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COUNTABLE COMPACTNESS MODULO AN IDEAL OF NATURAL NUMBERS

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Abstract: In this article, we introduce the idea of I-compactness as a covering property through ideals of $\mathbb{N}$ and regardless of the I-convergent sequences of points. The frameworks of s-compactness, compactness and sequential compactness are compared to the structure of I-compact space. We began our research by looking at some fundamental characteristics, such as the nature of a subspace of an I-compact space, then investigated its attributes in regular and separable space. Finally, various features resembling finite intersection property have been investigated, and a connection between I-compactness and sequential I-compactness has been established.

Keywords: Ideal, Open cover, Compact space, I-convergence.

1. Introduction

The concept of statistical convergence ([17], also [21]) depends on the idea of natural density of subsets of the set of natural number $\mathbb{N}$. The density of a set $S \subseteq \mathbb{N}$ is denoted by $\delta(S)$ and defined as

$$\delta(S) = \lim_{n\to\infty} \frac{1}{n}\big|\{k \le n : \ k \in S\}\big|.$$

Later on, Maio and Kočinak [14] redefine the statistical convergence for a topological space. On the recent days, one of the most significant study area in pure mathematics is the ideal convergence which is an extension of statistical convergence and other convergence concepts. To define the ideal convergence Kostyrko et al. [19] used the notion of an ideal which is defined as $I \subseteq \mathcal{P}(\mathbb{N})$ having the following properties:

(i) $\varnothing \in I$;
(ii) $A \cup B \in I$, for each $A, B \in I$;
(iii) for each $A \in I$ and $B \subseteq A \Rightarrow B \in I$.

In a topological space (X, τ), a sequence $x = (x_n)$ is said to be I-convergent if there exists a ℓ such that for every open neighbourhood U of ℓ, the set [20]

$$\{n \in \mathbb{N} : x_n \notin U\} \in I.$$

This idea is being studied and used broadly by many researchers [1, 10, 12, 13].

On the other hand, the study of different covering properties (some recent works [7, 9, 11]) has received a lot of attention in topology. A family $\mathcal{U}$ of open subsets of topological space X is called an open cover of X if $\cup\mathcal{U} = X$. A topological space X is called compact if every open cover of X has a finite subcover. More specifically, it has became very essential to explore the structure of compactness and its generalized versions for topological spaces.

The star operator was introduce by E.K. van Douwen in 1991 [15] as

$$\mathrm{St}\,(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \varnothing\}$$

where A is a subset of space X and $\mathcal{U}$ is a family of subsets of X. Using star operator the concept of compactness has been generalized in many ways and has been studied by many authors extensively [2–6]. In our study we make an attempt to expand this region with the help of Ideal.

In recent days sequential compactness via ideal has been introduced by Singha and Roy [22] under the name of I-compactness, and I-compactness module via an ideal defined on X has also been studied by Gupta and Kaur [18] under the same name.

The purpose of our study is to explore the concept of compactness via ideal of natural number $\mathbb{N}$. We also establish a relation between sequential I-compactness and I–compactness.

2. Preliminaries

Throughout the paper a space X means a topological space with the corresponding topology τ, '$\because$' stands for 'therefore' and for other symbols and notions we follow [16].

Definition 1 [16]. *A topological space X is called a compact space if every open cover of X has a finite subcover, i.e., if for every open cover $\{U_s\}_{s\in\mathbb{S}}$ of the space X there exists a finite set $\{s_1, s_2, ..., s_k\} \subset \mathbb{S}$ such that $X = U_{s_1} \cup U_{s_2} \cup ... \cup U_{s_k}$.*

Definition 2 [16]. *A topological space X is called a Lindelöf space if every open cover of X has a countable subcover.*

It is known that every compact space is Lindelöf but the converse is not true.

Definition 3 [16]. *A topological space (X,τ) is called a countably compact space if every countable open cover of X has a finite sub-cover.*

Every compact space is countably compact. But the space W_0 of all countable ordinals is countably compact but not compact [16]. The space $\mathbb{N}$ of all natural numbers equipped with discrete topology is Lindelöf but it is not countably compact.

