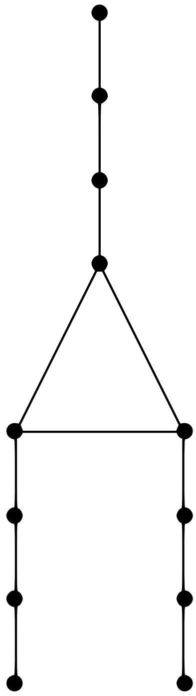

Figure 13. Ciliate $C(3,3)$.

Remark 2. $\gamma_h[C(p,q)] = p\,\gamma_h(P_q)$.

Definition 10. *A generalized ciliate $GC(p,s,t)$ is obtained by attaching t-copies of path P_s to each vertex of the cycle C_p.*

Proposition 2. $$\gamma_h[GC(3,s,t)]=\begin{cases} 2+3t\left\lfloor\frac{s}{3}\right\rfloor, & \text{if } s\equiv 0 \pmod 6;\\ 3+3t\left\lfloor\frac{s}{3}\right\rfloor, & \text{if } s\equiv 1,3 \pmod 6;\\ 3t\left\lfloor\frac{s}{3}\right\rfloor+1, & \text{if } s\equiv 4,5 \pmod 6;\\ 3+3t\left\lfloor\frac{s}{3}\right\rfloor, & \text{if } s\equiv 2 \pmod 6.\end{cases}$$

Theorem 9. $$\gamma_h[GC(4,s,t)]=\begin{cases} 2+4t\left\lfloor\frac{s}{3}\right\rfloor, & \text{if } s\equiv 0,1 \pmod 6;\\ 4+4t\left\lfloor\frac{s}{3}\right\rfloor, & \text{if } s\equiv 2 \pmod 6;\\ 4t\left\lfloor\frac{s}{3}\right\rfloor, & \text{if } s\equiv 3,4,5 \pmod 6.\end{cases}$$

P r o o f. Let us denote the vertices in the i^{th} copy of the path P_s as $\{v_1^i, v_2^i \dots v_s^i : 1 \le i \le t\}$. as shown in Fig. 14. Clearly, to hop dominate the leaves and its support vertices in every i^{th} copy of P_s, the vertices v_{s-2}^i and v_{s-3}^i $(1 \le i \le t)$ have to be chosen for any γ_h-set of $GC(4,s,t)$.

Case 1: $s \equiv 0, 1 \pmod 6$. To hop dominate v_1's and the vertices of the cycle, any γ_h-set includes u_2, u_3. The remaining vertices in each copy of P_s in GC(4,s,t) will induce a path, thus it is sufficient to add to $\{v_{s-2}^i, v_{s-3}^i, v_{s-8}^i, v_{s-9}^i \dots v_5^i, v_4^i\}$, $1 \le i \le s$ to γ_h-set of $GC(4,s,t)$. Thus,

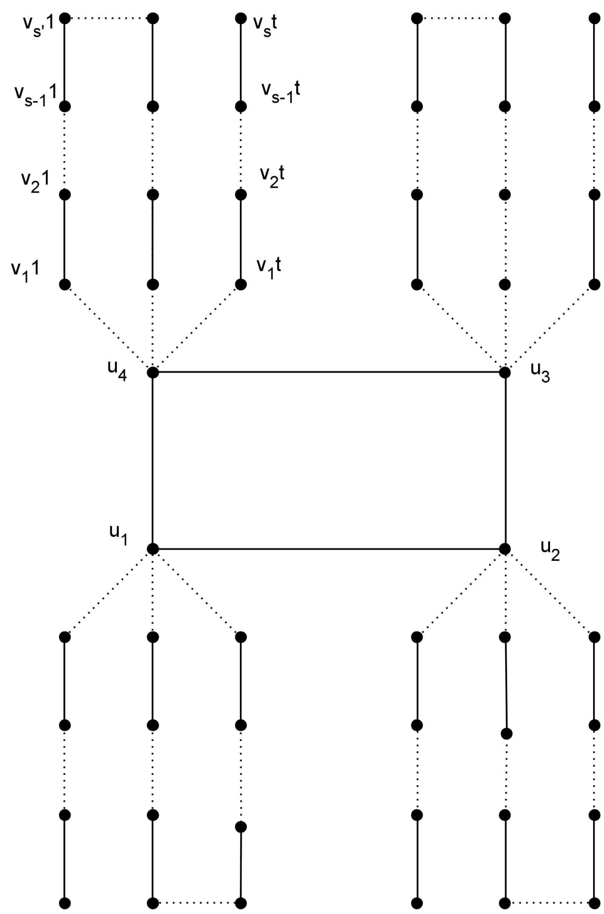

Figure 14. Generalized ciliate $GC(4,s,t)$.

$\gamma_h[GC(4,s,t)] \leq 2 + 4t\gamma_h(P_s) \leq 4t\lfloor s/3 \rfloor$ and it is easily seen that $\gamma_h[GC(4,s,t)] \geq 2 + 4t\gamma_h(P_s)$. Therefore,

$$\gamma_h[GC(4,s,t)] = 2 + 4t\gamma_h(P_s).$$

Case 2: $s \equiv 2 \pmod 6$. Any γ_h-set comprises u_1, u_2, u_3, u_4 to hop dominate v_1^i and v_2^i as well as the vertices of the cycle. Each copy's remaining vertices will induce a path on $(s-2)$ vertices. As a result, $\{v_{s-2}^i, v_{s-3}^i, v_{s-8}^i, v_{s-9}^i \dots v_6^i, v_5^i\}$ are required to form a γ_h-set of $GC(4,s,t)$. Thus, $\gamma_h[GC(4,s,t)] \leq 4 + 4t\lfloor s/3 \rfloor$ and it is easy to show that $\gamma_h[GC(4,s,t)] \geq 4 + 4t\lfloor s/3 \rfloor$. Therefore,

$$\gamma_h[GC(4,s,t)] = 4 + 4t\lfloor s/3 \rfloor.$$

Case 3: $s \equiv 3, 4, 5 \pmod 6$. When $s \equiv 3, 5 \pmod 6$, $H = \{v_{s-2}^i, v_{s-3}^i, v_{s-8}^i, v_{s-9}^i \dots v_3^i, v_2^i\}$ forms a γ_h-set of $GC(4,s,t)$, whereas for $s \equiv 4 \pmod 6$, $H = \{v_{s-2}^i, v_{s-3}^i, v_{s-8}^i, v_{s-9}^i \dots v_2^i, v_1^i\}$ forms a γ_h-set. Thus, $\gamma_h[GC(4,s,t)] = 4t\gamma_h(P_s) \leq 4t(\lfloor s/3 \rfloor + 1)$. It is easily seen that $\gamma_h[GC(4,s,t)] \geq 4t(\lfloor s/3 \rfloor + 1)$. Therefore,

$$\gamma_h[GC(4,s,t)] = 4t(\lfloor s/3 \rfloor + 1).$$

□

3. Conclusion

In this study, we computed hop domination number for some special families of graphs like triangular, quadrilateral, alternate triangular, alternate quadrilateral snake graphs and examined hop domination number for some generalized graph structures like generalized theta graph, glued path graph. In future, the result obtained for generalized ciliates $p = 3, 4$ may be extended to $p > 4$.

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ON A CLASS OF EDGE–TRANSITIVE DISTANCE–REGULAR ANTIPODAL COVERS OF COMPLETE GRAPHS[1]

Ludmila Yu. Tsiovkina

Krasovskii Institute of Mathematics and Mechanics,
Ural Branch of the Russian Academy of Sciences,
16 S. Kovalevskaya Str., Ekaterinburg, 620108, Russia

tsiovkina@imm.uran.ru

Abstract: The paper is devoted to the problem of classification of edge-transitive distance-regular antipodal covers of complete graphs. This extends the classification of those covers that are arc-transitive, which has been settled except for some tricky cases that remain to be considered, including the case of covers satisfying condition $c_2 = 1$ (which means that every two vertices at distance 2 have exactly one common neighbour).

Here it is shown that an edge-transitive distance-regular antipodal cover of a complete graph with $c_2 = 1$ is either the second neighbourhood of a vertex in a Moore graph of valency 3 or 7, or a Mathon graph, or a half-transitive graph whose automorphism group induces an affine 2-homogeneous group on the set of its fibres. Moreover, distance-regular antipodal covers of complete graphs with $c_2 = 1$ that admit an automorphism group acting 2-homogeneously on the set of fibres (which turns out to be an approximation of the property of edge-transitivity of such cover) are described.

A well-known correspondence between distance-regular antipodal covers of complete graphs with $c_2 = 1$ and geodetic graphs of diameter two that can be viewed as underlying graphs of certain Moore geometries, allows us to effectively restrict admissible automorphism groups of covers under consideration by combining Kantor's classification of involutory automorphisms of these geometries together with the classification of finite 2-homogeneous permutation groups.

Keywords: Distance-regular graph, Antipodal cover, Geodetic graph, Arc-transitive graph, Edge-transitive graph, 2-transitive group, 2-homogeneous group.

Introduction

A distance-regular antipodal cover of a complete graph can be defined as a connected graph whose vertex set admits a partition into n classes (called *fibres*) of the same size $r \geq 2$ such that each class induces a coclique, the union of any two distinct classes induces a perfect matching, and any two non-adjacent vertices from distinct classes have exactly $c_2 \geq 1$ common neighbours. According to [8], such a graph will be referred to as an (n, r, c_2)-*cover*. One can see that an (n, r, c_2)-cover is indeed a cover (or a covering graph) of the complete graph K_n in the topological sense (see [8] or [7]), and that its diameter is 3.

To date, almost all arc-transitive (n, r, c_2)-covers have been classified (see [14, 15, 19–22]), except for the following two tricky cases: when an arc-transitive automorphism group induces an affine permutation group on the set of fibres (see [22]) or $c_2 = 1$ (see a discussion below in this section). Note that an arc- or, more generally, edge-transitive automorphism group of an (n, r, c_2)-cover induces a 2-homogeneous action on its fibres. The purpose of this paper is to study the $(n, r, 1)$-covers whose automorphism group acts 2-homogeneously on the set of fibres, and to describe those that are edge-transitive.

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The afore-mentioned interplay between edge-transitivity and 2-homogeneity allows us to base our arguments on the classification of finite 2-homogeneous permutation groups, which follows from the classification of finite 2-transitive permutation groups and the Kantor's fundamental result [11]. To investigate admissible groups of automorphisms, we also exploit a remarkable correspondence between $(n, r, 1)$-covers and geodetic graphs of diameter two (see [2]) that are equivalent to certain Moore geometries. The classification of involutory automorphisms of these geometries that is due to Kantor [12] together with the Higman's technique for studying automorphisms of association schemes (e.g., see [4, Section 3.7]) turn out to be effective tools for their description.

Main results of this paper are presented by the following two theorems.

Theorem 1. *Let Δ be a $(k+1, r, 1)$-cover with $s := k - r + 1 > 1$, let Σ be the set of fibres of Δ, and $G = \mathrm{Aut}(\Delta)$. Denote by K and G^Σ the kernel and the image of the induced action of G on Σ, respectively. Then $k = cs$ and $r = cs - s + 1$ for some $c \in \mathbb{Z}$, and the following statements hold:*

(1) *if G^Σ is a 2-homogeneous, but not 2-transitive group, then*

$$G^\Sigma \le \mathrm{A\Gamma L}_1(q), \quad k + 1 = q \equiv 3 \pmod 4, \quad \sqrt{cs + (s/2 - 1)^2} \notin \mathbb{Z},$$

and either $K = 1$, $s = 2$ and $c = (q - 1)/2$, or s is odd;

(2) *if G^Σ is an almost simple 2-transitive group, then either $K = 1$, $s = 2$, $c = 2^{e-1}$, $\mathrm{Soc}(G) \simeq \mathrm{L}_2(2^e)$ and Δ is a Mathon graph, or G acts intransitively on vertices of Δ;*

(3) *if G^Σ is an affine 2-transitive group, then G acts intransitively on arcs of Δ.*

Theorem 2. *Suppose Δ is an edge-transitive $(k+1, r, 1)$-cover, let Σ be the set of fibres of Δ, and $G = \mathrm{Aut}(\Delta)$. Denote by K and G^Σ the kernel and the image of the induced action of G on Σ, respectively. Then either $k = r \in \{2, 6\}$ and Δ is the second neighbourhood of a vertex in a Moore graph of valency $k + 1$, or $k > r$ and one of the following statements holds:*

(1) *G^Σ is an almost simple 2-transitive group, $K = 1$, $\mathrm{Soc}(G) \simeq \mathrm{L}_2(2^e)$ and Δ is an arc-transitive Mathon graph of valency $k = 2^e$;*

(2) *G^Σ is an affine 2-homogeneous group and Δ is a half-transitive graph.*

Recall that the only Moore graphs of valency 3 or 7 are the Petersen graph or the Hoffman-Singleton graph, respectively (see [10]). Note that for each admissible k the resulting graph in Theorem 2 (1) is unique (up to isomorphism) and its construction is due to Mathon (e.g., see [3, Proposition 1.17.3]).

