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REDUCING GRAPHS BY LIFTING ROTATIONS OF EDGES TO SPLITTABLE GRAPHS

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Abstract: A graph G is splittable if its set of vertices can be represented as the union of a clique and a coclique. We will call a graph H a splittable ancestor of a graph G if the graph G is reducible to the graph H using some sequential lifting rotations of edges and H is a splittable graph. A splittable r-ancestor of G we will call its splittable ancestor whose Durfey rank is r. Let us set $s = (1/2)(\operatorname{sum}\operatorname{tl}(\lambda) - \operatorname{sum}\operatorname{hd}(\lambda))$, where $\operatorname{hd}(\lambda)$ and $\operatorname{tl}(\lambda)$ are the head and the tail of a partition λ. The main goal of this work is to prove that any graph G of Durfey rank r is reducible by s successive lifting rotations of edges to a splittable r-ancestor H and s is the smallest non-negative integer with this property. Note that the degree partition $\operatorname{dpt}(G)$ of the graph G can be obtained from the degree partition $\operatorname{dpt}(H)$ of the splittable r-ancestor H using a sequence of s elementary transformations of the first type. The obtained results provide new opportunities for investigating the set of all realizations of a given graphical partition using splittable graphs.

Keywords: Integer partition, Graphical partition, Degree partition, Splittable graph, Rotation of an edge.

1. Introduction

Everywhere we mean by a graph a simple graph, i. e., a graph without any loops and multiple edges. We will adhere to the terminology and notation from [1, 2, 6].

An *integer partition*, or simply, a *partition* is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers that contains only a finite number of non-zero components (see [1]).

Let $\operatorname{sum}\lambda$ denote the sum of all components of a partition λ and called it the *weight* of the partition λ. It is often said that a partition λ is a partition of the non-negative integer $n = \operatorname{sum}\lambda$. The *length* $\ell(\lambda)$ of a partition λ is the number of its non-zero components. For convenience, a partition λ will often be written as $\lambda = (\lambda_1, \dots, \lambda_t)$, where $t \geq \ell(\lambda)$, i. e., we will omit zeros, starting from some zero component without forgetting that the sequence is infinite.

We will say that the partition $(\lambda_1, \dots, \lambda_i - 1, \dots, \lambda_j + 1, \dots)$ is obtained from the partition $(\lambda_1, \dots, \lambda_i, \dots, \lambda_j, \dots)$ by an elementary transformation of the first type. An elementary transformation of the second type is a reduction of some partition component by 1.

A partition can conveniently be depicted as a Ferrers diagram, which can be thought of as a set of square boxes of the same size (see the example below in Fig. 1). We will use Cartesian notation for Ferrers diagrams.

For each partition λ, we will consider a *conjugate partition* λ^* whose components are equal to the number of boxes in the corresponding rows of the Ferrers diagram of the partition λ.

We determine the *rank* $r(\lambda)$ of the partition λ by setting $r(\lambda) = \max\{i | \lambda_i \geq i\}$. Obviously, the rank $r = r(\lambda)$ of a partition λ is equal to the number of boxes on the main diagonal of the Ferrers diagram of this partition.

As the *head* $\operatorname{hd}(\lambda)$ we take the partition that is obtained from the partition λ by reducing the first r components by the same number $r - 1$ and zeroing all components with numbers $r + 1$, $r + 2, \dots$ (for an example, see the diagram in Fig. 2).

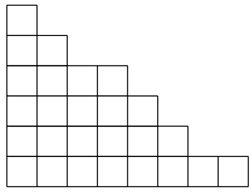

Figure 1. The Ferrers diagram of the partition $(6, 5, 4, 4, 3, 2, 1, 1)$.

As the *tail* $\mathrm{tl}(\lambda)$ we take a partition for which the Ferrers diagram of the conjugate partition is obtained from the Ferrers diagram of the partition λ by deleting the first r columns, i. e. the Ferrers diagram of the partition $\mathrm{tl}(\lambda)^*$ is located to the right of the Durfey square (see Fig. 2).

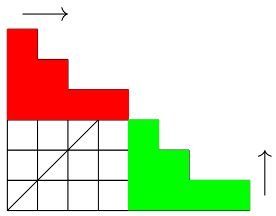

Figure 2. The head $\mathrm{hd}(\lambda) = (3, 2, 1, 1)$ and the tail $\mathrm{tl}(\lambda) = (4, 2, 1)$ of the partition $(6, 5, 4, 4, 3, 2, 1, 1)$.

The theory of partitions is one of the classical areas of combinatorics. Its foundations were laid by L. Euler. Information about the achievements of the theory of partitions can be found in [1].

This work continues the cycle of researches by V.A. Baransky, T.A. Koroleva, T.A. Senchonok and V.V. Zuev, which uses the method of elementary transformations for partitions and the associated method of rotating edges in graphs. Using these methods, new results were obtained on some details of the structure of the lattice of partitions and the properties of graphical partitions, including maximal graphical partitions. Results were also obtained on the connection of graphs with threshold graphs, and an important class of bipartite-threshold graphs was considered (a brief overview of the results obtained is contained in [2]).

