URAL MATHEMATICAL JOURNAL, Vol. 10, No. 1, 2024, pp. 84–98

DOI: 10.15826/umj.2024.1.008

ARTINIAN M-COMPLETE, M-REDUCED, AND MINIMALLY M-COMPLETE ASSOCIATIVE RINGS

Leonid M. Martynov

Omsk State University of Ways of Communication,
35 Marksa Ppr., Omsk, 644046, Russian Federation

Tatiana V. Pavlova

University of Tyumen,
6 Volodarskogo Str., Tyumen, 625003, Russian Federation

t.v.pavlova@utmn.ru

Abstract: In 1996, the first author defined analogs of the concepts of complete (divisible), reduced, and periodic abelian groups, well-known in the theory of abelian groups, for arbitrary varieties of algebras. In 2021, the first author proposed a modification of the concepts of completeness and reducibility, which is more natural in the case of associative rings. The paper studies the modification of these concepts for associative rings. Artinian **M**-complete, **M**-reduced rings, and minimally **M**-complete associative nilpotent rings, simple rings with unity, and finite rings are characterized.

Keywords: Associative ring, Artinian ring, Finite ring, Complete ring, Reduced ring.

1. Introduction

In the theory of abelian groups, the notions of complete (divisible), reduced, and periodic (in particular, primary) groups are of great importance. In [16] (see also [17, 19]), some analogs of these notions were defined for arbitrary varieties of algebras. In the mentioned papers, the concepts of (atomic) complete, (atomic) reduced, and solvable algebra [30] (see also [31]) were defined by means of the atoms of these varieties and the Malcev products for these atoms [15]. Furthermore, the notions of periodic and primary algebra were defined using notions of (atomic) completeness, (atomic) reducibility, and solvability. In particular, an algebra is called periodic if each of its monogenic (i.e., one generated) subalgebras is finitely reduced. Note that a group or semigroup is periodic as a universal algebra (in our sense) if and only if it is periodic as a group or semigroup in the ordinary sense.

It is different for associative rings. In the theory of associative rings, a ring is called periodic if its multiplicative semigroup is periodic (in the ordinary sense). Any finite nonprime field is a periodic ring in the ordinary sense. On the other hand, such a field is monogenic, but it is not a finitely reduced ring; i.e., it is not a periodic algebra in the sense of papers [16, 17, 19]. To remove this difference, paper [21] suggests modifying the concepts of complete, reducible, periodic, and primary associative rings. This is done by using a special set $\mathbf{M}$ of subvarieties of the variety As of associative rings, where $\mathbf{M}$ is the union of the set of lattice atoms of subvarieties of As and the set of all varieties, each of which is generated by some finite nonprime field. In this case, $\mathbf{M}$-periodic rings are rings with finite monogenic subrings (i.e., there is an analogy with groups and semigroups). Moreover, every finite field is both $\mathbf{M}$-periodic and $\mathbf{M}$-primary. Thus, the modification of the concepts discussed, given in [21], is more natural for associative rings.

In [21], properties of $\mathbf{M}$-periodic and $\mathbf{M}$-primary associative rings are studied. In addition, paper [21] characterizes the $\mathbf{M}$-periodic, $\mathbf{M}$-primary, and $\mathbf{M}$-reduced varieties of associative rings. From the results of paper [21] (in particular, Remark 5.17), it follows that the class $\mathcal{MC}$ of all $\mathbf{M}$-complete rings of As is closed with respect to homomorphic images, extensions, and direct sums in As. Furthermore, the class $\mathcal{MR}$ of all $\mathbf{M}$-reduced rings of As is closed with respect to subrings, direct products, and extensions in As. Besides, the variety As is transverbal (in the sense of [15]) with respect to any variety belonging to $\mathbf{M}$.

If we replace the set $At(L(As))$ by the set $\mathbf{M}$ in paper [18], the main result of [18] will change. From the modified result, we obtain that any ring R belonging to As contains the largest $\mathbf{M}$-complete subring $C_{\mathbf{M}}(R)$; $C_{\mathbf{M}}(R)$ is a two-sided ideal of the ring R; the factor ring $R/C_{\mathbf{M}}(R)$ is an $\mathbf{M}$-reduced ring. All of the above means that the basic properties of the modified concepts of completeness and reducibility for associative rings are saved. Further, let a mapping $r_{\mathbf{M}} : As \to As$ be such that $r_{\mathbf{M}}(R) = C_{\mathbf{M}}(R)$ for all $R \in As$. From the above, we obtain that $r_{\mathbf{M}}$ is a radical in the sense of Kurosh and Amitsur (see, for example, [1, p. 91] or [6, p. 27]). Here, $\mathcal{MC}$ is a radical class and $\mathcal{MR}$ is a semisimple class.

We say that the radical $r_{\mathbf{M}}$ is the $\mathbf{M}$-*complete radical* and the ideal $C_{\mathbf{M}}(R)$ of R is the $\mathbf{M}$-*complete radical of the ring* R. Note that $C_{\mathbf{M}}(R)$ contains any $\mathbf{M}$-complete subring of the ring R. Therefore, $r_{\mathbf{M}}$ is a *strict radical* in the sense of Kurosh [13] (see also [6, p. 148]).

In papers [10, 11, 20, 24, 25], the complete radical of an associative ring was studied. It is easy to verify that analogous main results of these papers also hold for the $\mathbf{M}$-complete radical. In papers [12, 22, 26–28], the structure of complete and reduced associative rings was studied. The main results of these papers are significantly modified if we replace the concepts of completeness and reducibility with the concepts of $\mathbf{M}$-completeness and $\mathbf{M}$-reducibility.

Recall that in the theory of abelian groups, the concept of a complete group coincides with that of a divisible group. Any minimal divisible abelian group is isomorphic to the (additive) quasi-cyclic group $\mathbb{C}_{p^\infty}$, where p is a prime, or to the additive group $\mathbb{Q}^+$ of the field $\mathbb{Q}$ of rational numbers. Minimal divisible abelian groups have significant importance since any divisible abelian group is a direct sum of minimal divisible abelian groups (see, for example, [4, Theorem 23.1, p. 124]). An associative ring R is called minimal $\mathbf{M}$-complete if it is a nonzero $\mathbf{M}$-complete ring, and all proper subrings of the ring R are $\mathbf{M}$-reduced rings.

This paper aims to characterize $\mathbf{M}$-complete, $\mathbf{M}$-reduced associative Artinian rings, and minimal $\mathbf{M}$-complete finite associative rings. Note that the class of Artinian rings contains all finite rings.

Nevertheless, in this paper, we do not limit ourselves to Artinian rings and provide some results, which are valid for all associative rings. We will use the results from papers [12, 22, 23, 26–29] on the study of complete and reduced associative rings and modify them for the concepts of $\mathbf{M}$-completeness and $\mathbf{M}$-reducibility. If the obtained results have fundamental changes, then proofs are provided. The modified formulations of the statements are given only with references to similar statements if the changes are insignificant. Before formulating and proving the basic results of the paper, we will provide and prove several lemmas. Some lemmas are of independent interest.

First, let us give some definitions, notations, and facts about associative rings.

2. Basic definitions, notations, and preliminary information

Further in the paper, by a *ring* we mean an associative ring (not necessarily with the unity), by an *ideal* we mean a two-sided ideal. Denote by $|M|$ the cardinal number of a set M. Positive integers are denoted by k, l, m, n (sometimes with subscripts), and primes are denoted by p and q. Denote by R^+ the additive group of the ring R. A ring with zero multiplication will be called an abelian ring. An abelian ring with an additive group R^+ is denoted by R^0. Denote by O the zero ideal of the ring R. A *simple ring* is a nonzero ring having no ideal besides O and itself. The smallest $n \in \mathbb{N}$ is said to be the *characteristic* of a ring R if $nR = O$ and is denoted by *char* R. If there is no such n, then *char* $R = 0$ is assumed.