We also remark here that in [6, Proposition 4] it was claimed that each $(k+1, r, 1)$-cover with $s > 1$ that possesses a group of automorphisms acting 2-homogeneously on the fibres necessarily has valency $k = 2^e$ and $s = 2$. Unfortunately, the proof of this result (see an exposition in [16]) is flawed; Theorem 1 shows that it holds under the additional assumption of arc-transitivity of the graphs under consideration. Thus, compared together with previous results (see [14, 15]), the classification of arc-transitive (n, r, c_2)-covers in the almost simple case is complete.

The organization of the paper is as follows. In Section 1 we recall some basic definitions and facts on $(n, r, 1)$-covers. In Section 2 we obtain general results on automorphisms of such a graph. Section 3 is devoted to the proofs of Theorems 1 and 2.

1. Preliminaries

Throughout the paper we consider only finite undirected graphs without loops or multiple edges. By a *subgraph* of a graph Γ we mean a vertex-induced subgraph, and we also identify a subset X of vertices of Γ with the subgraph of Γ that is induced by X. The distance between vertices x and y of a graph Γ is denoted by $\partial_\Gamma(x,y)$, or simply $\partial(x,y)$ if Γ is clear from the context. For a vertex a of a graph Γ, we denote by $\Gamma_i(a)$ the *i-th neighbourhood* of a, that is the subgraph of Γ induced by the set $\{b \in \Gamma \mid \partial_\Gamma(a,b) = i\}$. The number of neighbours of a vertex a, i.e., the size of $\Gamma_1(a)$, is the *valency* of a in Γ. For a fixed graph Γ and any its vertex a, the subgraph $\Gamma_1(a)$ is also denoted by $[a]$ if the graph Γ is clear from the context; we also put $a^\perp := \{a\} \cup [a]$. A graph is said to be *regular* if all its vertices have the same valency; a graph is said to be *biregular* if it is not regular and every of its vertices has one of two possible valencies.

A graph is *geodetic* if every two of its vertices are joined by a unique shortest path. A biregular geodetic graph of diameter two that is not contained in $a^\perp$ for any its vertex a is referred to as a BRG-*graph*.

A connected graph Γ of diameter d is called *distance-regular* if there are integers c_i, a_i and b_i, for all $i \in \{0, 1, \dots, d\}$, such that for each pair of vertices x and y with $\partial_\Gamma(x,y) = i$, the following equalities hold:

$$c_i = |\Gamma_{i-1}(x) \cap \Gamma_1(y)|, \quad a_i = |\Gamma_i(x) \cap \Gamma_1(y)| \quad \text{and} \quad b_i = |\Gamma_{i+1}(x) \cap \Gamma_1(y)|,$$

where $b_d = c_0 = 0$ by definition; in particular, $|\Gamma_1(x)| = b_0 = c_i + a_i + b_i$ holds for any $i \in \{0, 1, ..., d\}$. The sequence $\{b_0, b_1, \dots, b_{d-1}; c_1, \dots, c_d\}$ is called the *intersection array* of Γ.

A distance-regular graph of diameter 2 is also called *strongly regular*. A graph is said to be *edge regular* if it is regular and there is a non-negative integer λ such that every two adjacent vertices have exactly λ common neighbours; a graph is said to be *amply regular* if it is edge-regular and there is a non-negative integer μ such that every two vertices at distance 2 have exactly μ common neighbours.

If the binary relation "to be at distance 0 or d" on the set of vertices of a connected graph Γ of diameter d is an equivalence relation, then the graph Γ is called *antipodal*; the classes of this relation are called *antipodal classes* or *fibres* of Γ. We will say that an antipodal graph Γ is an *antipodal cover* of a graph Δ, if Γ is not a complete graph and the following three conditions are satisfied: (i) every fibre of Γ induces a coclique, (ii) the union of any two distinct fibres of Γ induces a coclique or a perfect matching, and (iii) Δ is isomorphic to the graph $\bar{\Gamma}$ defined on the fibres of Γ, in which two vertices are adjacent if and only if the union of corresponding fibres forms a matching in Γ. By the Smith's theorem [3, Theorem 4.2.1], non-cyclic distance-regular graphs fall into families of primitive, bipartite or antipodal graphs. Every graph of diameter d from the latter family is a complete graph or a complete multipartite graph with parts of equal sizes if $d = 1$ or 2, and it is an antipodal cover of a distance-regular graph of diameter $\lfloor d/2 \rfloor$ when $d \geq 3$ [5]. Hence distance-regular antipodal covers of complete graphs are precisely antipodal distance-regular graphs of diameter 3. They do not have a universal construction and form a large infinite class of graphs that is closely related to many interesting combinatorial objects, like projective planes or generalized quadrangles; we refer the reader to [3, 8, 16] for more background.

For a subset X of a group acting on a set Ω, by $\mathrm{Fix}_\Omega(X)$ we denote the set of points in Ω that are fixed by every element of X. When $X = \{g\}$, we write "$\mathrm{Fix}_\Omega(g)$" instead of "$\mathrm{Fix}_\Omega(\{g\})$". We also write $\mathrm{Fix}(X) = \mathrm{Fix}_\Omega(X)$ if Ω is clear from the context. In what follows, for a graph Γ and a subset $X \subseteq \mathrm{Aut}(\Gamma)$, we identify the set $\mathrm{Fix}(X)$ with the subgraph of Γ that is induced by $\mathrm{Fix}(X)$.

A graph is called *vertex-transitive* or *edge-transitive*, if its automorphism group acts transitively on the set of its vertices or on the set of its edges, respectively. A graph is called *arc-transitive*, if its automorphism group acts transitively on set of its arcs (ordered pairs of adjacent vertices). A graph is called *half-transitive*, if it is vertex- and edge-transitive, but not arc-transitive.

Our other terminology and notation are mostly standard and follow [1, 3].

Further in this section, we provide some auxiliary results that are used in the proofs of Theorems 1 and 2.

Throughout the rest of the paper, Δ is an $(k+1, r, 1)$-cover, Σ is the set of fibres of Δ and $s := k - r + 1$. By [8, Theorem 3.4] there is an integer c such that $cs = k$, the number $cs + 1$ is odd and $s \leq c$. Put

$$v = (cs+1)(cs-s+1) \quad \text{and} \quad D = cs - s + 1 + s^2/4 = cs + (s/2-1)^2.$$

Then v is the number of vertices of Δ and its distinct eigenvalues are

$$\theta_0 = cs, \quad \theta_1 = (s-2)/2 + \sqrt{D}, \quad \theta_2 = -1, \quad \theta_3 = (s-2)/2 - \sqrt{D}$$

of respective multiplicities

$$m_0 = 1, \quad m_1 = \frac{(cs+1)(cs-s)}{2}\Big(1 - \frac{s-2}{2\sqrt{D}}\Big), \quad m_2 = cs, \quad m_3 = \frac{(cs+1)(cs-s)}{2}\Big(1 + \frac{s-2}{2\sqrt{D}}\Big).$$

Due to a result of Gardiner [5, Proposition 5.1] the eigenvalues of Δ are integral if $s \neq 2$. Hence for odd s the number $2\sqrt{D}$ is an odd integer (since $D = cs + 1 - s + s^2/4$), while for even $s > 2$ already the number $\sqrt{D}$ is an integer.

Let us construct a graph $\widehat{\Delta}$ by adding a coclique A to Δ, whose vertices are identified with the fibres of Δ, together with a vertex $\hat{b}$ such that $\widehat{\Delta}(\hat{b}) = A$, and assuming that a vertex $F \in A$ is adjacent to just those vertices of Δ which belong to the fibre $F \in \Sigma$. Note that each vertex from A has valency $r+1$ in $\widehat{\Delta}$.

It is easy to see that $\widehat{\Delta}$ is a geodetic graph of diameter two and hence by [3, Theorem 1.17.1] either $\widehat{\Delta}$ is a strongly regular graph and $s = 1$, or $\widehat{\Delta}$ is a BRG-graph with valencies $r+1$ and $k+1$, $r < k$, $s \geq 2$ and the following statements hold:

(1) if A and B denote the sets of vertices of $\widehat{\Delta}$ of valencies $r+1$ and $k+1$, respectively, then A is a coclique, for each vertex $a \in A$ the subgraph $[a]$ is a coclique, and if x and y is a pair of adjacent vertices from B, then $|[x] \cap [y]| = k - r = s - 1$;

(2) $|\widehat{\Delta}| = (r+1)(k+1) + 1$.

Moreover, each geodetic graph of diameter two that has no vertex adjacent to all others, can be viewed as the underlying graph of a *Moore geometry*, i.e. an incidence system of points and lines which satisfies the following axioms:

(i) there is at least one line, and each line has at least two points;

(ii) two points are on at most one line;

(iii) no point is collinear with all others;

(iv) two non-collinear points are both collinear with exactly one common point;

(v) a point not in a line is collinear with at most one point of the line;

(vi) there are no triangles or quadrangles of lines.

Also, by [5, Proposition 5.2], if $s=1$, then $\widehat{\Delta}$ is a Moore graph (and $c \in \{2,6,56\}$). In what follows we assume that $s \geq 2$, so $\widehat{\Delta}$ is a BRG-graph, and its corresponding Moore geometry is said to have type $(cs+2, cs-s+3, s+1)$ in this case (see [12, p. 314]).

We say that $\widehat{\Delta}$ has *type* $\mathcal{D}_\alpha$, if there is a projective plane $(X,\mathcal{L})$ of order $\alpha = r+1$ with a polarity π such that $\widehat{\Delta}$ is isomorphic to the graph on X, in which two vertices x and y are adjacent if and only if $x \in y^\pi$ (wherein $k = r+1$ and A coincides with the set of absolute points of the polarity π).

Lemma 1. *The following statements hold*:

(1) *$c \geq 2$, the number $cs+1$ is odd, $s \leq c$, and the neighbourhood of each vertex in Δ is the disjoint union of c isolated cliques of size s*;

(2) *if $c=2$, then Δ is a unique distance-regular graph with intersection array $\{4,2,1;1,1,4\}$ (the line graph of the Petersen graph) and $\widehat{\Delta}$ has type $\mathcal{D}_4$*;

(3) *if $2 < c \leq 1000$, then either $s=2$ and $\widehat{\Delta}$ has type $\mathcal{D}_{2c}$, or the pair $(s;D)$ is one of*: (4; 25), (4; 49), (3; 169/4), (6; 100), (9; 625/4), (4; 81), (11; 1225/4), 4; 121), (18; 784), (4; 169), (35; 8649/4), (4; 225), (10; 676), (4; 289), (4; 361), (21; 7921/4),(46; 4900), (4; 441), (11; 5625/4), (4; 529), (4; 625), (26; 4356), (14; 2500), (4; 729), (4; 841), (4; 961), (4; 1089), (4; 1225), (8; 2601), (15; 20449/4), (4; 1369), (5; 7569/4), (4; 1521), (9; 14161/4), (4; 1681), (152; 70225), (4; 1849), (4; 2025), (20; 10201), (4; 2209), (4; 2401), (144; 93025), (4; 2601), (56; 38025), (44; 30625), (114; 81796), (4; 2809), (4; 3025), (4; 3249), (4; 3481), (7; 25281/4), (4; 3721), (4; 3969).

P r o o f. The first two statements follow by [3, Proposition 1.17.3] and [8, Theorem 3.4].

To prove the third statement, first observe that the number of cliques of size $s+1$ of Δ equals $(cs+1)(cs-s+1)c/(s+1)$. Then, for $2 \leq s \leq c \leq 1000$, the computer check in GAP (which uses integrality conditions for the eigenvalues of Δ and their multiplicities together with the condition of integrality of the number $(cs+1)(cs-s+1)c/(s+1)$) gives just those feasible pairs $(s;D)$ that are listed in (3). The lemma is proved. □

The above restrictions for parameters of Δ will be frequently used in following arguments, in particular, the list of feasible parameters from Lemma 1 (3) will be needed in Section 3 to rule out the existence of Δ in a series of special cases.

