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IMPROVED FIRST PLAYER STRATEGY FOR THE ZERO-SUM SEQUENTIAL UNCROSSING GAME

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Abstract: This paper deals with the known uncrossing zero-sum two-player sequential game, which is employed to obtain upper running time bound for the transformation of an arbitrary subset family of some finite set to an appropriate laminar one. In this game, the first player performs such a transformation, while the second one tries to slow down this process as much as possible. It is known that for any game instance specified by the ground set and initial subset family of size n and m respectively, the first player has a winning strategy of $O(n^4 m)$ steps. In this paper, we show that the first player has a more efficient strategy, which helps him (her) to win in $O\big(\max\{n^2, mn\}\big)$ steps.

Keywords: Laminar family, Uncrossing game, Efficient winning strategy.

1. Introduction

A *laminar family* of subsets for a given finite set is a well-known concept in discrete mathematics widely exploited in algorithm design for combinatorial optimization problems. For instance, in [5] and [4], laminar families are used to construct polynomial time approximation algorithms for several modifications of the Tree Augmentation Problem (TAP). Recently [1], these results have been extended to efficient approximation algorithms for the Leaf-to-Leaf Connectivity Augmentation Problem (CAP).

On the other hand, *strongly laminar instances* of the Asymmetric Traveling Salesman Problem (ATSP) belong to the main building blocks of the first constant-ratio approximation algorithm for the ATSP proposed by the authors of the breakthrough papers [9] and [10]. By relying on this algorithm the authors of recent works [3, 6, 7] proved constant-ratio polynomial time approximation for several related asymmetric combinatorial problems including Prize-Collecting TSP and Capacitated Vehicle Routing Problem.

An advantage of employment the laminar families in approximation algorithms is based on the following two observations:

(i) for any finite set V of size $|V| = n$, the size $|\mathcal{L}|$ of an arbitrary laminar family $\mathcal{L}$ of its subsets is $O(n)$ [8];

(ii) any non-necessary laminar subset family $\mathcal{F}$ of size m can be transformed to some equivalent laminar family (see, e.g. [2]) in polynomial time with respect to n and m.

To obtain an upper bound for the running time of such a transformation, the authors of [2] introduced a sequential two-person zero-sum *uncrossing game*, where the first player is aimed to make the initial family $\mathcal{F}$ laminar, and the second player tries to slow down this process as much as possible. For each step of the game, its state is specified by the current family. The game ends with the first player winning as soon as the state becomes laminar family.

In [2], the authors showed that the first player has a strategy that allows he (she) to win in polynomial number of steps.

Theorem 1. *In the uncrossing game, the first player has a strategy to make the initial family* $\mathcal{F}$ *laminar in* $O(n^4 m)$ *steps.*

Theorem 2. *In the uncrossing game, the first player has a strategy to make the family* $\mathcal{F}$ *laminar in* $O(\max\{nm, n^2\})$ *steps.*

The rest of our paper is structured as follows. In Section 2, we give formulation of the uncrossing game and recall some its properties essential to our own constructions. Section 3 contains the proof of Theorem 2. Finally, in Section 4, we summarize our results and overview some directions of future research.

2. Problem statement and preliminaries

Let V be an arbitrary non-empty finite set. In the sequel, we call it *ground set*.

Definition 1. *Subsets* $X, Y \in 2^V$ *are called crossing and denoted* $X \nparallel Y$*, if all of the sets* $X \backslash Y$*,* $Y \backslash X$*,* $X \cap Y$*,* $V \backslash (X \cup Y)$ *are non-empty. Otherwise the sets* X *and* Y *are called laminar and denoted* $X \parallel Y$*.*

A subset family $\mathcal{F} \subseteq 2^V$ is called *laminar* if it contains no crossing subsets.