Definition 4 [16]. *A topological space (X,τ) is said to be sequentially compact space if every sequence in X has a convergent subsequence.*

Definition 5 [15]. *A topological space (X,τ) will be called a star compact space (in short* St*-compact) if for every open cover $\mathcal{U}$ of X, there exists a finite subset $\mathcal{U}' = \{U_k : k = 1, 2, 3, \dots, m\}$ such that*

$$\mathrm{St}\Big(\bigcup_{n=1}^{m} U_n, \mathcal{U}\Big) = X.$$

Definition 6 [8]. *A topological space (X,τ) will be called a statistical compact (in short s-compact) space if for every countable open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of X, there exists a sub-cover $\mathcal{V} = \{U_{m_k} : k \in \mathbb{N}\}$ of $\mathcal{U}$ such that $\delta\big(\{m_k : U_{m_k} \in \mathcal{V}\}\big) = 0$.*

3. Compactness via ideal

Definition 7. *Let I be a non-trivial ideal defined on $\mathbb{N}$. A topological space (X,τ) will be called an I-compact space if for every countable open cover $\mathcal{U} = \{A_n : n \in \mathbb{N}\}$ of X, there exists a sub-cover $\mathcal{V} = \{A_{n_k} : k \in \mathbb{N}\}$ such that $\{n_k : A_{n_k} \in \mathcal{V}\} \in I$.*

Remark 1. Countable compactness is equivalent to I_{fin}-compactness where I_{fin} indicates the ideal of all finite subsets of $\mathbb{N}$.

Proposition 1. *Every I_{fin}-compact space is a s-compact space.*

P r o o f. Let (X,τ) be an I_{fin}-compact space and $\mathcal{U} = \{A_n : n \in \mathbb{N}\}$ be a countable open cover of X. Therefore there exists a sub-cover $\mathcal{V} = \{A_{n_k} : k \in \mathbb{N}\}$ of $\mathcal{U}$ with $\{n_k : n_k \in \mathcal{V}\} \in I_{fin}$. i.e. $\mathcal{V} = \{A_{n_k} : k \in \mathbb{N}\}$ is a finite sub-cover of X. But the finite set of indices $\{n_k : A_{n_k} \in \mathcal{V}\}$ has natural density zero, i.e. $\delta(\{n_k : A_{n_k} \in \mathcal{V}\}) = 0$, $\therefore X$ is a s-compact space. □

Example 1. Converse of Proposition 1 may not be true. Indeed there exists a s-compact space which is not I_{fin}-compact.

Let $X = (-1,1)$ and $\tau = \{(-\alpha,\alpha) : \alpha \in [0,1]\}$. Clearly (X,τ) is a topological space. Consider a countable open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of X. If $X \in \mathcal{U}$, then $X = U_p$ for some $p \in \mathbb{N}$ and $\mathcal{V} = \{U_p\}$ is a sub-cover of $\mathcal{U}$ with $\delta(\{k : U_k \subset \mathcal{V}\}) = \delta(\{p\}) = 0$ and we are done.

Now let $X \notin \mathcal{U}$ and $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is a non-trivial countable open cover of X. We consider the sub-cover $\mathcal{U}' = \{U_{n_k} : k \in \mathbb{N}\}$, where

$$U_{n_k} = \begin{cases} U_k, & k = 1, \\ \bigcup_{n \le n_k} U_n, & \text{when } \bigcup_{n \le n_k} U_n \text{ becomes a superset of } \quad U_{n_{k-1}} \text{ for } \quad k > 1. \end{cases}$$

Now, $\{U_{n_k} : k \in \mathbb{N}\}$ is an increasing sequence of open sets by means of inclusion $(\subseteq)$ and is an open cover of X. It also has a sub-cover $\mathcal{V} = \{U_{n_{k^2}} : k \in \mathbb{N}\}$ with $\delta(\{n_{k^2} : U_{n_{k^2}} \in \mathcal{V}\}) = 0$. Moreover $\mathcal{V}$ is a subset of $\mathcal{U}$, $\therefore X$ is a s-compact space.

