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ON ONE INEQUALITY OF DIFFERENT METRICS FOR TRIGONOMETRIC POLYNOMIALS

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Abstract: We study the sharp inequality between the uniform norm and $L^p(0,\pi/2)$-norm of polynomials in the system $\mathscr{C}=\{\cos(2k+1)x\}_{k=0}^{\infty}$ of cosines with odd harmonics. We investigate the limit behavior of the best constant in this inequality with respect to the order n of polynomials as $n\to\infty$ and provide a characterization of the extremal polynomial in the inequality for a fixed order of polynomials.

Keywords: Trigonometric cosine polynomial in odd harmonics, Nikol'skii different metrics inequality.

1. Problem statement, backgroung, and some preliminaries

1.1. Some notation

This paper considers classical spaces of complex-valued functions of one variable; in fact, we use not the spaces themselves but the norms of these spaces on some subspaces of polynomials. Let $I=[a,b]$ be an interval on the real axis, and let υ be a nonnegative, integrable function on I called a weight. For $0<p<\infty$, the space $L^p_\upsilon=L^p_\upsilon(I)$ consists of complex-valued, Lebesgue measurable on I functions f such that the function $\upsilon|f|^p$ is integrable on I. The functional

$$\|f\|_p=\|f\|_{L^p_\upsilon(I)}=\left(\frac{1}{b-a}\int_a^b|f(x)|^p\upsilon(x)\,dx\right)^{1/p},\quad f\in L^p_\upsilon, \tag{1.1}$$

is a norm in the space $L^p_\upsilon=L^p_\upsilon(I)$ for $1\le p<\infty$, but not for $0<p<1$. Nevertheless, for all $0<p<\infty$, we will refer to (1.1) as a norm or, more precisely, as a p-norm. The space $L^2_\upsilon=L^2_\upsilon(I)$ (here $p=2$) is a Hilbert space with the inner product

$$\langle f,g\rangle=\frac{1}{b-a}\int_a^b f(x)\overline{g(x)}\upsilon(x)\,dx,\quad f,g\in L^2_\upsilon.$$

In the case of unit weight $v(x) \equiv 1$, the weight symbol is omitted in the notation of spaces and their norms. By $L^\infty = L^\infty(I)$ we mean the space $C = C(I)$ of functions continuous (bounded) on the interval I with the uniform norm

$$\|f\|_\infty = \|f\|_{C(I)} = \max\big\{|f(x)|\colon x \in (I)\big\}.$$

The space $C = C(I)$ contains the subspace $C_0 = C(I)_0$ of functions f vanishing at the right end point of the interval: $f(b) = 0$. In what follows, the parameter a is equal to zero: $a = 0$, and b is 1, $\pi/2$, or π depending on the situation.

We define the spaces of 2π-periodic functions accordingly: the spaces $\mathcal{L}^p_{2\pi}$, $0 < p < \infty$, with p-norm

$$\|f\|_p = \|f\|_{\mathcal{L}^p_{2\pi}} = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^p dx\right)^{1/p}$$

and the space $C_{2\pi}$ of continuous 2π-periodic functions with the uniform norm

$$\|f\|_\infty = \|f\|_{C_{2\pi}} = \max\{|f(x)|\colon x \in \mathbb{R}\} = \max\{|f(x)|\colon x \in [-\pi, \pi]\}.$$

1.2. Nikol'skii C–L^p inequality: the classical case

Let $\mathcal{F}_n = \mathcal{F}_n(\mathbb{C})$, $n \geq 1$, be the set of trigonometric polynomials