The set of natural numbers is denoted by $\mathbb{N}$, and the set of primes is denoted by $\mathbb{P}$. Denote by $\mathbb{Z}$ the ring of integers, and by $\mathbb{Q}$ the field of rational numbers. Furthermore, let $\mathbb{Z}_n$ denote the ring of residue classes modulo $n > 1$. The finite field (Galois field) of p^m elements is denoted by $\mathbb{F}_{p^m}$. A *prime field* is a field, which has no proper subfields. Any prime field is isomorphic to the field $\mathbb{Q}$ of rational numbers or the finite field $\mathbb{F}_p$ of p elements.

If M is a nonempty subset of a ring R, then $\langle M\rangle$ and (M) denote the subring and the ideal of R generated by M, respectively. The subring generated by $a \in R$ is called *monogenic* and denoted by $\langle a\rangle$. An element $e \in R$ with the property $e^2 = e$ is called an *idempotent* of R. The idempotent $e \in R$ is called *basic* if $\sigma(e)$ is the unity of the factor ring $R/J(R)$, where $J(R)$ is the Jacobson radical of the ring R and σ is the natural homomorphism of R to $R/J(R)$. A ring R is called *idempotent* if $R^2 = R$, where $R^2 = \langle a \cdot b \mid a, b \in R\rangle$.

Denote by $M_n(R)$ the ring of square $n \times n$ matrices over a ring R. For a commutative ring R with unity, $R[x]$ denotes the ring of polynomials in x over R. Denote by $\mathbb{Z}\langle X\rangle$ a free (in *As*) ring over the infinite countable set $X = \{x_1, x_2, \dots\}$, i.e., the ring of polynomials with integer coefficients in noncommuting variables of X with zero free terms. An *identity* is a formal equality of the form $f(x_1, x_2, \dots, x_n) = 0$, where $f(x_1, x_2, \dots, x_n) \in \mathbb{Z}\langle X\rangle$.

Suppose that φ is the natural homomorphism of $\mathbb{Z}_{p^k}[x]$ onto $\mathbb{Z}_p[x]$ and $f(x) \in \mathbb{Z}_{p^k}[x]$ is a unitary polynomial of degree m such that $\varphi\big(f(x)\big)$ is an irreducible polynomial over $\mathbb{Z}_p$; then the factor ring $\mathbb{Z}_{p^k}[x]/\big(f(x)\big)$ is called a *Galois ring* of characteristic p^k and order p^{km}. The Galois ring, up to isomorphism, is defined by the numbers p, k, and n and is denoted by $GR(p^k, m)$. It is obvious that $GR(p^k, 1) \cong \mathbb{Z}_{p^k}$ and $GR(p, m) \cong \mathbb{F}_{p^m}$. Galois rings play a special role in the structural theory of finite associative rings.

Let $\operatorname{var}\Sigma$ denote the variety of rings defined by a system Σ of ring identities, and let $\operatorname{var}\mathcal{K}$ be the least variety of rings containing a class $\mathcal{K}$ of rings (in other words, $\operatorname{var}\mathcal{K}$ is the variety generated by $\mathcal{K}$). The free monogenic ring in *As* will be denoted by $\mathbb{Z}\langle x\rangle$. Below we use the following important notations:

$\mathcal{Z}_n^0 = \operatorname{var}\{nx = 0,\ xy = 0\} = \operatorname{var}\mathbb{Z}_n^0$;

$\mathcal{F}_{p^m} = \operatorname{var}\{px = 0,\ x^{p^m} = x\} = \operatorname{var}\mathbb{F}_{p^m}$.

Let $\mathcal{V}$ be a variety of rings. A ring R is called $\mathcal{V}$-*complete* if R has no homomorphisms onto nonzero rings from $\mathcal{V}$. Equivalently, $\mathcal{V}(R) = R$, where $\mathcal{V}(R)$ is the *verbal ideal* of R (i.e., $\mathcal{V}(R)$ is the least ideal in the set of all ideals I of the ring R such that the factor ring R/I belongs to $\mathcal{V}$). A ring is called $\mathcal{V}$-*solvable* if it has no nonzero $\mathcal{V}$-complete subrings.

Let $\mathbf{M}$ be the union of two sets $\mathbf{Z}$ and $\mathbf{F}$ of varieties of rings, where

$$\mathbf{Z} = \{\mathcal{Z}_p^0 \mid p \in \mathbb{P}\}, \quad \mathbf{F} = \{\mathcal{F}_{p^m} \mid p \in \mathbb{P},\ m \in \mathbb{Z}_+\},$$

i.e., $\mathbf{M} = \mathbf{Z} \cup \mathbf{F}$. Note that $\mathbf{M}$ contains the set $At(L(As))$, where $At(L(As))$ consists of the varieties $\mathcal{Z}_p^0$ and $\mathcal{F}_p$ for any prime p (see, for example, [9]). A ring R is called $\mathbf{M}$-*complete* if R is $\mathcal{M}$-complete for every $\mathcal{M} \in \mathbf{M}$. We call a ring R $\mathbf{M}$-*reduced* if R has no nontrivial $\mathbf{M}$-complete subrings. By analogy, the concepts of $\mathbf{Z}$-complete ($\mathbf{F}$-complete) ring and of $\mathbf{Z}$-reduced ($\mathbf{F}$-reduced) ring are defined. We point out the connection between the concepts of completeness and $\mathbf{M}$-completeness, as well as the concepts of reducibility and $\mathbf{M}$-reducibility. Obviously, any $\mathbf{M}$-complete ring is complete. But the converse statement, generally speaking, is incorrect. For example, any nonminimal finite field F is complete, while F is $\mathbf{M}$-reduced. On the other hand, any reduced ring is $\mathbf{M}$-reduced.

Recall that if a variety $\mathcal{V}$ is given by an identity system Σ, then the $\mathcal{V}$-verbal $\mathcal{V}(R)$ of a ring R coincides with the ideal of R generated by the values in R of all polynomials that are the left-hand sides of the identities of Σ. For varieties $\mathcal{Z}_p^0$ and $\mathcal{F}_{p^m}$ and a ring R, we indicate formulas to calculate the corresponding verbals:

$$\mathcal{Z}_p^0(R) = pR + R^2, \quad \mathcal{F}_{p^m}(R) = pR + R_{p^m},$$

where R_{p^m} is the ideal generated by the set

$$\{r^{p^m} - r \mid r \in R\}.$$

It is clear that a ring R is **M**-complete (**M**-reduced) if and only if $\mathcal{Z}_p^0(R) = R$ ($\mathcal{Z}_p^0(R) = O$) and $\mathcal{F}_{p^m}(R) = R$ ($\mathcal{F}_{p^m}(R) = O$) for all p and m, respectively.

It is clear that the **M**-complete radical $C_{\mathbf{M}}(R)$ of a ring R is equal to the sum of all **M**-complete subrings of R. A ring R is **M**-complete if and only if $C_{\mathbf{M}}(R) = R$. In particular, the **M**-complete radical $C_{\mathbf{M}}(R)$ of any ring R is an **M**-complete ideal of R. A ring R is **M**-reduced if and only if $C_{\mathbf{M}}(R) = O$. From a well-known fact for arbitrary radicals (see, for example, [1], Proposition 1, p. 91) it follows that the **M**-complete radical of a ring R is the intersection of all its ideals I such that the factor ring R/I is **M**-reduced.

