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CANONICAL APPROXIMATIONS IN IMPULSE STABILIZATION FOR A SYSTEM WITH AFTEREFFECT[1]

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Abstract: For optimal stabilization of an autonomous linear system of differential equations with aftereffect and impulse controls, the formulation of the problem in the functional state space is used. For a system with aftereffect, approximating systems of ordinary differential equations proposed by S.N. Shimanov and J. Hale are used. A method for constructing approximations for optimal stabilizing control of an autonomous linear system with aftereffect and impulse controls is proposed. Matrix Riccati equations are used to find approximating controls.

Keywords: Differential equation with aftereffect, Canonical approximation, Optimal stabilization, Impulse control.

1. Introduction

The control object is described as an autonomous linear system of differential equations with aftereffect and impulse control

$$\frac{dx(t)}{dt} = \int_{-\tau}^{0} [d_s\eta(s)]x(t+s) + Bu. \tag{1.1}$$

Here, $t \in \mathbb{R}^+ = (0, +\infty)$, $x : [-\tau, +\infty) \to \mathbb{R}^n$, $\tau > 0$, B is a constant matrix of dimension $n \times r$, the matrix function η has bounded variation on $[-\tau, 0]$, and $\eta(0) = 0$. Impulse controls are generalized functions defined by the formulas

$$u(t) = \frac{dv(t)}{dt}, \quad t \in \mathbb{R}^+,$$

in which control impulses $v\colon [0, +\infty) \to \mathbb{R}^r$ have bounded variations on any finite interval and $v(0) = 0$.

For any initial function $\varphi \in \mathbb{H}$, there is a unique solution $x(t,\varphi)$, $t \geq -\tau$, to equation (1.1) satisfying the condition $x(t,\varphi) = \varphi(t)$, $-\tau \leq t \leq 0$, and the integral equation

$$x(t) = \varphi(0) + \int_{0}^{t} \left(\int_{-\tau}^{0} [d_\xi \eta(\xi)] x(s+\xi) \right) ds + B\left(v(t) - v(+0)\right), \quad t \in \mathbb{R}^+.$$

Here, $\mathbb{H} = \mathbb{L}_2([-\tau, 0), \mathbb{R}^n) \times \mathbb{R}^n$ is a Hilbert space of functions with the scalar product

$$\langle \varphi, \psi \rangle_H = \psi^\top(0)\varphi(0) + \int_{-\tau}^{0} \psi^\top(\vartheta)\varphi(\vartheta)d\vartheta.$$

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Solutions to the integral equation are functions with bounded variations on any finite interval of the positive semi-axis $[0,+\infty)$. They define generalized solutions to the differential equation (1.1).

Need to find an impulse control formed according to the feedback principle, which ensures stable operation of system (1.1) and minimizes a given criterion for the quality of transient processes

$$J = \int_0^{+\infty} \Big(x^\top(t) C_x x(t) + v^\top(t) C_v v(t) \Big)\, dt, \tag{1.2}$$

where C_x and C_v are positive definite matrices.

The problems of optimal stabilization of autonomous linear systems of differential equations with aftereffects for non-impulse controls have been studied quite well [5, 8, 10, 11]. For impulse controls, they were studied in [1, 6, 21]. Constructive procedures for constructing optimal stabilizing controls are associated with finite-dimensional approximations of differential equations with aftereffects. In control problems and the theory of differential games for finite-dimensional approximations of equations with aftereffects, systems of ordinary differential equations proposed by Krasovskii are widely used. Approximations of optimal nonimpulse controls are constructed [4, 8, 12, 15]. An estimate of the accuracy of these approximations in the optimal stabilization problem for differential equations with concentrated delay was obtained by Bykov and Dolgii [2]. In [7], for the problem of optimal impulse stabilization, finite-dimensional approximations to a differential equation with aftereffect proposed by Krasovskii were used.