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

of order (at most) n with complex coefficients. In function theory and its applications, inequalities between two different norms of polynomials are of great importance. Such inequalities arose in Jackson's paper [19], but they were thoroughly investigated and applied by Nikol'skii [25]; [26, Ch. 3, Sect. 3.3], in this connection, they are called Nikol'skii inequalities or inequalities of different metrics. Large studies have been devoted to such inequalities, see [25]; [26, Ch. 3, Sect. 3.3]; [21, Ch. 3, Sects. 3.5–3.6]; [11, Ch. 8, Sect. 8.4]; [17, 24] and the bibliography therein. In this paper, the authors will need some information about the Nikol'skii inequalities

$$\|f_n\|_\infty \leq C(n)_p \|f_n\|_p, \quad f_n \in \mathcal{F}_n, \tag{1.2}$$

between the uniform norm and p-norm of polynomials (see, the same sources [21, Ch. 3, Sects. 3.5, 3.6]; [17, 24]). We assume that $C(n)_p$ is the best (the smallest possible) constant in this inequality. Employing harmonic analysis, it is easy to obtain (see, for example, [21, Ch. 3, Sect. 3.5, Theorem 3.5.1]) that if $p = 2$, then

$$C(n)_2 = \sqrt{2n+1} \tag{1.3}$$

and inequality (1.2) becomes an equality at the Dirichlet kernel

$$D_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \cos kx, \tag{1.4}$$

i.e., the Dirichlet kernel is an extremal polynomial. The exact values of $C(n)_p$ for $p \neq 2$ are unknown. There are constructive estimates for $C(n)_p$, $0 < p < \infty$, mostly upper ones; see [11, Ch. 8, Sect. 8.4]; [1]; [12, Sect. 7.2]; [15–18] and the bibliography therein. Note for the future the upper estimate of Badkov [12, Sect. 7.2, Theorem 7.2]

$$C(n)_p \leq 4n^{1/p}, \quad 0 < p < \infty. \tag{1.5}$$

It is not the best at the moment, but sufficient for us in what follows.

Much research has been devoted to inequality (1.2) for $p = 1$; information about the history and results related to this inequality can be found in [10, 15–17, 29, 30]. The following estimates are simple and quite rough:

$$n + 1 \le C(n)_1 \le 2n + 1. \tag{1.6}$$

The upper estimate follows from the representation of polynomials $f_n \in \mathcal{F}_n$ in the form of convolution

$$f_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f_n(t)D_n(x-t)dt, \quad f_n \in \mathcal{F}_n,$$

with the Dirichlet kernel (1.4). The Fejér kernel

$$F_n(t) = \sum_{k=0}^{n} D_k(t) =$$
$$= \frac{n+1}{2} + \sum_{k=1}^{n}(n+1-k)\cos kt = \frac{1}{2}\left(\frac{\sin(n+1)\frac{t}{2}}{\sin\frac{t}{2}}\right)^2, \quad t \ne 2\nu\pi, \quad \nu \in \mathbb{Z},$$

provides the former inequality in (1.6). This kernel is nonnegative (see, for example, [27, Vol. 2, Part 6, Sect. 3, Problem 18]) and such that

$$\|F_n\|_\infty = F_n(0) = \frac{(n+1)^2}{2}, \quad \|F_n\|_1 = \frac{1}{\pi}\int_0^{\pi} F_n(t)dt = \frac{n+1}{2}.$$

This implies the former inequality in (1.6).

Taikov in [29] gives the result of S.B. Stechkin that (for $p = 1$) there exists a constant $c > 0$ such that

$$C(n)_1 = c\,n + o(n), \quad n \to \infty. \tag{1.7}$$

Stechkin's proof of this result is given in [30]. Estimates (1.6) imply that $1 \le c \le 2$. Taikov [29] obtained substantially closer two-sided bounds for the constant c. Hörmander and Bernhardsson have obtained [14] the best estimates currently:

$$1.081857643 \le c \le 1.081857645.$$

Let $\mathcal{E}(\sigma)$ be the space of entire functions of exponential type (at most) $\sigma > 0$, and let $\mathcal{E}(\sigma)_p$ for $0 < p \le \infty$ be the space of functions $f \in \mathcal{E}(\sigma)$ belonging on the real axis to the spaces $L^p = L^p(\mathbb{R})$ with finite norms

$$\|f\|_p = \|f\|_{L^p(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(x)|^p dx\right)^{1/p}, \quad 0 < p < \infty,$$
$$\|f\|_\infty = \|f\|_{C(\mathbb{R})} = \sup\{|f(x)| \colon x \in \mathbb{R}\}, \quad p = \infty.$$

