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STATISTICAL CONVERGENCE IN TOPOLOGICAL SPACE CONTROLLED BY MODULUS FUNCTION

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Abstract: The notion of f-statistical convergence in topological space, which is actually a statistical convergence's generalization under the influence of unbounded modulus function is presented and explored in this paper. This provides as an intermediate between statistical and typical convergence. We also present many counterexamples to highlight the distinctions among several related topological features. Lastly, this paper is concerned with the notions of s^f-limit point and s^f-cluster point for a unbounded modulus function f.

Keywords: Asymptotic density, f-statistical convergence, f-statistical limit point, f-statistical cluster point.

1. Introduction

Statistical density was initially introduced in Zygmund's 1935 monograph [18]. Extending on the concept using statistical density, Fast [12] (along with Schohenberg [17]) in 1951 broadened the definition of convergence to include statistical convergence. Let $\mathbb{N}$ denote the set of natural numbers, and $A \subseteq \mathbb{N}$. The notation $\delta(A)$ signifies the natural density or asymptotic density of set A [12], defined as

$$\delta(A) = \lim_{n\to\infty} \frac{|\{k \leq n : k \in A\}|}{n}.$$

A real sequence $\{x_n : n \in \mathbb{N}\}$ is considered statistically convergent to a point l (see [17]) if, for every $\epsilon > 0$

$$\delta(\{n \in \mathbb{N} : |x_n - l| \geq \epsilon\}) = 0.$$

Subsequent to the contributions of Fridy [13] and Connor [9] in the realm of statistical convergence, other mathematicians have displayed considerable interest in this domain. In 2008, Maio and Kočinac [15] extended the notion to statistical convergence in topological spaces. In a topological space (X, τ), a sequence $\{x_n : n \in N\}$ is deemed statistically convergent to a point l if, for every neighborhood U of l,

$$\delta(\{n \in \mathbb{N} : x_n \notin U\}) = 0.$$

This form of convergence has proven to be highly valuable across various fields, particularly in the examination of open cover classes and selection principles [1–4, 10, 14, 16].

To establish a notion of convergence that sits between the ordinary convergence and statistical convergence many authors produced several approches. In 2012, Bhunia et al. [6] (see also Çolak et al. [7, 8]) enhanced the idea of s-convergence for real sequences by imposing a limitation on asymptotic density up to order α, where $0 < \alpha \leq 1$. Through the utilization of asymptotic density of order α, a more stringent convergence criterion is introduced, surpassing statistical convergence but remaining less stringent than the conventional convergence in a topological space. As a direct outcome of this exploration, a novel class of open covers, denoted as $s^\alpha - \Gamma$, emerges. With the same purpose the unbounded modulus function is used in this paper and the concept of f-statistical convergence has been extended to the topological point of view. The class of modulus functions, as described by the given conditions, is a set of functions from the positive real numbers to the positive real numbers. Let's break down the key properties:

1. *Zero at Zero.* The function $f(x)$ is equal to zero if and only if x is equal to zero. This implies that the function is zero only at the origin.

2. *Subadditivity.* For any positive real numbers x and y, the function satisfies the property $f(x+y) \leq f(x)+f(y)$. This condition is known as subadditivity, indicating that the function's values do not grow faster than the sum of its individual parts. It's a form of the triangle inequality.

3. *Monotonicity.* The function is increasing, meaning that as the input increases, the function values also increase. Mathematically, if $a<b$, then $f(a) \leq f(b)$.

4. *Right-Continuous at Zero.* The function $f(x)$ is continuous from the right at $x=0$. This implies that as the input approaches zero from the positive side, the function values approach the limit without any sudden jumps or discontinuities.

This class of modulus functions appears to capture functions that exhibit properties similar to those of the absolute value function. The conditions ensure a certain level of behavior for the function, making it well-behaved and suitable for various mathematical applications.

Function f is unbounded if

$$\lim_{x\to\infty} f(x)=\infty.$$

For an unbounded modulus function f, f-density of a set $A \subseteq \mathbb{N}$ is denoted by $\delta^f(A)$ and is defined as [5]

$$\delta^f(A)=\lim_{n\to\infty}\frac{f(|\{k\leq n: k\in A\}|)}{f(n)}.$$

In this paper we have explored that the function f is very useful to control the rate at which statistical convergence occurs. We extend the concept of s-convergence to s^f-convergence in topological environment and explore several attributes of this convergence criteria. In last section we investigate some properties of s^f-limit points and s^f-cluster points.