Similarly, the **Z**-complete radical $C_{\mathbf{Z}}(R)$ and the **F**-complete radical $C_{\mathbf{F}}(R)$ of a ring R are defined by the sets **Z** and **F**, respectively. Recall that a radical r is called strict if the radical $r(R)$ of a ring R contains every r-radical subring A (i.e., a subring with the property $r(A) = A$) of R. As above, the **M**-complete radical is strict. It is clear that **Z**-complete and **F**-complete radicals are also strict. A ring is **M**-complete if and only if it is simultaneously **Z**-complete and **F**-complete. Denote by $\mathfrak{M}$ ($\mathfrak{Z}$, $\mathfrak{F}$) the class of all rings belonging to the varieties of rings from the set **M** (**Z**, **F**), respectively. It is clear that **M**-complete (**Z**-complete, **F**-complete) radicals are upper radicals defined by the class $\mathfrak{M}$ ($\mathfrak{Z}$, $\mathfrak{F}$), respectively. In addition, for any prime p, we need the notation $\mathfrak{F}_p$ for the class of rings of characteristic p from the class $\mathfrak{F}$.

Recall that the transverbality of the variety As over the subvariety $\mathcal{V}$ means that, for any ring R and an ideal I of R, $\mathcal{V}(I)$ is an ideal of R. As already noted, the transverbality of the variety As over any variety from the set **M** is proved in paper [21].

In additive notation, the atoms of the lattice $L(Ab)$ of subvarieties of the variety Ab of all abelian groups Ab are the varieties $\mathcal{A}_p = \operatorname{var}\{px = 0\}$ for all primes p. Note that the $\mathcal{A}_p$-completeness of an abelian group A means the validity of the equality $\mathcal{A}_p(A) = pA = A$. Further, the divisibility of an abelian group A is equivalent to its $\mathcal{A}_p$-completeness over all primes p, i.e., the completeness of A. It is well known that in every abelian group A, a divisible subgroup is always a direct summand, in A there is the largest divisible subgroup $C(A)$, and A is the direct sum of its complete and reduced subgroups. Moreover, as noted above, every divisible abelian group is the direct sum of some sets of isomorphic copies of the additive group $\mathbb{Q}^+$ of rationales and copies of quasi-cyclic groups C_{p^∞} for some primes p.

Recall that a ring is called a left Artinian ring if any decreasing chain of its left ideals stabilizes. Equivalently, the ring satisfies the minimum condition of left ideals. Further, left Artinian rings will be called Artinian rings. It is well known (see, for example, [8, Theorem 1, p. 63]) that the Jacobson radical of an Artinian ring is nilpotent. In addition, by the Wedderburn–Artin Theorem (see, for example, [8, p. 65]), any Artinian semisimple (in the sense of Jacobson radical) ring is isomorphic to the direct sum of finitely many full matrix rings over skew fields. It is well known that a factor ring of an Artinian ring is Artinian. Also, if the ideal I and the factor ring R/I of a ring R are both Artinian, then R itself is Artinian.

In conclusion of this section, we give some well-known statements that do not relate to the concepts of **M**-completeness and **M**-reducibility but are needed for the sequel.

Theorem 1. [5, Theorem 122.7, p. 350] *Every Artinian ring R is the ring direct sum $R = S \oplus T_{p_1} \oplus \cdots \oplus T_{p_k}$ of some torsion-free Artinian ring S and a finite number of Artinian p_i-rings T_{p_i} corresponding to various primes p_i.*

Theorem 2. [32, Proposition 6] *Let R be a finite ring with unity of characteristic p^k and radical $J(R)$. Then R contains a subring Q isomorphic to a direct sum of matrix rings over Galois rings such that $Q/pQ \cong R/J(R)$ and a (Q,Q)-submodule M of $J(R)$ such that $R = Q + M$ with $Q \cap M = O$.*

Theorem 3. [7, Theorem 1.4.3, p. 35] *If an additive group R^+ of a left Artinian ring R is a torsion-free group, then R possesses a left unity.*

Proposition 1. [28, Lemma 12] *An Artinian ring R is finite if and only if $mR = O$ for some $m \in \mathbb{N}$ and the factor ring $R/J(R)$ is finite.*

Lemma 1. [10, Lemma 3] *If I is an ideal of a ring R and K is a field, then any homomorphism $\varphi : I \to K$ can be extended to a homomorphism $\bar{\varphi} : R \to K$.*

Lemma 2. [24, Lemma] *For any ideal I of a ring R, the relation $M_n(R)/M_n(I) \cong M_n(R/I)$ holds.*

3. Artinian M-complete rings

This section aims to obtain a characterization of **M**-complete Artinian rings. Let us first give several lemmas. Some of them are valid for arbitrary rings.

Lemma 3. *If R is an* **M***-complete ring, then R^2 is an* **M***-complete ring.*

P r o o f. Let R be an **M**-complete ring. Since R is a $\mathcal{Z}_p^0$-complete ring for any prime p, the relation $\mathcal{Z}_p^0(R) = pR + R^2 = R$ holds for all primes p; i.e., $R = R^2 + pR$. We have

$$R^2 = (R^2 + pR)(R^2 + pR) = R^4 + pR^3 + p^2R^2 = R^4 + pR(R^2 + pR) = R^4 + pR^2;$$

i.e., $\mathcal{Z}_p(R^2) = R^2$. This equality means that the ring R^2 is $\mathcal{Z}_p^0$-complete.

We now show that R^2 is $\mathcal{F}_{p^m}$-complete for all p and m. Since R is a $\mathcal{Z}_p^0$-complete ring for any prime p, for each x of R, we find elements a, a_i, and b_i ($i = 1, \dots, n$) of R such that

$$x = pa + \sum_{i=1}^{n} a_i b_i.$$

Then,

$$x^{p^m} - x = \Big(pa + \sum_{i=1}^{n} a_i b_i\Big)^{p^m} - \Big(pa + \sum_{i=1}^{n} a_i b_i\Big) = pz + \bigg(\Big(\sum_{i=1}^{n} a_i b_i\Big)^{p^m} - \Big(\sum_{i=1}^{n} a_i b_i\Big)\bigg)$$

for some $z \in R$. Therefore, $R_{p^m} \subseteq pR + (R^2)_{p^m}$. It follows that

$$(R_{p^m})^2 \subseteq (pR + (R^2)_{p^m})(pR + (R^2)_{p^m}) \subseteq pR^2 + (R^2)_{p^m}.$$

Considering this inclusion and the fact that the ring R is $\mathcal{F}_{p^m}$-complete, we get

$$R^2 = (pR + R_{p^m})(pR + R_{p^m}) \subseteq pR^2 + (R_{p^m})^2 \subseteq pR^2 + (R^2)_{p^m} = \mathcal{F}_{p^m}(R^2);$$

i.e., R^2 is $\mathcal{F}_{p^m}$-complete for all p and m. □

Corollary 1. *If R is a minimally* **M***-complete ring and $R^2 \neq O$, then $R^2 = R$.*

Lemma 4. *A simple ring R is either* **M***-complete and does not belong to the set $\mathfrak{M}$ or* **M***-reduced and isomorphic to a field $\mathbb{F}_{p^m}$ for some p and m.*

P r o o f. A simple ring R is a ring with nonzero multiplication, so $R \in \mathbf{F}$ for some prime p. As is well known, any nonzero ring of any variety from $\mathbf{F}$ is a subdirect product of finite fields (see, for example, [30]). □

Lemma 5. *Any nil ring R is an $\mathcal{F}$-complete ring.*

P r o o f. The homomorphic image of a nil ring is a nil ring and therefore cannot be a nonzero ring of a variety $\mathcal{F}_{p^m}$ for any p and m. □

Lemma 6. [26, Lemma 3] *If the ideal I of a ring R is contained in the kernel of any homomorphism of R onto rings from a variety $\mathcal{V}$ of rings, then R is a $\mathcal{V}$-complete ring if and only if R/I is a $\mathcal{V}$-complete ring.*

Repeating almost verbatim the proof of Lemma 5 from [11], one can verify the validity of the following statement.

