URAL MATHEMATICAL JOURNAL, Vol. 9, No. 2, 2023, pp. 36–45
DOI: 10.15826/umj.2023.2.003

ON SEQUENCES OF ELEMENTARY TRANSFORMATIONS IN THE INTEGER PARTITIONS LATTICE

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Abstract: An *integer partition*, or simply, a *partition* is a nonincreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of nonnegative integers that contains only a finite number of nonzero components. The *length* $\ell(\lambda)$ of a partition λ is the number of its nonzero components. For convenience, a partition λ will often be written in the form $\lambda = (\lambda_1, \dots, \lambda_t)$, where $t \geq \ell(\lambda)$; i.e., we will omit the zeros, starting from some zero component, not forgetting that the sequence is infinite. Let there be natural numbers $i, j \in \{1, \dots, \ell(\lambda) + 1\}$ such that (1) $\lambda_i - 1 \geq \lambda_{i+1}$; (2) $\lambda_{j-1} \geq \lambda_j + 1$; (3) $\lambda_i = \lambda_j + \delta$, where $\delta \geq 2$. We will say that the partition $\eta = (\lambda_1, \dots, \lambda_i - 1, \dots, \lambda_j + 1, \dots, \lambda_n)$ is obtained from a partition $\lambda = (\lambda_1, \dots, \lambda_i, \dots, \lambda_j, \dots, \lambda_n)$ by an *elementary transformation of the first type*. Let $\lambda_i - 1 \geq \lambda_{i+1}$, where $i \leq \ell(\lambda)$. A transformation that replaces λ by $\eta = (\lambda_1, \dots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \dots)$ will be called an *elementary transformation of the second type*. The authors showed earlier that a partition μ dominates a partition λ if and only if λ can be obtained from μ by a finite number (possibly a zero one) of elementary transformations of the pointed types. Let λ and μ be two arbitrary partitions such that μ dominates λ. This work aims to study the shortest sequences of elementary transformations from μ to λ. As a result, we have built an algorithm that finds all the shortest sequences of this type.

Keywords: Integer partition, Ferrers diagram, Integer partitions lattice, Elementary transformation.

1. Introduction

Everywhere below, by a graph, we mean a simple graph, i.e., a graph without loops and multiple edges. We will adhere to the terminology and notation from [6].

An *integer partition*, or simply a *partition*, is a nonincreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of nonnegative integers that contains only a finite number of nonzero components (see [1]). Let $\operatorname{sum}\lambda$ denote the sum of all components of a partition λ and call it the *weight* of the partition λ. It is often said that a partition λ is a partition of the nonnegative integer $n = \operatorname{sum}\lambda$. The *length* $\ell(\lambda)$ of a partition λ is the number of its nonzero components. For convenience, a partition λ will often be written in the form $\lambda = (\lambda_1, \dots, \lambda_t)$, where $t \geq \ell(\lambda)$; i.e., we will omit the zeros, starting from some zero component, not forgetting that the sequence is infinite.

Let IPL denote the lattice of all (integer) partitions of all nonnegative integers, and let $IPL(m)$ denote the lattice of all partitions of a given nonnegative integer m. On the lattices IPL and $IPL(m)$, where $m \in \mathbb{N}$, the well-known domination relation is considered [7].

We define two types of elementary transformations (see [2, 3]) of the partition

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_t, 0, 0, \dots),$$

where $t = \ell(\lambda) + 1$.

Let there be natural numbers $i, j \in \{1, \dots, t\}$ such that $1 \leq i < j \leq \ell(\lambda) + 1$ and

(1) $\lambda_i - 1 \geq \lambda_{i+1}$ (or, equivalently, $\lambda_i > \lambda_{i+1}$);
(2) $\lambda_{j-1} \geq \lambda_j + 1$ (or, equivalently, $\lambda_{j-1} > \lambda_j$);
(3) $\lambda_i = \lambda_j + \delta$, where $\delta \geq 2$.