Let (x, v, y) be a triple of different vertices of a graph $G = (V, E)$ such that $xv \in E$ and $vy \notin E$. We call such a triple

1) *lifting* if $\deg(x) \leq \deg(y)$,
2) *lowering* if $\deg(x) \geq 2 + \deg(y)$,
3) *preserving* if $\deg(x) = 1 + \deg(y)$.

A transformation φ of a graph G such that $\varphi(G) = G - xv + vy$, i. e., the edge xv is first removed from G and then the edge vy is added, is called *a rotation of the edge* (in the graph G *around vertex* v), corresponding to the triple (x, v, y).

The rotation of an edge in a graph G corresponding to the triple (x, v, y) is called

1) *lifting* if the triple (x, v, y) is lifting,
2) *lowering* if the triple (x, v, y) is lowering,
3) *preserving* if the triple (x, v, y) is preserving.

We will consider the cases where $\deg(x) = 1$ or $\deg(y) = 0$ are admissible, i. e., after the edge is rotated, an isolated vertex may appear, or the edge will rotate in the graph G with the addition of a new isolated vertex. Note that a rotation of an edge in a graph G is lifting if and only if its inverse rotation is lowering.

A graph G is called *splittable* (see, for example, [6]) if its set of vertices can be represented as the union of a clique and a coclique.

These graphs were introduced in [3], where it was shown, that G is splittable if and only if it does not have an induced subgraph isomorphic to one of the three forbidden graphs C_4, C_5, or $2K_2$.

R.I. Tyshkevich used splittable graphs to study unigraphic partitions, i.e., graphic partitions that have a unique realization up to isomorphism and isolated vertices [9].

Many other characterizations and properties of splittable graphs have been discovered (see [7, Ch. 8–9] and [8]). Among them is the fact that whether a graph G is splittable can be determined from its degree sequence $\mathrm{dpt}(G)$ [5]. In our terminology, such a condition is equivalent to the following equality $\mathrm{sum}(\mathrm{hd}(\lambda)) = \mathrm{sum}(\mathrm{tl}(\lambda))$, where $\lambda = \mathrm{dpt}(G)$ [2]. Note that the graph G is threshold if and only if $\mathrm{hd}(\lambda)) = \mathrm{tl}(\lambda)$.

We will call a graph H a splittable ancestor of a graph G if the graph G is reducible to the graph H using some sequential lifting rotations of edges and H is a splittable graph. Note that the graph G can be obtained from H by sequentially performing lowering rotations of edges. It is important to note that therefore $\mathrm{dpt}(H)$ can be obtained from $\mathrm{dpt}(G)$ using elementary transformations of the first type. This means that the partition $\mathrm{dpt}(H)$ lies above the partition $\mathrm{dpt}(G)$ in the lattice of all partitions of the weight $\mathrm{sum}(\mathrm{dpt}(G))$.

A *Durfey rank* of a graph G is the rank (i. e., Durfey rank) of its degree partition, i. e., the number of boxes on the main diagonal of the Ferrers diagram of $\mathrm{dpt}(G)$.

Let G be an arbitrary graph with vertex set V, r is the Durfey rank and n is the cardinality of vertices of G. Let q be a natural number such that $1 \leq q < n$.

An ordered pair (V_1, V_2) of subsets of a set V will be called a *2-decomposition of rank q* of the set V if $|V_1| = q$, $|V_2| = n - q$ and $V = V_1 \bigsqcup V_2$, i. e., V is the disjoint union of the sets V_1 and V_2 (here the sets V_1 and V_2 do not intersect). We will sometimes omit the words “rank q” if we know what rank we are talking about. The sets V_1 and V_2 will be called the first and second components of the 2-decomposition, respectively.

In this work, we will consider 2-decompositions of rank r of the set V, where r is the Durfey rank of G, i. e., at $q = r$.

Among 2-decompositions (V_1, V_2) of rank r of the set V we select special 2-decompositions, which we will call *principal 2-decompositions* of the graph G, for which all vertices of the set V_1 have degrees greater than or equal to r, and all vertices of the set V_2 have degrees less than or equal to r.

Let the degree partition of the graph G have the form

$$\lambda = \mathrm{dpt}(G) = (\lambda_1, \ldots, \lambda_r, \lambda_{r+1}, \ldots, \lambda_n),$$

r is the rank of the partition λ and n is the number of vertices. Let us order the set of vertices $V = \{v_1, \ldots, v_r, v_{r+1}, \ldots, v_n\}$ of the graph G in such a way that

$$\lambda_1 = \deg\, v_1 \geq \cdots \geq \lambda_r = \deg\, v_r \geq r \geq \lambda_{r+1} = \deg\, v_{r+1} \geq \cdots \geq \lambda_n = \deg\, v_n.$$

We can obtain the principal 2-decomposition (V_1, V_2) of the graph G by setting

$$V_1 = \{v_1, \ldots, v_r\} \quad \text{and} \quad V_2 = \{v_{r+1}, \ldots, v_n\}.$$

Let $u \in V_1$, $v \in V_2$ and vertices u and v have the same degrees equal to r. Let us move on to a new principal 2-decomposition (V_1', V_2') of the graph G by setting

$$V_1' = V_1 - u + v \quad \text{and} \quad V_2' = V_2 - v + u.$$

This procedure we will called a *procedure of exchanging vertices of degree r* from the sets V_1 and V_2. It transforms the principal 2-decomposition (V_1, V_2) to the principal 2-decomposition (V_1', V_2') of the graph G. It is clear that using sequences of exchanges of vertices of degree r from any principal 2-decomposition (V_1, V_2) one can obtain all principal 2-decompositions of the graph G. It is easy to see that the principal 2-decompositions of a graph G can differ from each other only by vertices of degree r in the first and the second components.