Canonical approximations were used in the problem of optimal stabilization of systems of differential equations with aftereffect and non-impulse controls in the works of Krasovskii and Osipov [13, 17], Markushin and Shimanov [16], Pandolfi [18, 19], Bykov and Dolgii [3]. In this work, when constructing approximations for optimal impulse stabilizing control, we use canonical approximations to the differential equation with aftereffect.

2. Stabilization problem in a Hilbert state space

When solving the problem, it is convenient, following Krasovskii [14, p. 162], to move from a finite-dimensional to an infinite-dimensional formulation, introducing functional elements

$$\mathbf{x}_t(\vartheta) = x(t+\vartheta), \quad \vartheta \in [-\tau, 0], \quad t \geq 0,$$

belonging to a separable Hilbert space $\mathbb{H}$ for solutions of system (1.1).

System (1.1) is associated with the differential equation

$$\frac{d\mathbf{x}_t}{dt} = \mathfrak{A}\mathbf{x}_t + \mathfrak{B}u, \quad t \in \mathbb{R}^+. \tag{2.1}$$

Here, $\mathfrak{A} : \mathbb{H} \to \mathbb{H}$ is an unbounded operator with the domain

$$D(\mathfrak{A}) = \big\{ \mathbf{x} \in \mathbb{H} : \mathbf{x} \in \mathbb{W}_2^1([-\tau, 0], \mathbb{R}^n) \big\}$$

defined by the formulas

$$(\mathfrak{A}\mathbf{x})(\vartheta) = \frac{d\mathbf{x}(\vartheta)}{d\vartheta}, \quad \vartheta \in [-\tau, 0), \quad (\mathfrak{A}\mathbf{x})(0) = \int_{-\tau}^{0} [d_s \eta(s)] \mathbf{x}(s).$$

A bounded operator $\mathfrak{B} : \mathbb{R}^r \to \mathbb{H}$ is defined by the formulas

$$(\mathfrak{B}\mathbf{u})(\vartheta) = 0, \quad \vartheta \in [-\tau, 0), \quad (\mathfrak{B}\mathbf{u})(0) = Bu.$$

The quality criterion for transient processes corresponding to (1.2) has the form

$$\mathbf{J} = \int\limits_0^{+\infty} \Big(\langle \mathbf{C}_x \mathbf{x}_t, \mathbf{x}_t \rangle_H + v^\top(t) C_v v(t) \Big)\, dt, \tag{2.2}$$

where a bounded self-adjoint nonnegative operator $\mathbf{C}_x : \mathbb{H} \to \mathbb{H}$ is defined by the formulas

$$(\mathbf{C}_x \mathbf{x})(\vartheta) = 0, \quad \vartheta \in [-\tau, 0), \quad (\mathbf{C}_x \mathbf{x})(0) = C_x \mathbf{x}(0).$$

Using the complexification of the space $\mathbb{H}$, we will consider the scalar product

$$\langle \mathbf{x}, \mathbf{y} \rangle_H = \mathbf{y}^*(0)\mathbf{x}(0) + \int\limits_{-\tau}^{0} \mathbf{y}^*(\vartheta)\mathbf{x}(\vartheta)\, d\vartheta.$$

The eigenvalues of the operator $\mathfrak{A}$ coincide with the roots of the characteristic equation

$$\delta(\lambda) = \det \Delta(\lambda) = 0, \quad \lambda \in \mathbb{C}, \tag{2.3}$$

where (see [14, p. 164])

$$\Delta(\lambda) = \lambda I_n - \int\limits_{-\tau}^{0} [d_s \eta(s)] \exp(\lambda s), \quad \lambda \in \mathbb{C}.$$

We will consider the nondegenerate case when the characteristic equation has a countable number of roots λ_k, $k \in \mathbb{N}$. To simplify further calculations, we will restrict ourselves to describing the canonical expansion procedure only for differential equations (2.1), all roots of the characteristic equations of which are simple. For any $\alpha \in \mathbb{R}$, a finite number of roots of equation (2.3) lie in the half-plane

$$\{\lambda \in \mathbb{C} : \mathrm{Re}\,(\lambda) > \alpha\}.$$

Consequently, they can be numbered in descending order of their real parts, and the numbers of complex conjugate roots must differ by one. The sequence of roots of the characteristic equation satisfies the condition $\mathrm{Re}\,(\lambda_n) \to -\infty$ as $n \to +\infty$. For the general case, the theory of canonical expansion is described in [9, 20].