2. Preliminaries

In this paper, a space X is defined as a topological space X with topology τ. No separation axioms have been granted, unless otherwise stated. For standard ideas, symbols, and terminology, we refer to [11]. For the convenience of the readers, this section includes certain required concepts.

The unbounded modulus functions defined on the set $\mathbb{N}$ of all natural numbers are the modulus functions taken into consideration in this study. Therefore, right continuous at zero and zero to zero property are disregarded. The modulus function $f:\mathbb{N}\to\mathbb{R}^+$, defined as $f(n)=\log(1+n)$, is regarded as such in the majority of the cases. It is obvious to note that this modulus function is unbounded.

Definition 1 [5]. *For an unbounded modulus function f, f-density of a set $A\subseteq\mathbb{N}$ is denoted by $\delta^f(A)$ and is defined as*

$$\delta^f(A)=\lim_{n\to\infty}\frac{f(|\{k\leq n: k\in A\}|)}{f(n)}.$$

Using the concept Bhardwaj et al. [5] extended the concept of statistical convergence of a real sequence upto s^f-convergence.

Definition 2 [5]. *A real sequence $\{x_n : n \in \mathbb{N}\}$ is considered s^f-convergent to a point l if, for every $\epsilon > 0$,*

$$\delta^f(\{n \in \mathbb{N} : |x_n - l| \geq \epsilon\}) = 0,$$

where f is an unbounded modulus function.

In [15], the concept of statistical dense sub-sequence, s-limit point of a sequence and s-cluster point of a sequence are discussed.

Definition 3 [15]. *A subsequence $\mathcal{V} = \{x_{n_k} : k \in \mathbb{N}\}$ of the sequence $\{x_n : n \in \mathbb{N}\}$ is called a statistically dense if*

$$\delta(\{n_k : x_{n_k} \in \mathcal{V}\}) = 1.$$

Definition 4 [15]. *A point x is said to be a statistical limit point of a sequence $\{x_n : n \in \mathbb{N}\}$ in a space X, if there is a set $\{n_1 < n_2 < ... < n_k < ...\} \subset \mathbb{N}$ whose asymptotic density is not zero (which means that it is greater than zero or does not exist) such that*

$$\lim_{k\to\infty} x_{n_k} = x.$$

Definition 5 [15]. *A point x is called a statistical cluster point of a sequence $\{x_n : n \in \mathbb{N}\}$ if for each neighborhood U of x the asymptotic density of the set $\{n \in \mathbb{N} : x_n \in U\}$ is positive.*

3. On f-statistical convergence

Definition 6. *Let $f : \mathbb{N} \to \mathbb{R}$ be an unbounded modulus function and (X, τ) be a topological space. A sequence $\{x_n : n \in \mathbb{N}\}$ in X will be called f-statistical convergent (in short s^f-convergent) to $x \in X$, if for every neighborhood U of x,*

$$\delta^f(\{n \in \mathbb{N} : x_n \notin U\}) = 0,$$

$$i.e., \quad \lim_{n\to\infty} \frac{f(|\{k \leq n : x_n \notin U\}|)}{\{f(n)\}} = 0.$$

From the study of Maio and Kočinac [15], we know that every convergent sequence is statistical convergent but converse is not true. Since for a finite set F, $\delta^f(F) = 0$, therefore usual convergence implies the s^f-convergence. For any unbounded modulus function f and $A \subseteq \mathbb{N}$, if $\delta^f(A) = 0$, then $\delta(A) = 0$. Thus every s^f-convergent sequence is statistical convergent. Moreover, concept of statistical convergence coincides with the concept of s^f-convergence if the unbounded modulus function under consideration is $f(n) = n$.

Example 1. There is a sequence in topological space which is statistical convergent but not s^f-convergent and a sequence which is s^f-convergent but not convergent.