We will say that the partition $\eta = (\lambda_1, \dots, \lambda_i - 1, \dots, \lambda_j + 1, \dots, \lambda_n)$ is obtained from a partition $\lambda = (\lambda_1, \dots, \lambda_i, \dots, \lambda_j, \dots, \lambda_n)$ by an *elementary transformation of the first type* (or a *box movement*). Conditions (1), (2), and (3) guarantee that a partition again will be obtained. Note that η differs from λ by precisely two components with numbers i and j. For the Ferrers diagram, this transformation means moving the top box of i-column to the right to the top of the j-column. We will use Cartesian notation for the Ferrers diagram: each k-column consists of λ_k boxes (see [6]).

It should be noted that a box can also be thrown to the zero component with the number $\ell(\lambda) + 1$. The fact that η is obtained from λ by moving the box will be briefly written in the form $\lambda \rightharpoondown \eta$. Note that an elementary transformation of the first type preserves the weight of the partition, while the length of the partition can be preserved or lifted by 1.

We now define elementary transformations of the second type for the partition $\lambda = (\lambda_1, \lambda_2, \dots)$.

Let $\lambda_i - 1 \geq \lambda_{i+1}$ (or, equivalently, $\lambda_i > \lambda_{i+1}$), where $i \leq \ell(\lambda)$. A transformation that replaces λ by $\eta = (\lambda_1, \dots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \dots)$ will be called an *elementary transformation of the second type* (or a *box removement*). As in the previous case, we will briefly write $\lambda \rightharpoondown \eta$. It should be noted that box removal reduces the weight of the partition exactly by 1, while the length of the partition can be preserved or lowered by 1.

It was shown in [2, 3] that a partition μ dominates a partition λ if and only if λ can be obtained from μ by sequentially applying a finite number (possibly a zero one) of elementary transformations of the pointed types.

Let λ and μ be two arbitrary partitions and $\lambda \leq \mu$. The height (μ, λ) of a partition μ over a partition λ is the number of transformations in a shortest sequence of elementary transformations transforming μ into λ.

The following theorem was proved in [4].

Theorem 1 [4, Theorem 1]. *Let $\mu \geq \lambda$ in IPL and $C = \operatorname{sum} \mu - \operatorname{sum} \lambda$. Then*

$$\operatorname{height}(\mu, \lambda) = \sum_{j=1,\, \mu_j > \lambda_j}^{\infty} (\mu_j - \lambda_j) = \frac{1}{2} C + \frac{1}{2} \sum_{j=1}^{\infty} |\mu_j - \lambda_j|.$$

Let μ and λ be some fixed partitions such that $\mu > \lambda$. Consider sequences of elementary transformations from μ to λ (both types of elementary transformations are admissible):

$$\mu = \xi_{(0)} \rightharpoondown \xi_{(1)} \rightharpoondown \cdots \rightharpoondown \xi_{(s)} = \lambda.$$

This paper aims to describe an algorithm (Algorithm 1) for constructing all possible shortest sequences of this kind. Algorithm 1 generalizes an algorithm constructed in [4] (see Algorithm 2).

2. Main results

Let $\mu = (\mu_1, \dots, \mu_t)$ and $\lambda = (\lambda_1, \dots, \lambda_t)$ be two nonzero partitions, where t is the maximum length of μ and λ.

Note that if $\mu \geq \lambda$, then $\operatorname{sum} \mu \geq \operatorname{sum} \lambda$; i.e., the integer $C = \operatorname{sum} \mu - \operatorname{sum} \lambda$ is nonnegative.

By definition of the dominance relation, a condition $\mu \geq \lambda$ is equivalent to the system of inequalities

$$\mu_1 + \cdots + \mu_k \geq \lambda_1 + \cdots + \lambda_k \quad (k = 1, \dots, t).$$

We write this system of inequalities in the following equivalent form:

$$\sum_{j=1,\, \mu_j > \lambda_j}^{k} (\mu_j - \lambda_j) \geq \sum_{j=1,\, \mu_j < \lambda_j}^{k} (\lambda_j - \mu_j) \quad (k = 1, \dots, t). \tag{2.1}$$

Here and below, in the case when no index satisfies a summation condition, we will assume that the corresponding sum is equal to 0.