Again suppose that (X,τ) is I_{fin}-compact and consider the countable open cover

$$\mathcal{W} = \Big\{W_n = \Big(-1 + \frac{1}{n}, 1 - \frac{1}{n}\Big) : \ n \in \mathbb{N}\Big\}.$$

Since X is I_{fin}-compact there exists a sub-cover of $\mathcal{W}$, say $\mathcal{W}' = \{W_{n_k} : k = 1,2,...,q\}$ with $\{n_k : W_{n_k} \in \mathcal{W}'\} \in I_{fin}$.

Suppose $n_{k_{max}} = \max\{n_k : n_k \in \mathcal{W}'\}$ then we have

$$\therefore \bigcup \mathcal{W}' = \Big(-1 + \frac{1}{n_{k_{max}}}, 1 - \frac{1}{n_{k_{max}}}\Big) \neq X,$$

which is a contradiction. So X is not I_{fin}-compact.

Corollary 1. *Every Lindelöf I_{fin}-compact space is a compact space.*

P r o o f. By Lindelöfness, every open cover has a countable sub-cover. By I_{fin}-compactness, that countable sub-cover will have a finite sub-cover. Hence it will be a compact space. □

Theorem 1. *Every closed subspace of an I-compact space is an I-compact.*

P r o o f. Let (A, τ_A) be an arbitrary closed subspace of a I-compact space (X, τ) and $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be a countably infinite cover of (A, τ_A). Then there exists a countable sequence $\mathcal{V} = \{V_n : n \in \mathbb{N}\} \in \tau$ such that $U_n = V_n \cap A$.

Now consider a countably infinite sequence $\mathcal{W} = \{(X \setminus A) \cup V_n : n \in \mathbb{N}\}$, which is a cover of X. Since (X, τ) is I-compact space, $\therefore \exists$ a sub-cover $\mathcal{W}' = \{(X \setminus A) \cup V_{n_k} : k \in \mathbb{N}\}$ of $\mathcal{W}$ with $\{n_k : (X \setminus A) \cup V_{n_k} \in \mathcal{W}'\} \in I$. Again, $\cup V_{n_k} \supseteq A$ then

$$A \cap (\cup V_{n_k}) = A \implies \cup(A \cap V_{n_k}) = A \implies \cup(U_{n_k}) = A, \quad \therefore \cup U_{n_k} \subseteq \cup U_n = \mathcal{U}$$

and $\mathcal{U}$ is a countably infinite cover of (A, τ_A). So $\{\cup U_{n_k} : k \in \mathbb{N}\}$ is a countably sub-cover of (A, τ_A) with $\{n_k : U_{n_k} \in \mathcal{U}\} \in I$.

Therefore (A, τ_A) is the I-compact space. □

Theorem 2. *Let (X, τ) be a I-compact space and (Y, σ) be a topological space. If $f : (X, \tau) \to (Y, \sigma)$ is the open continuous surjection mapping, then (Y, σ) is also the I-compact space.*

P r o o f. Let $f : (X, \tau) \to (Y, \sigma)$ be an open continuous surjection mapping and (X, τ) is I-compact space.

Let $\{U_n : n \in \mathbb{N}\}$ be a countable open cover of Y. So,

$$\cup\{U_n : n \in \mathbb{N}\} = Y \implies f^{-1}\big\{\cup\{U_n : n \in \mathbb{N}\}\big\} = f^{-1}(Y) \implies \cup\{f^{-1}(U_n) : n \in \mathbb{N}\} = X.$$