Definition 2. *A subset family* $\mathcal{I} \subseteq 2^V$ *is called cross-closed, if for any crossing sets* $X, Y \in \mathcal{I}$ *at least one of the pairs*

$$\{X \cap Y, X \cup Y\} \quad \textit{or} \quad \{X \backslash Y, Y \backslash X\} \tag{2.1}$$

belongs to $\mathcal{I}$ *as well.*

Obviously, for an arbitrary $X \nparallel Y$, both pairs (2.1) are laminar. To simplify description of the *uncrossing game*, define two elementary *uncrossing operations* as follows. The first operation $\xrightarrow{\cap,\cup}$ substitutes some crossing subsets $X, Y \in \mathcal{F}$ by the subsets $X \cap Y, X \cup Y \in \mathcal{I}$, while the second one $\xrightarrow{\backslash,\backslash}$ makes the similar replacement by the subsets $X \backslash Y, Y \backslash X \in \mathcal{I}$.

Definition 3. *The uncrossing game is a sequential two-player zero-sum game, whose instance is specified by some non-empty finite set* V *and a family* $\mathcal{F}$*. It is assumed that* $\mathcal{F} \subseteq \mathcal{I}$ *for some cross-closed family* $\mathcal{I} \subseteq 2^V$ *given implicitly by a membership oracle.*

Each step of the game is defined as follows. If the family $\mathcal{F}$ *is laminar, then the first player wins. Otherwise, it performs one uncrossing operation for some subsets* $X, Y \in \mathcal{F}$*,* $X \nparallel Y$*. In turn, the second player returns back to* $\mathcal{F}$ *one of the subsets* X *or* Y*.*

In [2] several properties of the uncrossing game were proved. We remind some of them which are necessary for our own results.

Statement 1. *Let* $X, Y, Z \in \mathcal{F}$ *such that* $X \nparallel Y$*,* $X \parallel Z$*, and* $Y \parallel Z$*. The relation* $Q \parallel Z$ *holds for any* $Q \in \{X \backslash Y, Y \backslash X, X \cup Y, X \cap Y\}$*.*

Statement 1 allows to simplify the uncrossing game at each step, when the current family $\mathcal{F}$ contains a subset Z laminar to each other $X \in \mathcal{F}$.

Lemma 1. *The first player has a strategy to reduce the initial game instance to* $O(nm)$ *subinstances of the uncrossing game, each of them is of the smaller size.*

As it follows from the proof of Lemma 1, each of the obtained subinstances is equivalent to the uncrossing game specified by the ground set W and the family $\mathcal{R}$, where

$$W = \overline{1,r}, \quad \text{and} \quad \mathcal{R} = \{\overline{1,2},\ \overline{1,3}, \dots, \overline{1,r-2},\ \overline{2,r-1}\} \tag{2.2}$$

for some $r \leq n$, and the appropriate cross-closed family $\mathcal{I}^*$ induced by the family $\mathcal{I}$. Hereinafter, we use the standard notation $\overline{i,j}$ for the integer interval $\{i, i+1, \dots, j-1, j\}$, whose entries are taken modulo r. In turn, it is easy to verify that this game is equivalent to the game

$$CG(r): \begin{cases} W = \overline{1,r}, \\ \mathcal{R} = \{\overline{1,2},\ \overline{2,3},\ \overline{2,4}, \dots, \overline{2,r-1}\}, \end{cases} \tag{2.3}$$

which is the main object of the subsequent discussion.

Indeed, we can obtain game $CG(r)$ (see Fig. 1) by the cyclic shift on -1 to form a family

$$\mathcal{R} = \{\overline{r,1},\ \overline{1,2},\ \overline{1,3}, \dots, \overline{1,r-2}\}$$

and taking the complement $W \setminus \overline{r,1} = \overline{2,r-1}$. The reverse transformation can be obtained by supplementing the complement $W \setminus \overline{1,2} = \overline{3,r}$ producing the family $\{\overline{3,r},\ \overline{2,3},\ \overline{2,4}, \dots, \overline{2,r-1}\}$ by the cyclic shift on -1 resulting in the family

$$\mathcal{R} = \{\overline{1,2},\ \overline{1,3},\ , \dots, \overline{1,r-2}, \overline{2,r-1}\}.$$

The game $CG(r)$ belongs to the known class of *cyclic* uncrossing games (see, e.g. [2]), each of them has the following properties:

(i) the family $\mathcal{R}$ is partitioned into two laminar subfamilies $\mathcal{R} = \mathcal{L}_1 \dot\sqcup \mathcal{L}_2$, such that $\overline{1,i} \in \mathcal{L}_1$ for $2 \leqslant i \leqslant r-2$ and $\overline{2,j} \in \mathcal{L}_2$;
(ii) the game is invariant to cyclic shifts and replacing each interval by its complement;
(iii) any two neighboring elements $i, i+1$ (modulo r) are *separated* by some set $X \in \mathcal{R}$, i.e. $|X \cap \{i, i+1\}| = 1$.