For some integer $C \geq 0$, a condition $\mu \geq \lambda$ is equivalent to the system

$$\begin{cases} \mu_1 + \cdots + \mu_t & = \lambda_1 + \cdots + \lambda_t + C, \\ \mu_1 + \cdots + \mu_{k-1} & \geq \lambda_1 + \cdots + \lambda_{k-1} \quad (k = 2, \ldots, t). \end{cases} \tag{2.2}$$

Since the following equalities are true:

$$\mu_1 + \cdots + \mu_{k-1} + \mu_k + \cdots + \mu_t = \lambda_1 + \cdots + \lambda_{k-1} + \lambda_k + \cdots + \lambda_t + C \quad (k = 2, \ldots, t),$$

system (2.2) is equivalent to the system

$$\begin{cases} \mu_1 + \cdots + \mu_t & = \lambda_1 + \cdots + \lambda_t + C, \\ \mu_k + \cdots + \mu_t & \leq \lambda_k + \cdots + \lambda_t + C \quad (k = 2, \ldots, t). \end{cases} \tag{2.3}$$

Let us rewrite system (2.3) in the equivalent form:

$$\begin{cases} \displaystyle\sum_{j=1,\, \mu_j > \lambda_j}^{t} (\mu_j - \lambda_j) & = \displaystyle\sum_{j=1,\, \mu_j < \lambda_j}^{t} (\lambda_j - \mu_j) + C, \\ \displaystyle\sum_{j=k,\, \mu_j > \lambda_j}^{t} (\mu_j - \lambda_j) & \leq \displaystyle\sum_{j=k,\, \mu_j < \lambda_j}^{t} (\lambda_j - \mu_j) + C \quad (k = 2, \ldots, t). \end{cases} \tag{2.4}$$

For $i = 1, 2, \ldots, t$, we say that an i-component of a partition μ has an *i-hill* (or simply a *hill*) with respect to the partition λ if $\mu_i > \lambda_i$. In the case when the condition $\mu_i > \lambda_i$ is satisfied, we assume that upper $\mu_i - \lambda_i$ boxes of the i-column of the Ferrers diagram of the partition μ form the i-hill of height $\mu_i - \lambda_i$.

For $j = 1, 2, \ldots, t$, we say that a j-component of a partition μ has a *j-pit* (or simply a *pit*) with respect to the partition λ if $\mu_j < \lambda_j$. In the case when the condition $\mu_j < \lambda_j$ is satisfied, we assume that there is the j-pit of depth $\lambda_j - \mu_j$ over the j-column of the Ferrers diagram of the partition μ.

Let us reformulate conditions (2.1) as follows.

For any $k = 1, \ldots, t$, the sum of the heights of all i-hills such that $1 \leq i \leq k$ is greater than or equal to the sum of the depths of all j-pits such that $1 \leq j \leq k$.

Respectively, system (2.4) is equivalent to the following statement.

For some nonnegative integer C:

- the sum of the heights of all hills is equal to the integer C plus the sum of the depths of all pits;
- for any $k = 2, \ldots, t$, the sum of the heights of all i-hills such that $i \geq k$ does not exceed the integer C plus the sum of the depths of all j-pits such that $j \geq k$.

In what follows, we will assume that $\mu \geq \lambda$.

We will say that a j-pit is *admissible* if $\mu_{j-1} > \mu_j$. Note that the admissibility of a j-pit is a necessary condition for the possibility of moving the box to the j-column of the partition μ from some column with a number less than j.

We will call an i-hill of a partition μ *open* if $\mu_i > \mu_{i+1}$. Note that the openness of i-hill is a necessary condition for the possibility of moving the box from i-column of the partition μ to some column with a greater number than i or for removal of the box from i-column of the partition μ.

For $k = 1, 2, \ldots, t$, we say that a number k is a *separator* for the partition μ with respect to the partition λ if

$$\mu_1 + \cdots + \mu_k = \lambda_1 + \cdots + \lambda_k.$$

This means that

$$\sum_{j=1,\,\mu_j>\lambda_j}^{k} (\mu_j - \lambda_j) = \sum_{j=1\,\mu_j<\lambda_j}^{k} (\lambda_j - \mu_j).$$

This condition is equivalent to the following statement.

The sum of the heights of all i-hills such that $1 \le i \le k$ is equal to the sum of the depths of all j-pits such that $1 \le j \le k$.