Let (X, τ) be a topological space where $X = \{1, 2\}$ and $\tau = \mathcal{P}(X)$. Consider the sequence $\{x_n : n \in \mathbb{N}\}$ where

$$x_n = \begin{cases} 1, & \text{if} \quad n = m^2 \text{ for some } m \in \mathbb{N}. \\ 2, & \text{otherwise.} \end{cases}$$

Let the function $f(n) = \log(1 + n)$ be the unbounded modulus function under consideration. Neighborhoods of 2 are $U_1 = \{2\}$ and $U_2 = X$. Here,

$$\delta(\{n \in \mathbb{N} : x_n \notin U_1\}) = \delta(\{m^2 : m \in \mathbb{N}\}) = 0$$

and

$$\delta(\{n \in \mathbb{N} : x_n \notin U_2\}) = \delta(\emptyset) = 0.$$

Therefore,

$$x_n \overset{s-\lim}{\longrightarrow} 2.$$

But for the neighborhood $U_1 = \{2\}$ of 2, we have

$$\delta^f(\{n \in \mathbb{N} : x_n \notin U_1\}) = \frac{1}{2} \neq 0.$$

Also, for the neighborhood $V = \{1\}$ of 1, we have

$$\delta^f(\{n \in \mathbb{N} : x_n \notin V\}) = 1 \neq 0.$$

So, $\{x_n : n \in \mathbb{N}\}$ is neither s^f-convergent to 1 nor s^f-convergent to 2.

In the same space consider the sequence $\{y_n : n \in \mathbb{N}\}$ where

$$y_n = \begin{cases} 1, & \text{if} \quad n = m^m \text{ for some } m \in \mathbb{N}, \\ 2, & \text{otherwise.} \end{cases}$$

Then $\{y_n : n \in \mathbb{N}\}$ is s^f-convergent to 2 but not convergent.

Thus we have the following diagram (see Fig. 1).

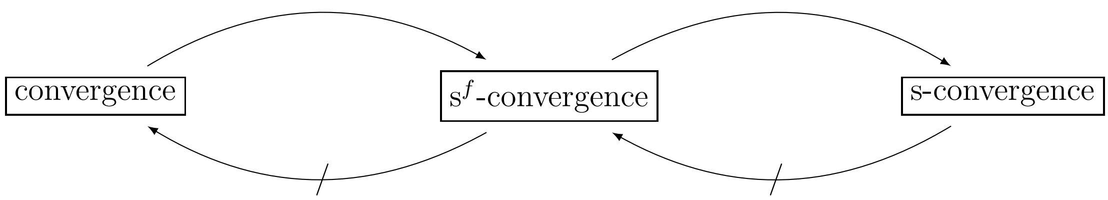

Figure 1. Types of convergence and the relationship between them.

Definition 7. *A sequence $\{x_n : n \in \mathbb{N}\}$ in a topological space X is said to s^f_* convergent to $x_0 \in X$ if there exists $A \subseteq \mathbb{N}$ with $\delta^f(A) = 1$ such that*

$$\lim_{m\to\infty,\ m\in A} x_n = x_0.$$

Example 2. There exists a sequence $\{x_n : n \in \mathbb{N}\}$ which is s^f_*-convergent but not s^f-convergent.

Let us assume a topological space (X, τ) where $X = \{a, b\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}\}$. Again we construct a sequence $\{x_n : n \in \mathbb{N}\}$ where

$$x_n = \begin{cases} a, & \text{if} \quad n \in 2\mathbb{N}, \\ b, & \text{otherwise.} \end{cases}$$

Let $f(n) = \log(1+n)$ be the modulus function under consideration then, for the neighbourhood $\{a\}$ of 'a', we have

$$\delta^f(\{n \in \mathbb{N} : x_n \notin \{a\}\}) = \delta^f(\{\mathbb{N} \setminus 2\mathbb{N}\}) = \lim_{n\to\infty} \frac{f(n+1)}{f(2n+1)} = \lim_{n\to\infty} \frac{\log(n+2)}{\log(2n+2)} = 1.$$

And for the neighbourhood $\{b\}$ of 'b', we have

$$\delta^f(\{n \in \mathbb{N} : x_n \notin \{a\}\}) = \delta^f(\{2\mathbb{N}\}) = \lim_{n\to\infty} \frac{f(n)}{f(2n)} = \lim_{n\to\infty} \frac{\log(n+1)}{\log(2n+1)} = 1.$$

Therefore $\{x_n : n \in \mathbb{N}\}$ neither s^f-convergent to a nor s^f-convergent to 'b'.