In the sequel, we propose a first player winning strategy for the $CG(r)$ based on transformation of the initial game to some simpler one. To this end, each time when the transformed family contains a pair $\{i, i+1\}$ violating property (iii), we can *contract* the interval $\overline{i,i+1}$ to a single point, which makes ground set W smaller and accordingly transforms $\mathcal{R}$ and $\mathcal{I}^*$.

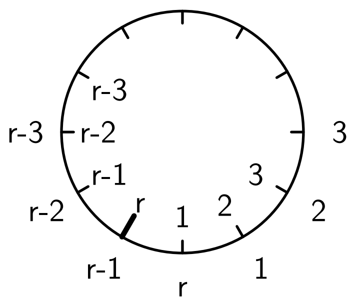

Figure 1. Towards the equivalence of game (2.2) and $CG(r)$.

Evidently, for $r < 4$ the game $CG(r)$ has a trivially solvable by the first player, since the initial family $\mathcal{R}$ is laminar. Consider the first non-trivial game $CG(4)$.

Lemma 2. *The first player can win in the $CG(4)$ after the single step.*

P r o o f. By construction of $CG(4)$, the family $\mathcal{R}$ consists of the only crossing pair $\overline{1,2}$ and $\overline{2,3}$. Since $\mathcal{I}^*$ is cross-closed, it contains either $\{1\}$ and $\{3\}$ or $\{2\}$ and $\{4\}$ (equivalent to $\overline{1,3}$). Then, the first player at the first step performs either the uncrossing operation $\overline{1,2} \nparallel \overline{2,3} \xrightarrow{\setminus,\setminus} \{1\},\{3\}$ or $\overline{1,2} \nparallel \overline{2,3} \xrightarrow{\cap,\cup} \{2\},\{4\}$, respectively. Thus, regardless to the behavior of the second player, the family $\mathcal{R}$ becomes laminar to the second step, since singletons are laminar to any other subset and can be excluded. Therefore, Lemma 2 follows.

3. Proof of Theorem 2

For subsequent discussions we will need to introduce one more family of cyclic uncrossing games. Each such a game has the form

$$CG(r,q): \begin{cases} W = \overline{1,r}, \\ \mathcal{R} = \{\overline{1,2},\dots,\overline{1,q},\ \overline{2,q+1},\dots,\overline{2,r-1}\}, \end{cases}$$

where $2 \le q \le r-2$ and the family $\mathcal{I}^*$ satisfies the additional constraints

$$\{1\},\{2\} \in \mathcal{I}^*, \quad \{3\},\dots,\{q\},\ \{r\} \notin \mathcal{I}^*. \tag{3.1}$$

First, consider the special case, where $q = r-2$.

Lemma 3. *For an arbitrary $r \ge 4$, in the game $CG(r, r-2)$, the first player has a winning strategy within at most $r-3$ steps.*

P r o o f can be obtained by induction on r. In the base case $r = 4$, the game $CG(4,2)$ allows a single step winning strategy of the first player. To prove the induction step, assume that in $CG(r-1, r-3)$ there exists a winning strategy of the first player within $r-4$ steps and show that the claim holds for the game $CG(r, r-2)$.