Let us make a simple remark that, for any i-hill, the number i is not a separator for the partition μ with respect to λ.

Indeed, by the condition $\mu \ge \lambda$, we have

$$\mu_1 + \cdots + \mu_{i-1} \ge \lambda_1 + \cdots + \lambda_{i-1}.$$

Since $\mu_i > \lambda_i$, it follows $\mu_1 + \cdots + \mu_i > \lambda_1 + \cdots + \lambda_i$. Consequently, the number i is not a separator.

Lemma 1. *Assume that $\mu \ge \lambda$, the partition μ has an i-hill with respect to the partition λ, where $1 \le i \le t$, and a partition μ' is obtained from the partition μ by applying an elementary transformation of the second type, which consists in removing the top box from the i-hill. Then*

(1) *if, for the partition μ, there is a separator k such that $i < k \le t$, then the condition $\mu' \ge \lambda$ is not satisfied*;

(2) *if, for the partition μ, there are no separators k such that $i < k \le t$, then the condition $\mu' \ge \lambda$ is satisfied.*

P r o o f. 1. A separator k satisfies the condition

$$\mu_1 + \cdots + \mu_i + \cdots + \mu_k = \lambda_1 + \cdots + \lambda_i + \cdots + \lambda_k.$$

For the partition μ', we have $\mu'_i = \mu_i - 1$ and $\mu'_p = \mu_p$ for $p \ne i$ and $p = 1, \dots, k$. Hence, we have the inequality

$$\mu'_1 + \cdots + \mu'_i + \cdots + \mu'_k < \lambda_1 + \cdots + \lambda_i + \cdots + \lambda_k.$$

Therefore, μ' does not dominate λ, i.e., the condition $\mu' \ge \lambda$ is not satisfied.

2. For any number k such that $i \le k \le t$, since it is not a separator for the partition μ, the condition

$$\mu_1 + \cdots + \mu_i + \cdots + \mu_k > \lambda_1 + \cdots + \lambda_i + \cdots + \lambda_k$$

is true. Since, for the partition μ', we have $\mu'_i = \mu_i - 1$ and $\mu'_p = \mu_p$ for $p \ne i$ and $p = 1, \dots, k$, it follows that

$$\mu'_1 + \cdots + \mu'_i + \cdots + \mu'_k \ge \lambda_1 + \cdots + \lambda_i + \cdots + \lambda_k.$$

In addition, for any $k = 1, \dots, i-1$, the condition

$$\mu'_1 + \cdots + \mu'_k \ge \lambda_1 + \cdots + \lambda_k$$

is true.

Therefore, μ' dominates λ, i.e., the condition $\mu' \ge \lambda$ is true. $\square$

Assume that $\mu \ge \lambda$, the partition μ has an i-hill with respect to the partition λ, where $1 \le i \le t$, and the partition μ' is obtained from the partition μ by applying an elementary transformation of the second type, which consists in removing the top box from the i-hill. We will call such an elementary transformation of the second type *proper* for a partition μ with respect to a partition λ if, for a partition μ, there are no separators k such that $i < k \le t$. Lemma 1 states that $\mu' \ge \lambda$ holds if and only if the corresponding elementary transformation of the second type is proper.

Lemma 2. *Assume that $\mu \geq \lambda$, the partition μ has an i-hill and a j-pit with respect to the partition λ, where $1 \leq i < j \leq t$, and a partition μ' is obtained from the partition μ by applying an elementary transformation of the first type, which consists in moving the upper box from the i-hill to the j-pit. Then*

(1) *if the partition μ has a separator k such that $i < k < j$, then the condition $\mu' \geq \lambda$ is not satisfied;*
(2) *if, for the partition μ, there are no a separators k such that $i < k < j$, then the condition $\mu' \geq \lambda$ is satisfied.*

P r o o f. 1. It can be proven in exactly the same way as (1) in Lemma 1.