On the other hand $2\mathbb{N} \subseteq \mathbb{N}$ such that $\delta^f(2\mathbb{N}) = 1$ and $\{x_n : n \in \mathbb{N}\} = \{a, a, \dots\}$

$$\lim_{n\to\infty,\ n\in 2\mathbb{N}} x_n = a, \quad \Rightarrow x_n \overset{s_*^f-\lim}{\to} a.$$

Although s_*^f-convergence does not imply the s^f-convergence of a sequence, the s^f-convergence of a sequence implies its s_*^f-convergence in a first countable space.

Theorem 1. *In a first countable space, if a sequence $\{x_n : n \in \mathbb{N}\}$ in X s^f-converges to x, then this sequence s_*^f-converges to x.*

P r o o f. Let (X, τ) be a first countable topological space and $\{x_n : n \in \mathbb{N}\}$ be a sequence in X which s^f-converges to x. Since X is first countable, there exists countable decreasing local base $U_{1,x} \supseteq U_{2,x} \supseteq U_{3,x} \supseteq \dots$ at the point x. Now consider a set $A_i = \{n \in \mathbb{N} : x_n \in U_{i,x}\}$ for every $i \in \mathbb{N}$. Then we have $A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$.

Again we know that sequence $\{x_n : n \in \mathbb{N}\}$ is s^f-convergent then

$$\begin{aligned} &\delta^f(\{n \in \mathbb{N} : x_n \notin U_{i,x}\}) = 0, \quad \forall i \in \mathbb{N} \\ &\Longrightarrow \delta^f(\{A_i^c\}) = 0, \quad \forall i \in \mathbb{N} \\ &\Longrightarrow \delta^f(\{A_i\}) = 1, \quad \forall i \in \mathbb{N}. \end{aligned}$$

Let $m_1 \in A_1$ be arbitrary. Since $\delta^f(A_2) = 1$, we can find a $m_2 \in A_2$ such that $m_2 > m_1$ and such that

$$\frac{f(|\{A_2(n)\}|)}{f(n)} = \frac{f(|\{k \in A_2 : k \leq n\}|)}{f(n)} > \frac{1}{2} = 1 - \frac{1}{2}, \quad \text{for all} \quad n \geq m_2.$$

In similar way if we obtain $m_1 < m_2 < \dots < m_i \in A_i$, such that for every $n \geq m_i$ then,

$$\frac{f(|\{A_i(n)\}|)}{f(n)} = \frac{f(|\{k \in A_i : m \leq n\}|)}{f(n)} > 1 - \frac{1}{i}.$$

Now we define a set $A \subseteq \mathbb{N}$ as for each $m \leq m_1$ and $m \in A$; if $i \geq 1$ and for $m_i < m \leq m_{i+1}$, $m \in A$ if and only if $m \in A_i$. Let $A = \{n_1 < n_2 < \dots\}$. For all $n \in \mathbb{N}$ such that $m_i \leq n \leq m_{i+1}$, we have

$$\begin{aligned} \frac{f(|\{A(n)\}|)}{f(n)} &\geq \frac{f(|\{A_i(n)\}|)}{f(n)} \geq 1 - \frac{1}{i}, \\ \textit{i.e.,} \quad \lim_{n\to\infty} \frac{f(|\{A(n)\}|)}{f(n)} &\geq \lim_{n\to\infty} \frac{f(|\{A_i(n)\}|)}{f(n)} \geq \lim_{n\to\infty} 1 - \frac{1}{i}, \\ \textit{i.e.,} \quad \lim_{n\to\infty} \frac{f(|\{A(n)\}|)}{f(n)} &= 1, \ \Rightarrow \delta^f(A) = 1. \end{aligned}$$

Now, let V be a neighbourhood of x and $U_i \subseteq V$. If we take $n \in A$, $n \geq m_i$ then there exists $j \geq i$ such that we get $m_j \leq n \leq m_{j+1}$. So by the construction of A, $n \in A_j$. Therefore, for each $n \in A$, $n \geq m_i$ we get $x_n \in U_j$ and $x_n \in U_j \subseteq U_i \subseteq V$,

$$\textit{i.e.,} \quad \lim_{n\to\infty,\ n\in A} x_n = x.$$

Thus $\{x_n : n \in \mathbb{N}\}$ s^f_*-converges to x. $\square$

Example 3. s^f-limit of an s^f-convergent sequence may not be unique.