Lemma 7. *A nilpotent ring R is* **M***-complete if and only if its additive group R^+ is divisible.*

We omit the proof of the following statement analogous to Lemma 2 from [26], which corresponds almost verbatim to the proof of that lemma and uses the results mentioned in Section 1 on the Jacobson radical of Artinian rings, semisimple Artinian rings, the structure of divisible abelian groups, and Lemma 7.

Lemma 8. *The following conditions are equivalent for a ring R:*

(1) $R^+ \cong \bigoplus_{i=1}^{n} C_{p_i^\infty}$;
(2) *R is an* **M***-complete abelian Artinian ring;*
(3) *R is an* **M***-complete Artinian nilpotent ring.*

The main result of this section is a modification of two statements of Theorems 1 and 2 of [26].

Theorem 4. *An Artinian ring R is* **M***-complete if and only if the following conditions hold for its ideal R^2:*

(1) *R^2 is an idempotent Artinian ring and if $R^2 \neq O$, then*

$$R^2/J(R^2) \cong \bigoplus_{i=1}^{k} M_{n_i}(K_i),$$

where K_i is a skew field and $M_{n_i}(K_i) \ncong \mathbb{F}_{p^m}$ for any prime p, $m \in \mathbb{N}$, and $i = 1, \ldots, k$;
(2) *if $R^2 \neq R$, then*

$$R/R^2 \cong \bigoplus_{j=1}^{n} C^0_{p_j^\infty}.$$

P r o o f. Let R be an $\mathbf{M}$-complete Artinian ring. It follows from Theorem 1 that R is the direct sum of its ideals S and T, where S is an $\mathbf{M}$-complete torsion-free Artinian ring and T is an $\mathbf{M}$-complete Artinian periodic ring. Therefore, it is enough to consider both rings separately.

By Theorem 3, the ring S possesses a left unity. Therefore, $S^2 = S$. Since S is an $\mathbf{M}$-complete ring, it follows that the factor ring $S/J(S)$ is an $\mathbf{M}$-complete ring. From the Wedderburn–Artin theorem and Lemma 4, it follows that the factor ring $S/J(S)$ is isomorphic to a direct sum of finitely many full matrix rings over skew fields and does not contain summands isomorphic to a finite field $\mathbb{F}_{p^m}$ for any prime p and $m \in \mathbb{N}$.

Further, consider a decreasing chain of ideals in T: $T \supseteq T^2 \supseteq T^3 \supseteq \dots$ Since T is an Artinian ring, we have $T^n = T^{n+1}$ for some n; i.e., the ideal T^n is an idempotent ring. Then, $\overline{T} = T/T^n$ is an $\mathbf{M}$-complete Artinian nilpotent ring and, by Lemma 8, $\overline{T}$ is an abelian ring. Hence, $xy \in T^n$ for all $x, y \in T$ and, therefore, $T^2 = T^n$; i.e., T^2 is an idempotent ring. In addition, by Lemma 8,

$$T/T^2 \cong \bigoplus_{j=1}^{n} C^0_{p_j^\infty}.$$

Let us show that the ideal T^2 is an Artinian ring. Let I be a left ideal of T^2. The group T^+ is periodic, therfore, for any $i \in I$, there exists $m \in \mathbb{N}$ such that $mi = 0$. By Lemma 8,

$$T/T^2 \cong \bigoplus_{i=1}^{k} C^0_{p_i^\infty},$$

it follows that there exists $t_1 \in T$ for any $t \in T$ such that $m\bar{t_1} = \bar{t}$ in the factor ring $\overline{T} = T/T^2$. Since $t - mt_1 \in T^2$, we have

$$ti = (t - mt_1 + mt_1)i = (t - mt_1)i + mt_1 i = (t - mt_1)i \in I.$$

This means that I is a left ideal of T.

Thus, T^2 is an Artinian and $\mathbf{M}$-complete ring by Lemma 3. Therefore, $T^2/J(T^2)$ also is $\mathbf{M}$-complete ring. It follows that

$$T^2/J(T^2) \cong \bigoplus_{i=1}^{k} M_{n_i}(K_i),$$

where K_i is a skew field and $M_{n_i}(K_i) \ncong \mathbb{F}_{p^m}$ for any prime p and $m \in \mathbb{N}$.

Conversely, let for the ideal R^2 of an Artinian ring R, conditions (1) and (2) of the theorem be satisfied. Four cases are possible:

(i) $R = R^2 = O$. Then, R is an $\mathbf{M}$-complete ring by definition.

(ii) $R \neq R^2 = O$ and $R/R^2 \cong \bigoplus\limits_{j=1}^{n} C^0_{p_j^\infty}$. Then, $R = R/R^2$ is an $\mathbf{M}$-complete ring by Lemma 8.

(iii) $R = R^2 \neq O$ and $R/J(R) = R^2/J(R^2) \cong \bigoplus\limits_{i=1}^{k} M_{n_i}(K_i)$, where K_i is a skew field and $M_{n_i}(K_i) \ncong \mathbb{F}_{p^m}$ for any prime p and $m \in \mathbb{N}$. In this case, R is a nonzero idempotent Artinian ring and $R/J(R)$ is an $\mathbf{M}$-complete ring. Then, $J(R)$ is an $\mathbf{M}$-complete ring by Lemma 5. This means that $J(R)$ is contained in the kernel of any homomorphism onto rings from a variety $\mathcal{F}_{p^m}$ for any p and m. Then, by Lemma 6, $R/J(R)$ is an $\mathcal{F}$-complete ring if and only if R is an $\mathcal{F}$-complete ring. The ring $R = R^2$ also is $\mathcal{Z}$-complete. Hence, R is an $\mathbf{M}$-complete ring.

(iv) $R \neq R^2$ and $R^2 \neq O$, where $R/R^2 \cong \bigoplus\limits_{j=1}^{n} C^0_{p_j^\infty}$ and the ideal R^2 is an idempotent Artinian ring. Then, $R^2/J(R^2) \cong \bigoplus\limits_{i=1}^{k} M_{n_i}(K_i)$, where K_i is a skew field and $M_{n_i}(K_i) \ncong \mathbb{F}_{p^m}$ for any prime p and $m \in \mathbb{N}$. The **M**-completeness of R^2 is proved similar to case (iii). The factor ring R/R^2 is **M**-complete by Lemma 8. In this case, it follows from Lemma 2.2 of [21] that the extension of the **M**-complete ring R^2 by the **M**-complete ring R/R^2 is an **M**-complete ring. Besides, it is known that an extension of an Artinian ring by an Artinian ring is also Artinian.

□

A special case of Theorem 4 is a modification of the result on complete finite rings from [12].

Corollary 2. *A finite nonzero ring R is* **M***-complete if and only if the following conditions hold*:

(1) $R^2 = R$;
(2) $R/J(R)$ *is an* **M***-complete ring and* $R/J(R) \cong \oplus_{i=1}^{n} M_{n_i}(\mathbb{F}_{p_i{}^{m_i}})$, *where* $n_i > 1$ *for all* $i = 1, 2, \ldots, n$, $m_i \in \mathbb{N}$, *and* p_i *are primes.*

4. Artinian M-reduced rings

This section aims to characterize **M**-reduced Artinian rings.

The following statements are modifications for **M**-reduced rings of lemmas for reduced rings from [27]. Their proofs are easy to obtain if we replace the field $\mathbb{F}_p$ by the field $\mathbb{F}_{p^m}$ for any p and m.

Lemma 9. [27, Lemma 1] *For an Artinian nilpotent ring R, the following conditions are equivalent*:

(1) $mR = O$ *for some* $m \in \mathbb{N}$;
(2) R *is a finite ring*;
(3) R *is an* **M***-reduced ring.*

From Lemma 9, it follows that all nilpotent **M**-reduced Artinian rings are finite.

Lemma 10. [27, Lemma 1] *Any Artinian* **M***-reduced ring has characteristic* $m > 0$.

From Lemma 10 and Theorem 1, it follows that it is sufficient to characterize Artinian **M**-reduced rings of characteristic p^k.