Let (V_1, V_2) be an arbitrary 2-decomposition of the set V of vertices of a graph G. Then the set E of all edges of the graph G connecting vertices from V_1 with vertices from V_2 will be called a *section* of the graph G corresponding to the 2-decomposition (V_1, V_2). We will call a bipartite graph (V_1, E, V_2) the *sandwich subgraph* of the graph G corresponding to the 2-decomposition (V_1, V_2).

We have $\operatorname{hd}(\lambda) \leq \operatorname{tl}(\lambda)$ by virtue of the ht-criterion [2], where $\operatorname{hd}(\lambda)$ and $\operatorname{tl}(\lambda)$ are the head and the tail of the partition λ, and the integer $\operatorname{sum}\operatorname{tl}(\lambda) - \operatorname{sum}\operatorname{hd}(\lambda)$ is even. Let us set

$$s = \frac{1}{2}(\operatorname{sum}\operatorname{tl}(\lambda) - \operatorname{sum}\operatorname{hd}(\lambda)).$$

Let r is the Durfey rank of a graph G. A *splittable r-ancestor* of G we will called its splittable ancestor whose Durfey rank is r.

The main goal of this study is to prove the following theorem.

Theorem 1. *Let G be an arbitrary graph whose Durfey rank is equal to r and $\lambda = \operatorname{dpt} G$.*

1. *Let (V_1, V_2) be a principal 2-decomposition of the graph G. Then the graph G is reduced to a splittable r-ancestor $H' = (K(V_1), E', V_2)$ by means of some sequential execution of s lifting edge rotations, and s is the smallest non-negative integer with this property.*
2. *Let (V_1', V_2') be a non-principal 2-decomposition of the set of vertices V of the graph G and the graph G is reducible to some splittable r-ancestor of the form $H' = (K(V_1'), E', V_2')$ by sequentially performing of t lifting rotations of edges. Then $t > s$.*

We see that any graph of Durfey rank r is reducible by s successive lifting rotations of edges to a splittable graph of Durfey rank r, and s is the smallest non-negative integer with this property.

Let a splittable graph $H' = (K(V_1), E', V_2)$ be obtained from a graph G using some sequential execution of s lifting edge rotations, where (V_1, V_2) is some 2-decomposition of the set of vertices of the graph G. Then the graph G can be obtained from the splittable graph H' using an inverse sequence consisting of s lowering edge rotations. Therefore, the degree partition $\operatorname{dpt}(G)$ of the graph G can be obtained from the degree partition $\operatorname{dpt}(H')$ of the graph H' using a sequence of s elementary transformations of the first type [2].

Let r is the Durfey rank of a graph G. Its *closest splittable r-ancestor* is a splittable graph H', which has Durfey rank r and which can be obtained from the graph G by some sequential execution of s lifting rotations of edges.

Next, we present an algorithm (see Algorithm 1 and Lemma 6) for finding all closest splittable r-ancestors of a graph G.

Corollary 1. *Let G be a graph of Durfey rank r. Then the graph G is obtainable from some of its closest splittable r-ancestor using a sequence consisting of*

$$s = \frac{1}{2}(\operatorname{sum}\operatorname{tl}(\operatorname{dpt}(G)) - \operatorname{sum}\operatorname{hd}(\operatorname{dpt}(G)))$$

lowering rotations of edges, and the degree partition $\operatorname{dpt}(G)$ of the graph G is obtainable from the degree partition $\operatorname{dpt}(H')$ of the graph H' using a sequence of s elementary transformations of the first type.

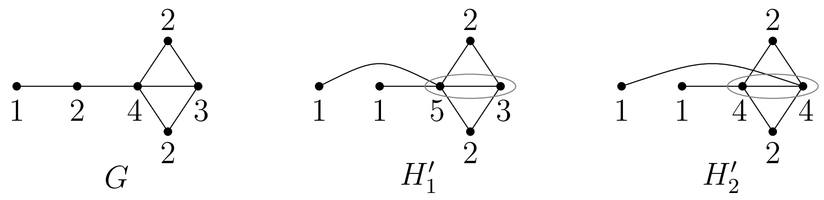

Figure 3. The graph G with Durfey rank 2 and its two non-isomorphic closest splittable 2-ancestors.

Example 1. Figure 3 shows an example of a graph G of Durfey rank 2 and its two non-isomorphic closest splittable 2-ancestors, each obtained from G by a single lifting rotation of an edge. Note that here $s = 1$ and the clique V_1 in the graphs H'_1 and H'_2 is two-element. It is easy to check that for each $i = 1, 2$ the degree partition $\mathrm{dpt}(G)$ of the graph G is obtained from the degree partition $\mathrm{dpt}(H'_i)$ of the graph H'_i using one elementary transformation of the first type.