Lemma 2. *Let Φ be an amply regular graph with $\mu = 1$ and suppose there is an automorphism g of Φ such that for a $\langle g \rangle$-orbit Ψ each vertex $x \in \Psi$ is adjacent to x^g. Then Ψ is a cycle or a clique.*

P r o o f. Suppose Ψ is not a clique. Denote by i the least number in $\{2,...,|\Psi|-1\}$ such that the vertices x and x^{g^i} are not adjacent. Then $\{x^g, x^{g^{i-1}}\} \subseteq [x] \cap [x^{g^i}]$ and hence $i=2$. Now let j denote the least number in $\{3,...,|\Psi|-1\}$ such that the vertices x and x^{g^j} are adjacent. Then $\{x^{g^{-1}}, x^{g^j}\} \subseteq [x] \cap [x^{g^{j-1}}]$ and hence $j = |\Psi|-1$. Thus, we conclude that Ψ is a cycle.

The lemma is proved. □

Recall that (x,y) denotes the greatest common divisor of x and y.

Lemma 3 [16, Lemma 2.2.1]. *The graph Δ has exactly $cs+1$ fibres, each of size $cs-s+1$, and the following statements hold*:

(1) *$s+1$ divides $c(c-1)(c-2)$ and each odd prime divisor of D divides $(s-2,c)(4c+1,s-4)(c-1,s^2+4)$;*

(2) *if $c>2$, then there is a divisor d of cs such that $s = d(d-2)/(c-d)$ and $\sqrt{D} = (d+c(d-2)/(c-d))/2$, and if an odd prime p divides $(D, s-2)$, then the p-part of d is less than p-part of c;*

(3) *if $cs = 2^n$, then $s=2$;*

(4) *if $s=2$, then $\widehat{\Delta}$ has type $\mathcal{D}_{2c}$.*

P r o o f. Note that Δ has exactly $cs+1$ fibres, each of size $cs-s+1$.

(1) Since there are exactly $c(cs+1)(cs-s+1)/(s+1)$ cliques of size $s+1$ in Δ, $s+1$ divides $c(c-1)(c-2)$. Let p be an odd prime divisor of D. Then p divides $(cs+1)(cs-s)(s-2)$, $(p,c) = (p,s-2)$ and $(p,cs+1) = (p,s^2/4-s) = (p,4c+1)$. So, we conclude $(p,c-1) = (p,1+s^2/4)$.

(2) Let $c>2$. Put $D=y^2$ and $y-s/2+1=d$. Then

$$y^2-(s/2-1)^2 = cs, \quad y+s/2-1 = cs/d.$$

Further,

$$y = (d+cs/d)/2, \quad s/2-1 = (cs/d-d)/2,$$

hence

$$s = d(d-2)/(c-d), \quad y = (d+c(d-2)/(c-d))/2.$$

Suppose an odd prime p divides $(D, s-2)$. As

$$s-2 = (d^2-2c)/(c-d),$$

we get that p-part of d is less than p-part of c.

(3) Let $cs = 2^n$. Suppose $s>2$. Then

$$2^n + (s/2-1)^2 = D$$

is a square of a positive integer y. Hence

$$y-s/2+1 = 2^l, \quad y+s/2-1 = 2^{n-l}, \quad y = 2^{l-1}+2^{n-l-1}, \quad s = 2^{n-l}-2^l+2.$$

Since s is a power of 2, we find $l=n-l$ or $l=1$. If $l=1$, then $s=2^{n-1}$, which implies $c=s=2$, while if $l=n-l$, then $s=2$, a contradiction in both cases.

(4) If $s=2$, then by [3, Proposition 1.17.2] $\widehat{\Delta}$ has type $\mathcal{D}_{2c}$ (and thus it can be constructed on the points of a projective plane of order $q=2c=D$ with a polarity π, whose absolute points form a line A).

The lemma is proved. □

2. Automorphisms of $(k+1, r, 1)$-covers

In this section, we prepare some technical results about automorphisms of Δ, which will be needed for the proof of Theorems 1 and 2.

The permutation representation of a group $G \leq \mathrm{Aut}(\Delta)$ in its natural action on the vertex set of Δ gives rise to a matrix representation $G \to \mathrm{GL}_v(\mathbb{C})$. Recall that $\mathbb{C}^v$ is the orthogonal direct sum of the eigenspaces $W_0, \ldots, W_3$ of the adjacency matrix of Δ, where W_i corresponds to the eigenvalue θ_i. As each W_i is a G-invariant subspace, it affords a character, say χ_i, of G. We can calculate values of this character using the theory of association schemes (see [4, Section 3.7]). Namely, let Q be the second eigenmatrix of Δ. (We assume that the first column of Q consists of the multiplicities m_i's.) Then, for an element $g \in G$, one has

$$\chi_i(g) = \frac{1}{v}\sum_{j=0}^{3} Q_{ij}\alpha_j(g),$$

where $\alpha_j(g)$ denotes the number of vertices x of Δ such that $\partial(x, x^g) = j$. Recall that every character value must be an algebraic integer; in particular, if the value is rational, then it is an integer. The second eigenmatrix Q for Δ was determined in [16].

Lemma 4 [16, Lemma 2.2.2]. *If $g \in \mathrm{Aut}(\Delta)$, then*

$$\chi_1(g) = \frac{(\sqrt{D} - s/2 + 1)}{(cs - s + 1)(2cs\sqrt{D})}\Big((c^2s^2 - cs^2 + s/2 + \sqrt{D} - 1)\alpha_0(g) + (s/2 + \sqrt{D} - 1)\times$$
$$\times(cs - s + 1)\alpha_1(g) - (cs - s/2 - \sqrt{D} + 1)\alpha_3(g)\Big) - (cs + 1)/(2\sqrt{D}),$$
$$\chi_2(g) = \frac{\alpha_0(g) + \alpha_3(g)}{cs - s + 1} - 1.$$

Lemma 5. *If $\alpha_3(g) = v$ for an element $g \in \mathrm{Aut}(\Delta)$, then $s > 2$.*

P r o o f. Suppose $\alpha_3(g) = v$ for an element $g \in \mathrm{Aut}(\Delta)$. Then by Lemma 4 we have

$$\chi_1(g) = -\frac{(\sqrt{D} - s/2 + 1)(cs + 1)}{2\sqrt{D}}.$$

Now if $s = 2$, then $\chi_1(g) = -(cs + 1)/2 \in \mathbb{Z}$, but $cs + 1$ is odd, a contradiction. The lemma is proved. □

Lemma 6. *If $\alpha_2(g) = v$ for an element $g \in \mathrm{Aut}(\Delta)$, then $s = 2$.*

P r o o f. Suppose $\alpha_2(g) = v$ for an element $g \in \mathrm{Aut}(\Delta)$. Then by Lemma 4 we have

$$\chi_1(g) = -\frac{cs + 1}{2\sqrt{D}}.$$

If s is even and $s \geq 4$, then 2 divides $cs + 1$, which is an odd number by Lemma 1, a contradiction. Suppose s is odd. Then c is even and $4D = 4(cs + 1) + s(s - 4)$ divides $(cs + 1)^2$, hence

$$(4(cs + 1) + s(s - 4))x = (cs + 1)^2$$

for an integer x. Put $y = (x, cs+1)$. Then $x = ay, cs+1 = by$, $(s, by) = 1$ and $b > a$. Thus

$$4ay + \frac{s(s-4)a}{b} = by,$$

and b divides $s-4$. Hence $c \geq s > a$ and

$$4a + s\frac{(s-4)a}{by} = b,$$

implying by divides $(s-4)a$. But then $s^2+1 \leq cs+1 \leq s(s-4)$, a contradiction. The lemma is proved. □

In Lemmas 7–9 it is supposed that there is an element $g \in \mathrm{Aut}(\Delta)$ of prime order p and $\Omega = \mathrm{Fix}(g)$. For a vertex $x \in \Delta$, we put

$$R_i(x) = \{y \in [x] \mid \partial(y, y^g) = i\},$$

where $i = 0, 1, 2, 3$, and by $F(x)$ we denote the fibre of Δ containing x.

Lemma 7 (cf. [16, Lemma 2.2.3], [12, Theorem 4.10 (i)–(ii)]). *Suppose $\Omega = \varnothing$. Then*

$$\alpha_3(g) = (cs - s + 1)t$$

with $t = |\mathrm{Fix}_\Sigma(g)|$, and $s = 2$ or the number $\alpha_1(g) + st/2 - cs - 1$ is a multiple of $\sqrt{D}$; moreover, the following statements hold:

(1) *if $p = 2$, then st is odd, $\alpha_1(g) = (cs-s+1)(cs+1-t)$, and if $\alpha_3(g) < v$, then $t = s(c-1)$ and the set $\{x \in \Delta \mid \partial(x, x^g) = 1\}$ is the disjoint union of $cs-s+1$ isolated cliques of size $s+1$;*

(2) *if $p = 3$ and $(3, s+1) = 1$, then $\alpha_1(g) = 0$, the number $cs-s+1$ is divisible by 3,*

$$\chi_1(g) = \frac{-(cs+1)(\sqrt{D} - s/2 + 1) + 3l(\sqrt{D} - s/2)}{2\sqrt{D}},$$

where $l = (cs+1-t)/3$, and $(cs+1)(s/2-1) - 3ls/2$ is a multiple of $\sqrt{D}$.

P r o o f. First, note that $(cs-s+1, cs+1) = 1$.

If $\partial(u, u^g) = 3$ for a vertex u, then $F(u) = F(u^g)$ and hence p divides $cs-s+1$ (the size of a fibre). In particular, if s is even, then $p > 2$.

By the integrality of $\chi_2(g)$, it follows that $\alpha_3(g) = (cs-s+1)t$ for a non-negative integer t. Further,

$$\chi_1(g) = \frac{\alpha_1(g) - (\sqrt{D} - s/2)t - (cs+1)}{2\sqrt{D}},$$

and if $s > 2$, then $\alpha_1(g) - (\sqrt{D} - s/2)t - (cs+1)$ is divisible by $2\sqrt{D}$.

(1) Let $p = 2$ and

$$\Phi = \{x \in \Delta \mid \partial(x, x^g) = 3\}.$$

Note that $\partial(x, x^g) \neq 2$ for any vertex x of Δ (as otherwise $[x] \cap [x^g] \subseteq \Omega$, which contradicts our assumption), so that

$$\Delta \setminus \Phi = \{x \in \Delta \mid \partial(x, x^g) = 1\}$$

and $\alpha_2(g) = 0$. Since $cs+1$ is odd, g fixes a fibre of Δ and st is odd. Also, we have $\alpha_1(g) = (cs-s+1)(cs+1-t)$ and

$$\chi_1(g) = \frac{(cs-s+1)(cs+1-t) - (\sqrt{D}-s/2)t - (cs+1)}{2\sqrt{D}} \in \mathbb{Z}.$$

Suppose $x \in \Phi$. Then $[x] = R_1(x) \cup R_3(x)$ and $|R_3(x)| = t-1$. Note that, for each edge $\{y,z\} \subset [x]$, we have $z \neq y^g$ (otherwise $y^g \in [x] \cap [x^g]$, which is impossible by assumption).

If $[x]$ contains an edge $\{y,z\} \subset \Delta \setminus \Phi$, then $[x^g]$ contains the edge $\{y^g, z^g\} \subset \Delta \setminus \Phi$, and, since $\{y,z\} \subset R_1(x)$, we get that $\{z, y, y^g, z^g\}$ is a 4-cycle or a clique, a contradiction. Hence each $(s+1)$-clique that contains an edge from $\Delta \setminus \Phi$, is contained in $\Delta \setminus \Phi$ itself. Since $\Delta \setminus \Phi$ is a regular graph of valency $cs-t$ and $\alpha_1(g) = |\Delta \setminus \Phi|$, we conclude that $\Delta \setminus \Phi$ is an edge regular graph with $\lambda_{\Delta \setminus \Phi} = s-1$ and the number of its edges equals $(cs+1-s)(cs+1-t)(cs-t)/2$. It follows that there are exactly $(cs+1-s)(cs+1-t)(cs-t)/((s+1)s)$ cliques of size $s+1$ in $\Delta \setminus \Phi$.