2. Due to the remark made before Lemma 1 and the conditions of this lemma, the numbers k such that $i \leq k < j$ are not separators for the partition μ with respect to the partition λ, hence, for such numbers k, we have

$$\mu_1 + \dots + \mu_i + \dots + \mu_k > \lambda_1 + \dots + \lambda_i + \dots + \lambda_k.$$

For the partition μ', we have $\mu'_i = \mu_i - 1$ and $\mu'_p = \mu_p$ for $p \neq i$ and $p = 1, \dots, k$. Consequently, we get

$$\mu'_1 + \dots + \mu'_i + \dots + \mu'_k \geq \lambda_1 + \dots + \lambda_i + \dots + \lambda_k.$$

Further, we note that the following condition is true for any $k = 1, \dots, i-1$:

$$\mu'_1 + \dots + \mu'_k \geq \lambda_1 + \dots + \lambda_k.$$

For any $k \geq j$, the following condition holds:

$$\mu_1 + \dots + \mu_i + \dots + \mu_j + \dots + \mu_k \geq \lambda_1 + \dots + \lambda_i + \dots + \lambda_j + \dots + \lambda_k.$$

Consequently,

$$\mu_1 + \dots + (\mu_i - 1) + \dots + (\mu_j + 1) + \dots + \mu_k \geq \lambda_1 + \dots + \lambda_i + \dots + \lambda_j + \dots + \lambda_k.$$

Hence, we have

$$\mu'_1 + \dots + \mu'_i + \dots + \mu'_j + \dots + \mu'_k \geq \lambda_1 + \dots + \lambda_i + \dots + \lambda_j + \dots + \lambda_k.$$

Therefore, the condition $\mu' \geq \lambda$ is satisfied. □

Assume that $\mu \geq \lambda$, the partition μ has an i-hill and has a j-pit with respect to the partition λ, where $1 \leq i < j \leq t$, and a partition μ' is obtained from the partition μ by applying an elementary transformation of the first type, which consists in moving the upper box from i-hill to the j-pit. Such an elementary transformation of the first type will be called *proper* for the partition μ with respect to the partition λ if, for μ, there are no separators k such that $i < k < j$. Lemma 2 states that $\mu' \geq \lambda$ holds if and only if the corresponding elementary transformation of the first type is proper.

Lemma 3. *Let $\mu \geq \lambda$. Then*

(1) *for every pit of the partition μ, there is a hill to the left of that pit;*
(2) *if there is a pit of the partition μ, then also there is an admissible pit.*

P r o o f. Let the partition μ have a j-pit with respect to the partition λ for some $j \in \{1, \dots, t\}$.

Since $\mu \geq \lambda$, by conditions (2.1), there exists an i-hill such that $1 \leq i < j$. We assume that the i-hill is the nearest hill to the left from the j-pit, where $1 \leq i < j$. Then there are no hills between that i-hill and the j-pit; i.e., there are no s-hills such that $i < s < j$.

Then a pit closest to that i-hill on the right is admissible. Indeed, let such a pit be located in the k-column, i.e., it is a k-pit, where $i < k \leq j$. Then $\mu_{k-1} \geq \lambda_{k-1} \geq \lambda_k > \mu_k$, i.e., $\mu_{k-1} > \mu_k$. □

Lemma 4. *Let $\mu \geq \lambda$, and let an i-hill be the nearest hill to the left for an admissible j-pit, where $1 \leq j \leq t$. Then this i-hill is open, $\mu_i \geq 2 + \mu_j$, and there are no separators k for the partition μ such that $i < k < j$.*

P r o o f. Note first that $\mu_i > \lambda_i \geq \lambda_{i+1} \geq \mu_{i+1}$, so $\mu_i > \mu_{i+1}$, i.e., the i-hill is open. Moreover, $\mu_i > \lambda_i \geq \lambda_j > \mu_j$, hence $\mu_i \geq 2 + \mu_j$.

Let us show that there are no separators k for the partition μ such that $i < k < j$.