For a graph G of Durfey rank r, consider the *family of all closest splittable r-ancestors*, consisting of pairwise non-isomorphic graphs without isolated vertices. Let us denote this family by $CSrA(G)$.

Let $\lambda = \mathrm{dpt}(G)$. Let $CSrA(\lambda)$ denote a family of graphs that is equal to the union of families $CSrA(G)$, when G runs through all realizations of the partition λ on the set V, i. e., this is the set of all closest splittable r-ancestors of all realizations (up to isomorphism and isolated vertices) of the partition λ.

It is useful to note that the operation swap of switching edges in alternating 4-cycles does not change the set V of vertices of the graph [4], therefore all realizations of the partition λ can be considered up to isomorphism and isolated vertices on some single set V.

Note that it would be interesting to find a fairly simple description of the family $CSrA(\lambda)$, since from the graphs of this family one can obtain, by Corollary 1, all realizations of the partition λ using sequences consisting of s lowering rotations of edges. This fact makes it possible to study the family of all realizations of the partition λ without using switching edges operation [4] in graphs.

2. Proof of the main results

We first present four auxiliary lemmas and one algorithm.

Lemma 1. *Let $H = (K(V_1), E_1, V_2)$ be a splittable graph, $\mu = \mathrm{dpt}(H)$, V_1 be a clique of cardinality $r = r(\mu)$, consisting of elements of degrees $\mu_1, \ldots, \mu_r$ greater than or equal to r, and V_2 be a coclique consisting of elements of degrees $\mu_{r+1}, \ldots, \mu_n$ less than or equal to r, where n is the number of elements of the graph H. Then* $\mathrm{sum}\,\mathrm{hd}(\mu) = \mathrm{sum}\,\mathrm{tl}(\mu)$.

P r o o f. Let us remove all edges of the form $e = uv$ from the graph H, where $u, v \in V_1$. We obtain a bipartite graph $H_1 = (V_1, E_1, V_2)$, which is a sandwich subgraph of the graph H and for which $\mathrm{dpt}_{H_1}(V_1) = \mathrm{hd}(\mu)$ and $\mathrm{dpt}_{H_1}(V_2) = \mathrm{tl}^*(\mu)$ (see [2]). Therefore, we have $\mathrm{sum}\,\mathrm{hd}(\mu) = |E_1| = \mathrm{sum}\,\mathrm{tl}^*(\mu) = \mathrm{sum}\,\mathrm{tl}(\mu)$. □

Lemma 2. *Let (V_1, V_2) be an arbitrary principal 2-decomposition of a graph G whose Durfey rank is r. Let $e = vx$ be an edge of the graph G such that $v, x \in V_2$. Then there is a vertex $y \in V_1$ for which the triple (x, v, y) is lifting (see Fig. 4). Let us denote by H the graph that obtainable from the graph G using the lifting rotation of edge corresponding to this triple. Then for the graph H we have*

- $\deg u \geq r$ *for any vertex* $u \in V_1$;

- $\deg u \le r$ *for any vertex* $u \in V_2$;
- *the Durfey rank of the graph* H *is equal to* r *and the pair* (V_1, V_2) *remains a principal 2-decomposition for graph* H;
- $\operatorname{sum} \operatorname{hd}(\lambda) + 1 = \operatorname{sum} \operatorname{hd}(\eta)$, $\operatorname{sum}\ \operatorname{tl}(\lambda) - 1 = \operatorname{sum} tl(\eta)$, *where* $\eta = \operatorname{dpt}(H)$.

P r o o f. Since $v \in V_2$, we have $\deg v \le r$. If v is adjacent to all vertices from V_1, then, taking into account the edge $e = vx$, we obtain $\deg v \ge r + 1$ which is contradictory. Therefore, there is a vertex $y \in V_1$ that is not adjacent to v and for which obviously holds $\deg y \ge r \ge \deg x$ (see Fig. 4). Therefore, the triple (x, v, y) is lifting. It is clear that for a graph H obtained from a graph G using

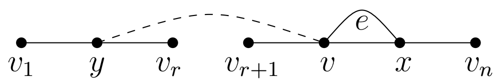

Figure 4. The lifting rotation of the edge $e = vx$ in the graph G.

the lifting rotation corresponding to this triple, the conclusions of the lemma are satisfied in the obvious way. □

Lemma 3. *Let* (V_1, V_2) *be an arbitrary principal 2-decomposition of a graph* G *whose Durfey rank is equal to* r. *Let the vertices* $y, v \in V_1$ *of the graph* G *be distinct and not adjacent. Then there is a vertex* $x \in V_2$ *for which the triple* (x, v, y) *is lifting (see Fig. 5). Let us denote by* H *the graph that obtainable from the graph* G *using the lifting rotation of edge corresponding to this triple. Then for the graph* H *holds*

- $\deg u \ge r$ *for any vertex* $u \in V_1$;
- $\deg u \le r$ *for any vertex* $u \in V_2$;
- *Durfey rank of the graph* H *is equal to* r *and the pair* (V_1, V_2) *remains a principal 2-decomposition for graph* H;
- $\operatorname{hd}(\lambda) + 1 = \operatorname{sum} \operatorname{hd}(\eta)$, $\operatorname{sum} \operatorname{tl}(\lambda) - 1 = \operatorname{sum} \operatorname{tl}(\eta)$, *where* $\eta = \operatorname{dpt}(H)$.