At the first step of this game, the first player performs the uncrossing operation

$$\overline{1,r-2} \nparallel \overline{2,r-1} \xrightarrow{\setminus,\setminus} \{1\},\{r-1\},$$

which is admissible since $\{r\}$ does not belong to the cross-closed family $\mathcal{I}^*$. By construction, the second player has two options, to return to $\mathcal{R}$ either $\overline{1,r-2}$ or $\overline{2,r-1}$. In the first case, $\mathcal{R}$ becomes laminar and the first player wins immediately. Otherwise, the family $\mathcal{R}$ takes the form $\{\overline{1,2},\dots\overline{1,r-3},\overline{2,r-1}\}$. Here elements $r-2$ and $r-1$ no longer separated. After the contraction, we obtain the game

$$CG(r-1,r-3): \begin{cases} W = \overline{1,r-1}, \\ \mathcal{R} = \{\overline{1,2},\dots,\overline{1,r-3},\overline{2,r-2}\}. \end{cases}$$

Therefore, in the initial game $CG(r, r-2)$ the first player has a winning strategy of at most

$$1 + (r-4) = r-3$$

steps. Lemma 3 is proved. □

Lemma 4. *For the game $CG(r,q)$ first player has a strategy that allows him to win in at most $2r-q-5$ steps.*

P r o o f. The main idea of our proof consists of the following steps:

(i) we introduce an auxiliary edge-weighted digraph H with the node set

$$\mathcal{G} = \{CG(r,q)\colon 2 \leq q \leq r-2\}.$$

An ordered pair $\big(CG(r_1,q_1), CG(r_2,q_2)\big)$ is an arc of the graph H, if the first player has a strategy to reduce $CG(r_1,q_1)$ to the game $CG(r_2,q_2)$ without visiting any other element of $\mathcal{G}$. Each arc is weighted with the appropriate number of steps. It is convenient to illustrate this graph on the integer lattice (Fig. 2).

(ii) we define linear order on the node set of the graph H as follows:

$$CG(r_2,q_2) \prec CG(r_1,q_1) \iff (r_2 < r_1) \vee ((r_1 = r_2) \wedge (q_2 > q_1)); \tag{3.2}$$

(iii) we show that this order is consistent to the digraph H in the following way: for an arbitrary arc $\big(CG(r_1,q_1), CG(r_2,q_2)\big)$ it holds; $CG(r_2,q_2) \prec CG(r_1,q_1)$.

(iv) finally, we complete the proof by induction regarding the order (3.2).