Lemma 11. [27, Lemma 2] *Any ring R of characteristic p^k, where $k \in \mathbb{N}$ and p is a prime, is $\mathcal{Z}_q^0$-complete and $\mathcal{F}_{q^m}$-complete for any prime $q \neq p$ and $m \in \mathbb{N}$.*

Lemma 12. [27, Lemma 3] *A ring R of characteristic p^k for a prime p and $k \in \mathbb{N}$, is an* **M***-complete ring if and only if the ring R/pR is* **M***-complete.*

The following lemma describes the **M**-complete radical of an Artinian ring of characteristic p^k for a prime p and $k \in \mathbb{N}$.

Lemma 13. *For an Artinian ring R of characteristic p^k, where p is a prime and $k \in \mathbb{N}$, there exists $m, n \in \mathbb{N}$ such that the $\mathcal{F}$-complete radical $C_{\mathbf{F}}(R) = \mathcal{F}_{p^m}(R)$ and the* **M***-complete radical $C_{\mathbf{M}}(R) = \mathcal{F}_{p^m}^n(R)$, where n is the idempotent degree of the verbal $\mathcal{F}_{p^m}(R)$; i.e., $\mathcal{F}_{p^m}^n(R) = \mathcal{F}_{p^m}^{n+1}(R)$.*

P r o o f. Consider the set of all verbals $\mathcal{F}_{p^d}(R)$ of the ring R, where $d \in \mathbb{N}$. Note that, for any $t \in \mathbb{N}$ and $x \in R$,

$$x^{p^{td}} - x = \left(x^{p^d} - x\right) \cdot \sum_{i=0}^{i=s} x^{p^{td} - p^d - i(p^d - 1)}, \quad \text{where} \quad s = \sum_{j=0}^{j=t} p^{(t-j)d}.$$

This means that if h is divisible by d, then $\mathcal{F}_{p^h}(R) \subseteq \mathcal{F}_{p^d}(R)$.

Since R is an Artinian ring, R contains the minimal verbal $\mathcal{F}_{p^m}(R)$. At the same time, $\mathcal{F}_{p^m}(R) \subseteq \mathcal{F}_{p^d}(R)$ for all $d \in \mathbb{N}$ (assuming that this is not the case, we get that the verbal $\mathcal{F}_{p^m}(R)$ is not minimal since it contains the verbal $\mathcal{F}_{p^l}(R) \subseteq \mathcal{F}_{p^m}(R) \cap \mathcal{F}_{p^d}(R) \neq \mathcal{F}_{p^m}(R)$, where l is the least common multiple of d and m).

By Lemma 11, the ring R of characteristic p^k is $\mathcal{F}_{q^t}$-complete for any prime $q \neq p$ and $t \in \mathbb{N}$. It follows from Lemma 1 that any ideal of an $\mathcal{F}_{q^t}$-complete ring, in particular, the ideal $\mathcal{F}_{p^m}(R)$, is also an $\mathcal{F}_{q^t}$-complete ring. From the same lemma and the fact that $\mathcal{F}_{p^m}(R) \subseteq \mathcal{F}_{p^k}(R)$ for all $k \in \mathbb{N}$, it follows that $\mathcal{F}_{p^m}(R)$ is an $\mathcal{F}_{p^k}$-complete ring for all $k \in \mathbb{N}$. Therefore, $\mathcal{F}_{p^m}(R)$ is an $\mathcal{F}$-complete ring; i.e., $\mathcal{F}_{p^m}(R) \subseteq C_{\mathbf{F}}(R)$; hence, $\mathcal{F}_{p^m}(R) = C_{\mathbf{F}}(R)$.

The decreasing chain of ideals $\mathcal{F}_{p^m}(R) \supseteq \mathcal{F}^2_{p^m}(R) \supseteq \mathcal{F}^3_{p^m}(R) \supseteq \ldots$ of the ring R stabilizes at some step n. That is, $\mathcal{F}^n_{p^m}(R)$ is an idempotent ring; so it is $\mathcal{Z}$-complete. In addition, $\mathcal{F}^n_{p^m}(R)$ is an $\mathcal{F}$-complete ring by Lemma 1. So, $\mathcal{F}^n_{p^m}(R) \subseteq C_{\mathbf{M}}(R)$. Conversely, since $C_{\mathbf{M}}(R) \subseteq \mathcal{F}_{p^m}(R)$, we have $C^n_{\mathbf{M}}(R) \subseteq \mathcal{F}^n_{p^m}(R)$. Hence, $C_{\mathbf{M}}(R) = C^n_{\mathbf{M}}(R) \subseteq \mathcal{F}^n_{p^m}(R)$ and therefore $C_{\mathbf{M}}(R) = \mathcal{F}^n_{p^m}(R)$. $\square$

Lemma 14. *A nonnilpotent Artinian ring R of characteristic p^k, where p is a prime and $k \in \mathbb{N}$, is an $\mathbf{M}$-reduced ring if and only if R is a finite ring and $C_{\mathbf{F}}(R) = J(R)$. In addition, the factor ring $R/J(R)$ is isomorphic to a finite direct sum of fields $\mathbb{F}_{p^{k_i}}$ for $k_i \in \mathbb{N}$.*

P r o o f. First, we show that $C_{\mathbf{F}}(R) = J(R)$. It follows from Lemma 5 that $J(R)$ is an $\mathcal{F}$-complete ring, so $J(R) \subseteq C_{\mathbf{F}}(R)$. Conversely, by Lemma 13, $C_{\mathbf{F}}(R) = \mathcal{F}_{p^m}(R)$ for some $m \in \mathbb{N}$. Since R is an $\mathbf{M}$-reduced ring, we have $C^n_{\mathbf{F}}(R) = \mathcal{F}^n_p(R) = O$ for some $n \in \mathbb{N}$. Thus, $\mathcal{F}_{p^m}(R)$ is a nilpotent ideal; hence, $C_{\mathbf{F}}(R) = \mathcal{F}_{p^m}(R) \subseteq J(R)$.

The factor ring $R/J(R)$ is isomorphic to a direct sum of finitely many full matrix rings over skew fields. Since $\mathcal{F}_{p^m}(R) = J(R)$, the factor ring $R/J(R)$ belongs to the variety $\mathcal{F}_{p^m}$. Therefore, each of these summands belongs to the variety $\mathcal{F}_{p^m}$ and is isomorphic to $\mathbb{F}_{p^{k_i}}$ for some $k_i \in \mathbb{N}$ by Lemma 4. Also, since $R/J(R)$ is a finite ring, R is also a finite ring by Proposition 1.

Conversely, if R is a finite ring, then $J(R)$ is an $\mathbf{M}$-reduced ring by Lemma 9. Hence, the ring R is $\mathbf{M}$-reduced as an extension of the $\mathbf{M}$-reduced ring $J(R)$ by the $\mathbf{M}$-reduced ring $R/J(R)$. $\square$

The following statement, similar to Teorem 2 of [27], describes the structure of Artinian $\mathbf{M}$-reduced rings.

Theorem 5. *An Artinian ring R is an $\mathbf{M}$-reduced if and only if R is a finite ring with $\mathcal{F}$-complete radical $C_{\mathbf{F}}(R) = J(R)$ and either $R = J(R)$ or $R/J(R) \cong \bigoplus_{i=1}^{n} \mathbb{F}_{p_i}^{k_i}$, where p_i is a prime and $k_i \in \mathbb{N}$.*

P r o o f. By Lemma 10, for any Artinian $\mathbf{M}$-reduced ring R, there exists $m \in \mathbb{N}$ such that $mR = O$. Let $m = p_1^{k_1} \cdot p_2^{k_2} \cdot \ldots \cdot p_n^{k_n}$ be the canonical representation of the number m. Then, by Theorem 1, the ring R is a finite direct sum of its ideals R_i, where $p_i^{k_i} R_i = O$ for all $1 \leq i \leq n$.