Now suppose that $x \in \Delta \setminus \Phi$. Then $[x] = R_1(x) \cup R_3(x)$, $|R_3(x)| = t$ and, as it was proved above, $[x] \cap [x^g] \subset \Delta \setminus \Phi$. Note that for each vertex $y \in R_1(x) \setminus \{x^g\}$ we have $y^g \in [x^g] \cap [y]$ and since $\{x, y, y^g, x^g\}$ cannot be a 4-cycle, we get $\{y, y^g\} \subset [x] \cap [x^g]$. This implies $|R_1(x)| = s$. On the other hand, $|R_1(x)| = cs-t$, which, by the preceding equality, implies that $s(c-1) = t$. Hence, there are exactly $cs+1-s$ cliques of size $s+1$ in $\Delta \setminus \Phi$ and $\alpha_1(g) = (cs-s+1)(s+1)$, which implies that $\Delta \setminus \Phi$ is the disjoint union of $cs-s+1$ isolated cliques of size $s+1$.

(2) Let $p=3$ and $(3, s+1) = 1$. Then $\alpha_1(g) = 0$ (otherwise there is a (unique) $(s+1)$-clique L that contains a 3-cycle $\{u, u^g, u^{g^2}\}$, yielding $L = L^g$, which contradicts the assumption $\Omega = \varnothing$). If 3 divides $cs+1$, then $\alpha_2(g) = v$ and by Lemma 6 we obtain $s=2$, a contradiction. Assume that there are exactly $3l$ fibres that are not fixed by g. Then $\alpha_2(g) = 3l(cs-s+1)$ and $\alpha_3(g) = (cs+1-3l)(cs-s+1)$. Hence

$$\chi_1(g) = \frac{-(cs+1)(\sqrt{D}-s/2+1) + 3l(\sqrt{D}-s/2)}{2\sqrt{D}} \in \mathbb{Z}$$

and $(cs+1)(s/2-1) - 3ls/2$ is a multiple of $\sqrt{D}$. The lemma is proved. □

Remark 1. Note that Lemma 7 (1) specifies statements of [12, Theorem 4.10 $(i),(ii)$], and Lemma 7 (2) corrects [16, Lemma 2.2.3] (namely, the condition $(3, s+1) = 1$ is missing there).

Lemma 8 (see [16, Lemma 2.2.4], [12, Theorem 4.10 (iv)–(vi)]). *Suppose $\Omega \neq \varnothing$ and $p=2$. Then one of the following statements holds*:

(1) *Ω is a fibre of Δ, and either $s=2$, or $cs^2 - s^2/2 + s - \alpha_1(g)$ is a multiple of $\sqrt{D}$*;

(2) *Ω is an $(s+1)$-clique and $c = s = 2$*;

(3) *Ω is an $(c's'+1, c's'-s'+1, 1)$-cover, and the parameters c, s, c' and s' satisfy the following equality*:

$$(cs-s+1)(cs-c's') = (c's'+1)(c's'-s'+1)\Big(cs - c's + \frac{c'(s-s')}{(s'+1)}\Big);$$

moreover, (i) $s'=2$ and $2c = (2c')^2$ if $s=2$, and (ii) $cs-s+1 = (c's'-s'+1)^2$ if $s > s' > 1$.

P r o o f. Let $p = 2$ and $a \in \Omega$. Note that, for each vertex $e \in \Omega \cap \Delta_3(a)$, the valency of e in Ω coincides with that of a. Indeed, for each vertex $x \in \Omega_1(a)$, there is a unique vertex $x' \in [x] \cap [e]$, and $x' \in \Omega$. Conversely, $\{x\} = [x'] \cap [a]$.

(1) Suppose that all vertices in Ω are at pairwise distance 3 in Δ and $|\Omega| = \omega$. Suppose further that for some vertex $a \in \Omega$ we have $F(a) \setminus \Omega \neq \varnothing$ and let $u \in F(a) \setminus \Omega$. Then $\partial(u, u^g) = 3$ and for each vertex $x \in [u]$ we get $x^\perp \subset \Delta \setminus \Omega$, hence $\partial(x, x^g) \neq 2$. If $\partial(x, x^g) = 3$ for all $x \in [u]$, then by [8, Corollary 6.3] $g = 1$, a contradiction. It follows that there is a vertex $x \in [u]$ such that $\partial(x, x^g) = 1$ and $s = |[x] \cap [x^g]| + 1$ is odd. Hence ω is even. Now if, for a vertex $y \in [u]$, $\partial(y, y^g) = 3$, then g fixes a vertex in $F(y)$ that has a neighbour in Ω, which contradicts our assumption. It follows that $R_1(u) = [u]$. Since $s \geq 3$, we may assume that $[u]$ contains an edge $\{x, y\}$. Then $[u^g]$ contains the edge $\{x^g, y^g\}$ and, as $[u^g] \cap [u] = \varnothing$, we get that $\{x^g, x, y, y^g\}$ is a 4-cycle, a contradiction.

Hence, $\omega = cs - s + 1$, that is $\Omega = F(a)$. Then $\alpha_3(g) = 0$, $\alpha_2(g)$ is divisible by $2s$, $\alpha_1(g) = cs\omega - \alpha_2(g)$ is divisible by $2s$, and

$$
\begin{aligned}
\chi_1(g) \;\; = \;\; & \frac{(\sqrt{D} - s/2 + 1)}{(2cs\sqrt{D})}\Big((c^2s^2 - cs^2 + s/2 + \sqrt{D} - 1) + (s/2 + \sqrt{D} - 1)\alpha_1(g)\Big) - \\
& (cs + 1)/(2\sqrt{D}) = ((cs - s)(\sqrt{D} - s/2 + 1) + \alpha_1(g) - cs)/(2\sqrt{D}).
\end{aligned}
$$

Thus, $s = 2$ or $2\sqrt{D}$ divides $s(c - 1)(\sqrt{D} - s/2 + 1) + \alpha_1(g) - cs$.

(2) Suppose Ω is an ω-clique. Then $1 < \omega \leq s + 1$. Suppose further that there is a vertex $x \in \Delta \setminus \Omega$ that has no neighbours in Ω. Clearly, $\partial(x, x^g) \neq 2$, and if $\partial(x, x^g) = 1$, then, since $|[a] \setminus \Omega| = cs - s$ is even, we get that s is even and $[x] \cap [x^g]$ contains a vertex from Ω, a contradiction. Hence $\partial(x, x^g) = 3$. Furthermore, each vertex of Ω has exactly $cs - \omega + 1$ neighbours in $\Delta \setminus \Omega$, among which there are exactly $cs - s$ vertices that do not belong to the maximal clique of Δ containing Ω. Hence there are exactly $s - \omega + 1 + \omega(cs - s)$ vertices in $\Delta \setminus \Omega$ that have a neighbour in Ω. Thus, $\alpha_1(g) + \alpha_2(g) = s + 1 - \omega + \omega(cs - s)$ and $\alpha_3(g) = \omega(cs - s)$. Then $v = \omega + s + 1 - \omega + \omega(cs - s) + \omega(cs - s)$, which implies $c = s = 2$ and $\omega = 3$.

(3) Suppose Ω contains a pair of vertices a and b such that $\partial(a, b) = 2$. Put $[a] \cap [b] = \{c\}$. Then $[a]$ contains a unique vertex $e \in \Delta_3(b)$ (which, obviously, belongs to Ω) and $\Omega_1(b)$ contains a unique vertex $f \in \Delta_3(a)$. Further,

$$|\Omega_1(a) \cap \Delta_2(b) \setminus c^\perp| = |\Omega_1(b) \cap \Delta_2(a) \setminus c^\perp|.$$

Let $X_1, ..., X_n$ denote the fibres that intersect Ω. Then a vertex in $X_1 \cap \Omega$ has a unique neighbour in each of the fibres $X_2, ..., X_n$, hence Ω is a regular graph of valency $n - 1$ and $|\Omega| = n|X_i \cap \Omega|$. Moreover, Ω is a $(|\Omega|/n)$-cover of an n-clique, in which any two non-adjacent vertices from distinct fibres, say $\Omega \cap X_i$ and $\Omega \cap X_j$, have exactly one common neighbour. It follows by [8, Lemma 3.1] that Ω is an $(c's' + 1, c's' - s' + 1, 1)$-cover, where $c's' = n - 1$ and, clearly,

$$s' - 1 = |\Omega_1(x) \cap \Omega_1(y)| \equiv s - 1 \pmod{p}.$$

Note that there are exactly $n(n - s')(cs - c's')$ edges between Ω and $\Delta \setminus \Omega$, and there are exactly $c'n(n - s')/(s' + 1)$ maximal cliques in Ω. Hence we find that the number of vertices of Δ that have exactly $s' + 1$ neighbours in Ω equals

$$\tau_{s'+1} := c'n(n - s')(s - s')/(s' + 1),$$

and the number of vertices of Δ that have exactly one neighbour in Ω equals

$$\tau_1 := n(n - s')(cs - c's).$$

Clearly, if there is a vertex $x \in \Delta \setminus \Omega$ that has no neighbour in Ω, then $F(a) \not\subset \Omega$ for all $a \in \Omega$, and, as above, we obtain $\partial(x, x^g) \neq 2$. Put

$$\Phi = \{y \in \Delta \mid \partial(y, y^g) = 1, \quad [y] \subset \Delta \setminus \Omega\}.$$

First we prove that $|\Omega| + n = s(c-1)$ in the case $\Phi \neq \varnothing$. Suppose $x \in \Phi$. Since g fixes the subgraph $[x] \cap [x^g]$ and $[x] \subset \Delta \setminus \Omega$, it follows that s is odd. We have $|R_1(x)| = s$, $|R_3(x)| = n$ (since $\partial(w, w^g) = 3$ if and only if g fixes a vertex in $F(w)$) and $|R_2(x)| = cs - |R_1(x)| - n$.

Let us compare the sizes of the sets $R_2(x)$ and $R_3(x)$. As Ω contains no vertices from $F(x) \cup F(x^g)$, we get that Ω contains a vertex $b \in \Delta_2(x) \cap \Delta_2(x^g)$, and, since the number $p^1_{22} = s(c-1)(cs-2)$ is even, the number of vertices in $\Delta_2(x) \cap \Delta_2(x^g) \cap \Omega$ is also even. For the vertex $y \in [b] \cap [x]$ we have $y^g \in [b] \cap [x^g]$ and $\partial(y, y^g) = 2$ (otherwise $\{x, x^g, y, y^g\}$ is a clique and $[b] \cap [x]$ contains y, y^g, which is impossible). Pick a vertex $w \in [x]$. If $\partial(w, w^g) = 2$, then $w, w^g \in [a]$ for a vertex $a \in \Omega$, x has a unique neighbour $u \in F(a)$, x^g has a unique neighbour $u^g \in F(a) = F(u)$ and $\{a, w, x, x^g, w^g\}$ is a 5-cycle. If $\partial(w, w^g) = 3$, then $w \in F(w^g)$ and, for each vertex $a \in \Omega$ such that $a \in \Delta_2(x) \cap \Delta_2(x^g) \cap F(w)$, we get that $\{a, u, x, x^g, u^g\}$ is a 5-cycle, where $\{u\} = [a] \cap [x]$. Since for each vertex $w \in R_3(x)$ there are exactly $c's' - s' + 1$ vertices in $\Omega \cap F(w)(\cap \Delta_2(x))$, there are exactly $c's' - s' + 1$ vertices y such that $\{y\} = [x] \cap [a] \subset R_2(x)$, where $a \in \Omega \cap F(w)$. Hence,

$$|R_2(x)| = (c's' - s' + 1)|R_3(x)|,$$

which implies

$$s = cs - n(c's' - s' + 2), \quad \text{and} \quad cs - s = n(c's' - s' + 2),$$

that is

$$|\Omega| + n = s(c-1).$$

Now consider the BRG-graph $\widehat{\Delta}$ and note that its corresponding Moore geometry $\mathcal{G}$ has type $(cs+2, cs-s+3, s+1)$ (in notation of [12, p. 314]). Since $\Omega \not\subseteq z^\perp$ for any vertex $z \in \widehat{\Delta}$, $\{\hat{b}\} \cup \Omega$ induces a subgeometry of $\mathcal{G}$ (recall, $\hat{b}$ denotes the vertex of $\widehat{\Delta}$ isolated in B), so by Lemma 7 and [12, Theorem 4.10] we obtain that one of the following three possibilities occurs: (i) $s' = 1$; (ii) $s = s'$; (iii) $s > s' > 1$ and $cs - s + 1 = (c's' - s' + 1)^2$ (or equivalently, $s(c-1) = s'(c'-1)(c's' - s' + 2)$).