P r o o f. If v is not adjacent to all vertices from V_2, then by virtue of the equality $|V_1| = r$ we have $\deg v < r$, which is contradictory. Therefore, there is a vertex $x \in V_2$ that is adjacent to v and for which it obviously holds $\deg y \ge r \ge \deg x$ (see Fig. 5).

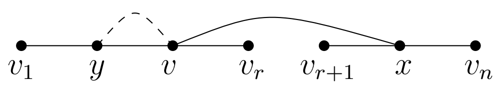

Figure 5. The lifting rotation of edge $e = vx$ in the graph G.

Therefore, the triple (x, v, y) is lifting. It is clear that for a graph H obtained from a graph G using the lifting rotation corresponding to this triple, the conclusions of the lemma are satisfied in the obvious way. □

Let (V_1, V_2) be an arbitrary 2-decomposition of rank r of the set V of vertices of a graph G.

Let by $W_1(G, V_1, V_2)$ we denote the set of all pairs of non-adjacent distinct vertices from V_1, and by $W_2(G, V_1, V_2)$ we denote the set of all pairs of adjacent vertices from V_2, i. e., the number of pairs of vertices from V_2 that are connected by edges of the graph G.

Through

$$w(G, V_1, V_2) = w_1(G, V_1, V_2) + w_2(G, V_1, V_2)$$

we will denote the *weight* of the 2-decomposition (V_1, V_2) of the graph G, where

$$w_1 = |W_1(G, V_1, V_2)| \text{ and } w_2 = |W_2(G, V_1, V_2)|.$$

Let (V_1, V_2) be any principal 2-decomposition of a graph G. Based on Lemmas 2 and 3, the following algorithm obviously leads to the construction of a splittable r-ancestors of the graph G of the form $H' = (K(V_1), E', V_2)$ using $t = w(G, V_1, V_2)$ lifting rotations of edges.

Algorithm 1. Let G be an arbitrary graph of Durfey rank r, $\lambda = \deg G$ and (V_1, V_2) be a principal 2-decomposition.

1. Let $H_0 = G$.
2. Let the graph H_i be constructed from the graph H_0 using i lifting rotations of edges, where

$$0 \le i < w(G, V_1, V_2) \text{ and } w(H_i, V_1, V_2) = w(G, V_1, V_2) - i.$$

   Perform any of the following two actions (a) or (b).

   (a) If there is an edge $e = vx$ of the graph H_i such that $v, x \in V_2$, then by Lemma 2 there is a vertex $y \in V_1$ for which the triple (x, v, y) is lifting. Let us denote by H_{i+1} the graph that is obtained from the graph H_i using the lifting rotation of edge corresponding to this triple.
   (b) If there are two distinct non-adjacent vertices $y, v \in V_1$ of the graph H_i, then by Lemma 3 there is a vertex $x \in V_2$ for which the triple (x, v, y) is lifting. Let us denote by H_{i+1} the graph that is obtained from the graph H_i using the lifting rotation of edge corresponding to this triple.

3. Step 2 perform t times, where $t = w(G, V_1, V_2)$. As a result, a splittable r-ancestor $H_t = (K(V_1), E_t, V_2)$ of the graph G will be constructed.

Proving Theorem 1, we will establish along the way that using Algorithm 1 we can find all the closest splittable r-ancestors of the graph G.

Let (V_1, V_2) be an arbitrary 2-decomposition of rank r of the set V of vertices of a graph G. Then $|V_1| = r$ and $|V_2| = n - r$, where n is the number of vertices of the graph G.

Let $u \in V_1$ and $v \in V_2$. Let by $w(G, u \in V_1, v \in V_2)$ we denote the sum of the number of vertices from V_1 that are not adjacent to u and distinct from u, as well as the number of vertices from V_2 adjacent to v. We will call this integer by a *contribution* of the pair of vertices u and v to the weight $w(G, V_1, V_2)$ of the 2-decomposition (V_1, V_2) of the graph G.

For an arbitrary vertex z of the graph G, let $D_1(z)$ and $D_2(z)$ denote, respectively, the number of vertices from V_1 and V_2 adjacent to vertex z. Let us also put $d_1(z) = |D_1(z)|$ and $d_2(z) = |D_2(z)|$. Then obviously $\deg z = d_1(z) + d_2(z)$.

Lemma 4. *Let (V_1', V_2') be an arbitrary 2-decomposition of rank r of the set V of vertices of a graph G, where r is the Durfey rank of this graph. Let $u \in V_1'$ and $v \in V_2'$. Let us put $V_1'' = V_1' - u + v$ and $V_2'' = V_2' - v + u$. (This procedure we will call, as before, the exchanging vertices in 2-decomposition.) Then the 2-decomposition (V_1'', V_2'') has rank r and it holds*

1) *if $\deg u < \deg v$ in graph G, then $w(G, V_1'', V_2'') < w(G, V_1', V_2')$;*
2) *if $\deg u = \deg v$ in graph G, then $w(G, V_1'', V_2'') = w(G, V_1', V_2')$.*

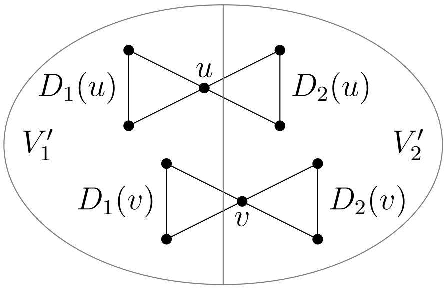

Figure 6. Sets $D_1(u)$ and $D_1(v)$, as well as sets $D_2(u)$ and $D_2(v)$ may intersect.