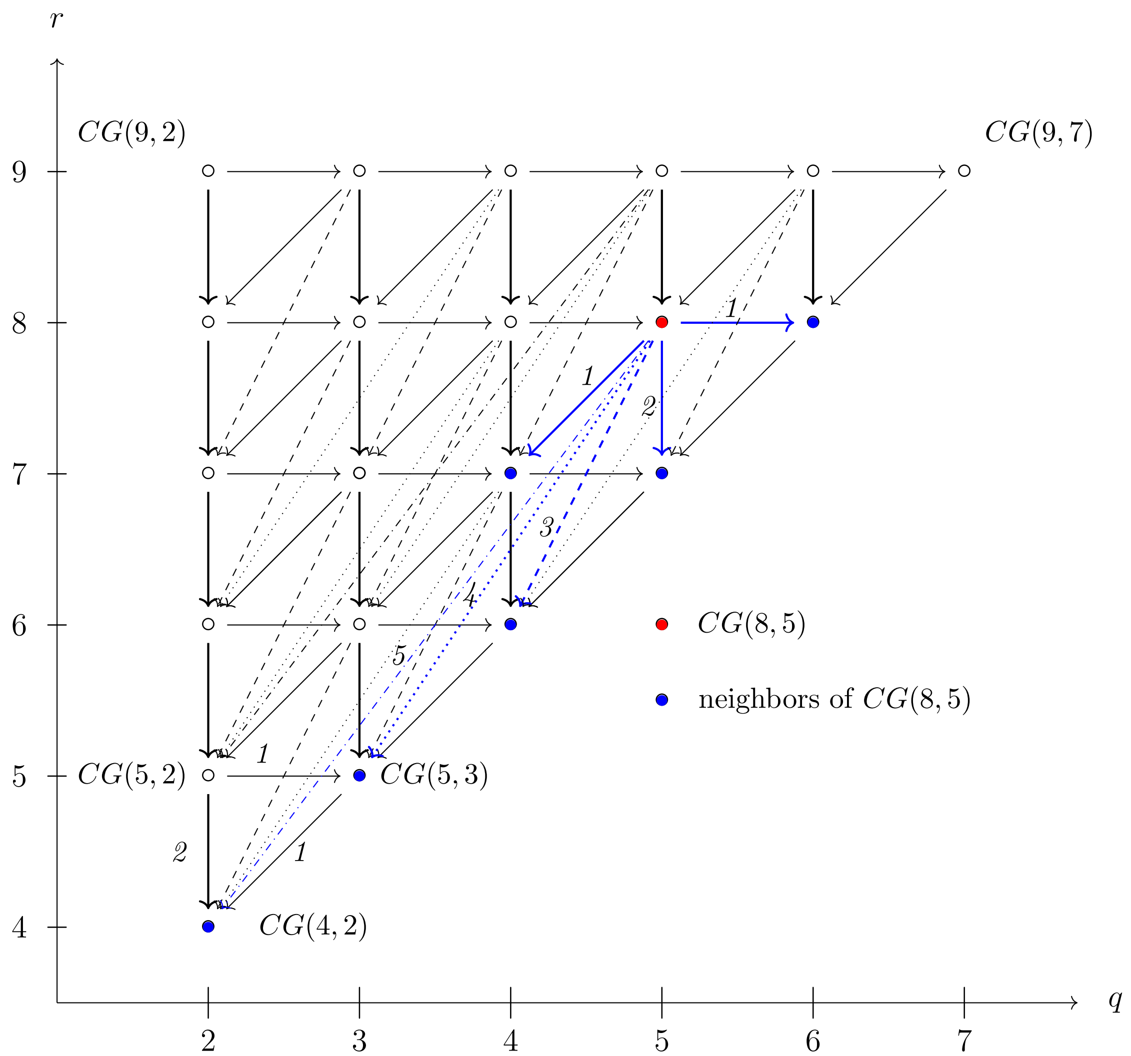

Figure 2. An auxiliary digraph H. Outgoing arcs and neighbors of $CG(8,5)$ are highlighted.

Discuss the aforementioned points in detail.

Point (i). To define arc set of the graph H, fix an arbitrary node $CG(r,q) \in \mathcal{G}$ and show that its outgoing neighbors (i.e. the nodes $CG(r',q') \in \mathcal{G}$, for which $(CG(r,q), CG(r',q'))$ is an arc) are as follows:

$$\begin{aligned}\mathcal{N}(CG(r,q)) = \big\{&CG(r-1,q-1), CG(r,q+1), CG(r-1,q),\\ &CG(r-2,q-1),\dots,CG(r-q+1,2)\big\}\end{aligned} \tag{3.3}$$

In addition, we calculate a weight assigned to each arc connecting $CG(r,q)$ with its neighbors (Fig. 2).

By construction, for the family $\mathcal{I}^*$ specifying the game $CG(r,q)$ there exist two options, $\mathcal{I}^*$ can contain or not the singleton $\{q+1\}$.

Case $\{q+1\} \in \mathcal{I}^*$. At the first step of the game, the first player makes the uncrossing operation

$$\overline{1,q} \nparallel \overline{2,q+1} \xrightarrow{\setminus,\setminus} \{1\}, \{q+1\}.$$

If the second player returns $\overline{1,q}$, i. e.

$$\mathcal{R} = \{\overline{1,2},\ \overline{1,q},\ \overline{2,q+2},\dots,\overline{2,r-1}\}$$

and then, after the contacting $\{q+1,q+2\}$ the initial game is transformed to

$$CG(r-1,q): \begin{cases} W = \overline{1,r-1},\\ \mathcal{R} = \{\overline{1,2},\dots,\overline{1,q},\ \overline{2,q+1},\dots,\overline{2,r-2}\},\end{cases}$$

since this actions obviously keep all the additional constraints

$$\{1\},\{2\} \in \mathcal{I}^* \quad \text{and} \quad \{3\},\dots,\{q\},\{r-1\} \notin \mathcal{I}^* \tag{3.4}$$

for the transformed family $\mathcal{I}^*$. Thus, $CG(r-1,q)$ is an outgoing neighbor of the node $CG(r,q)$.