Choose a positive integer N that satisfies requirement (A):

$$\mathrm{Re}\,(\lambda_n) < 0, \quad n > N.$$

Let $\mathbb{H}^N$ be the linear span of the eigenfunctions of the operator $\mathfrak{A}$ corresponding to its eigenvalues belonging to the set

$$\sigma_N = \{\lambda_1, \ldots, \lambda_N\} \subset \sigma(\mathfrak{A}),$$

where $\lambda_k \in \mathbb{C}$, $k = \overline{1, N}$, and $\sigma(\mathfrak{A})$ is the set of eigenvalues of the operator $\mathfrak{A}$. The projector $\mathfrak{P}_N \left(\mathfrak{P}_N \mathbb{H} = \mathbb{H}^N\right)$ defines the canonical decomposition of the space $\mathbb{H}$ into a direct sum, in which an element $\mathbf{x} \in \mathbb{H}$ uniquely defines the elements $\mathbf{x}^N \in \mathbb{H}$ and $\mathbf{z}^N \in (I - \mathfrak{P}_N)\,\mathbb{H}$ such that $\mathbf{x} = \mathbf{x}^N + \mathbf{z}^N$.

When constructing canonical approximations to the stabilization problem, the projection method scheme is used. We use the complexification of state space elements $\mathbf{x} \in \mathbb{H}$ and controls $u \in \mathbb{C}^r$. Applying the projector $\mathfrak{P}_N$ to equation (2.1) and taking into account the equalities

$$\mathfrak{P}_N \mathfrak{A} = \mathfrak{A} \mathfrak{P}_N = \mathfrak{A} \mathfrak{P}_N^2, \quad \mathbf{x}^N = \mathfrak{P}_N \mathbf{x},$$

we obtain the approximating equation

$$\frac{d\mathbf{x}_t^N}{dt} = \mathfrak{A}_N \mathbf{x}_t^N + \mathfrak{B}_N u, \quad t \in \mathbb{R}^+, \tag{2.4}$$

where finite-dimensional operators $\mathfrak{A}_N : \mathbb{H}^N \to \mathbb{H}^N$ and $\mathfrak{B}_N : \mathbb{C}^r \to \mathbb{H}^N$ are defined by the formulas $\mathfrak{A}_N = \mathfrak{A}\mathfrak{P}_N$ and $\mathfrak{B}_N = \mathfrak{P}_N \mathfrak{B}$.

The new quality criterion corresponding to (2.2) has the form

$$\mathbf{J}_N = \int_0^{+\infty} \left(\langle \mathbf{C}_x \mathbf{x}_t^N, \mathbf{x}_t^N \rangle_H + v^*(t) C_v v(t) \right) dt. \tag{2.5}$$

3. Finite-dimensional optimal stabilization problem

The subspace $\mathbb{H}^N$ is topologically equivalent to the finite-dimensional Hilbert space $\mathbb{C}^N$ with the inner product $z^* y$, where $y, z \in \mathbb{C}^N$. Let the topological isomorphism be given by the mapping

$$\pi_N : \mathbb{H}^N \to \mathbb{C}^N, \quad x^N = \pi_N \mathbf{x}^N, \quad \mathbf{x}^N \in \mathbb{H}^N, \quad x^N \in \mathbb{C}^N.$$