Consider a number k such that $i < k < j$. Since $\lambda_j > \mu_j$ and there are no s-hills such that $k < s < j$, we successively obtain

$$\sum_{p=1,\, \mu_p > \lambda_p}^{k} (\mu_p - \lambda_p) = \sum_{p=1,\, \mu_p > \lambda_p}^{j} (\mu_p - \lambda_p) \geq \sum_{p=1,\, \mu_p < \lambda_p}^{j} (\lambda_p - \mu_p) > \sum_{p=1,\, \mu_p < \lambda_p}^{k} (\lambda_p - \mu_p).$$

Therefore, the condition with the number k from (2.1) is a strict inequality, i.e., the number k is not a separator for the partition μ. □

Let $\mu \geq \lambda$, and let an i-hill be the nearest hill to the left for an admissible j-pit, where $1 \leq j \leq t$. Then, by Lemma 4, the following conditions are true:

(1) $\mu_i - 1 \geq \mu_{i+1}$;
(2) $\mu_{j-1} \geq \mu_j + 1$;
(3) $\mu_i \geq 2 + \mu_i$.

This is a necessary condition for a possibility of applying the box movement from i-column to the j-column of the partition μ.

The corresponding elementary transformation of the first type will be call a *moving the upper box into an admissible pit from the hill closest to it on the left.* By Lemma 2, such a transformation is proper.

Corollary 1. *Let μ' be a partition obtained from a partition μ by moving the upper box into an admissible pit from the hill closest to it on the left. Then $\mu' \geq \lambda$.*

Lemma 5. *Assume that $\mu \geq \lambda$, the last hill of the partition μ with respect to the partition λ has a number i, and μ' is a partition obtained from the partition μ by removing the upper box from i-hill. If $C = \operatorname{sum} \mu - \operatorname{sum} \lambda > 0$, then the last i-hill of the partition μ is open and $\mu' \geq \lambda$.*

P r o o f. Note that $\mu_i > \lambda_i \geq \lambda_{i+1} \geq \mu_{i+1}$; i.e., $\mu_i > \mu_{i+1}$, i.e., the i-column of the Ferrers diagram of the partition μ is open, and, therefore, an elementary transformation of the second type is applicable, which consists in removing of the upper box from i-hill. Let $C > 0$.

When passing from μ to μ' and replacing C by $C - 1$, the first condition of system (2.4) is preserved, since the sums decrease by 1 on the left and on the right.

When passing from μ to μ' and replacing C with $C - 1$, the second condition of system (2.4) for $k \leq i$ is preserved for the same reason.

Let $k > i$. Then the partition μ satisfies

$$0 = \sum_{j=k,\, \mu_j > \lambda_j}^{t} (\mu_j - \lambda_j) \leq \sum_{j=k,\, \mu_j < \lambda_j}^{t} (\lambda_j - \mu_j) + C.$$

Since $C > 0$ and $\mu_j = \mu'_j$ for $j \geq k$, we get

$$0 = \sum_{j=k,\, \mu'_j > \lambda_j}^{t} (\mu'_j - \lambda_j) \leq \sum_{j=k,\, \mu'_j < \lambda_j}^{t} (\lambda_j - \mu'_j) + (C - 1).$$

□

Lemmas 1 and 5 imply

Corollary 2. *Let μ' be a partition obtained from a partition μ by removing the top box from the last hill with number i. Then, for the partition μ, there are no separators k such that $i < k \leq t$.*

We now fix partitions μ and λ such that $\mu > \lambda$. Consider sequences of elementary transformations from μ to λ (both types of elementary transformations are admissible):

$$\mu = \xi_{(0)} \rightharpoondown \xi_{(1)} \rightharpoondown \cdots \rightharpoondown \xi_{(s)} = \lambda. \tag{2.5}$$

Let us construct now an algorithm for finding all possible shortest sequences of this kind.

Let $C = \operatorname{sum} \mu - \operatorname{sum} \lambda$. Since $\mu > \lambda$, we have $C \geq 0$. Obviously, in the sequence (2.5), there are exactly C elementary transformations of the second type, since elementary transformations of the second type reduce the weight of the partition by 1, and elementary transformations of the first type preserve the weight of the partition.

On the other hand, if some sequence of elementary transformations transforms μ into λ, then each box contained in any of the hills must be removed or moved. Therefore, the number of elementary transformations in such a sequence is not less than the sum of the heights of all hills. By (2.4), the sum of the heights of all hills is equal to C plus the sum of the depths of all pits. It is clear that all pits must be eliminated when passing from μ to λ in accordance with (2.5). Therefore, in sequence (2.5), there are at least p movements of the boxes to pits, where p is equal to the total depth of all pits. This implies that $s \geq C + p$.