Let us assume a topological space (X, τ) where $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{b, c\}, \{a\}\}$ and $f(n) = \log(1+n)$ be the unbounded modulus function under consideration. We construct a sequence $\{x_n : n \in \mathbb{N}\}$ where

$$x_n = \begin{cases} a, & \text{if} \quad n = m^m \quad \text{for some} \quad m \in \mathbb{N}, \\ b, & \text{otherwise.} \end{cases}$$

Open neighbourhoods of b are $U_1 = X$ and $U_2 = \{b, c\}$.
For the neighbourhood U_1, $\{n \in \mathbb{N} : x_n \notin U_1\} = \emptyset$. So, $\delta^f(\{n \in \mathbb{N} : x_n \notin U_1\}) = 0$.
For the neighbourhood U_2 we have $\{n \in \mathbb{N} : x_n \notin U_2\} = \delta^f(\{n^n : n \in \mathbb{N}\})$.

Therefore

$$\delta^f(\{n^n : n \in \mathbb{N}\}) = \lim_{n\to\infty} \frac{\log(n+1)}{\log(n^n+1)}$$
$$= \lim_{n\to\infty} \frac{n^n+1}{(n+1)n^n(1+\log(n))} = \lim_{n\to\infty} \frac{n^n(1+1/n^n)}{n^n n(1+1/n)(1+\log(n))} = 0.$$

Therefore, $\{x_n : n \in \mathbb{N}\}$ is f-statistical convergent sequence and $x_n \overset{s^f-\lim}{\to} b$.

Since, neighbourhood of b is the only neighbourhood of c, we can say for every neighbourhood U of c also

$$\delta^f(\{n \in \mathbb{N} : x_n \notin U\}) = 0.$$

Thus

$$x_n \overset{s^f-\lim}{\to} c.$$

Thus the limit of an s^f-convergent sequence may not be unique.

Theorem 2. *In a Hausdorff space any s^f-convergent sequence has a unique limit.*

P r o o f. Let $\{a_n : n \in \mathbb{N}\}$ be a s^f-convergent sequence in a topological space (X, τ) and

$$a_n \overset{s^f-\lim}{\to} a, \quad a_n \overset{s^f-\lim}{\to} b.$$

Since, X is Hausdorff space then there exist open sets G and H such that $a \in G, b \in H$ and $G \cap H = \emptyset$. But $\{a_n : n \in \mathbb{N}\}$ is an s^f-convergent sequence which s^f-converges to both a and b. Therefore

$$\delta^f(\{n \in \mathbb{N} : a_n \notin G\}) = 0, \quad \delta^f(\{n \in \mathbb{N} : a_n \notin H\}) = 0.$$

Since, $G \cap H = \emptyset$ and $H \subseteq X \setminus G$. Now,

$$\delta^f(\{n \in \mathbb{N} : a_n \in H\}) \le \delta^f(\{n \in \mathbb{N} : a_n \in X \setminus G\}) = \delta^f(\{n \in \mathbb{N} : a_n \notin G\}) = 0.$$

So,

$$\delta^f(\{n \in \mathbb{N} : a_n \in H\}) = 0$$

and hence

$$\delta^f(\{n \in \mathbb{N} : a_n \notin H\}) = 1.$$

This contradicts the fact that

$$\delta^f(\{n \in \mathbb{N} : a_n \notin H\}) = 0.$$

Hence in Hausdorff space any s^f-convergent sequence has the unique limit. $\square$

Proposition 1. *In a discrete topological space (X,τ), let $p, q \in X$. $h : \mathbb{N} \to \mathbb{N}$ be an one-one function and f be the unbounded modulus function under consideration. Then the sequence*

$$x_n = \begin{cases} p, & \text{if} \quad n = h(k) \quad \text{for some} \quad k \in \mathbb{N}, \\ q, & \text{otherwise} \end{cases}$$

is s^f-convergent to q if

$$\lim_{n\to\infty} \frac{f(n)}{f \circ h(n)} \to 0$$

and does not converges otherwise.

P r o o f. In the mentioned topological space, $\{q\}$ is the smallest neighbourhood of q. To show the s^f-convergence of the sequence $\{x_n : n \in \mathbb{N}\}$, it is enough to show that

$$\delta^f(\{n \in \mathbb{N} : x_n \notin \{q\}\}) = 0.$$

Now,

$$\delta^f(\{n \in \mathbb{N} : x_n \notin \{q\}\}) = \lim_{n\to\infty} \frac{f(|\{k \leq n : x_n \notin \{q\}\}|)}{f(n)}$$
$$= \lim_{n\to\infty} \frac{f(|\{h(k) \leq n : k \in \mathbb{N}\}|)}{f(n)} = \lim_{n\to\infty} \frac{f(n)}{f \circ h(n)}.$$

Hence the proposition is true. □

Example 4. Subsequence of an s^f-convergent sequence may not be s^f-convergent.