Using the mapping π_N, we replace equation (2.4) in the spaces $\mathbb{H}^N$ with an equivalent equation in the space $\mathbb{C}^N$

$$\frac{dx^N}{dt} = A_N x^N + B_N u, \quad t \in \mathbb{R}^+, \tag{3.1}$$

where finite-dimensional operators $A_N : \mathbb{C}^N \to \mathbb{C}^N$ and $B_N : \mathbb{C}^r \to \mathbb{C}^N$ are defined by the formulas

$$A_N = \pi_N \mathfrak{A}_N \pi_N^{-1}, \quad B_N = \pi_N \mathfrak{B}_N.$$

The equivalent quality criterion corresponding to (2.5) has the form

$$J_N = \int_0^{+\infty} \left(x^{N*}(t) C_x^N x^N(t) + v^*(t) C_v v(t) \right) dt, \tag{3.2}$$

where a finite-dimensional operator $C_x^N : \mathbb{C}^N \to \mathbb{C}^N$ is defined by the formula

$$C_x^N = \pi_N^{-1*} \mathbf{C}_x \pi_N^{-1}.$$

Using the substitutions

$$u(t) = \frac{dv(t)}{dt}, \quad y^N(t) = x^N(t) - B_N v(t), \quad t \in \mathbb{R}^+, \tag{3.3}$$

we replace the finite-dimensional problem of optimal impulse stabilization (3.1), (3.2) with the finite-dimensional problem of optimal nonimpulse stabilization. It is posed for the system of differential equations

$$\frac{dy^N}{dt} = A_N y^N + A_N B_N v, \quad t \in \mathbb{R}^+, \tag{3.4}$$

with new nonimpulse controls v and quality criterion corresponding to (3.2) of the form

$$\hat{J}_N = \int_0^{+\infty} \left(y^{N*}(t) C_{yy}^N y^N(t) + 2 y^{N*}(t) C_{yv}^{N*} v^N(t) + v^*(t) C_{vv}^N v(t) \right) dt, \tag{3.5}$$

where

$$C_{yy}^N = C_x^N, \quad C_{yv}^N = C_x^N B_N, \quad C_{vv}^N = C_v + B_N^* C_x^N B_N.$$

Assume that, for the problem of optimal non-impulse stabilization (3.4), (3.5) the matrix Riccati equation,

$$\begin{gathered} K^N A_N + A_N^* K^N + C_x^N - \left(K^N A_N + C_x^N\right) \tilde{C}_{vv}^N \left(A_N^* K^N + C_x^N\right) = 0, \\ \tilde{C}_{vv}^N = B_N \left(C_{vv}^N\right)^{-1} B_N^*, \end{gathered} \tag{3.6}$$

has a unique positive definite solution K^N. Then the optimal stabilizing control of problem (3.4), (3.5) is defined by the formula

$$v^{No}[y^N] = -\left(C_{vv}^N\right)^{-1} B_N^* \left(A_N^* K^N + C_x^N\right) y^N, \quad y^N \in \mathbb{C}^N. \tag{3.7}$$

Using formula (3.7), we find optimal stabilizing impulse controls of problem (3.1), (3.2).

Theorem 1. *Let the matrix Riccati equation* (3.6) *have a unique positive definite solution* K^N *and*

$$\det\left(C_v - B_N^* A_N^* K^N B_N\right) \neq 0.$$

Then the optimal stabilizing impulse control of problem (3.1), (3.2) *is defined by the formula*

$$u^{No}[t, x_0^N, x^N] = -\left(C_{vv}^N\right)^{-1} B_N^* \left(A_N^* K^N + C_x^N\right) \left(x_0^N \delta(t) + A_N x^N\right), \quad x^N \in \mathbb{C}^N, \tag{3.8}$$

where $\delta(\cdot)$ *is the Dirac function.*

P r o o f. Using formulas (3.7) and (3.3), we obtain

$$v^N(t) = -\left(C_{vv}^N\right)^{-1} B_N^* \left(A_N^* K^N + C_x^N\right) \left(x^N(t) - B_N v^N(t)\right), \quad t \in \mathbb{R}^+, \quad x^N \in \mathbb{C}^N,$$

or

$$\begin{gathered} \left(I_r - \left(C_{vv}^N\right)^{-1} B_N^* \left(A_N^* K^N + C_x^N\right) B_N\right) v^N(t) = \\ -\left(C_{vv}^N\right)^{-1} B_N^* \left(A_N^* K^N + C_x^N\right) x^N(t), \quad t \in \mathbb{R}^+, \quad x^N \in \mathbb{C}^N. \end{gathered}$$