Else, if the second player returns $\overline{2,q+1}$, performing the similar contraction of $\{q,q+1\}$ transforms $CG(r,q)$ to the game

$$CG(r-1,q-1): \begin{cases} W = \overline{1,r-1},\\ \mathcal{R} = \{\overline{1,2},\ \overline{1,q-1},\ \overline{2,q},\dots,\overline{2,r-2}\},\end{cases}$$

since constraints (3.4) are also remain valid.

In the *case* $\{q+1\} \notin \mathcal{I}^*$, we have $\overline{1,q+1}, \overline{2,q} \in \mathcal{I}^*$, since the crossing intervals $\overline{1,q}$ and $\overline{2,q+1}$ belong to the cross-closed family $\mathcal{I}^*$. By carrying out the similar argument for $\overline{1,q-1} \nparallel \overline{2,q}$, $\overline{1,q-2} \nparallel \overline{2,q-1}$ and so on and relying on equation (3.1), we obtain that in this case all the intervals $\overline{2,3}, \overline{2,4},\dots,\overline{2,q}$ belong to $\mathcal{I}^*$.

At the first step of the game $CG(r,q)$, the first player performs the uncrossing operation

$$\overline{1,2} \nparallel \overline{2,q+1} \xrightarrow{\cap,\cup} \{2\}, \{1,q+1\}.$$

Next, if the second player returns $\overline{1,2}$ the game $CG(r,q)$ is transformed to

$$CG(r,q+1): \begin{cases} W = \overline{1,r},\\ \mathcal{R} = \{\overline{1,2},\dots,\overline{1,q},\ \overline{1,q+1},\ \overline{2,q+2},\dots,\overline{2,r-1}\},\end{cases}$$

since constraints (3.1) together with $\{q+1\} \notin \mathcal{I}^*$ hold.

Otherwise, if the second player returns $\overline{2,q+1}$ we have

$$\mathcal{R} = \{\overline{1,3},\dots,\overline{1,q+1},\ \overline{2,q+1},\dots,\overline{2,r-1}\}.$$

By contracting the interval $\{2,3\}$ to the singleton $\{2\}$, we got a new game G' specified by the novel ground set $W = \overline{1,r-1}$, subset family

$$\mathcal{R}' = \{\overline{1,2},\dots,\overline{1,q},\ \overline{2,q},\dots,\overline{2,r-2}\}$$

and the transformed cross-closed family $\mathcal{I}^*$ containing the singleton $\{2\}$. Observe that, although the constraints (3.4) are still satisfied, game G' does not belong to the game family $\mathcal{G}$.

Further, if $q = 2$, then the game G' coincides with $CG(r-1,2)$. Otherwise, we need to proceed with the playing.

At the second step, the first player makes the uncrossing operation

$$\overline{1,2} \nparallel \overline{2,q} \xrightarrow{\cap,\cup} \{2\}, \overline{1,q}.$$

If the second player returns $\overline{1,2}$, the family $\mathcal{I}^*$ is kept unchanged and we obtain the game

$$CG(r-1,q) : \begin{cases} W = \overline{1,r-1}, \\ \mathcal{R} = \{\overline{1,2},\dots,\overline{1,q},\ \overline{2,q+1},\dots,\overline{2,r-2}\} \end{cases}$$

since constraints (3.4) remain valid.

Else, if the second player returns $\overline{2,q}$ and transforms the family $\mathcal{R}$ to the form $\{\overline{1,3},\ \dots,\overline{1,q},\ \overline{2,q},\ \overline{2,q+1},\dots,\overline{2,r-2}\}$, as in the previous step, by contracting interval $\{2,3\}$ to the singleton $\{2\}$, we obtain again the game

$$G' : \begin{cases} W = \overline{1,r-2}, \\ \mathcal{R}' = \{\overline{1,2},\dots,\overline{1,q-1},\ \overline{2,q-1},\dots,\overline{2,r-3}\} \end{cases}$$

satisfying the conditions $\{1\},\{2\} \in \mathcal{I}^*$ and $\{3\},\dots,\{q-1\},\{r-2\} \notin \mathcal{I}^*$ for the transformed family $\mathcal{I}^*$. Again, if $q = 3$, game G' is equivalent to $CG(r-2,q-1)$. Else, we proceed with playing the game G'.