Since f is a continuous surjection mapping and U_n is a countable open cover of Y then

$$\{f^{-1}(U_n) : n \in \mathbb{N}\} = \mathcal{V}$$

is a countable open cover of X. Again (X, τ) is I-compact space then there exists a sub-cover, $\{f^{-1}(U_{n_1}), f^{-1}(U_{n_2}), ...\}$ of $\mathcal{V}$ where $\{n_k : k \in \mathbb{N}\} \in I$,

$$\cup\{f^{-1}(U_{n_k}) : k \in \mathbb{N}\} = X,$$
$$f\big[\cup\{f^{-1}(U_{n_k}) : k \in \mathbb{N}\}\big] = f(X) = Y,$$
$$\cup\{U_{n_k} : k \in \mathbb{N}\} = Y, \quad \because f[f^{-1}(U_{n_k})] = U_{n_k},$$

$\therefore \{U_{n_k} : k \in \mathbb{N}\}$ is a countable sub-cover of $\{U_n : n \in \mathbb{N}\}$ where $\{n_k : k \in \mathbb{N}\} \in I$, (y, σ) is also I-compact space. □

Definition 8. *Let (X, τ) be a topological space and I be a ideal on $\mathbb{N}$. A subset $A \subseteq X$ will be called I-compact subset of X if for every countable cover $\{U_n : n \in \mathbb{N}\}$ of A by elements of τ there exists a $S \in I$ such that $A \subseteq \bigcup_{n \in S} U_n$.*

Theorem 3. *In a regular space (X, τ), if A is countable I_{fin}-compact subset of X, then for every closed set B disjoint from A there exists $U, V \in \tau$ such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \varnothing$.*

P r o o f. Let $A = \{x_n : n \in \mathbb{N}\}$ be a countable I_{fin}-compact subset of a regular space (X, τ) and B be an arbitrary closed set disjoint from A, $\therefore$ for every $x_n \in A$, $x_n \notin B$. But X is a regular space. Therefore there exists $U_n, V_n \in \tau$ such that $x_n \in U_n$, $B \subseteq V_n$ and $U_n \cap V_n = \varnothing$ $\forall n \in \mathbb{N}$.

It is obvious that $\{U_n : n \in \mathbb{N}\}$ is a countable open cover of A by the elements of τ. But A is an I_{fin}-compact subset of X, $\therefore$ there exist $S \in I_{fin}$ such that $A \subseteq \bigcup_{n \in S} U_n$.

Now $S \in I_{fin}$ is a finite set, $\therefore U = \bigcup_{n\in S} U_n \in \tau$ and $V = \bigcap_{n\in S} V_n \in \tau$ and $A \subseteq U$ and $B \subseteq V$. So we have to show that $U \cap V = \varnothing$. On the contrary suppose $U \cap V \neq \varnothing$ and $p \in U \cap V$

$$\begin{aligned} &\implies p \in (\bigcup_{n\in S} U_n) \cap (\bigcap_{n\in S} V_n) \implies p \in \bigcup_{n\in S} U_n \quad \text{and} \quad p \in \bigcap_{n\in S} V_n \\ &\implies p \in U_k \quad \text{for some} \quad k \in S \quad \text{and} \quad p \in V_k \quad \forall k \in S \\ &\implies p \in U_k \quad \text{and} \quad p \in V_k \quad \text{for some} \quad k \in S \\ &\implies p \in U_k \cap V_k \quad \text{for some} \quad k \in S \subseteq \mathbb{N}, \end{aligned}$$

which is a contradiction to the fact that

$$U_n \cap V_n = \varnothing \quad \forall n \in \mathbb{N}, \quad \therefore U \cap V = \varnothing.$$

Hence the theorem is proven. □

Corollary 2. *If A is a countable I_{fin}-compact subset of a Hausdörff space X, then for every $x \notin A$ there exist $U, V \in \tau$ such that $A \subseteq U, x \in V$ and $U \cap V = \varnothing$.*

P r o o f. In a Hausdörff space, every singleton set $\{x\}$ is a closed set. So by Theorem 3 the result follows directly. □

Definition 9. *A topological space (X, τ) will be called sequentially I-compact if every sequence of elements of X has a I-convergent subsequence.*

Theorem 4. *A separable I_{fin}-compact space is a st-compact space*

P r o o f. Let (X, τ) be a separable I_{fin}-compact space. Therefore there exists a countable dense subset $A = \{x_n : n \in \mathbb{N}\}$ of X and $\mathcal{U}$ being an arbitrary open cover of X. Using the elements of $\mathcal{U}$ we construct a sequence of open sets $\{U_n : n \in \mathbb{N}\}$ where $U_n = \bigcup\{U \in \mathcal{U} : x_n \in U\}$ for all $n \in \mathbb{N}$. But A is a dense subset of X, $\therefore A \cap U \neq \varnothing \quad \forall U \in \mathcal{U}$.