Taking into account the equality

$$I_N - \left(C_{vv}^N\right)^{-1} B_N^* \left(A_N^* K^N + C_x^N\right) B_N = \left(C_{vv}^N\right)^{-1} \left(C_v - B_N^* A_N^* K^N B_N\right)$$

and the condition

$$\det\left(C_v - B_N^* A_N^* K^N B_N\right) \neq 0,$$

we get

$$\begin{gathered} v^N(t) = -\left(C_v - B_N^* A_N^* K^N B_N\right)^{-1} B_N^* \left(A_N^* K^N + C_x^N\right) x^N(t), \\ t \in \mathbb{R}^+, \quad v^N(0) = 0, \quad x^N \in \mathbb{C}^N. \end{gathered}$$

The control v^N is differentiable on the positive semi-axis $\mathbb{R}^+$ and has a unique discontinuity point of the first kind $t = 0$ with a limit value

$$v^N(+0) = -\left(C_v - B_N^* A_N^* K^N B_N\right)^{-1} B_N^* \left(A_N^* K^N + C_x^N\right) x_0^N.$$

As a result, the impulse control of problem (3.1), (3.2) is defined by the formula

$$u^N(t) = -\left(C_v - B_N^* A_N^* K^N B_N\right)^{-1} B_N^* \left(A_N^* K^N + C_x^N\right) \left(x_0^N \delta(t) + \frac{dx^N(t)}{dt}\right), \quad t \geq 0, \quad x^N \in \mathbb{C}^N.$$

Using (3.1), we obtain the equality

$$\begin{gathered} u^N(t) = -\left(C_v - B_N^* A_N^* K^N B_N\right)^{-1} B_N^* \left(A_N^* K^N + C_x^N\right) \left(x_0^N \delta(t) + A_N x^N(t) + B_N u^N(t)\right), \\ t \geq 0, \quad x^N \in \mathbb{C}^N. \end{gathered}$$

This explains the validity of formula (3.8), which completes the proof of the theorem. $\square$

4. Stabilizing impulse control of a system of differential equations with aftereffect

Using formula (3.8) and the connection between elements of the spaces $\mathbb{H}^N$ and $\mathbb{C}^N$, we find a stabilizing control for an autonomous linear system of differential equations with aftereffect.

Theorem 2. *Let requirement (A) and the conditions of Theorem 1 be satisfied. Then the control*

$$u^{No}[t,\varphi,\mathbf{x}_t] = -\left(C_{vv}^N\right)^{-1} B_N^* \left(A_N^* K^N + C_x^N\right)\left(\pi_N\varphi\delta(t) + A_N\pi_N\mathbf{x}_t\right), \quad \varphi,\mathbf{x}_t \in \mathbb{H}, \quad t>0, \tag{4.1}$$

is stabilizing for the system of differential equations with aftereffect (1.1).

P r o o f. For control (4.1), the differential equation (2.1) takes the form

$$\frac{d\mathbf{x}_t}{dt} = \left(\mathfrak{A} - \mathfrak{D}_N A_N \pi\right)\mathbf{x}_t - \mathfrak{D}_N\pi\varphi\delta(t), \quad t \in \mathbb{R}^+.$$

Here

$$\left(\mathfrak{D}_N v\right)(\vartheta) = 0, \quad \vartheta \in [-\tau, 0), \quad \left(\mathfrak{D}_N v\right)(0) = B_N\left(C_{vv}^N\right)^{-1} B_N^*\left(A_N^* K^N + C_x^N\right) v, \quad v \in \mathbb{C}^N.$$