For any $0 < p < \infty$ on $\mathcal{E}(\sigma)_p$, we have the inequality

$$\|f\|_\infty \le A_p \sigma^{1/p}\|f\|_p, \quad f \in \mathcal{E}(\sigma)_p, \tag{1.8}$$

ascending to Nikol'skii [25]; [26, Ch. 3, Sect. 3.3], in which A_p is a finite constant depending only on the parameter p; see details in [15, 16]. In what follows, we assume that A_p is the least possible, i.e., best constant in (1.8). The exact value of this quantity is currently known only for $p = 2$; namely, $A_2 = 1/\sqrt{\pi}$; see, for example, [31, Ch. IV, Sect. 4.9, Subsect. 4.9.53, (28)].

Gorbachev [16] obtained a significantly more informative assertion in comparison with (1.7). Namely, he proved that the following limit relation is true:

$$\lim_{n\to\infty} \frac{C(n)_1}{2\pi n} = A_1. \tag{1.9}$$

Actuality, Gorbachev obtained [16] an even more precise result; namely, he proved the following two-sided inequality:

$$A_1 \le \frac{C(n)_1}{2\pi n} \le \frac{n+1}{n} A_1,$$

which entails, in particular, (1.9).

The following statement belongs to Levin and Lubinsky [22, Theorem 2.1, (2.1)]; it means that a relation similar to (1.9) holds for all $0 < p < \infty$.

Theorem A. *The following limit relation is valid for* $0 < p < \infty$:

$$\lim_{n\to\infty} \frac{C(n)_p}{(2\pi n)^{1/p}} = A_p. \tag{1.10}$$

This statement is part of a more general result of Ganzburg and Tikhonov [15, Theorem 1.5, (1.19)]. Nikol'skii inequalities and, more generally, Bernstein–Nikol'skii inequalities

$$\|f_n^{(r)}\|_q \le C^{(r)}(n)_{p,q} \|f_n\|_p, \quad f_n \in \mathcal{F}_n, \tag{1.11}$$

for trigonometric polynomials and similar inequalities for entire functions are a broad area of function theory. Ganzburg and Tikhonov, in the already cited paper [15], and Gorbachev and Mart'yanov in [18] studied the relationship between exact constants in Bernstein–Nikol'skii (C, L^p)-inequalities and, more generally, (L^q, L^p)-inequalities for polynomials and entire functions of exponential type. We presented here in Theorem A only some results on this topic, which we will need in what follows; for a rich overview of these studies, see [17].

Over the last century, extensive investigations have been carried out on sharp inequalities, i.e., the study of exact constants and extremal functions in inequalities (1.11) for trigonometric polynomials, as well as for algebraic polynomials and entire functions of exponential type; for specific results and further references, see [1, 2, 10, 12, 13, 15, 17, 21, 23, 24, 28, 31].

1.3. Nikol'skii inequality between the uniform norm and L^p-norm on the interval $[0, \pi/2]$ for polynomials in the cosine system with odd harmonics

1.3.1. Nikol'skii inequality for $\mathscr{C}_n$-polynomials

Let $\mathscr{C}_n = \mathscr{C}_n(\mathbb{C})$, $n \ge 0$, be the set of polynomials

$$\phi_n(x) = \sum_{k=0}^{n} a_k \cos(2k+1)x \tag{1.12}$$

with complex coefficients in the cosine system with odd harmonics

$$\mathscr{C} = \{\cos(2k+1)x\}_{k=0}^{\infty}. \tag{1.13}$$

The functions (1.12) will be called $\mathscr{C}_n$-polynomials or $\mathscr{C}$-polynomials of order n. The functions (1.12) are trigonometric polynomials; as trigonometric polynomials they have order $2n+1$. Note that the functions (1.12) vanish at the point $x = \pi/2$: $\varphi_n(\pi/2) = 0$, so none of them is the identical unity.