Hence $\Phi = \varnothing$ and each vertex in $\Delta \setminus \bigcup\limits_j X_j$ has exactly one or $s'+1$ neighbours in Ω and, for all vertices $x \in \Delta$ such that $x^\perp \subset \Delta \setminus \Omega$ we have $\partial(x, x^g) = 3$. Thus,

$$\alpha_3(g) = v - |\Omega| - \tau_1 - \tau_{s'+1},$$

and, on the other hand,

$$\alpha_3(g) = (c's' + 1)(cs - s + 1 - c's' + s' - 1),$$

which together give

$$(cs - s + 1)(cs - c's') = (c's' + 1)(c's' - s' + 1)\Big(cs - c's + \frac{c'(s - s')}{(s' + 1)}\Big).$$

In particular, if $s = 2$, then, since $s' \leq s$ and

$$s' - 1 = |\Omega_1(x) \cap \Omega_1(y)| \equiv s - 1 \pmod 2,$$

we get $s' = 2$ and

$$cs - s + 1 = (c's' + 1)(c's' - s' + 1),$$

so that
$$2c-1=(2c'+1)(2c'-1) \quad \text{and} \quad 2c=(2c')^2.$$

The lemma is proved. □

Remark 2. Note that Lemma 8 specifies statements of [12, Theorem 4.10 (iv)–(vi)] and of [16, Lemma 2.2.4]. Also, the proof of Lemma 8 fills a gap in the proof of [16, Lemma 2.2.4 (3)], in which the case $\Phi=\varnothing$ was not excluded properly.

Lemma 9 [16, Lemma 2.2.5]. *If $\Omega\neq\varnothing$ and $p>2$, then one of the following statements holds:*

(1) *Ω is contained in a fibre of Δ;*

(2) *Ω is an ω-clique and $\omega\leq s+1$;*

(3) *Ω is an $(c's'+1, c's'-s'+1, 1)$-cover, where $c's'+1$ is the number of fibres of Δ intersecting Ω, and $s'-1=|\Omega_1(x)\cap\Omega_1(y)|\equiv s-1 \pmod p$.*

P r o o f. Let $a\in\Omega$. Then for each vertex $e\in\Omega\cap\Delta_3(a)$ we have $|\Omega_1(a)|=|\Omega_1(e)|$.

Clearly, if Ω consists of vertices that are at pairwise distance 3 in Δ, then the statement (1) is true, while if Ω is a ω-clique, then $\omega\leq s+1$ and the statement (2) holds.

Now let Ω contain two vertices a and b such that $\partial(a,b)=2$. Then $[a]$ contains a unique vertex that belongs to $\Delta_3(b)$ (and, obviously, to Ω) and $\Omega_1(b)$ contains a unique vertex that belongs to $\Delta_3(a)$. Put $[a]\cap[b]=\{x\}$. Then $|\Omega_1(a)\cap\Delta_2(b)\setminus x^{\perp}|=|\Omega_1(b)\cap\Delta_2(a)\setminus x^{\perp}|$.

Let $X_1,...,X_n$ denote the fibres of Δ that intersect Ω. Then each vertex in $X_1\cap\Omega$ has a unique neighbour in each of the fibres $X_2,...,X_n$, hence Ω is a regular graph of valency $n-1$ and $|\Omega|=n|X_i\cap\Omega|$. Moreover, Ω is a $(|\Omega|/n)$-cover of a n-clique, in which any two non-adjacent vertices from distinct fibres, say $\Omega\cap X_i$ and $\Omega\cap X_j$, have exactly one common neighbour. Thus, the remaining claims of (3) follow by [8, Lemma 3.1].

The lemma is proved. □

3. Proofs of Theorems 1 and 2

In this section, we prove Theorems 1 and 2.

From now on we assume that there is a subgroup $G\leq \mathrm{Aut}(\Delta)$ that induces a 2-homogeneous permutation group G^{Σ} on the set Σ of fibres of Δ and we denote by K the kernel of the induced action of G, so that $G/K\simeq G^{\Sigma}$. We also put $m:=|\Sigma|=cs+1$.

Note that by [8, Corollary 6.3] K is semiregular, in particular, $\alpha_3(g)=v$ for each non-trivial element $g\in K$. It also implies that K acts semiregularly both on the set of arcs of Δ and on the set of its cliques of size $s+1$. For each subgroup $X\leq K$, we denote by Δ^X the graph on the set of X-orbits, whose edges are the (unordered) pairs of X-orbits that are joined by an edge of Δ. In particular, if $1<|X|<r$, then by [8, Theorem 6.2, Corollary 6.3] Δ^X is a non-bipartite $(m,(cs-s+1)/|X|,|X|)$-cover.

First, in Lemma 10, we consider the case, when the group G^{Σ} is 2-homogeneous, but not 2-transitive.

Lemma 10. *If the group G^{Σ} is not 2-transitive, then $G^{\Sigma}\leq \mathrm{A\Gamma L}_1(q)$, $|\Sigma|=q\equiv 3 \pmod 4$, $\sqrt{D}\notin\mathbb{Z}$, and either $s=2$, $c=(q-1)/2$ and $\widehat{\Delta}$ is a graph of type $\mathcal{D}_{q-1}$, or s is odd.*

P r o o f. Suppose that G induces 2-homogeneous, but not 2-transitive permutation group G^Σ on Σ. As $m \geq 5$ is odd, it follows by [11, Theorem 1] that $G/K \simeq G^\Sigma \leq \mathrm{A\Gamma L}_1(q)$ and $|\Sigma| = q \equiv 3 \pmod 4$. If $\sqrt{D} \in \mathbb{Z}$, then $cs = D - (s/2-1)^2$ is an even difference of squares, which is impossible in this case. Hence $\sqrt{D} \notin \mathbb{Z}$ and either $s = 2$ or $c = (q-1)/2$ is odd, or s is odd. Finally, if $s = 2$, then by Lemma 3 we obtain that $\widehat{\Delta}$ is a graph of type $\mathcal{D}_{q-1}$. The lemma is proved. □

Further in Lemmas 11–14, we assume that the group G^Σ is 2-transitive; in this case by a Burnside's theorem, the group G^Σ is either almost simple, or affine, and we consider the corresponding cases in course, basing our argument on the classification of finite 2-transitive permutation groups (e.g. see [9, Theorem 2.9]).

Lemma 11. *Suppose the group G^Σ is almost simple. Then either $c = 2^{n-1}, s = 2$, $\widehat{\Delta}$ is a graph of type $\mathcal{D}_{2^n}$, $\mathrm{Soc}(G) \simeq \mathrm{L}_2(2^n)$ and Δ is a Mathon graph, or the group G acts intransitively on the vertices of Δ.*

P r o o f. Suppose that G induces an almost simple permutation group G^Σ on Σ. Then the socle H of G^Σ is a non-abelian simple group. In view of Lemma 1, we may assume that $s = 2$ (and the size of a fibre is $m-2$) or $|\Sigma| \geq 25$. Fix $F \in \Sigma$ and $a \in F$.

If $H = \mathrm{Sp}_{2n}(2)$, then the number $m \in \{2^{n-1}(2^n \pm 1)\}$ is even, a contradiction.

In the case $H = {}^2\mathrm{G}_2(q)$ we have $q = 3^{2e+1}$ and $m = q^3+1$ is even, a contradiction.

In the case $H = \mathrm{U}_3(q)$ we have $m = q^3+1$, hence $q = 2^e$. By Lemma 3 we have $s = 2$ and $\widehat{\Delta}$ is a graph of type $\mathcal{D}_{2^{3e}}$. Then by Lemma 5 we have $K = 1$ and, since $G_{\{F\}}$ contains no subgroup of index q^3-1, G cannot act transitively on the vertices of Δ.

In the case $H = {}^2\mathrm{B}_2(q)$ we have $q = 2^{2e+1}$ and $m = q^2+1$. By Lemma 3 we get $s = 2$ and $\widehat{\Delta}$ has type $\mathcal{D}_{2^{2(2e+1)}}$. By Lemma 5 it follows that $K = 1$ and, since $G_{\{F\}}$ contains no subgroup of index q^2-1, G cannot act transitively on the vertices of Δ.

If H is a Mathieu group M_m, then (since m is odd) $m \in \{11, 23\}$ and, by Lemma 1 we have $s = 2$.

If the pair (H, m) is one of $(\mathrm{L}_2(11), 11)$, $(\mathrm{M}_{11}, 12)$, $(\mathrm{Alt}_7, 15)$, $(\mathrm{L}_2(8), 28)$, $(\mathrm{HiS}, 176)$, or $(\mathrm{Co}_3, 276)$, then (since m is odd) $m = 11$, $H = \mathrm{L}_2(11)$ or $m = 15$, $H = \mathrm{Alt}_7$ and, by Lemma 1 we have $s = 2$.

If $m = 23$, then by [8, Theorem 5.4] $(-1)^{c-1}(2c-1) = 21 \equiv z^2 \pmod{11}$ for some $z \in \mathbb{Z}$ and by the Euler's criterion $21^5 \equiv 1 \pmod{11}$, a contradiction.

Similarly, for $m = 15$ by [8, Theorem 5.4] we get $(-1)^{c-1}(2c-1) = 13 \equiv z^2 \pmod 7$ for some $z \in \mathbb{Z}$ and by the Euler's criterion $13^3 \equiv 1 \pmod 7$, a contradiction.

If $m = 11$, then by Lemma 5 we have $G^\Sigma \simeq G$, and either $H = \mathrm{L}_2(11)$ and $H_{\{F\}} \simeq \mathrm{Alt}_5$, or $H = \mathrm{M}_{11}$ and $H_{\{F\}} \simeq \mathrm{Alt}_6 : \mathrm{Z}_2$. But in both cases G contains no subgroup of index 99, hence G cannot act transitively on the vertices of Δ.

It remains to consider "alternating" and "linear" cases.

1. Let $H = \mathrm{Alt}_m$. Then H contains an involution g that is a product of two independent transpositions and the number of the fibres that are fixed by g equals $m-4$.

Note that for $m = 5$ we have $c = s = 2$ and by Lemma 5 we have $\mathrm{L}_2(4) \simeq \mathrm{Alt}_5 \trianglelefteq G^\Sigma \simeq G$, and, moreover, if G is vertex-transitive, then it has a single orbit on arcs of Δ as well. So let $m \geq 7$.

1.1. First, suppose $K = 1$. Then $G \simeq G^\Sigma$ and we may identify H with the socle of G.

Put $\Omega = \mathrm{Fix}(g)$. If $\Omega = \varnothing$, then by Lemma 7 we obtain $m - 4 = s(c-1)$, that is $s = 3, m \geq 13$. In this case, H contains an involution g' that is a product of four independent transpositions and the number of the fibres that are fixed by g' equals $m-8$, and by Lemma 7 we have $\Omega' = \mathrm{Fix}(g') \neq \varnothing$ (otherwise $m - 8 = 3(c-1)$, which is impossible). Hence Ω' is distance-regular and its parameters satisfy the equality given in Lemma 8(3), which, in view of Lemma 1, contradicts the restriction $m \geq 13$.

Hence by Lemma 8 we obtain that Ω is an $(c's'+1, c's'-s'+1, 1)$-cover,

$$4(m-s) = (m-4)(m-s')\Big(m-1-c's+\frac{c'(s-s')}{(s'+1)}\Big)$$

and $m = 7$, that is $c's' = 2$ and Ω is a 6-cycle. But $s-1 \equiv s'-1 \pmod 2$ and hence Δ has intersection array $\{6,5,1;1,1,6\}$, which contradicts the assumption $s > 1$.

1.2. Now let $K > 1$. If $|K|$ is odd or coincides with the size of a fibre (so that $G_{\{F\}} = K : G_a$), then there are involutions $g \in G \setminus K$ with $|\mathrm{Fix}_\Sigma(g)| = m-4$ or $m-8$, and we proceed as in the subcase 1.1.