It follows from the properties of a finite direct sum of rings that the rings R_i for all $1 \le i \le n$ are **M**-reduced Artinian rings. If the ring R_i is nonnilpotent, then it satisfies the conditions of Lemma 14, otherwise R_i satisfies the conditions of Lemma 9. If R_i is nilpotent, then $C_{\mathbf{F}}(R_i) = R_i$ by Lemma 5. In each case, R_i is a finite ring and $C_{\mathbf{F}}(R_i) = J(R_i)$. Thus, R is a finite ring and its $\mathcal{F}$-complete radical

$$C_{\mathbf{F}}(R) = \bigoplus_{i=1}^{n} C_{\mathbf{F}}(R_i) = \bigoplus_{i=1}^{n} \mathcal{F}_{p_i^{m_i}}(R_i) = \bigoplus_{i=1}^{n} J(R_i) = J(R).$$

Moreover, if $R \neq J(R)$, then the factor ring $R/J(R)$ is a finite direct sum of ideals isomorphic to finite fields.

Conversely, any finite ring R satisfying the conditions of the theorem is an extension of the **M**-reduced ring $J(R)$ by the **M**-reduced ring $R/J(R)$. This means that R is an **M**-reduced ring. $\square$

5. Minimally M-complete Artinian rings

This section aims to characterize minimally **M**-complete Artinian rings. Before proving the main result, we formulate analogs of auxiliary statements from [22] and [28] and prove some of them.

The proofs of the following several statements almost verbatim correspond to the proofs of their analogs, so, we omit them.

Proposition 2. [25, Proposition 1] *For a basic idempotent e of a nonnilpotent Artinian ring R, $C_{\mathbf{M}}(eRe) = eC_{\mathbf{M}}(R)e$.*

For an **M**-complete radical, the requirement that the idempotent e is the basic idempotent of a nonnilpotent Artinian ring A is essential. For example, in the **M**-complete ring $R = M_2(\mathbb{F}_p)$, for the idempotent

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

the subring $eRe \cong \mathbb{F}_p$ is **M**-reduced; i.e., $C_{\mathbf{M}}(eRe) = O$. However, $eC_{\mathbf{M}}(R)e = eRe \neq O$.

Corollary 3. [25, Corollary 2] *A nonnilpotent minimally* **M***-complete Artinian ring contains a unit.*

Lemma 15. [22, Lemma 11] *If any decreasing chain of ideals of a ring R contained in the ideal I of this ring stabilizes at some finite step, then the* **M***-reducibility of the ring R implies the* **M***-reducibility of the ring R/I.*

Corollary 4. *Any homomorphic image of an Artinian* **M***-reduced ring is an* **M***-reduced ring.*

Corollary 5. *The homomorphic image of a minimally* **M***-complete finite ring is a minimally* **M***-complete ring.*

Lemma 16. [28, Lemma 11] *A finite idempotent ring R of characteristic p^k is minimally* **M***-complete if and only if R/pR is a minimally* **M***-complete ring.*

Lemma 17. [22, Lemma 15] *If I is a nilpotent ideal of a ring R of characteristic p^k and K is an* $\mathbf{M}$*-reduced subring of the ring R, then the homomorphic image $\overline{K}$ in the ring $\overline{R} = R/I$ is also an* $\mathbf{M}$*-reduced ring.*

Corollary 6. *If R is a minimally* $\mathbf{M}$*-complete Artinian ring of characteristic p^k, then the factor ring $R/J(R)$ is also a minimally* $\mathbf{M}$*-complete ring.*

Lemma 18. [22, Lemma 1] *A minimally* $\mathbf{M}$*-complete nilpotent ring is isomorphic to the ring $\mathbb{Q}^0$ or the ring $C^0_{p^\infty}$ for some prime p.*

Lemma 19. [22, Lemma 3] *A skew field K of characteristic zero is minimally* $\mathbf{M}$*-complete if and only if it is isomorphic to the field $\mathbb{Q}$ of rational numbers.*

Corollary 7. *The ring $\mathbb{Z}$ of integers and any of its subrings are* $\mathbf{M}$*-reduced.*

Proposition 3. [24, Proposition] *For any ring R and $n > 1$, the ring $M_n(R)$ is* $\mathbf{M}$*-complete if and only if the ring R is* $\mathbf{Z}$*-complete.*

Corollary 8. *For any idempotent ring R and $n > 1$, the ring $M_n(R)$ is* $\mathbf{M}$*-complete.*

The description of minimally $\mathbf{M}$-complete skew field of prime characteristic p differs significantly from the description of complete skew field of prime characteristic p obtained in Lemma 4 of [22].

Lemma 20. *A skew field K of prime characteristic p is minimally* $\mathbf{M}$*-complete if and only if K is isomorphic to the algebraic closure $\widehat{\mathbb{F}}_p$ of the field $\mathbb{F}_p$.*

P r o o f. Let K be the minimally $\mathbf{M}$-complete skew field of prime characteristic p. Then K contains the field $\mathbb{F}_p$ that obviously lies in the center of K.

Just as in the proof of Lemma 4 of [22], it can be shown that the existence of an element in K that is transcendent with respect to the field $\mathbb{F}_p$ is impossible.

Therefore, all elements of the skew field K are algebraic with respect to the field $\mathbb{F}_p$. It is clear that elements of K are algebraic with respect to a field $\mathbb{F}_{p^m}$ for any $m > 1$. Recall that the field $\mathbb{F}_{p^m}$ is $\mathbf{M}$-reduced by Lemma 4. Taking into account the well-known facts that, for any finite field $\mathbb{F}_q$ in the ring $\mathbb{F}_q[x]$, there exists an irreducible polynomial of any positive degree (see, for example, [14, Corollary 2.11, p. 70]) and an algebraic extension of $\mathbb{F}_q$ containing any of its roots is again a finite field, it is easy to understand that K must coincide with the union of a countable infinite strictly increasing sequences of corresponding finite fields, i.e., K is isomorphic to the algebraic closure $\widehat{\mathbb{F}}_p$ of the field $\mathbb{F}_p$.

Conversely, if a skew field K of prime characteristic p is the algebraic closure of the field $\mathbb{F}_p$, i.e., $A \cong \widehat{\mathbb{F}}_p$, then any proper nonzero subring F of K is a finite field $\mathbb{F}_{p^m}$ for some m and therefore it is $\mathbf{M}$-reduced. Thus, K is a minimally $\mathbf{M}$-complete ring. $\square$

An analog of Lemma 5 from [22] for the $\mathbf{M}$-completeness also has significant changes.

Lemma 21. *The full ring $M_n(K)$ of matrices over a skew field K is minimally* $\mathbf{M}$*-complete if and only if $M_n(K)$ is isomorphic either to the field $\mathbb{Q}$ of rational numbers or the algebraic closure $\widehat{\mathbb{F}}_p$ of the field $\mathbb{F}_p$ or a ring $M_2(\mathbb{F}_p)$ for some prime p.*

P r o o f. Let $M_n(K)$ be a minimally **M**-complete ring of matrices of order n over a skew field K. Being **M**-complete, the ring $M_n(K)$ does not belong to any variety of **M** since the rings of the latter are **M**-reduced.

Let $n = 1$. In this case, $M_1(K) \cong K$. If $char\ K = 0$, then $K \cong \mathbb{Q}$ by Lemma 19. If $char\ K = p$, then $K \cong \widehat{\mathbb{F}}_p$ by Lemma 20.