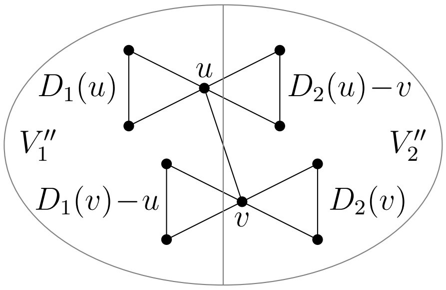

Figure 7. Sets $D_1(u)$ and $D_1(v) - u$, as well as sets $D_2(u) - v$ and $D_2(v)$ may intersect.

P r o o f. Let us first consider two cases.
1 case. Let vertices u and v be not adjacent in the graph G. Then (see Fig. 6)

$$w(G, u \in V_1', v \in V_2') = r - 1 - d_1(u) + d_2(v) = r - 1 - d_1(u) + \deg v - d_1(v),$$
$$w(G, v \in V_1'', u \in V_2'') = r - 1 - d_1(v) + d_2(u) = r - 1 - d_1(v) + \deg u - d_1(u).$$

2 case. Let vertices u and v be adjacent in the graph G. Then (see Fig. 7)

$$w(G, u \in V_1', v \in V_2') = r - 1 - d_1(u) + d_2(v) = r - 1 - d_1(u) + \deg v - d_1(v),$$
$$w(G, v \in V_1'', u \in V_2'') = r - 1 - (d_1(v) - 1) + (d_2(u) - 1)$$
$$= r - 1 - d_1(v) + 1 + \deg u - d_1(u) - 1 = r - 1 - d_1(v) + \deg u - d_1(u).$$

Thus, in each of the two cases considered, following equalities are satisfied

$$w(G, u \in V_1', v \in V_2') = r - 1 - d_1(u) + \deg v - d_1(v),$$
$$w(G, v \in V_1'', u \in V_2'') = r - 1 - d_1(v) + \deg u - d_1(u).$$

Finally, let's look at two cases.

1. Let $\deg u < \deg v$ in the graph G. Then, by virtue of the two equalities obtained, we have

$$w(G, u \in V_1', v \in V_2') > w(G, v \in V_1'', u \in V_2''),$$

   i. e., the contribution of vertices u and v decreased when moving from the 2-decomposition (V_1', V_2') to the 2-decomposition (V_1'', V_2''). This implies $w(G, V_1'', V_2'') < w(G, V_1', V_2')$.
2. Let $\deg u = \deg v$ in graph G. Then we have

$$w(G, u \in V_1', v \in V_2') = w(G, v \in V_1'', u \in V_2'').$$

   This implies $w(G, V_1'', V_2'') = w(G, V_1', V_2')$. □

Lemma 4 has a simple meaning: if you exchange a vertex of a lower degree from V_1' in a 2-decomposition (V_1', V_2') with a vertex of a higher degree from another component of this 2-decomposition, then the weight will decrease when moving from a 2-decomposition (V_1', V_2') to a new 2-decomposition (V_1'', V_2'').

Lemma 4 implies

Corollary 2. 1. *Non-negative integers $w(G, V_1, V_2)$ are the same for all principal 2-decompositions (V_1, V_2) of the graph G.*

2. *Non-negative integer $w(G, V_1, V_2)$ for principal 2-decomposition (V_1, V_2) of the graph G is less than the same form integer for any non-principal 2-decomposition of rank r.*

P r o o f. It is enough to note that principal 2-decompositions can differ only in the location of vertices of degree r in their components. In addition, any non-principal 2-decomposition of rank r comes to a principal 2-decomposition of rank r using a certain sequence of operations of exchanging vertices. □

Lemma 5. *Let (V_1, V_2) be an arbitrary 2-decomposition of rank q of the set V of vertices of a graph G, where $1 \leq q < n$ and n is the cardinality of V. Then any rotation of an edge in the graph G can change the weight $w(G, V_1, V_2)$ of the 2-decomposition (V_1, V_2) by no more than 1 when moving to a new graph.*

P r o o f. Let the rotation of the edge $e = xv$ correspond to a triple (x, v, y). Vertices v and y are different and not adjacent. The old edge $e = xv$ and the new edge $f = vy$ cannot lie in different sets V_1 and V_2, since they have a common vertex v incident to them. This obviously implies the statement of the lemma. □

P r o o f of Theorem 1. Let (V_1, V_2) be a principal 2-decomposition of a graph G. Let $w_1 = w_1(G, V_1, V_2)$ be the number of all pairs of distinct non-adjacent vertices from V_1, $w_2 = w_2(G, V_1, V_2)$ be the number of all pairs of adjacent vertices from V_2. Then

$$w = w(G, V_1, V_2) = w_1(G, V_1, V_2) + w_2(G, V_1, V_2) = w_1 + w_2.$$

Algorithm 1 reduce the graph G to a splittable r-ancestor of the form $H' = (K(V_1), E', V_2)$ by using w lifting rotations of edges.