Suppose that $1 < |K| < cs - s + 1$ and G acts transitively on vertices of Δ. Then

$$\mathrm{Alt}_{m-1} \leq (G_{\{F\}})^\Sigma (\simeq G_{\{F\}}/K) \leq \mathrm{Sym}_{m-1},$$

and the graph Δ^K admits a vertex-transitive action of G/K, and the size of a fibre in $\Delta_1 = \Delta^K$ is $r' = (cs-s+1)/|K|$. If $r' = 2$, then G/K is a distance-transitive group of automorphisms of Δ_1 with $\mathrm{Alt}_{m-1} \leq G_aK/K \leq (\mathrm{Aut}(\Delta_1))_x$ for some vertex $x \in \Delta_1$, which implies that Δ_1 is bipartite, a contradiction. Hence $r' \geq 3$. But the degree of a minimal permutation representation of Alt_{m-1} is $m-1$ unless $m \leq 5$, so we obtain either $m = 5$ and $s = 1$, or $m \geq 7$ and

$$\mathrm{Alt}_{m-1} \leq G_aK/K \leq (\mathrm{Aut}(\Delta_1))_x$$

for some vertex $x \in \Delta_1$ (and hence $r' = |G_{\{F\}}/K : G_aK/K| \leq 2$), a contradiction in both cases.

2. Next we assume $H = \mathrm{L}_d(q)$. Then Σ can be regarded as the set of 1-dimensional subspaces of $V = \mathbb{F}_q{}^d$. Note that, since

$$m = \frac{(q^d-1)}{(q-1)}$$

must be odd, q is even or d is odd.

Let $d = 2$. Then $q = 2^n, m = q+1$ and by Lemma 3 we have $s = 2$, which by Lemma 5 implies $\mathrm{L}_2(q) \trianglelefteq G \leq \mathrm{P\Gamma L}_2(q)$. Note if the group G is vertex-transitive, then it has a single orbit on arcs of Δ, and moreover, its socle is also arc-transitive (otherwise Δ would be bipartite or disconnected, which is impossible), and hence Δ is a Mathon graph (see [3, Proposition 12.5.3]).

Suppose further that $d \geq 3$ and fix a basis $e_1, e_2, ..., e_d$ of V.

2.1. Assume first that $K = 1$. In the argument below, we identify H with the socle of G and consider various involutions $g \in H$ and subgraphs $\Omega = \mathrm{Fix}(g)$ of their fixed points.

2.1.1. Suppose q is odd. Then d is odd and there is an involution $g \in H$ such that its preimage in $\mathrm{SL}(V)$ fixes e_d and, for all $i \in \{1, 2, ..., d-1\}$, it maps e_i to $-e_i$, so that

$$|\mathrm{Fix}_\Sigma(g)| = \frac{(q^{d-1}-1)}{(q-1)} + 1.$$

If $\Omega = \varnothing$, then s is odd and by Lemma 7 we have

$$\frac{(q^{d-1}-1)}{(q-1)} + 1 = s(c-1).$$

But then

$$s = sc + 1 - s(c-1) - 1 = \frac{(q^d-1)}{(q-1)} - \frac{(q^{d-1}-1)}{(q-1)} - 2 = q^{d-1} - 2$$

and
$$\frac{(q^{d-1}-1)}{(q-1)}+1=(q^{d-1}-2)(c-1),$$
which contradicts the condition $c \geq s > 2$.

Hence by Lemma 8 we have that Ω is an $((q^{d-1}-1)/(q-1)+1, (q^{d-1}-1)/(q-1)-s', 1)$-cover and
$$(q^{d-1}-1)\Big(\frac{(q^d-1)}{(q-1)}-s\Big)=\Big(\frac{(q^{d-1}-1)}{(q-1)}+1\Big)\Big(\frac{(q^{d-1}-1)}{(q-1)}-s'+1\Big)\Big(cs-c's+\frac{c'(s-s')}{(s'+1)}\Big).$$
Hence $(q^{d-1}-1)/(q-1)+1$ divides $(q-1)(s+q-1)$, $d=3$ and $s=(q+11)/3$. But $(q+11, 3q(q+1))$ divides 330, which implies that the corresponding equation has no solution in natural numbers, a contradiction.

2.1.2. Now let q be even.

2.1.2 (a). If $d=2f+1$, then we assume that a preimage of g in $\mathrm{SL}(V)$ fixes e_d and, for $1 \leq i \leq d-1$, interchanges e_i with e_{d-1-i}. Then
$$|\mathrm{Fix}_\Sigma(g)|=\frac{(q^{f+1}-1)}{(q-1)}.$$
If $\Omega=\varnothing$, then s is odd and by Lemma 7 we have
$$\frac{(q^{f+1}-1)}{(q-1)}=s(c-1),$$
that is
$$s=q^{d-1}+...+q^{f+1}-1$$
and
$$\frac{(q^{f+1}-1)}{(q-1)}=(q^{d-1}+...+q^{f+1}-1)(c-1),$$
which contradicts the condition $c>2$.

Hence, by Lemma 8 we have that Ω is an $((q^{f+1}-1)/(q-1), (q^{f+1}-1)/(q-1)-s'-1, 1)$-cover and
$$\Big(\frac{(q^{2f+1}-1)}{(q-1)}-s\Big)q^{f+1}\frac{(q^f-1)}{(q-1)}=\frac{(q^{f+1}-1)}{(q-1)}\cdot\Big(\frac{(q^{f+1}-1)}{(q-1)}-s'\Big)\Big(cs-c's+\frac{c'(s-s')}{(s'+1)}\Big).$$
Hence
$$s=q^{f-1}+...+q+1, \quad s+q^f=sq+1, \quad c=q(q^f+1)$$
and
$$q^{2f+1}\frac{(q^f-1)}{(q-1)}=\Big(\frac{(q^{f+1}-1)}{(q-1)}-s'\Big)\Big(cs-c's+\frac{c'(s-s')}{(s'+1)}\Big),$$
and, since $q^f > s \geq s'$, we get $f=s=1$, a contradiction.

2.1.2 (b). For $d=2f$, we assume that a preimage of g in $\mathrm{SL}(V)$ fixes both e_{d-1} and e_d, and interchanges e_i with e_{d-2-i}, so that
$$|\mathrm{Fix}_\Sigma(g)|=\frac{(q^{f+1}-1)}{(q-1)}.$$

If $\Omega = \varnothing$, then s is odd and by Lemma 7 we have

$$\frac{(q^{f+1}-1)}{(q-1)} = s(c-1),$$

that is

$$s = q^{d-1} + ... + q^{f+1} - 1$$

and

$$\frac{(q^{f+1}-1)}{(q-1)} = (q^{d-1} + ... + q^{f+1} - 1)(c-1),$$

which contradicts the condition $c > 2$.

Hence, by Lemma 8, Ω is an $((q^{f+1}-1)/(q-1), (q^{f+1}-1)/(q-1) - s', 1)$-cover and

$$\Big(\frac{(q^{2f}-1)}{(q-1)} - s\Big)q^{f+1}\frac{(q^{f-1}-1)}{(q-1)} = \frac{(q^{f+1}-1)}{(q-1)} \cdot \Big(\frac{(q^{f+1}-1)}{(q-1)} - s'\Big)\Big(cs - c's + \frac{c'(s-s')}{(s'+1)}\Big).$$

If

$$\Big(\frac{(q^{f+1}-1)}{(q-1)}, \frac{(q^{f-1}-1)}{(q-1)}\Big) = 1,$$

then f is even, $q^f + ... + q + 1$ divides $sq^2 + q + 1$, $sq^2 = x(q^f + ... + q + 1) - q - 1$, q^2 divides $x - 1$ and

$$s \geq (q^2+1)(q^{f-2} + ... + q + 1) + q + 1.$$

But $s \leq c - 1$ and $cs = q^{2f-1} + ... + q^2 + q$, a contradiction.

Hence, f is odd and

$$\Big(\frac{(q^{f+1}-1)}{(q-1)}, \frac{(q^{f-1}-1)}{(q-1)}\Big) = q + 1.$$

It follows that $(q^{f+1}-1)/(q^2-1)$ divides $sq^2 + q + 1$, $sq^2 = x(q^f + ... + q + 1) - q - 1$, q divides $x - 1$, q^2 divides $x - (q+1)$ and $x = zq^2 + q + 1$ for a positive integer z. But

$$\frac{(q^{2f}-1)}{(q-1)} > s^2 > \frac{z^2(q^{f+1}-1)^2}{(q^2-1)^2} > q^{2f},$$

again a contradiction.

2.2. Now let $K > 1$. If $|K|$ is odd or coincides with the size of a fibre, then there are involutions $g \in G \backslash K$ with $|\mathrm{Fix}_\Sigma(g)|$ as in the subcase 2.1, and by a similar argument we come to a contradiction.

Suppose that $1 < |K| < cs - s + 1$, s is odd and G acts transitively on vertices of Δ.

Let $\widetilde{H}$ denote the full preimage of $H (= \mathrm{Soc}(G^\Sigma))$ in G and put $t = |\widetilde{H}_{\{F\}} : \widetilde{H}_a|$. We have

$$G^\Sigma \leq \mathrm{P\Gamma L}_d(q), \quad \widetilde{H}_{\{F\}}/K \simeq (\widetilde{H}_{\{F\}})^\Sigma \simeq \mathrm{E}_{q^{d-1}} \cdot \mathrm{SL}_{d-1}(q) \cdot \mathrm{Z}_{(q-1)/(d,q-1)}$$

(e.g. see [18]), and $(\widetilde{H}_a)^{[a]}$ is permutation isomorphic to $(\widetilde{H}_a K/K)^{\Sigma \backslash \{F\}}$. So

$$\frac{(cs-s+1)}{t}|G_a : \widetilde{H}_a| = |G : \widetilde{H}| = |G^\Sigma : H^\Sigma|$$

divides $(q-1)e$ (where $q = p^e$ for a prime p), and $|\widetilde{H}_{\{F\}} : \widetilde{H}_a K| = t/|K|$. Hence $\widetilde{H}_a \simeq \widetilde{H}_a K/K$ is isomorphic to a subgroup of $\mathrm{E}_{q^{d-1}} \cdot \mathrm{SL}_{d-1}(q) \cdot \mathrm{Z}_{(q-1)/(d,q-1)}$ with index $t/|K|$ dividing

$$(\frac{(q^d-1)}{(q-1)} - s)/|K|.$$

Since $\widetilde{H}_{\{F\}}/K$ contains a normal elementary abelian group R of size q^{d-1} that corresponds to a subgroup of $\mathrm{SL}(V)$, generated by transvections g with $g(e_1) = e_1$ and $g(u) - u \in \langle e_1 \rangle$ for all $u \in V$, we may assume $R \cap \widetilde{H}_a K/K \neq 1$ (otherwise q^{d-1} divides

$$t/|K| = \frac{(q^d - 1)}{(q-1)} - s = q^{d-1} + \frac{(q^{d-1} - 1)}{(q-1)} - s,$$

a contradiction).

Let q and d be odd. Then there is an element $h \in \widetilde{H}_a$ of order p such that

$$|\mathrm{Fix}_\Sigma(h)| = \frac{(q^{d-1} - 1)}{(q-1)},$$

and, since the valency of $\Omega = \mathrm{Fix}(h)$ is odd, by Lemma 9 Ω is a clique of size

$$\frac{(q^{d-1} - 1)}{(q-1)} \leq s + 1,$$

a contradiction.

Let q be even. Then there is an involution $h \in \widetilde{H}_a$ such that

$$|\mathrm{Fix}_\Sigma(h)| = \frac{(q^{d-1} - 1)}{(q-1)},$$

and, by Lemma 8 $\Omega = \mathrm{Fix}(h)$ is an $((q^{d-1} - 1)/(q-1), (q^{d-1} - 1)/(q-1) - s', 1)$-cover. Finally, it is easy to check that the equality given by Lemma 8(3) is not satisfied in this case, a contradiction.

The lemma is proved. □

Lemma 12. *If the group G^Σ is affine and the group G acts transitively on arcs of Δ, then $s+1$ divides $c(p^e, c-1)$, $s > 2$, $|K| = cs - s + 1$ and $|K|$ is divisible by $1 + lp$, where $cs + 1 = p^e$, p is a prime and l is a positive integer.*

P r o o f. Let $cs + 1 = p^e$ for a prime p, and denote by T the full preimage of $\mathrm{Soc}(G^\Sigma)$ in G. Since K acts semiregularly on each fibre, $(|K|, cs+1) = 1$ and hence each element $g \in T$ of order p has no fixed points. Besides, K has a complement T_0 in T that is an elementary abelian group of order p^e, and $K = O^p(T)$. Hence, by [1, 37.7], $N_T(T_0) = C_T(T_0)$.