Let us assume a topological space (X,τ) where $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{b, c\}, \{a\}\}$ and a sequence $\{x_n : n \in \mathbb{N}\}$ where,

$$x_n = \begin{cases} b, & \text{if} \quad n = m^m \quad \text{for some} \quad m \in \mathbb{N}, \\ a, & \text{otherwise.} \end{cases}$$

Now for every open neighbourhood U of a , we get

$$\{n \in \mathbb{N} : x_n \notin U\} \subseteq \{m^m : m \in \mathbb{N}\}.$$

Let us consider the unbounded modulus function $f(x) = \log(1 + x)$ then we have,

$$\delta^f(\{n \in \mathbb{N} : x_n \notin \{a\}\}) = 0.$$

So, $\{x_n : n \in \mathbb{N}\}$ is an s^f-convergent sequence. Again we construct a subsequence

$$x_{n_i} = \begin{cases} x_{i^i}, & \text{if} \quad i \quad \text{is odd,} \\ x_{((i-1)^{i-1}+1)}, & \text{if} \quad i \quad \text{is even.} \end{cases}$$

Now for the open neighbourhoods of $\{a\}$ of a, we have

$$\delta^f(\{i \in \mathbb{N} : x_{n_i} \notin \{a\}\}) = \delta^f(\{2n : n \in \mathbb{N}\}) = 1 \neq 0.$$

Similarly for the open neighbourhood of $\{b, c\}$ we have

$$\delta^f(\{i \in \mathbb{N} : x_{n_i} \notin \{b, c\}\}) = \delta^f(\{1, 3, 5, \dots\}) = 1 \neq 0.$$

Therefore,

$$x_n \overset{s^f-\lim}{\not\twoheadrightarrow} a, \quad x_n \overset{s^f-\lim}{\not\twoheadrightarrow} b \quad \text{and} \quad x_n \overset{s^f-\lim}{\not\twoheadrightarrow} c.$$

So, $\{x_{n_i} : i \in \mathbb{N}\}$ is not s^f-convergent sequence. □

Definition 8. *A subsequence $B = \{x_{n_k} : k \in \mathbb{N}\}$ of any sequence $A = \{x_n : n \in \mathbb{N}\}$ is called statistically f-dense (or s^f-dense) if $\delta^f(n_k : x_{n_k} \in B) = 1$.*

Theorem 3. *In a topological space (X, τ), a sequence $\{x_n : n \in \mathbb{N}\}$ is s^f-convergent if and only if each of its s^f-dense subsequence is s^f-convergent.*

P r o o f. Suppose (X, τ) be a topological space and $\{x_n : n \in \mathbb{N}\}$ be a sequence for which every s^f-dense subsequence is s^f-convergent. But

$$\lim_{n\to\infty} \frac{f(|\{k \le n : x_k \in \{x_n : n \in \mathbb{N}\}\}|)}{f(n)} = \lim_{n\to\infty} \frac{f(n)}{f(n)} = 1$$

for every unbounded modulus function f. So $\{x_n : n \in \mathbb{N}\}$ is s^f-dense in itself. Therefore, $\{x_n : n \in \mathbb{N}\}$ is s^f-convergent.

Conversely, let $\{x_n : n \in \mathbb{N}\}$ be a s^f-convergent sequence of a topological space (X, τ) and a subsequence $\{x_{n_k} : k \in \mathbb{N}\}$ is s^f-dense but not s^f-convergent. Therefore, there exists a point $p \in X$ and a neighbourhood U of p such that $\delta^f(\{k \in \mathbb{N} : x_{n_k} \notin U\}) \neq 0$. Now,

$$\begin{aligned}\lim_{n\to\infty} \frac{f(|\{n \in \mathbb{N} : x_n \notin U\}|)}{f(n)} &\ge \lim_{n\to\infty, k\to\infty} \frac{f(|\{n_k \in \mathbb{N} : x_{n_k} \notin U\}|)}{f(n)} \\ &= \lim_{k\to\infty} \frac{f(|\{n_k \in \mathbb{N} : x_{n_k} \notin U\}|)}{f(|n_k|)} \times \lim_{n\to\infty} \frac{f(|n_k|)}{f(n)} \neq 0.\end{aligned}$$