Let now $n > 1$. The ring $M_n(K)$, in this case, contains the skew field K as its proper subring. Due to the minimal **M**-completeness of the ring $M_n(K)$, the skew field K must be **M**-reduced. By Lemma 4, being a simple ring, K must be isomorphic to a finite field $\mathbb{F}_{p^m}$ for some p and m. It is obvious that the ring $M_n(\mathbb{F}_{p^m})$ for $n \geq 3$ contains a proper subring isomorphic to an **M**-complete ring $M_2(\mathbb{F}_{p^m})$, and therefore is not **M**-minimally complete. Hence, $n = 2$. If $m > 1$ then the ring $M_2(\mathbb{F}_{p^m})$ contains the **M**-complete proper subring $M_2(\mathbb{F}_p)$. Thus, $m = 1$.

Finally, we show that the ring $M_2(\mathbb{F}_p)$ is minimally **M**-complete. Consider any proper nonzero **M**-complete subring R of a ring $M_2(\mathbb{F}_p)$. Being finite, and therefore Artinian, a semisimple factor ring $R/J(R)$, by the Wedderburn-Artin theorem, is a direct sum of a finite number of matrix rings over suitable skew fields. It is clear that, in our case, these skew fields must be finite fields. But then the orders of the matrices included in the decomposition of the ring must be equal to 1. Being a direct sum of **M**-reduced fields, by Lemma 2.5 of [21], the ring $R/J(R)$ must also be **M**-reduced. But then it is clear that the subring R is not **M**-complete. Thus, the ring $M_2(\mathbb{F}_p)$ has no proper nonzero **M**-complete subrings; therefore, it is a minimally **M**-complete ring. □

Note that, in [23], it is indicated that Lemma 5 of [22] describing minimally complete rings $M_n(K)$ of all $(n \times n)$-matrices over a skew field K, in the end, mistakenly states that such is the ring $M_2(\mathbb{F}_p)$ for any p. That this is not the case follows from the well-known representation of finite fields by matrices (see, e.g., [14], p. 90): elements of a finite field $\mathbb{F}_{p^n}$ of order p^n can be represented by square matrices of order n over the field $\mathbb{F}_p$. Consequently, the ring $M_2(\mathbb{F}_p)$ contains a subring isomorphic to the complete field $\mathbb{F}_{p^2}$ and, therefore, is not a minimally complete ring. As a consequence, rings of matrices of the form $M_2(\mathbb{F}_p)$ for all p should be excluded from the formulations of Lemma 5 and condition (2) of the theorem from [22]. Rings R for which the factor ring R/pR is isomorphic to $M_2(\mathbb{F}_p)$ also must be excluded from the formulation of condition (3) of the same theorem. The exact formulation of the theorem from [22] is Theorem 4 of paper [29].

Corollary 9. *A semisimple Artinian ring is minimally* **M***-complete if and only if it is isomorphic either to the field* $\mathbb{Q}$ *of rational numbers or the algebraic closure* $\widehat{\mathbb{F}}_p$ *of the field* $\mathbb{F}_p$ *or a ring* $M_2(\mathbb{F}_p)$ *for some prime* p.

Lemma 22. *A minimally* **M***-complete finite ring* R *of a prime characteristic is semisimple by Jacobson.*

P r o o f. A ring R satisfying the conditions of Lemma 22 is an algebra of finite dimension over a field $\mathbb{F}_p$. The ring $R/J(R)$ is minimally **M**-complete by Corollary 5. Therefore, by Corollary 9, $R/J(R)$ is the ring $M_2(\mathbb{F}_p)$ of square matrices of order 2 over a finite field $\mathbb{F}_p$ for some p; i.e., $R/J(R)$ is a central simple algebra. In any case, we get that the algebra is a separable algebra over a field $\mathbb{F}_p$. The field $\mathbb{F}_p$ is perfect; therefore, according to the Wedderburn–Maltsev theorem (see, for example, Theorem 13.18 in [3], p. 575), $R = J(R) \oplus S$, where S is a subalgebra of R and S is isomorphic to $R/J(R)$. Since the ring $R/J(R)$ is **M**-complete and R is minimally **M**-complete, we have $R = S$; i.e., R is a semisimple ring. □

Corollary 10. *In a minimally* **M***-complete finite ring* R *of characteristic* p^k*, the equality* $J(R) = pR$ *is valid.*

P r o o f. In the ring R, the ideal pR is nilpotent; therefore, $pR \subseteq J(R)$. On the other hand, the ring R/pR is minimally **M**-complete by Lemma 5 and semisimple by Lemma 22. Hence, $J(R) \subseteq pR$; i.e., $J(R) = pR$. □

Lemma 23. *A minimally* **M***-complete ring R of all matrices of some order over the Galois ring is isomorphic to the ring $M_2(\mathbb{Z}_{p^k})$ for some prime p and $k \in \mathbb{N}$.*

P r o o f. For every Galois ring $GR(p^k, m)$, the factor ring

$$GR(p^k, m)/pGR(p^m, k) \cong GR(p, m) = \mathbb{F}_{p^m}.$$

Note, that any Galois ring has a unit and therefore is an idempotent ring. By Corollary 8, the matrix ring $M_n(GR(p^k, m))$ is **M**-complete for any $n \geq 2$. It follows that if $R = M_n(GR(p^k, m))$ is the minimally **M**-complete ring, then $n = 2$. By Lemma 2,

$$R/pR \cong M_2(GR(p^k, m))/M_2(pGR(p^k, m)) \cong M_2(GR(p, m)) = M_2(\mathbb{F}_{p^m}).$$

It follows from Lemma 21 that

$$R/pR \cong M_2(\mathbb{F}_p) \cong M_2(GR(p, 1)).$$

But then

$$R \cong M_2(GR(p^k, 1)) \cong M_2(\mathbb{Z}_{p^k}).$$

Conversely, let $R = M_2(\mathbb{Z}_{p^k})$ for some prime p and $k \in \mathbb{N}$. Then, R is a finite ring, for which $R^2 = R$ and $p^k R = O$. By Lemma 16, the ring R is minimally **M**-complete if and only if the ring R/pR is minimally **M**-complete. Since $R/pR \cong M_2(\mathbb{F}_p)$ is minimally **M**-complete by Lemma 21, we see that the ring $R = M_2(\mathbb{Z}_{p^k})$ is minimally **M**-complete. □

The main result of this section is the following modification of Theorem 1 from [22].

Theorem 6. (1) *Any minimally* **M***-complete nilpotent ring is isomorphic to the ring $\mathbb{Q}^0$ or the ring $C^0_{p^\infty}$ for some prime p.*

(2) *A simple ring with unit is minimally* **M***-complete if and only if it is isomorphic to the field $\mathbb{Q}$ of rational numbers or the algebraic closure $\widehat{\mathbb{F}}_p$ of the field $\mathbb{F}_p$ or a ring $M_2(\mathbb{F}_p)$ for some prime p.*

(3) *A finite ring is minimally* **M***-complete if and only if it is isomorphic to a matrix ring $M_2(\mathbb{Z}_{p^k})$ for some prime p and $k \in \mathbb{N}$.*

P r o o f. (1) This statement of Theorem 6 is the content of Lemma 18.

(2) Let a simple ring with unit is **M**-minimally complete. Then, it is an Artinian ring (see, for example, Corollary 4, [2], p. 196). But a simple Artinian ring is isomorphic to the ring of matrices $M_n(K)$ for some skew field K and a natural number n by the Wedderburn–Artin theorem. The rest follows from Lemma 9.

(3) Let R be a finite minimally **M**-complete ring. It follows that the additive group of the ring R is bounded. Then, by Theorem 1, R is a ring of characteristic p^k for some prime p. The ring R is nonnilpotent by Lemma 9. It follows that R is a ring with unity by Corollary 3.

By Corollary 6, the ring $R/J(R)$ is also minimally **M**-complete. Then, $R/J(R) \cong M_2(\mathbb{F}_p)$ by Corollary 9. By Theorem 2, the ring R contains a subring S isomorphic to the direct sum of full matrix rings over Galois rings such that $S/J(S) \cong R/J(R)$. In the ring S, the equality $J(S) = pS$

is valid. This means that $S/pS \cong M_2(\mathbb{F}_p)$. We obtain that S is a minimally **M**-complete subring of the ring R by Lemma 16. Therefore, $R = S$.