Using the canonical expansion of the space $\mathbb{H}$, we obtain the system of differential equations

$$\frac{d\mathbf{x}_t^N}{dt} = \left(\mathfrak{A}P_N - P_N\mathfrak{D}_N A_N\pi\right)\mathbf{x}_t^N - P_N\mathfrak{D}_N\pi\varphi\delta(t),$$
$$\frac{d\mathbf{z}_t^N}{dt} = \mathfrak{A}\left(I - P_N\right)\mathbf{z}_t^N - \left(I-P_N\right)\mathfrak{D}_N A_N\pi\mathbf{x}_t^N - \left(I-P_N\right)\mathfrak{D}_N\pi\varphi\delta(t), \quad t \geq 0$$

with the initial conditions

$$\mathbf{x}_0^N = P_N\varphi, \quad \mathbf{z}_0^N = \left(I - P_N\right)\varphi.$$

The control used guarantees exponential boundedness of the solutions of the first subsystem with negative exponents. The evolutionary operator $T_N(t)$, $t \in \mathbb{R}^+$, of the homogeneous part of the first subsystem is exponentially bounded with a negative exponent, according to the chosen canonical expansion [9, p. 170].

The solution of the second subsystem is defined by the formula [9, p. 185]

$$\mathbf{z}_t^N = T_N(t)\left(I-P_N\right)\varphi - \int_0^t T_N(t-s)\left(I-P_N\right)\mathfrak{D}_N\left(\mathfrak{A}_N\pi\mathbf{x}_s^N - \pi\varphi\delta(s)\right)ds$$
$$= T_N(t)\left(I-P_N\right)\left(\varphi - \mathfrak{D}_N\pi\varphi\right) - \int_0^t T_N(t-s)\left(I-P_N\right)\mathfrak{D}_N\mathfrak{A}_N\pi\mathbf{x}_s^N ds, \quad t\in\mathbb{R}^+.$$

This implies that the solutions of the second subsystem with negative exponents are exponentially bounded, which completes the proof of the theorem. □

Let us consider the eigenfunctions φ^i, $i=\overline{1,N}$, corresponding to the eigenvalues λ_i, $i=\overline{1,N}$, of the operator $\mathfrak{A}$. Due to their linear independence, they define the basis of the subspace $\mathbb{H}^N$. The eigenfunctions of the operator $\mathfrak{A}$ are defined by the formulas

$$\varphi^k(\vartheta) = \exp(\lambda_k\vartheta)\hat{\varphi}^k, \quad \vartheta \in [-\tau, 0],$$

where $\hat{\varphi}^k$ are nontrivial solutions to the algebraic system

$$\left(\lambda_k I_N - \int_{-\tau}^{0} [d_s\eta(s)] \exp(\lambda_k s)\right)\hat{\varphi}^k = 0, \quad k = \overline{1, N}.$$

To find a coordinate representation of the projector $\mathfrak{P}_N$ in the selected basis, it is necessary to consider for it a biorthogonal system of functions $\{\psi^j\}_{j=1}^N$. The unbounded operator $\mathfrak{A}$ has a dense domain in the space $\mathbb{H}$. Therefore, there is an unbounded conjugate operator $\mathfrak{A}^* : \mathbb{H} \to \mathbb{H}$ with the domain

$$D(\mathfrak{A}^*) = \big\{\mathbf{y} \in \mathbb{H} : \tilde{\mathbf{y}} \in \mathbb{W}_2^1([-\tau, 0], \mathbb{C}^n),\ \tilde{\mathbf{y}}(\vartheta) = \mathbf{y}(\vartheta) - \eta^\top(\vartheta)\mathbf{y}(0),$$
$$\vartheta \in [-\tau, 0],\ \tilde{\mathbf{y}}(-\tau) + \eta^\top(-\tau)\mathbf{y}(0) = 0\big\}.$$