Since,

$$\lim_{k\to\infty} \frac{f(|\{k \in \mathbb{N} : x_{n_k} \notin U\}|)}{f(k)} \neq 0$$

and $\{x_{n_k} : k \in \mathbb{N}\}$ is s^f-dense then

$$\text{i.e.} \quad \lim_{n\to\infty} \frac{f(|n_k|)}{f(n)} = 1.$$

So, we get

$$\delta^f(\{n \in \mathbb{N} : x_n \notin U\}) \ge \delta^f(\{k \in \mathbb{N} : x_{n_k} \notin U\}) \neq 0.$$

Therefore $\{x_n : n \in \mathbb{N}\}$ is not s^f-convergent sequence, which is a contradiction. So $\{x_{n_k} : k \in \mathbb{N}\}$ must be s^f-convergent. □

4. f-statistical limit point, f-statistical cluster point

In this section we extend the concept of statistical limit point to s^f-limit point by incorporating an unbounded modulus function f.

Definition 9. *In a topological space (X, τ), a point x_0 is called a f-statistical limit point (in short s^f-limit point) of a sequence $\{x_n : n \in \mathbb{N}\}$ if there exists a subsequence $\mathcal{V} = \{x_{n_k} : k \in \mathbb{N}\}$ such that $\delta^f\{n_k : k \in \mathbb{N}$ and $x_{n_k} \in \mathcal{V}\} > 0$ and*

$$\lim_{k\to\infty} x_{n_k} = x_0.$$

Definition 10. *In a topological space (X, τ), a point x_0 is called f-statistical cluster point (in short s^f-cluster point) of any sequence $\{x_n : n \in \mathbb{N}\}$ if for each neighbourhood U of x_0, $\delta^f\{n \in \mathbb{N} : x_n \in U\} > 0$.*

We denote the set of all f-statistical limit points and f-statistical cluster points by Λ_f and Θ_f, respectively.

Theorem 4. *For a sequence $\{x_n : n \in \mathbb{N}\}$ in a topological space (X, τ), $\Lambda_f(x_n) \subset \Theta_f(x_n)$.*

P r o o f. Let (X, τ) be a topological space, $\{x_n : n \in \mathbb{N}\}$ be a sequence and any point p be f-statistical limit point. Therefore, $p \in \Lambda_f(x_n)$. Then there exists a subsequence $\{x_{n_k} : k \in \mathbb{N}\}$, where $\{n_k : k \in \mathbb{N}\}$ have a positive δ^f-density and

$$\lim_{k\to\infty} x_{n_k} = p.$$

Now,

$$\delta^f(\{n_k : k \in \mathbb{N}\}) = \alpha \text{ (say)} > 0$$

and for every neighbourhood U of p, $\{n_k : x_{n_k} \notin U\} = F$ (say) is finite. But,

$$(\{n \in \mathbb{N} : x_n \in U\}) \supset (\{n_k : k \in \mathbb{N}\}) \setminus F.$$

Since F is a finite set,

$$\lim_{n\to\infty} \frac{f(|F|)}{f(n)} = 0$$

and f is a modulus function,

$$\begin{aligned}\delta^f(\{k_k : k \in \mathbb{N}\} \setminus F) &= \lim_{n\to\infty} \frac{f(|\{n_k : k \in \mathbb{N}\} \setminus F|)}{f(n)} = \lim_{n\to\infty} \frac{f(|\{n_k : k \in \mathbb{N}\} \setminus F|)}{f(n)} + \lim_{n\to\infty} \frac{f(|F|)}{f(n)} \\ &\geq \lim_{n\to\infty} \frac{f(|\{n_k : k \in \mathbb{N}\} \setminus F| + |F|)}{f(n)} = \lim_{n\to\infty} \frac{f(|\{n_k : k \in \mathbb{N}\}|)}{f(n)} = \delta^f(\{n_k : k \in \mathbb{N}\}).\end{aligned}$$

Therefore,

$$\delta^f(\{n \in \mathbb{N} : x_n \in U\}) \geq \delta^f(\{n_k : k \in \mathbb{N}\}) = \alpha > 0.$$