The main goal of this paper is to study the sharp inequality

$$\|\phi_n\|_\infty \le M(n)_p \|\phi_n\|_p, \quad \phi_n \in \mathscr{C}_n, \tag{1.14}$$

between the uniform norm

$$\|\phi_n\|_\infty = \|\phi_n\|_{C[0,\pi/2]} = \max\big\{|\phi_n(x)|\colon t \in [0,\pi/2]\big\},$$

and integral p-norm $(0 < p < \infty)$

$$\|\phi_n\|_p = \|\phi_n\|_{L^p(0,\pi/2)} = \left(\frac{2}{\pi}\int_0^{\pi/2} |\phi_n(x)|^p dx\right)^{1/p}$$

of $\mathscr{C}_n$-polynomials on the interval $[0,\pi/2]$. Inequality (1.14) appeared in the authors' paper [8] in connection with the study of a variant of the generalized translation in system (1.13) on an interval.

In connection with inequality (1.14), the question naturally arises of the sharp pointwise inequality

$$|\varphi_n(t)| \le M(n,t)_p \|\varphi_n\|_{L^p(0,\pi/2)}, \quad \varphi_n \in \mathscr{C}_n, \tag{1.15}$$

for points $t \in [0,\pi/2]$. Especially important, as will be seen from what follows, is the inequality (1.15) at the end point $t = 0$:

$$|\varphi_n(0)| \le M(n,0)_p \|\varphi_n\|_{L^p(0,\pi/2)}, \quad \varphi_n \in \mathscr{C}_n. \tag{1.16}$$

The study of inequalities (1.14), (1.15), and (1.16) includes, in particular, the study of the properties of extremal polynomials at which the inequalities turn into equalities. It is clear that if the polynomial ϕ_n^* is extremal in one of these inequalities, then, for any constant c, the polynomial $c\,\phi_n^*$ is also extremal. If any extremal polynomial of this inequality has the form $c\,\phi_n^*$ with some constant c, then ϕ_n^* is said to be the unique extremal polynomial.

The following statement is proved in the authors' paper [8, Theorem 4].

Theorem B. *For $1 \le p < \infty$ and $n \ge 0$, the following statements hold.*

(1) *The best constants in inequalities (1.14) and (1.16) coincide:*

$$M(n)_p = M(n,0)_p. \tag{1.17}$$

(2) *The polynomial φ_n^* extremal in inequality (1.16) attains its uniform norm at the point 0 and is also extremal in inequality (1.14).*

The authors do not know whether equality (1.17) holds for $0 < p < 1$; in this case, we can only state that $M(n,0)_p \le M(n)_p$.

1.3.2. Approximation interpretation of inequalities

The problems of studying inequalities (1.14), (1.15), and (1.16) can be reformulated as approximation problems; we will do this only for inequality (1.16). Consider the set

$$\mathscr{C}_n[0] = \{\phi_n \in \mathscr{C}_n\colon \phi_n(0) = 1\} \tag{1.18}$$

of polynomials with fixed value at the point 0: $\phi_n(0) = 1$. On this set, we define the value

$$E_n[0]_p = \inf\big\{\|\phi_n\|_{L^p(0,\pi/2)}\colon \phi_n \in \mathscr{C}_n[0]\big\} \tag{1.19}$$

of the least deviation from zero of the class of polynomials (1.18) in the space $L^p(0,\pi/2)$. It is clear that

$$E_n[0]_p = 1/M(n,0)_p.$$

Moreover, extremal polynomials in problem (1.19) and inequality (1.16) coincide. More precisely, (every) extremal polynomial in (1.19) is also extremal in (1.16); conversely, if φ is an extremal polynomial of inequality (1.16), then the polynomial $\varphi/\varphi(0)$ is extremal in (1.19). Thus, the problem of sharp inequality (1.16) is equivalent to problem (1.19) on the least deviation from zero of the class (1.18).