Finally, to the beginning of the $(q-1)$-th step, the game G' will have the form

$$G' : \begin{cases} W = \overline{1,r-q+2}, \\ \mathcal{R}' = \{\overline{1,2},\overline{1,3},\ \overline{2,3},\dots,\overline{2,r-q+1}\}, \end{cases}$$

where $\{1\},\{2\} \in \mathcal{I}^*$ and $\{3\},\{r-q+2\} \notin \mathcal{I}^*$. The first player makes the uncrossing operation

$$\overline{1,2} \nparallel \overline{2,3} \xrightarrow{\cap,\cup} \{2\}, \overline{1,3}.$$

If the second player returns $\overline{1,2}$, then we obtain the game $CG(r-q+2,3)$. Otherwise, if he (she) returns $\overline{2,3}$ the game G' is transformed to the game $CG(r-q+1,2)$. Indeed, after the second player move, we have $\mathcal{R} = \{\overline{1,3},\overline{2,3},\dots,\overline{2,r-q+1}\}$, which can be easily transformed to the form $\mathcal{R} = \{\overline{1,2},\overline{2,4},\dots,\overline{2,r-q+2}\}$ by contracting of the interval $\overline{2,3}$.

To this end, we proved that the set of outgoing neighbors of the node $CG(r,q)$ coincides with the set $\mathcal{N}(CG(r,q))$ defined by formula (3.3). As it follows from the argument above, weights of the appropriate arcs are as follows:

$$\begin{gathered} w\big(CG(r,q),CG(r,q+1)\big) = w\big(CG(r,q),CG(r-1,q-1)\big) = 1, \\ w\big(CG(r,q),CG(r-1,q)\big) = 2, \\ w\big(CG(r,q),CG(r-2,q-1)\big) = 3, \\ \dots, \\ w\big(CG(r,q),CG(r-q+1,2)\big) = q \end{gathered} \tag{3.5}$$

$$\begin{tikzcd}
CG(r,q) \arrow[r, "1"] \arrow[dr, dashed, "1"] \arrow[ddr, "1"'] \arrow[ddd, "1"] & CG(r,q+1) \\
& CG(r-1,q) \\
& CG(r-1,q-1) \\
\circ \arrow[r, "1"] \arrow[d, "1"] & CG(r-1,q) \\
\circ \arrow[r, "1"] \arrow[d, "1"] & CG(r-2,q-1) \\
\cdots \arrow[d, "1"] & \\
\circ \arrow[r, "1"] & CG(r-q+1,2)
\end{tikzcd}$$

Figure 3. Illustration of the argument of point (i).

(see, also Fig. 3). Generally speaking, there are two parallel arcs connecting the nodes $CG(r,q)$ and $CG(r-1,q)$, but, to obtain the upper bound we take the longest one. Point (i) is proved.

In turn, point (iii) immediately follows from (3.2) and (3.3).

To complete the proof, we use $CG(4,2)$ as a base case of the induction. Since this game is a special case of the game $CG(4)$, the first player can win in a single step, by Lemma 2.

Induction step follows from the point (iii) and the following simple observation. Indeed, denote by $L(r,q)$ the number of steps in the first player winning strategy in the game $CG(r,q)$. By construction,

$$L(r,q) = \max\left\{w\big(CG(r,q), CG(r',q')\big) + L(r',q') \colon CG(r',q') \in \mathcal{N}(CG(r,q))\right\}.$$

By induction hypothesis, $L(r',q') = 2r' - q' - 5$. Taking into account (3.5), it is easy to verify that $L(r,q) \leq 2r - q - 5$. Lemma 4 is proved. $\square$

Let's get back to the game $CG(r)$.

Lemma 5. *For an arbitrary $r \geq 4$, in the game $CG(r)$ first player has a winning strategy of at most $2r-7$ steps.*

P r o o f. By construction, the game $CG(r)$ has the following form:

$$CG(r) : \begin{cases} W = \overline{1,r}, \\ \mathcal{R} = \{\overline{1,2},\ \overline{2,3}, \ldots, \overline{2,r-1}\} \end{cases}$$

There exist two possible options.