Minimally **M**-complete rings of all matrices of some order over Galois rings are described in Lemma 23. Hence, we get that $R \cong M_2(\mathbb{Z}_{p^k})$ for some prime p and $k \in \mathbb{N}$. $\square$

6. Conclusion

The paper characterizes associative Artinian **M**-complete (Theorem 4), **M**-reduced (Theorem 5), and some classes of minimally **M**-complete associative Artinian rings (Theorem 6). For an exhaustive description of minimally **M**-complete Artinian rings, it is necessary to consider the remaining unexplored case of Artinian rings of characteristic p^k containing a subring isomorphic to the algebraic closure $\widehat{\mathbb{F}}_p$ of the field $\mathbb{F}_p$. As examples show, such rings exist for any prime p and $k \in \mathbb{N}$.

Acknowledgements

The preparation of this paper was overshadowed by the death of its first author. The second author dedicates this work to his memory, as a sign of gratitude for many years of fruitful collaboration. Leonid Matveevich Martynov always had an active life position, he was distinguished by high professionalism, decency, and love of life.

REFERENCES

1. Andrunakievich V. A., Ryabukhin Yu. M. *Radikaly algebr i strukturnaya teoriya* [Radicals of Algebra and Structure Theory]. Moscow: Nauka, 1979. 496 p. (in Russian)
2. Bourbaki N. *Algebra. Moduli, kol'ca, formy* [Algebra. Modules, Rings, Forms]. Moscow: Mir, 1966. 556 p.(in Russian)
3. Feis K. *Algebra: kol'ca, moduli i kategorii* [Algebra: Rings, Modules and Categories]. Vol. 1. Moscow: Mir, 1977. 688 p. (in Russian)
4. Fuchs L. *Beskonechnye abelevy gruppy* [Infinite Abelian Groups]. Vol. 1. Moscow: Mir, 1974. 335 p. (in Russian)
5. Fuchs L. *Beskonechnye abelevy gruppy* [Infinite Abelian Groups]. Vol. 2. Moscow: Mir, 1977. 416 p. (in Russian)
6. Gardner B. J., Wiegandt R. *Radical Theory of Rings.* New York: Marcel Dekker, 2004. 408 p. DOI: 10.1201/9780203913352
7. Herstein I. N. *Nekommutativnye kol'ca.* [Noncommutative Rings]. Moscow: Mir, 1972. 190 p. (in Russian)
8. Jacobson N. *Stroenie kolec.* [Structure of Rings]. Moscow: Izdat. Inostran. Liter., 1961. 392 p. (in Russian)
9. Kalicki J., Scott D. Equational completness of abstract algebras. *Indag. Math.*, 1955. Vol. 17. P. 650–659. DOI: 10.1016/S1385-7258(55)50089-8
10. Kornev A. I. Complete radicals of some group rings. *Sib. Math. J.*, 2007. Vol. 48. P. 857–862. DOI: 10.1007/s11202-007-0087-0
11. Kornev A. I., Pavlova T. V. Characterization of one radical of group rings over finite prime fields. *Sib. Math. J.*, 2004. Vol. 45. P. 504–512. DOI: 10.1023/B:SIMJ.0000028615.28396.f2
12. Kornev A. I., Pavlova T. V. Finite complete associative rings. In: *Matematika i Informatika: Nauka i Obrazovanie: collection of works.* Omsk: OmGPU, 2002. Vol. 1. P. 43–45. (in Russian)
13. Kurosh A. G. Radicals in the group theory. *Sib. Mat. Zh.*, 1962. Vol. 3, No. 6. P. 912–931. (in Russian)
14. Lidl R., Niederreiter G. *Konechnye polya* [Finite Fields]. Vol. 1. Moscow: Mir, 1988. 430 p. (in Russian)
15. Mal'tsev A. I. Multiplication of classes of algebraic systems. *Sib. Math. J.*, 1967. Vol. 8, No. 2. P. 254–267. DOI: 10.1007/BF02302476

16. Martynov L. M. On notions of completeness, solvability, primarity, reducibility and purity for arbitrary algebras. In: *Int. Conf. on Modern Algebra and Its Applications (Vanderbilt University, Tennessee, May 14–18, 1996. Schudule and Abstracts).* Nashville, 1996. P. 79–80.
17. Martynov L. M. On concepts of completeness, reducibility, primarity, and purity for arbitrary algebras. In: *Universal Algebra and its Applications: collection of works. Volgograd, September 6–11, 1999.* Volgograd: Peremena, 2000. P. 179–190. (in Russian)
18. Martynov L. M. One radical algebras with a property of transversality by minimal varieties. *Herald of Omsk University*, 2004. Vol. 2. P. 19–21. URL: https://elib.omsu.ru/issues/124/5973.php (in Russian)
19. Martynov L. M. Completeness, reducibility, primarity and purity for algebras: results and problems. *Sib. Èlektron. Mat. Izv.*, 2016. Vol. 13. P. 181–241. DOI: 10.17377/semi.2016.13.016 (in Russian)
20. Martynov L. M. On the complete radical of a monoid ring. *Herald of Omsk University*, 2017. Vol. 84, No. 2. P. 8–13.
21. Martynov L. M. About one modification of concepts of completely, reducibility, periodicity and primarity for associative rings. *Herald of Omsk University*, 2021. Vol. 26, No. 2. P. 12–22. DOI: 10.24147/1812-3996.2021.26(2).12-22
22. Martynov L. M., Pavlova T. V. On minimally complete associative rings. *Herald of Omsk University*, 2016. Vol. 79, No. 1. P. 6–13. (in Russian)
23. Martynov L. M., Pavlova T. V. Letter to the editor. *Herald of Omsk University*, 2019. Vol. 24, No. 2. P. 23–24. (in Russian)
24. Pavlova T. V. Complete radical of full ring of matrices over an arbitrary ring. In: *Matematika i Informatika: Nauka i Obrazovanie: collection of works.* Omsk: OmGPU, 2011. Vol. 10, P. 16–19. (in Russian)
25. Pavlova T. V. On Artinian rings with a split-off complete radical. *Herald of Omsk University*, 2018. Vol. 23, No. 4. P. 37–43. (in Russian)
26. Pavlova T. V. Complete associative Artinian rings. *Herald of Omsk University*, 2005. Vol. 1. P. 17–19. (in Russian)
27. Pavlova T. V. On reduced associative Artinian rings. In: *Problemy i perspektivy fiz-mat. i tekhn. obrazovaniya: collection of works.* Ishim, Filial TyumGU v Ishime, 2014. P. 40–46. (in Russian)
28. Pavlova T. V. Minimally complete associative Artinian rings. *Sib. Èlektron. Mat. Izv.*, 2017. Vol. 14. P. 1238–1247. DOI: 10.17377/semi.2017.14.105 (in Russian)
29. Pavlova T. V. Corrigendum: Minimally complete associative Artinian rings. *Sib. Èlektron. Mat. Izv.*, 2019. Vol. 16. P. 1913–1915. DOI: 10.33048/semi.2019.16.136 (in Russian)
30. Shevrin L. N., Martynov L. M. Attainable classes of algebras. *Siberian Math. J.*, 1971. Vol. 12, No. 6. P. 986–998.
31. Ševrin L. N., Martynov L. M. Attainability and solvability for classes of algebras. In: *Semigroups. Vol. 39: Coll. Math. Soc., J. Bolyai, G. Poliak, St. Schwarz, O. Steinfeld (eds.).* Amsterdam, Oxford, New York, 1985. P. 397–459.
32. Wilson R. S. On structure of finite rings. II *Pasific J. Math.*, 1974. Vol. 51, No. 1. P. 317–325. DOI: 10.2140/pjm.1974.51.317