The following algorithm constructs all shortest sequences of length $C + p$ of elementary transformations from μ to λ.

Algorithm 1. Let $\mu > \lambda$ and $C = \operatorname{sum} \mu - \operatorname{sum} \lambda$.

1. We set $\eta = \mu$ and $C' = C$.

2. To a current partition η and a number C', we apply any of the following proper elementary transformations for the partition η with respect to the partition λ:

   - if η has a pit, then we replace η with the partition obtained from η by moving the upper box from some open i-hill to some admissible j-pit for which there are no separators k such that $i < k < j$;

   - if $C' > 0$, then we replace C' with $C' - 1$ and replace the partition η with the partition obtained from η by removing the top box from some i-hill for which there are no separators k such that $i < k \leq t$.

3. Do step 2 as long as possible. The process will definitely end. The performed transformations will form the shortest sequence of elementary transformations from μ to λ. Its length is equal to the sum of the heights of all hills of the partition μ with respect to the partition λ and it is equal to
$$C + p = \frac{1}{2}C + \frac{1}{2}\sum_{j=1}^{\infty} |\mu_j - \lambda_j|.$$

Theorem 2. *Algorithm* 1 *is correct. Every shortest sequence of elementary transformations of the form* (2.5) *can be obtain by appropriate application of this algorithm.*

P r o o f. By Lemmas 1 and 2, all the constructed partitions η satisfy the condition $\eta \geq \lambda$.

If the current partition η satisfies $\eta > \lambda$, then, by Lemma 4 and Corollary 2, step 2 of the algorithm can be continued. It is clear that the algorithm will complete its work and the shortest sequence of the form (2.5) will be constructed. Its length is equal to the sum of the heights of all the hills of the partition μ with respect to the partition λ, i.e., it is equal to $C+p$. It is not difficult to see that
$$C + p = \frac{1}{2}C + \frac{1}{2}\sum_{j=1}^{\infty} |\mu_j - \lambda_j|.$$

Let sequence (2.5) be the shortest sequence of length $C + p$. There are C removals of boxes in this sequence. It must be eliminated all hills with the total height $C + p$, where p is equal to the total depth of all pits. Therefore, that sequence (2.5) consists of exactly C removals of boxes from hills and exactly p moves of boxes from hills. Because it must be eliminated all pits with the total depth p, every transformation $\xi_{(k-1)} \rightharpoondown \xi_{(k)}$ $(k = 1, \dots, s)$ consists in removing the box from a hill or consists in moving a box from a hill to a pit. Due to the conditions $\xi_{(k-1)} \geq \xi_{(k)} \geq \lambda$ and by Lemmas 1 and 2, every elementary transformation $\xi_{(k-1)} \rightharpoondown \xi_{(k)}$ is a proper elementary transformation with respect to the partition λ.

Therefore, a simple execution of Algorithm 1 along the shortest sequence (2.5) is correct. □

For applications the following special case of Algorithm 1 is useful which generally speaking does not construct all shortest sequences of the form (2.5).

Algorithm 2. [4] Let $\mu > \lambda$ and $C = \operatorname{sum}\mu - \operatorname{sum}\lambda$.

1. We set $\eta = \mu$ and $C' = C$.

2. Apply any of the following possible elementary transformations to a current partition η and an integer C':

   • if η has a pit, then we replace η with the partition obtained from η by moving the upper box into some admissible j-pit from the hill closest to it on the left;

   • if $C' > 0$, then we replace C' with $C' - 1$ and replace the partition η with the partition obtained from η by removing the top box from the last hill.

3. Do step 2 as long as possible. The process will definitely end. The performed transformations will form the shortest sequence of elementary transformations from μ to λ. Its length is equal to the sum of the heights of all hills of the partition μ with respect to the partition λ and is equal to
$$C + p = \frac{1}{2}C + \frac{1}{2}\sum_{j=1}^{\infty} |\mu_j - \lambda_j|.$$

Example 1. Let $\mu = (12, 8, 8, 4, 3, 2, 2, 2, 0, 0)$ and $\lambda = (10, 10, 6, 3, 3, 2, 1, 1, 1, 1)$. We have $\operatorname{sum}\mu = 41$, $\operatorname{sum}\lambda = 38$, and $C = 3$.