Therefore $p \in \Theta_f(x_n)$. So, $\Lambda_f(x_n) \subset \Theta_f(x_n)$. $\square$

Theorem 5. *In a topological space (X, τ), for a sequence $\{x_n : n \in \mathbb{N}\}$, the set $\Theta_f(x_n)$ is a closed set.*

P r o o f. Let $\{x_n : n \in \mathbb{N}\}$ be a sequence in a topological space (X, τ). Let U be an arbitrary neighbourhood of point the point $x_0 \in \overline{\Theta_f(x_n)}$. So $U \cap \Theta_f(x_n) \setminus \{x_0\} \neq \emptyset$. Then we can choose another point $x_0' \in U \cap \Theta_f(x_n)$, where x_0' is a f-statistical cluster point. Then there exist a neighbourhood V of a point x_0' such that $V \subset U$ and

$$\delta^f(\{n \in \mathbb{N} : x_n \in V\}) = \alpha > 0.$$

Obviously,

$$\{n \in \mathbb{N} : x_n \in U\} \supset \{n \in \mathbb{N} : x_n \in V\}$$

and hence

$$\delta^f(\{n \in \mathbb{N} : x_n \in U\}) \supset \delta^f(\{n \in \mathbb{N} : x_n \in V\}) = \alpha > 0.$$

It means that $\delta^f(\{n \in \mathbb{N} : x_n \in U\})$ is not a set that has zero δ^f-density, i.e., $x_0 \in \Theta_f(x_n)$. So $\overline{\Theta_f(x_n)} = \Theta_f(x_n)$. Hence $\Theta_f(x_n)$ is a closed set. $\square$

Theorem 6. *In a topological space (X,τ), if there exist two sequence $\{x_n : n \in \mathbb{N}\}, \{y_n : n \in \mathbb{N}\}$ such that $\delta^f(\{n \in \mathbb{N} : x_n \neq y_n\}) = 0$, then $\Theta_f(x_n) = \Theta_f(y_n)$ and $\Lambda_f(x_n) = \Lambda_f(y_n)$.*

P r o o f. Let $\{x_n : n \in \mathbb{N}\}$ and $\{y_n : n \in \mathbb{N}\}$ be two sequence of a topological space (X,τ). Suppose that q be any f-statistical cluster point with respect to $\{x_n : n \in \mathbb{N}\}$ sequence. So, for every neighbourhood U of q,

$$\delta^f(\{n \in \mathbb{N} : x_n \in U\}) > 0.$$

We have

$$\lim_{n\to\infty} \frac{f(|\{n \in \mathbb{N} : x_n \in U\}|)}{f(n)} > 0$$

and

$$\{n \in \mathbb{N} : x_n \in U\} \setminus \{n \in \mathbb{N} : x_n \neq y_n\} \subseteq \{n \in \mathbb{N} : y_n \in U\}.$$

Since $\delta^f\{n \in \mathbb{N} : x_n \neq y_n\} = 0$ then we get $\delta_f(\{n \in \mathbb{N} : y_n \in U\}) > 0$. This means that the set $\{n \in \mathbb{N} : y_n \in U\}$ is not a set that has zero δ_f-density so q is also f-statistical cluster point with respect to $\{y_n : n \in \mathbb{N}\}$ sequence. Therefore $\Theta_f(x_n) \subset \Theta_f(y_n)$. It is easy to see that $\Theta_f(y_n) \subset \Theta_f(x_n)$ from symmetry. Finally we have $\Theta_f(x_n) = \Theta_f(y_n)$. The equality $\Lambda_f(x_n) = \Lambda_f(y_n)$ can be shown in a similar way. $\square$

5. Conclusion

An unbounded modulus function can help to manage the rate of statistical convergence up to a great extend. In a first countable space, s^f_*-convergence does not entail s^f-convergence, although s^f-convergence requires s^f_*-convergence. An s^f-convergent sequence posses a unique limit in a Hausdörff space. A sequence is s^f-convergent if and only if each of its s^f-dense subsequence is s^f-convergent. The set $\Lambda_f(x_n)$ of all f-statistical limit points of a sequence $\{x_n\}$ is a subset of the set $\Theta_f(x_n)$ of all f-statistical cluster points of that sequence. Moreover the collection of all f-statistical cluster points forms a closed set in related topological space.

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