$\therefore \mathcal{U}' = \{U_n : n \in \mathbb{N}\}$ is a countable open cover of X. But X is I_{fin}-compact, $\therefore \exists S \in I_{fin}$ such that $\bigcup_{n_k\in S} U_{n_k} = X$. But $S = \{U_{n_1}, U_{n_2}, ..., U_{n_p}\}$ is a finite subset of $\mathbb{N}$. Therefore

$$\bigcup_{k=1}^{n} U_{n_k} = X.$$

But $x_{n_k} \in U_{n_k} \quad \forall k = 1, 2, ..., p$, therefore $F = \{x_{n_k} : k = 1, 2, ..., p\}$ is a finite subset of X and

$$\begin{gathered} \mathrm{St}\,(F, \mathcal{U}') \supseteq \bigcup_{k=1}^{n} U_{n_k} = X, \\ \mathrm{St}\,(F, \mathcal{U}') \supseteq \bigcup_{k=1}^{n} \Big\{ \bigcup\{U \in \mathcal{U} : x_{n_k} \in U\} \Big\} = X, \\ \mathrm{St}\,(F, \mathcal{U}) \supseteq \mathrm{St}\,(F, \mathcal{U}') \supseteq X, \\ \mathrm{St}\,(F, \mathcal{U}) = X, \end{gathered}$$

$\therefore X$ is a St-compact space. □

Definition 10. *Let I be a ideal on $\mathbb{N}$. A family $\mathcal{F} = \{F_n : n \in \mathbb{N}\}$ of subsets of a space X is said to have I-intersection property if $\mathcal{F} \neq \varnothing$ and $\bigcap_{n\in S} F_n \neq \varnothing$ for all $S \in I$.*

Theorem 5. *For a topological space (X, τ) and for a non trivial ideal I the following statements are equivalent*:

(1) *For a family $\mathcal{G} = \{G_n : n \in \mathbb{N}\}$ of open sets of X, if for every $S \in I$, $\mathcal{G}_S = \{G_{n_\alpha} : n_\alpha \in S\}$ fails to cover X, then $\mathcal{G}$ can not cover X.*
(2) *X is an I-compact space.*
(3) *Every family of countable closed subsets of X with I-intersection property has non-empty intersection.*
(4) *For a family $\mathcal{F} = \{F_n : n \in \mathbb{N}\}$ of closed subsets of X, if $\bigcap \mathcal{F} = \varnothing$, then there exists at least one $S \in I$ such that $\bigcap_{n \in S} F_n = \varnothing$.*

P r o o f. (1) $\Leftrightarrow$ (2): The statement (1) is the contrapositive statement of the definition of I-compact space, $\therefore$ statement (1) and (2) are equivalent.

(2) $\Leftrightarrow$ (3): Let X be an I-compact space, $\mathcal{H} = \{H_n : n \in \mathbb{N}\}$ be a arbitrary family of closed subsets of X having I-intersection property and suppose that $\bigcap \mathcal{H} = \varnothing$.