By Lemmas 2 and 3 we have

$$\operatorname{sum}\operatorname{hd}(\lambda) + (w_1 + w_2) = \operatorname{sum}\operatorname{hd}(\mu),$$
$$\operatorname{sum}\operatorname{tl}(\lambda) - (w_1 + w_2) = \operatorname{sum}\operatorname{tl}(\mu),$$

where $\mu = \operatorname{dpt}(H')$.

Since the splittable graph H' satisfies the conditions of Lemma 1, we obtain

$$\operatorname{sum}\operatorname{hd}(\mu) = \operatorname{sum}\operatorname{tl}(\mu),$$

which implies

$$\operatorname{sum}\operatorname{hd}(\lambda) + (w_1 + w_2) = \operatorname{sum}\operatorname{tl}(\lambda) - (w_1 + w_2).$$

Therefore,

$$2(w_1 + w_2) = \operatorname{sum}\operatorname{tl}(\lambda) - \operatorname{sum}\operatorname{hd}(\lambda) = 2s,$$

i. e., $s = w_1 + w_2 = w$.

Let (V_1', V_2') be an arbitrary 2-decomposition of the graph G. Suppose that the graph G is reduced to a splittable graph $H' = (K(V_1'), E_1, V_2')$ by t lifting rotations of edges, where V_1' is a clique of cardinality r and V_2' is a coclique. Then obviously $w(H', V_1', V_2') = 0$.

As t lifting rotations of the edges change the weight of 2-decomposition (V_1', V_2') from $w(G, V_1', V_2')$ to 0, by Lemma 5 the following holds:

$$t \geq w(G, V_1', V_2').$$

Let's look at two cases.

1 case. If (V_1', V_2') is the principal 2-decomposition of the graph G, then the resulting inequality, due to the fact that $w = s$, gives $t \geq s$ and the proof of statement 1) of the theorem is completed.

2 case. Let (V_1', V_2') be a non-principal 2-decomposition of the graph G. Then, taking into account Corollary 2, we obtain

$$t \geq w(G, V_1', V_2') > w(G, V_1, V_2) = s,$$

where (V_1, V_2) is an arbitrary of the principal 2-decompositions of the set V of vertices of the graph G. The proof of statement 2) is also completed. □

Lemma 6. *Any closest splittable r-ancestor of a graph G can be obtained by some application of Algorithm 1.*

P r o o f. Let $H' = (K(V_1'), E', V_2')$ be some closest splittable r-ancestor of the graph G, i. e., it can be obtained from the graph G using a sequence of s lifting rotations of edges.

Then, by virtue of what was established in the proof of the theorem, the 2-decomposition (V_1', V_2') is the principal 2-decomposition of the graph G (here $t = s$). It is clear that in a sequence of s lifting rotations of edges transforming G to H', each lifting rotation must decrease the weight of the 2-decomposition (V_1', V_2') by exactly 1, i. e., it must be performed in accordance with step 2 of Algorithm 1. □

Algorithm 1 we will call *the algorithm for reducing a graph G to a closest splittable r-ancestor*. Of course, different implementations of this algorithm may produce different closest splittable r-ancestors of the original graph G (see, for example, Fig. 3).

3. Conclusion

In conclusion, we note that in connection with Corollary 2 the following two problems are of interest.

Firstly, we give a necessary definition. Let μ and λ be graphical partitions of the same weight $2m$ such that μ dominates λ. Let $\mathrm{height}(\mu, \lambda)$ denote the height of the partition μ over the partition λ in the lattice of all partitions of weight $2m$, which is equal to the length of the shortest sequence of elementary transformations of the first type transforming μ into λ (see [2]).

Problem 1. *Let λ be a graphical partition of rank r. Find all graphical partitions μ of rank r that dominate partition λ such that $\mathrm{sum}\,\mu = \mathrm{sum}\,\lambda$,*

$$\mathrm{sum}\,\mathrm{hd}(\mu) = \mathrm{sum}\,\mathrm{tl}(\mu) \quad \textit{and} \quad \mathrm{height}(\mu, \lambda) = \frac{1}{2}(\mathrm{sum}\,\mathrm{tl}(\lambda) - \mathrm{sum}\,\mathrm{hd}(\lambda)).$$

Note that the condition $\operatorname{sum}\operatorname{hd}(\mu) = \operatorname{sum}\operatorname{tl}(\mu)$ means that any realization of the partition μ is a splittable graph (see, for example, [2]). The condition

$$\operatorname{height}(\mu, \lambda) = \frac{1}{2}(\operatorname{sum}\operatorname{tl}(\lambda) - \operatorname{sum}\operatorname{hd}(\lambda))$$

means that using some s lowering rotations of edges for any realization of the partition μ leads to a realization of the partition λ, where

$$s = \frac{1}{2}(\operatorname{sum}\operatorname{tl}(\lambda) - \operatorname{sum}\operatorname{hd}(\lambda)).$$