Suppose that G acts transitively on arcs of Δ. Pick $F \in \Sigma$. Then for each vertex $a \in F$ the group G_a acts transitively on $[a]$. We have $T_{\{F\}} = K$ and $|T : T_a| = (cs+1)|K|$. Hence T acts transitively on the vertices of Δ if and only if K acts transitively on F.

Suppose that K acts intransitively on F. Since T-orbits comprise an imprimitivity system of G, each T-orbit is a coclique (otherwise, a T-orbit containing an edge induces a subgraph of valency cs in Δ, which is impossible by assumption). Hence $\alpha_2(g) = v$ for each element $g \in T$ of order p, and by Lemmas 6 and 5 we have $s = 2$ and $K = 1$.

Then $|F| = 2c-1$, T is a subgroup of order p^e, normal in G, and each T-orbit contains a unique vertex from every fibre of Δ. Note that T acts semiregularly on the set of 3-cliques of Δ. Hence the number of 3-cliques of Δ is divisible by $2c+1$, which implies 3 divides $c(c-2)$, and $p > 3$.

Further, there are exactly t ($t < cs - s + 1$) T-orbits that intersect $[a]$, so that $cs = tj$ for some positive integer j. Let us show that $t = c$. Indeed, G_a acts transitively on the set of non-trivial elements of T and the group TG_a induces 2-transitive permutation group of degree p^e on a^T. Now, since each vertex of $[a]$ is adjacent to exactly $j-1$ vertices from $a^T \setminus \{a\}$ and there are exactly

$cs(j-1)$ edges between $[a]$ and $a^T \setminus \{a\}$, each vertex from $a^T \setminus \{a\}$ is adjacent "on the average" to $j-1$ vertices from $[a]$. Hence $j-1=1$ and $cs=2t$, that is $t=c$.

Denote by $a_1, ..., a_c$ the vertices of F that have a neighbour in $a^T \setminus \{a\}$. Then the set $\sigma = \{[a_i] \cap a^T\}_{i=1}^c$ comprises an imprimitivity system of G_a on $a^T \setminus \{a\}$, and the set $\pi = \{[a] \cap {a_i}^T\}_{i=1}^c$ comprises an imprimitivity system of G_a on $[a]$.

For a block $\{x,y\} \in \pi$, there is a 2-element $h \in G_a$ that interchanges the vertices x and y, so $x = y^h$ and $h^2 \in G_{a,x,y}$, and hence, G_a contains an involution z that fixes $\{x,y\}$.

Suppose $c > 2$. Put $\Omega = \mathrm{Fix}(z)$. Then $F \setminus \{a, a_1, ..., a_c\} \neq \varnothing$ and by Lemma 8 we have either $\Omega = F$, or Ω is an $\big(2c'+1, 2c'-1, 1\big)$-cover and $2c = (2c')^2$. But in the second case by Lemma 4 we obtain $\chi_1(g) = -(2c+1)/(4c') \in \mathbb{Z}$ for an element $g \in T$ of order p, a contradiction. Hence $\langle z \rangle \leq G_F \trianglelefteq G_{\{F\}}$ and $G_{a,x} (= G_{a,x,y})$ is a $2'$-group, that is $h^2 = 1$ and we may assume that $h = z$. It implies that h also fixes the block $\{x', y'\} \in \pi$ with $\{x'\} = [a] \cap [x]$ and $\{y'\} = [a] \cap [y]$, and $(|G|)_2$ divides $2c$.

Thus, for each edge $\{x, x'\} \subset [a]$, there is a (unique) edge $\{y, y'\} \subset [a]$ such that $\{x,y\} \in \pi$ and $\{x', y'\} \in \pi$. On the other hand, for a block $\{x,y\} \in \pi$, there is an element $h \in G_a$ such that $\partial(x, x^h) = 1$, and, since $[a]$ is the disjoint union of edges, $h^2 \in G_{a,x,y}$. Without loss of generality, we may assume that h is a 2-element. But $G_{a,x}$ is a $2'$-group, that is h is an involution of G_a such that $\Omega = \mathrm{Fix}(h) \neq F$ (since h interchanges distinct orbits x^T and $(x')^T$). By Lemma 8 it follows that Ω is an $\big(2c'+1, 2c'-1, 1\big)$-cover and $2c = (2c')^2$, which together with Lemma 4 imply $\chi_1(g) = -(2c+1)/(4c') \in \mathbb{Z}$ for an element $g \in T$ of order p, a contradiction.

Hence $c = s = 2$ and again by Lemma 4 we obtain $\chi_1(g) = -(2c+1)/(2\sqrt{2c}) \in \mathbb{Z}$ for an element $g \in T$ of order 5, a contradiction.

Now suppose that K acts transitively on F. Then $G_{\{F\}} = K : G_a$ and, by Lemma 5 we have $s > 2$. Since K acts semiregularly on the set of cliques of size $s+1$ of Δ, the number of cliques of size $s+1$ of Δ is divisible by $cs - s + 1$ and hence $s+1$ divides $c(p^e, c-1)$.

Further, in view of Lemmas 6 and 2, for each element $g \in T$ of order p there is a $\langle g \rangle$-orbit that is a cycle or a clique. Besides, $|\mathrm{Syl}_p(T)|$ divides $|K|$. If $|\mathrm{Syl}_p(T)| = 1$, then $T_0 = O_p(T)$ and, hence, $T_0 \trianglelefteq G$. But in this case, each T_0-orbit contains an edge and hence induces a subgraph of valency cs of Δ, a contradiction. Therefore, $|K|$ is divisible by $1 + lp$ for some positive integer l. The lemma is proved. □

Further for a finite group X we denote by $X^{(\infty)}$ the last term of the commutator series of X.

Lemma 13. *Suppose that the group G^Σ is affine. Then the group G acts intransitively on arcs of Δ or $G^\Sigma \leq \mathrm{A\Gamma L}_1(q)$.*

P r o o f. Note that G^Σ can be identified with a subgroup of $\mathrm{A\Gamma L}_d(q)$, where q is a power of an odd prime p. Thus, the socle T of G^Σ is regarded as the additive group of a linear space V with dimension d over $\mathbb{F}_q$, $cs = q^d - 1$, and the stabiliser G_0 of the zero vector in G^Σ acts transitively on the set of non-zero vectors of V. Fix a basis $e_1, ..., e_d$ of V.

First, the case ${G_0}^{(\infty)} = G_2(q)$, as well as the cases $G_0 \in \{\mathrm{Alt}_6, \mathrm{Alt}_7, \mathrm{U}_3(3)\}$ are immediately ruled out, since $m = |\Sigma|$ must be odd.

Suppose further that G acts transitively on arcs of Δ. Take $F \in \Sigma$. Then by Lemma 5 we have $|K| = cs - s + 1$, $s > 2$ and $G_0 \simeq G_{\{F\}}/K \simeq G_a$ for each vertex $a \in F$, thus we may identify the groups ${G_a}^{[a]}$ and ${G_0}^{V \setminus \{0\}}$ in what follows.

Since K acts regularly on each fibre, $(|K|, cs+1) = 1$. Hence the full preimage T of $\mathrm{Soc}(G^\Sigma)$ in G contains an element y of order p that has no fixed points, and by Lemma 2 we obtain that each $\langle y \rangle$-orbit containing an edge is a cycle or a clique, while by Lemma 6 we have $\alpha_2(y) < v$.

Note if some $\langle y\rangle$-orbit is a clique, then y fixes an $(s+1)$-clique containing it, implying p divides $(c-1,s+1)$.

1. Suppose $G_0{}^{(\infty)} = \mathrm{SL}_d(q)$ or $\mathrm{Sp}_d(q)$, where $d \geq 2$, and take an involution $g \in G_0{}^{(\infty)}$ that maps e_i to $-e_i$ for $1 \leq i \leq 2$ and fixes e_i when $i \geq 3$ (in case $G_0{}^{(\infty)} = \mathrm{Sp}_d(q)$ we assume that (e_1, e_2) is a hyperbolic pair). Then the number of fibres that are fixed by g equals q^{d-2}. Put $\Omega = \mathrm{Fix}(g)$.

By Lemma 8 we obtain that Ω is an $\big(c's'+1, c's'-s'-1, 1\big)$-cover, $c's' = q^{d-2}-1$ and

$$(q^2-1)(q^d-s) = (q^{d-2}-s')\Big(cs - c's + \frac{c'(s-s')}{(s'+1)}\Big).$$

Thus, $q^{d-2}-s'$ divides $(q^2-1)(q^2s'-s)$ and $3 \neq d \leq 8$.

For $5 \leq d \leq 8$, there is an involution $g' \in G_0{}^{(\infty)}$ that maps e_i to $-e_i$ if $1 \leq i \leq 4$ and fixes e_i if $i \geq 5$ (in case $G_0{}^{(\infty)} = \mathrm{Sp}_d(q)$ we assume that (e_1, e_2) and (e_3, e_4) are hyperbolic pairs). The number of fibres that are fixed by g' equals q^{d-4}, and, again by Lemma 8, $\Omega' = \mathrm{Fix}(g')$ is an $\big(c''s''+1, c''s''-s'-1, 1\big)$-cover, $c''s'' = q^{d-4}-1$ and

$$(q^4-1)(q^d-s) = (q^{d-4}-s'')\Big(cs - c''s + \frac{c''(s-s'')}{(s''+1)}\Big),$$

which contradicts the assumption $5 \leq d \leq 8$.

If $d = 4$, then there is an involution $g' \in G_0{}^{(\infty)}$ that fixes both e_3 and e_4, and, for $1 \leq i \leq 2$, maps e_i to $-e_i$, so that there are exactly q^2 fibres that intersect $\Omega' = \mathrm{Fix}(g')$, and, by Lemma 8, Ω' is an $\big(q^2, q^2-1-s', 1\big)$-cover, $q^2-1 = c''s''$ and

$$(q^4-s)(q^2-1) = (q^2-s'')\Big(q^4-1-c''s + \frac{c''(s-s'')}{(s''+1)}\Big).$$

If $s'' = 1$, then $c'' = q^2-1$ and $q^4 - s = q^4 - 1 - (q^2-1)s + (q^2-1)(s-1)/2$, a contradiction. So $s'' > 1$, and

$$(q^4-s)(q^2-1) > (q^2-s'')\Big(q^4-1-c''s + \frac{c''(s-s'')}{(s''+1)}\Big),$$

again a contradiction.

Hence, $d = 2$, $Z(G_0^{(\infty)})$ contains a unique involution g that, for $1 \leq i \leq 2$, maps e_i to $-e_i$, Δ is an $(q^2, q^2-s, 1)$-cover and, by Lemma 8, $\Omega = \mathrm{Fix}(g)$ is a fibre. This implies $\alpha_0(g) = cs - s + 1$ and, by Lemma 4 we have

$$\chi_1(g) = \frac{(cs-s)(\sqrt{D}-s/2) - s + \alpha_1(g)}{2\sqrt{D}} \in \mathbb{Z}.$$

Suppose s is odd. Then $\alpha_1(g) = 0$ (otherwise g fixes a vertex in $[a]$ for some $a \in \Omega$, which is impossible) and c is even. In this case the odd number $2\sqrt{D}$ divides

$$\sqrt{D}s + s^2(c-1)/2 + s = s(\sqrt{D} + s(c-1)/2 + 1).$$

Since $(s, 2\sqrt{D}) = 1$, $2\sqrt{D}$ divides $cs - s + 2 = q^2 - s + 1$. Put $x = (cs+1, 2\sqrt{D})$. By Lemma 3 we have that x equals 1 or is a power of 3, and, since $(cs-s+2, cs-s) = 1$ and $(x, s-2) = 1$, $2\sqrt{D}$ divides $x(c, s-2)$. If $x = 1$, then $s - 2 < 2\sqrt{D} \leq (c, s-2)$, a contradiction. Hence $q^2 = 3^e$ and there is an element of order 3 in $G \setminus K$ that has no fixed points. But $(3, s+1) = 1$ and, by Lemma 7 we obtain that 3 divides $cs - s + 1 = q^2 - s$, which contradicts the fact $(|K|, cs+1) = 1$.