Let

$$\mathcal{G} = \{G_n = X \setminus H_n : n \in \mathbb{N}\},$$

then

$$\bigcup \mathcal{G} = \bigcup_{n \in \mathbb{N}} (G_n = X \setminus H_n) = X \setminus \bigcap_{n \in \mathbb{N}} H_n = X \setminus \varnothing = X,$$

$$\therefore \mathcal{G} = \{G_n : n \in \mathbb{N}\}$$

is an countable open cover of X. But X is I-compact. Therefore, there exists a $S \in I$ such that

$$\bigcup_{n \in S} G_n = X,$$

$$\implies \bigcup_{n \in S} \{X \setminus H_n\} = X \quad [\because G_n = X \setminus H_n \quad \forall n \in \mathbb{N}] \implies X \setminus \bigcap_{n \in S} H_n = X \implies \bigcap_{n \in S} H_n = \varnothing,$$

which is a contradiction to the fact that $\mathcal{H}$ has I-intersection property, $\therefore \bigcap \mathcal{H} \neq \varnothing$.

Conversely let every countable family of closed subsets with I-intersection property has non-empty intersection.

Let $\mathcal{U} = \{B_n : n \in \mathbb{N}\}$ is an arbitrary countable open cover of X, $\therefore \mathcal{F} = \{F_n = X \setminus B_n : n \in \mathbb{N}\}$ is a family of closed subsets of X and

$$\bigcap \mathcal{F} = \bigcap_{n \in \mathbb{N}} X \setminus B_n = X \setminus \bigcup_{n \in \mathbb{N}} B_n = X \setminus X = \varnothing.$$

Thus the countable family $\mathcal{F}$ of closed subsets of X has empty intersection. So it can not have I-intersection property by our assumption. Therefore, there exists a $S \in I$ such that

$$\bigcap_{n \in S} F_n = \varnothing \implies \bigcap_{n \in S} X \setminus B_n = \varnothing \implies X \setminus \bigcup_{n \in S} B_n = \varnothing \implies \bigcup_{n \in S} B_n = \varnothing,$$

$\therefore \mathcal{V} = \{B_n : n \in S\}$ is a sub-cover of $\mathcal{U}$ and $S \in I$. Therefore X is an I-compact space.

(3) $\Leftrightarrow$ (4): Statement (3) and statement (4) are contrapositive to each other. Therefore, statement (3) and statement (4) are equivalent. □

Theorem 6. *Sequentially I_{fin}-compactness implies I_{fin}-compactness.*

P r o o f. Let (X, τ) be an I_{fin}-compact space. Then for every sequence $\{x_n : n \in \mathbb{N}\}$ of elements of X, there exists is a subsequence $\{x_{n_k} : n \in \mathbb{N}\}$ which is I-convergent. Let

$$I - \lim_{n \to \infty} x_n = \epsilon \in X.$$

On the other hand let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be a countable open cover of X. So $\exists U_m \in \mathcal{U}$ such that $\epsilon \in U_m$. Also $\{n_k \in \mathbb{N} : x_{n_k} \notin U_m\} \in I_{fin}$, suppose $\{n_{k_1}, n_{k_2}, ..., n_{k_p}\} = \{n_k \in \mathbb{N} : x_{n_k} \notin U_m\}$.

But $\mathcal{U}$ is a open cover of X. Therefore there exist

$$x_{n_{k_1}} \in U_{q_1} \in \mathcal{U},$$
$$x_{n_{k_2}} \in U_{q_2} \in \mathcal{U},$$
$$...$$
$$x_{n_{k_p}} \in U_{q_p} \in \mathcal{U}.$$

Now, the collection $\{U_m, U_{q_1}, U_{q_2}, ..., U_{q_p}\}$ is a sub-cover of $\mathcal{U}$ and $\{m, q_1, q_2, ..., q_p\}$ is a finite subset of $\mathbb{N}$, i.e. $\{m, q_1, q_2, ..., q_p\} \in I_{fin}$, $\therefore (X, \tau)$ is a I_{fin}-compact space. □

4. Conclusion

The paper reveals that I_{fin}-compactness is stronger covering property than statistical compactness, closed subspace of an I-compact space is I-compact, open continuous surjection of an I-compact space is I-compact, a separable I-compact space is a star-compact space. A topological space is I-compact if and only if every family of countable closed subsets of the space which has I-intersection property has non-empty intersection. This study can further be extended for the covering properties like Mengerness and Rothbergerness in the context of modulo an ideal of natural numbers.

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