It is defined by the formulas

$$(\mathfrak{A}^*\mathbf{y})(\vartheta) = -\frac{d\tilde{\mathbf{y}}(\vartheta)}{d\vartheta}, \quad \vartheta \in [-\tau, 0), \quad (\mathfrak{A}^*\mathbf{y})(0) = \tilde{\mathbf{y}}(0).$$

The eigenfunctions of the operator $\mathfrak{A}^*$ corresponding to its eigenvalues $\bar{\lambda}_k$, $k \in \mathbb{N}$, are defined by the formulas

$$\psi^k(\vartheta) = \exp(-\bar{\lambda}_k\vartheta)\left(\bar{\lambda}_k I_N - \int_{\vartheta}^{0} [d_s\eta^\top(s)] \exp(\bar{\lambda}_k s)\right)\hat{\psi}^k,$$
$$\vartheta \in [-\tau, 0), \quad \psi^k(0) = \hat{\psi}^k,$$

where $\hat{\psi}^k$ are nontrivial solutions to the algebraic system

$$\left(\bar{\lambda}_k I_N - \int_{-\tau}^{0} [d_s\eta^\top(s)] \exp(\bar{\lambda}_k s)\right)\hat{\psi}^k = 0, \quad k = \overline{1, N}.$$

The requirement of simplicity of the eigenvalues of the operator $\mathfrak{A}$ imposed above generates the biorthogonality of the system of eigenfunctions $\{\psi^j\}_{j=1}^N$ of the operator $\mathfrak{A}^*$ with respect to the system of eigenfunctions $\{\varphi^i\}_{i=1}^N$ of the operator $\mathfrak{A}$. For the fulfilment of the conditions $\langle\varphi^i, \psi^j\rangle_H = \delta_{ij}$, where δ_{ij}, $i, j = \overline{1, N}$, is the Kronecker symbol, it is necessary that

$$1 = \langle\varphi^i, \psi^i\rangle_H = \hat{\psi}^{i*}\left(I_n - \int_{-\tau}^{0} [d_s\eta^\top(s)] s \exp(\lambda_i s)\right)\hat{\varphi}^i, \quad i = \overline{1, N}.$$

These normalization conditions can be ensured by freedom in choosing the vectors $\hat{\psi}^i$, $i = \overline{1, N}$.

Let us define a coordinate representation of the projector $\mathfrak{P}_N$ by the formulas

$$\mathfrak{P}_N\mathbf{x} = \sum_{k=1}^{N} y_k\varphi^k = \mathbf{x}^N = \sum_{k=1}^{N}\langle\mathbf{x}^N, \psi^k\rangle_H\varphi^k, \quad \mathbf{x} \in \mathbb{H}, \quad \mathbf{x}^N \in \mathbb{H}^N, \quad \{y_k\}_{k=1}^N = \mathbf{y}^N \in \mathbb{C}^N.$$

The topological isomorphism $\pi_N : \mathbb{H}^N \to \mathbb{C}^N$ is defined by the formulas

$$\pi_N\mathbf{x}^N = \{\langle\mathbf{x}^N, \psi^k\rangle_H\}_{k=1}^N = \mathbf{y}^N, \quad \pi_N^{-1}\mathbf{y}^N = \sum_{k=1}^{N} y_k\varphi^k = \mathbf{x}^N, \quad \mathbf{x} \in \mathbb{H}, \quad \mathbf{x}^N \in \mathbb{H}^N, \quad \mathbf{y}^N \in \mathbb{C}^N.$$

We have the estimates
$$\|\pi_N\| \le \left(\sum_{k=1}^{N} \|\psi^k\|^2\right)^{1/2}, \quad \|\pi_N^{-1}\| \le \lambda_{max},$$
where λ_{max} is the spectral radius of the matrix $\{\langle\varphi^k, \varphi^m\rangle_H\}_{k,m=1}^N$.