Thus s is even and $|K|$ is odd. It follows that K is solvable and K' is a normal subgroup of G that is properly contained in K. Hence the graph $\Delta^{K'}$ admits an arc-transitive action of G/K', and the abelian group K/K' can be considered as the group of all automorphisms of $\Delta^{K'}$ fixing its fibres. But the size of a fibre in $\Delta^{K'}$ coincides with $|K/K'|$ and $(|K/K'|, cs+1) = 1$, a contradiction to [9, Theorem 2.5].

2. Let either $m = p^2$ and $p \in \{5, 7, 11, 23, 19, 29, 59\}$, or $m = 3^6$, or $m = 3^4$. As it was shown above, $s > 2$.

If $m = p^2$, $\mathrm{SL}_2(5) \trianglelefteq G_0$ and $p \in \{11, 19, 29, 59\}$, then we may assume by Lemma 1 that the triple $(c; s; D)$ is one of (30;4;121), (90;4;361), (210;4;841) or (870;4;3481).

If $m = 3^6 = 729$ and $\mathrm{SL}_2(13) \trianglelefteq G_0$, then by Lemma 1 $(c; s; D) = (182; 4; 729)$.

If G^Σ is solvable, $m = p^2$, $\mathrm{SL}_2(3) \triangleleft G_0$ and $p \in \{5, 7, 11, 23\}$, then by Lemma 1 we may assume that the triple $(c; s; D)$ is one of (6;4;25), (12;4;49), (30;4;121), or (132;4;529).

If $m = 3^4 = 81$ and G_0 contains a normal extraspecial subgroup H of order 32, then $(c; s; D) = (20; 4; 81)$.

Since in all these cases $s = 4$, there is an $\langle y\rangle$-orbit that is a cycle or $p = 5$ and 5 divides $c - 1$. Then $|K| = m - 4$, 5 divides $c(c - 1, p^e)$ and $|\operatorname{Syl}_p(T)| = 1 + lp$ divides $|K|$. It implies $m = p^2$ and $p \in \{5, 11, 19, 29, 59\}$ or $m = 3^4$.

Let $m = p^2$. We have $p^2 - 4 = t(1 + lp)$ and $(t; p) = (1; 5)$ (otherwise, $t = t'p - 4 > 1$ and $p = t'(1 + lp) - 4l$, which is impossible). Hence $|K| = 21$. If K is cyclic, then by [8, Theorem 9.2], 21 divides m, a contradiction. Hence, the subgroup $K' \simeq \mathrm{Z}_7$ of K is normal in G. Then the graph $\Delta^{K'}$ admits an arc-transitive action of G/K', and K/K' can be considered as a group of all automorphisms of $\Delta^{K'}$ fixing each its fibre. But $K/K' \simeq \mathrm{Z}_3$ and hence, by [8, Theorem 9.2], 3 divides m, a contradiction.

Let $p = 3$. Then K is a cyclic group of order 77 and, by [8, Theorem 9.2], 77 divides m, a contradiction.

Thus, the only remaining possibility is $G^\Sigma \leq \mathrm{A\Gamma L}_1(q)$. The lemma is proved. □

Lemma 14. *Suppose $G^\Sigma \leq \mathrm{A\Gamma L}_1(q)$, where $q = p^e$ for a prime p. Let H_1 be the stabiliser of a fibre F in G^Σ, $H = H_1 \cap \mathrm{AGL}_1(q)$, $\tilde f$ and $\tilde g$ be two elements of H, whose orders are $2'$-part and 2-part of $|H|$, respectively, and let $\tilde z$ be an involution in $\langle \tilde g\rangle$. Denote by z, f and g some representatives of the preimages of the elements $\tilde z, \tilde f$ and $\tilde g$ in G, respectively. Then* $\mathrm{Fix}(z)$ *is a fibre,*

$$\alpha_0(z) = q - s, \quad \alpha_3(z) = 0, \quad \chi_1(z) = (\alpha_1(z) + (\sqrt{D} - s/2 + 1)(cs - s) - cs)/(2\sqrt{D})$$

and the following statements hold.

(1) *If $K = 1$, then s is even, $s = 2$ or cs is divisible by 4, $(\sqrt{D}, s) \leq 2$ and $|fg|$ divides $\alpha_1(z)$, and, in particular,*

 (*i*) *if $s = 4$, then $c = d(d+2)/4$, $\sqrt{D} = d+1$ for some even integer d, $\alpha_1(z) = 2(p^{e/2}l + 2)$, where l is an even integer, and $\alpha_1(z)$ is divisible by $(q - 1)/(e, q - 1)$;*

 (*ii*) *if $p = 3$, then $s + 1$ is divisible by 3.*

(2) *G acts intransitively on arcs of Δ.*

P r o o f. First we show that $\tilde g \neq 1$. On the contrary, suppose that $\tilde g = 1$. Then $|H|$ is odd, $H_1/H \leq \mathrm{Z}_e$ and since H_1 is transitive on $\Sigma \setminus \{F\}$, e is even and $(e)_2 \geq (q - 1)_2 \geq 4$. If $p - 1$ is divisible by 4, then $(p^e - 1)_2 = (e)_2(p - 1)_2 > (e)_2$, a contradiction. Hence, $(p - 1)_2 = 2$ and the number $p^2 - 1$ is divisible by 4 and divides $p^e - 1$, and again $(p^e - 1)_2 = (e/2)_2(p^2 - 1)_2 > (e)_2$, a contradiction.

(1) Let $K = 1$. Let z, f and g denote some representatives of the preimages of $\tilde{z}, \tilde{f}$ and $\tilde{g}$ in G, respectively. Then the involution $z \in G_{\{F\}}$ does not fix any fibre from $\Sigma \setminus \{F\}$. If $\mathrm{Fix}(z) = \varnothing$, then $\alpha_2(z) = 0$ and $\alpha_1(z) = cs(cs+1-s)$, that is z fixes an $(s+1)$-clique and by Lemma 7 we have $cs = s + 1 = 2$, a contradiction. It follows by Lemma 8 that $\mathrm{Fix}(z)$ is a fibre, that is $\mathrm{Fix}(z) = F$, $\alpha_0(z) = q - s$ and $\alpha_3(z) = 0$.

Since for each nontrivial element $h \in \langle fg \rangle$ we have $\alpha_0(h) + \alpha_3(h) = cs - s + 1$,

$$\chi_1(h) = \frac{(\sqrt{D} - s/2 + 1)(\alpha_0(h) - 1) + \alpha_1(h) - cs}{2\sqrt{D}},$$

hence

$$\chi_1(z) = \frac{(\sqrt{D} - s/2)(cs - s) + \alpha_1(z) - s}{2\sqrt{D}}.$$

Suppose s is odd. Then $\alpha_1(z) = 0$ and c is even. In this case, $\chi_1(z) \in \mathbb{Z}$ and the odd number $2\sqrt{D}$ divides

$$\sqrt{D}s + s^2(c-1)/2 + s = s(\sqrt{D} + s(c-1)/2 + 1).$$

As $(s, 2\sqrt{D}) = 1$, we also get that $2\sqrt{D}$ divides $cs - s + 2 = q - s + 1$. By repeating the argument from Lemma 13, we obtain that 3 divides $(s - 1, 2\sqrt{D})$, $q = 3^e$ and there is an element of order 3 in G that has no fixed points. But $(3, s + 1) = 1$ and, by Lemma 7 we conclude that 3 divides $cs - s + 1 = q - s$, a contradiction.

Thus, s is even.

If cs is not divisible by 4, then c is odd. But then

$$D = cs + (s/2 - 1)^2 \equiv 2 \pmod 4,$$

which yields $s = 2$. If s is divisible by 4, then D is odd. Hence $(\sqrt{D}, s) \leq 2$.

As the element fg does not fix any vertex u such that $u^z \in [u]$ and centralizes z, we get that $|fg|$ divides $\alpha_1(z)$.

($1i$) Let $s = 4$. Then

$$c = d(d+2)/4, \quad q = (d+1)^2 \quad \text{and} \quad \sqrt{D} = d + 1.$$

Since $\chi_1(z) \in \mathbb{Z}$, $2(d+1)$ divides $\alpha_1(z) - 4$ and hence $\alpha_1(z) = 2(p^{e/2}l + 2)$ is divisible by $(q - 1)/(e, q-1)$.

Suppose that $\alpha_1(z)$ is not divisible by 4. As $c > 2$, we get $\alpha_1(z) > 0$ and $2(q-1, e)_2 \geq (q-1)_2$. If $p - 1$ is divisible by 4, then $(p^e - 1)_2 = (e)_2(p-1)_2 > 2(e)_2$, a contradiction. Hence, $(p-1)_2 = 2$ and the number $p^2 - 1$ is divisible by 4 and divides $p^e - 1$, and

$$(p^e - 1)_2 = (e/2)_2(p^2 - 1)_2 > 2(e)_2,$$

a contradiction. Thus l is even.

($1ii$) If y is an element of order p of the socle of G, then $\alpha_i(y) = 0$ for $i = 0, 3$ and $\chi_1(y) = (\alpha_1(y) - q)/(2\sqrt{D})$. In the case $p = 3$ in view of Lemma 7 we conclude that 3 divides $s + 1$.

(2) Suppose that G acts transitively on arcs of Δ. Then by Lemma 12 we have $|K| = cs - s + 1$ and $G_a \simeq G_{\{F\}}/K$ $(a \in F)$. Let z, f and g denote some representatives of the preimages of the elements $\tilde{z}, \tilde{f}$ and $\tilde{g}$ in G_a. Then we may assume that z is an involution and it does not fix any fibre from $\Sigma \setminus \{F\}$. Hence by Lemma 8, $\mathrm{Fix}(z)$ is a fibre, $\alpha_0(z) = q - s$ and $\alpha_3(z) = 0$.

Since for each nontrivial element $h \in \langle fg \rangle$ we have $\alpha_0(h) + \alpha_3(h) = cs - s + 1$,

$$\chi_1(z) = \frac{(\sqrt{D} - s/2)(cs - s) + \alpha_1(z) - s}{2\sqrt{D}}.$$

Suppose s is odd. Then $\alpha_1(z) = 0$ and c is even. In this case, $\chi_1(z) \in \mathbb{Z}$, and the odd number $2\sqrt{D}$ divides

$$\sqrt{D}s + s^2(c-1)/2 + s = s(\sqrt{D} + s(c-1)/2 + 1).$$

Again, by repeating the argument from Lemma 13, we obtain a contradiction to Lemma 7.

Thus s is even and $|K|$ is odd. It follows that K is solvable and $K' < K$. Hence the graph $\Delta^{K'}$ is a non-bipartite antipodal distance-regular graph of diameter 3 that admits an arc-transitive action of G/K', and the abelian group K/K' can be considered as the group of all automorphisms of $\Delta^{K'}$ fixing its fibres. But the size of a fibre in $\Delta^{K'}$ coincides with $|K/K'|$ and $(|K/K'|, cs+1) = 1$, a contradiction to [9, Theorem 2.5]. The lemma is proved. □

P r o o f of Theorem 1 follows immediately from Lemmas 10, 11, 13 and 14. □

P r o o f of Theorem 2. Assume Δ is not a 6-cycle. First note that each edge-transitive group of automorphisms of Δ induces a 2-homogeneous permutation group on Σ. It is also clear that if there is an edge-transitive group of automorphisms of Δ, then it acts transitively on its vertices as well, because Δ is a non-bipartite graph whenever $(c; s) \neq (2; 1)$. The case $(c; s) = (56; 1)$ cannot occur, as otherwise the order of G would be divisible by $57 \cdot 56$, which is impossible by [17] (see also [13]). Thus, it remains to apply Theorem 1. □

4. Open Questions

We conclude with few open questions.

1. Is there a half-transitive $(n, r, 1)$-cover?
2. Is there any $(n, r, 1)$-cover with $n - r > 1$ that possesses a group of automorphisms acting 2-homogeneously on the fibres when $n - 1$ is not a power of 2 or $n - r > 2$?

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Contact Information

16 S. Kovalevskaya str., Ekaterinburg, Russia, 620990

Phone: +7 (343) 375-34-73

Fax: +7 (343) 374-25-81

Email: secretary@umjuran.ru

Web-site: https://umjuran.ru

N.N. Krasovskii Institute of Mathematics and Mechanics
of the Ural Branch of Russian Academy of Sciences

Ural Federal University named after the first President of Russia B.N. Yeltsin

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