Option 1. $\{1\},\{3\} \in \mathcal{I}^*$ or $\{2\},\{r\} \in \mathcal{I}^*$.

$$
\begin{tikzcd}
CG(r) \arrow[r, "1"] \arrow[d, "1"] \arrow[dr, "0"] & \text{win} \\
 & CG(r,2) \text{ type} \\
CG(r-1) \arrow[r] \arrow[d] \arrow[dr] & \text{win} \\
 & CG(r-1,2) \text{ type} \\
\cdots \arrow[d] & \\
CG(4) &
\end{tikzcd}
$$

Figure 4. Scheme of possible transitions for game $CG(r)$.

It is easy to show that these conditions are symmetrical and can be considered in a similar way. Consider the case $\{1\}, \{3\} \in \mathcal{I}^*$.

At the first step, the first player makes the following uncrossing operation:

$$\overline{1,2} \nparallel \overline{2,3} \xrightarrow{\backslash,\backslash} \{1\}, \{3\}.$$

If the second player returns $\overline{2,3}$, then the first player wins immediately. Else, if he (she) returns $\overline{1,2}$, we obtain the game

$$CG(r-1) : \begin{cases} W = \overline{1, r-1}, \\ \mathcal{R} = \{\overline{1,2},\ \overline{2,3}, \dots, \overline{2, r-2}\} \end{cases}$$

after the contacting of $\{3, 4\}$.

Option 2. In this case we necessarily have $\{1\}, \{2\} \in \mathcal{I}^*$, $\overline{1,3} \in \mathcal{I}^*$, and $\{3\}, \{r\} \notin \mathcal{I}^*$. These conditions leads that the considered game is $CG(r, 2)$.

Complete the proof by induction by r (see Fig. 4). The base case is $CG(4)$, for which the claim follows from Lemma 2. Let Lemma 5 is valid for $r-1$. Prove it for r. Indeed, as it follows from the above argument, the game $CG(r)$ is either coincides with $CG(r, 2)$ or can be reduced in one step to the game $CG(r-1)$. In the first case, the first player has a winning strategy of at most $2r - 2 - 5 = 2r - 7$, by Lemma 4, while in the second case such a strategy has at most

$$1 + 2(r-1) - 7 \leq 2r - 7$$

steps. Lemma 5 follows. $\square$

To the moment, we proved all the necessary technical lemmas and can return to proof of Theorem 2.

P r o o f of Theorem 2. The proposed strategy of the first player for the general uncrossing game specified by the triple $(V, \mathcal{F}, \mathcal{I})$, where $|V| = n$, $|\mathcal{F}| = m$ and $\mathcal{I}$ is given implicitly by the membership oracle, consists of following two stages.

First, we solve the cyclic games $CG(r, 2)$ and $CG(r)$ for $r \in \overline{4, n}$. As it follows from Lemma 4 and Lemma 5, all these games can be solved by at most $O(n^2)$ steps.

At the second stage, we employ Lemma 1, who guarantees that the first player has a strategy to reduce the initial game to $O(mn)$ games of form $CG(r)$ (for some $r \in \overline{4, n}$). For each such cyclic game, we take the solution (laminar family) obtained at the first stage (in constant time).

Thus, the overall strategy has $B = O(mn + n^2)$ steps. It is clear that

$$B = \begin{cases} O(mn), & \text{if } m \geq n, \\ O(n^2), & \text{otherwise.} \end{cases}$$

Theorem 2 is proved. □

It should be noticed that our strategy of the first player differs from the framework proposed in [2], where cyclic games obtained in Lemma 1 were solved ad hoc. By following to this framework and applying our Lemma 4 and 5, we can obtain another winning strategy of the first player, but with worse running time upper bound $O(n^2m)$.

4. Conclusion

In this paper we proposed a more efficient first player winning strategy for the well-known uncrossing game. Proof of our result is entirely constructive and provides an algorithm to make laminar a given set family efficiently. Therefore, incorporating our result to approximation algorithms for combinatorial optimization problems relying on laminar set families can increase their performance.

In Lemma 5, for some special type of cyclic uncrossing games, we showed that the first player can win within at most linear number of steps. It seems interesting to generalize this result to more wide class of uncrossing games.

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