Consider the component-wise difference between μ and λ:

$$\begin{array}{rcccccccccc} \mu = & (12, & 8, & 8, & 4, & 3, & 2, & 2, & 2, & 0, & 0)^3, \\ \lambda = & (10, & 10, & 6, & 3, & 3, & 2, & 1, & 1, & 1, & 1), \\ \delta = \mu - \lambda = & (+2, & -2, & +2, & +1, & 0, & 0, & +1, & +1, & -1, & -1), \\ \Delta = & (2, & \underline{0}, & 2, & 3, & 3, & 3, & 4, & 5, & 4, & 3). \end{array}$$

Here, we have five hills and three pits, $t = 10$, and the height(μ, λ) is equal to the sum of the heights of all hills, i.e., it is equal to 7. At the end of the notation of μ at the top, the number 3 is indicated, which is equal to the number of boxes to be removed when working Algorithm 1.

Here, the sequence $\Delta = (\Delta_1, \Delta_2, \ldots, \Delta_{10})$ is given by the condition

$$\Delta_k = (\mu_1 + \cdots + \mu_k) - (\lambda_1 + \cdots + \lambda_k),$$

i.e., $\Delta_k = \Delta_{k-1} + \delta_k$. It is clear that the condition $\mu \geq \lambda$ is equivalent to the fact that $\Delta_k \geq 0$ for any $k = 1, 2, \ldots, t$. The sequence Δ has only one zero $\Delta_2 = 0$. This zero underlined below in Δ. Hence, there is exactly one separator (the integer 2) for μ with respect to λ.

Note that the 1-hill is open. Using Algorithm 1, we cannot remove boxes from the 1-hill ("move them across the separator"). We can move boxes from the 1-hill only to the admissible 2-pit.

The 9-pit is admissible but the 10-pit is not. We can move a box from any hills with numbers 3, 4, and 8 to the 9-pit since they are open and there are no separators between these hills and the 9-pit. Note that the 7-hill is not open.

Let us move the box from the open 3-hill to the admissible 9-pit:

$$\begin{array}{rcccccccccc} \mu = & (12, & 8, & 7, & 4, & 3, & 2, & 2, & 2, & 1, & 0)^3, \\ \lambda = & (10, & 10, & 6, & 3, & 3, & 2, & 1, & 1, & 1, & 1), \\ \delta = \mu - \lambda = & (+2, & -2, & +1, & +1, & 0, & 0, & +1, & +1, & 0, & -1), \\ \Delta = & (2, & \underline{0}, & 1, & 2, & 2, & 2, & 3, & 4, & 4, & 3). \end{array}$$

Note that, after such a transformation, the 10-pit became admissible.

Now let us remove the box from the 3-hill, which remained open:

$$\begin{array}{rcccccccccc} \mu = & (12, & 8, & 6, & 4, & 3, & 2, & 2, & 2, & 1, & 0)^2, \\ \lambda = & (10, & 10, & 6, & 3, & 3, & 2, & 1, & 1, & 1, & 1), \\ \delta = \mu - \lambda = & (+2, & -2, & 0, & +1, & 0, & 0, & +1, & +1, & 0, & -1), \\ \Delta = & (2, & \underline{0}, & \underline{0}, & 1, & 1, & 1, & 2, & 3, & 3, & 2). \end{array}$$

Note that another separator has appeared — the integer 3. In addition, we have replaced the counter value of the number of boxes to be deleted by 2.

Continuing to apply elementary transformations in the same spirit according to Algorithm 1, we will find some shortest sequence of elementary transformations of length 7 that transforms μ into λ.

Note that we can remove a box from an open hill at any step of the algorithm execution if the value of the counter of deleted boxes is greater than zero; it is only important that there is no any separator to the right of the hill.

An example of an operation of Algorithm 2 see in [4].

3. Conclusion

We note that finding the shortest chains of elementary transformations is an important problem in studying the properties of graphic partitions. The use of Algorithm 2 allowed us to obtain several interesting properties of graphic partitions (see, for example, [5, 6]). Using Algorithm 1 opens up more possibilities.

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