Theorem 3. *If the conditions of Theorem 2 hold, then the stabilizing controls for the system of differential equations with aftereffect (1.1) are defined by the formulas*

$$u^{No}[t,\varphi,\mathbf{x}_t] = -\left(C_{vv}^N\right)^{-1} B^\top \sum_{i,j=1}^{N} \hat{\psi}^i \left(\bar{\lambda}_i K_{ij}^N + \hat{\varphi}^{i*} C_x \hat{\varphi}^j\right)\left(\langle\varphi,\psi^j\rangle_H \delta(t) + \lambda_j \langle \mathbf{x}_t, \psi^j\rangle_H\right), \\ \varphi, \mathbf{x}_t \in \mathbb{H}, \quad t>0, \tag{4.2}$$

where
$$C_{vv}^N = C_v + B^\top \sum_{i,j=1}^{N} \hat{\psi}^i \hat{\varphi}^{i*} C_x \hat{\varphi}^j \hat{\psi}^{j*} B.$$

P r o o f. Using the coordinate representations of the projector $\mathfrak{P}_N$ and the topological isomorphism π_N, we find the following coordinate representations for the operators:

$$\mathfrak{A}_N \mathbf{x}^N = \mathfrak{A}\mathfrak{P}_N \mathbf{x}^N = \sum_{i=1}^{N} \langle \mathbf{x}^N, \psi^i\rangle_H \mathfrak{A}\varphi^i = \sum_{i=1}^{N} \lambda_i \langle \mathbf{x}^N, \psi^i\rangle_H \varphi^i, \quad \mathbf{x}^N \in \mathbb{H}^N,$$
$$A_N \mathbf{y}^N = \pi_N \mathfrak{A}_N \pi_N^{-1} \mathbf{y}^N = \pi_N \sum_{i=1}^{N} \lambda_i \langle \sum_{k=1}^{N} y_k \varphi^k, \psi^i\rangle_H \varphi^i = \sum_{i=1}^{N} \lambda_i y_i \pi_N \varphi^i$$
$$= \sum_{i=1}^{N} \lambda_i y_i \{\langle \varphi^i, \psi^k\rangle_H\}_{k=1}^N = \{\lambda_k y_k\}_{k=1}^N, \quad \mathbf{y}^N \in \mathbb{C}^N,$$
$$B_N u = \pi_N \mathfrak{B}_N u = \pi_N \mathfrak{P}_N \mathfrak{B} u = \pi_N \sum_{i=1}^{N} \hat{\psi}^{i*} B u \pi_N \varphi^i$$
$$= \sum_{i=1}^{N} \hat{\psi}^{i*} B u \{\langle \varphi^i, \psi^k\rangle_H\}_{k=1}^N = \{\hat{\psi}^{k*} B u\}_{k=1}^N, \quad u \in \mathbb{C}^r,$$
$$C_x^N \mathbf{y}^N = \pi_N^{-1*} \mathbf{C}_x \pi_N^{-1} \mathbf{y}^N = \left\{\langle \mathbf{C}_x \pi_N^{-1} \mathbf{y}^N, \varphi^i\rangle_H\right\}_{i=1}^N$$
$$= \left\{\hat{\varphi}^{i*}\left(C_x \pi_N^{-1} \mathbf{y}^N\right)(0)\right\}_{i=1}^N = \left\{\sum_{k=1}^{N} \hat{\varphi}^{i*} C_x \hat{\varphi}^k y_k\right\}_{i=1}^N, \quad \mathbf{y}^N \in \mathbb{C}^N.$$

Using these formulas, from (4.1) we obtain (4.2), which completes the proof of the theorem. □

As the positive integer N increases, the constructed stabilizing controls approximate the optimal impulse controls for the autonomous linear system of differential equations with aftereffect (1.1).

5. Conclusion

Approximations to an optimal impulse stabilizing control for an autonomous linear system of differential equations with aftereffect have been constructed. Evaluating the accuracy of approximations to an optimal impulse stabilizing control is a challenging problem.

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