Problem 2. *Let λ be a graphical partition of rank r.*

1. *For a given graph G of Durfey rank r, find the family $CSrA(G)$ of all its closest splittable r-ancestors.*
2. *Find the family $CSrA(\lambda)$ of all splittable graphs, each of which is the closest splittable r-ancestor for some realization of the partition λ.*
3. *Find a family of closest splittable r-ancestors of some realizations of the partition λ such that*
    - *every realization of the partition λ can be obtain (up to isomorphism and isolated vertices) from element of this family by sequentially applying s lowering rotations of edges where $s = (1/2)(\operatorname{sum}\operatorname{tl}(\lambda) - \operatorname{sum}\operatorname{hd}(\lambda))$,*
    - *this family has the smallest possible number of elements.*

The work [2] gives an example of a partition $\lambda = (4, 3, 2, 2, 2, 1)$, for which $r = 2$ and $s = 1$ such that each of its realizations can be obtained from a common splittable 2-ancestor using a single lowering rotation of an edge.

In conclusion, let us give another example that shows that one splittable r-ancestor may not be sufficient to obtain all realizations of a given partition λ of rank r by sequentially applying s lowering rotations of edges.

Example 2. Let $\lambda = (3, 3, 2, 2, 1, 1)$. Then $r = 2$, $\operatorname{hd}(\lambda) = (2, 2)$, $\operatorname{tl}(\lambda) = (4, 2)$, $s = 1$.

It is easy to check that the partition λ has 5 pairwise non-isomorphic realizations G_1, G_2, G_3, G_4, G_5 without isolated vertices and these realizations have exactly 2 non-isomorphic closest splittable 2-ancestors H_1' and H_2' (see Fig. 8 and Fig. 9). Here V_1 consists of two vertices of the highest degree, and V_2 consists of four remaining vertices (note that for the graph H_1' in V_2 there is one vertex of zero degree).

In G_2 we have $t_1 = 1$ and $t_2 = 0$, and in G_1, G_3, G_4, G_5 we have $t_1 = 0$ and $t_2 = 1$.

It is easy to check that with respect to the principal 2-decomposition (V_1, V_2)

- graph G_1 has H_1' as exactly one closest splittable 2-ancestor;
- graphs G_2 and G_3 have exactly 2 closest splittable 2-ancestors H_1' and H_2';
- graphs G_4 and G_5 have H_2' as exactly one closest splittable 2-ancestor.

Note also that graph H_1' can be obtained from graph H_2' using a single lifting rotation of an edge, and $\deg H_2' = (4, 3, 2, 1, 1, 1)$ can be obtained from $\deg H_1' = (4, 3, 2, 2, 1)$ using one elementary transformation of the first type.

Note that the graph H_1' is a threshold graph [6], since the partition $(4, 3, 2, 2, 1)$ has the same tail and head [2], and the graph H_2' is not a threshold graph, since its degree partition $(4, 3, 2, 1, 1, 1)$ is not a maximum graphical partition.

It is clear that to obtain all realizations of the partition $\lambda = (3, 3, 2, 2, 1, 1)$ by applying a single lowering rotation of an edge, we need to use both graphs H_1' and H_2'. Here H_1' and H_2' are not common closest splittable 2-ancestors of graphs G_1 and G_5.

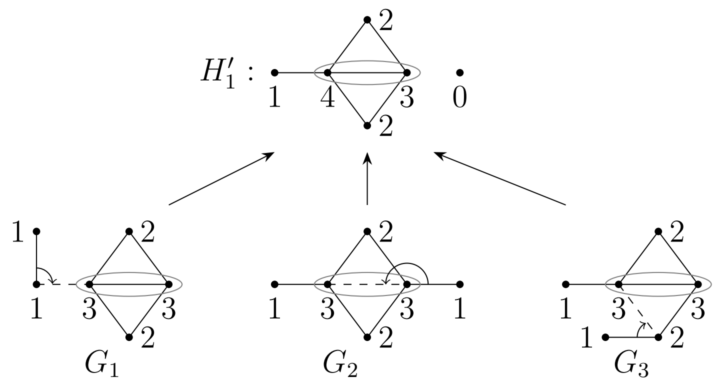

Figure 8. The common closest splittable 2-ancestor of graphs G_1, G_2 and G_3.

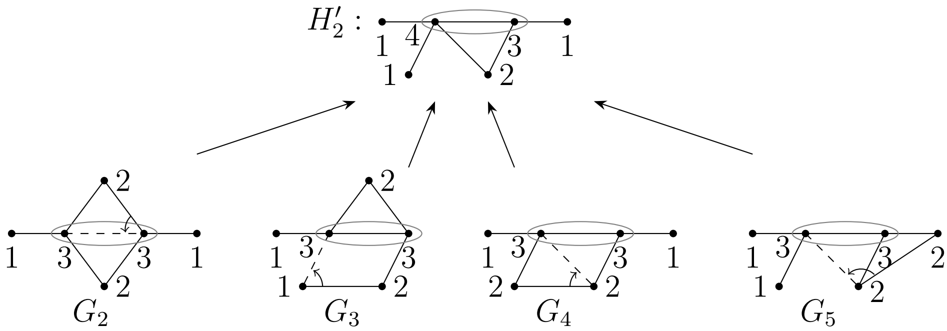

Figure 9. The common closest splittable 2-ancestor of graphs G_2, G_3, G_4 and G_5.

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