1.3.3. Christoffel–Darboux kernel for system (1.13)

Sometimes, we will use the following shorter notation for the functions of system (1.13):

$$\eta_k(x) = \cos(2k+1)x, \quad x \in [0, \pi/2]. \tag{1.20}$$

This system of functions is orthogonal with respect to the inner product

$$\langle f, g\rangle = \frac{2}{\pi}\int_0^{\pi/2} f(t)\overline{g(t)}\, dt.$$

More precisely, as is easy to see, for $k, m \geq 1$, the inner products

$$\delta_{k,m} = \langle \eta_k, \eta_m\rangle = \frac{2}{\pi}\int_0^{\pi/2} \cos(2k+1)t\, \cos(2m+1)t\, dt$$

have the following values: $\delta_{k,m} = 0$, $k \neq m$, and $\delta_{k,k} = 1/2$.

Due to the orthogonality of the system $\{\eta_k\}_{k\geq 1}$, the coefficients of the polynomial

$$\phi(x) = \sum_{k=0}^{n} a_k \cos(2k+1)x \tag{1.21}$$

are expressed in terms of the polynomial itself by the formulas $a_k = 2\langle \phi, \eta_k\rangle$. Substituting these expressions into (1.21), we obtain

$$\phi(x) = \sum_{k=0}^{n} a_k\eta_k(x) = 2\sum_{k=0}^{n}\langle \phi, \eta_k\rangle \eta_k(x) = \Big\langle \phi, 2\sum_{k=0}^{n}\eta_k(x)\eta_k\Big\rangle,$$

which can be written in the form

$$\phi(x) = \frac{2}{\pi}\int_0^{\pi/2} \phi(t)\mathcal{K}_n(x,t)dt,$$

where

$$\mathcal{K}_n(x,t) = 2\sum_{k=0}^{n}\eta_k(x)\eta_k(t) \tag{1.22}$$

is the Christoffel–Darboux kernel for the system $\mathscr{C}_n$. Convolute this kernel. Using the formula

$$2\cos a\, \cos b = \cos(a+b) + \cos(a-b), \tag{1.23}$$

we find

$$\mathcal{K}_n(x,t) = 2\sum_{k=0}^{n}\eta_k(x)\eta_k(t) = 2\sum_{k=0}^{n}\cos\left((2k+1)x\right)\cos\left((2k+1)t\right) =$$
$$= \sum_{k=0}^{n}\left(\cos\left((2k+1)(x+t)\right) + \cos\left((2k+1)(x-t)\right)\right).$$

Using the formula (see, for example, [27, Part 6, Sect. 3, Problem 16])

$$\sum_{k=0}^{n} \cos(2k+1)\theta = \frac{\sin 2(n+1)\theta}{2\sin\theta}, \quad \theta \neq \nu\pi, \quad \nu \in \mathbb{Z},$$

we obtain the following representation for the Christoffel–Darboux kernel:

$$\mathcal{K}_n(x,t) = \frac{1}{2}\left(\frac{\sin 2(n+1)(x+t)}{\sin(x+t)} + \frac{\sin 2(n+1)(x-t)}{\sin(x-t)}\right).$$

The orthogonality of the system of functions (1.20) implies that, for a pair of polynomials

$$\phi(x) = \sum_{k=0}^{n} a_k \cos(2k+1)x, \quad \psi(x) = \sum_{k=0}^{n} b_k \cos(2k+1)x,$$

a generalized version of Parceval's identity holds:

$$2\langle \phi, g\rangle = \sum_{k=1}^{n} a_k \overline{b}_k.$$

In particular, the norm of the polynomial $\phi \in \mathscr{C}_n$ is expressed in terms of its Fourier coefficients $\{a_k\}$ by Parceval's identity

$$2\|\phi\|^2_{L_2(0,\pi/2)} = \sum_{k=0}^{n} |a_k|^2.$$

Using this equality and Hölder's inequality, we obtain the inequality

$$|\phi(0)| = \left|\sum_{k=0}^{n} a_k\right| \leq \sqrt{n+1}\left(\sum_{k=0}^{n} |a_k|^2\right)^{1/2} = \sqrt{2(n+1)}\,\|\phi\|_{L_2(0,\pi/2)},$$

which at the kernel (1.22) turns into an equality. Thus, in the space $L_2(0,\pi/2)$, we have

$$M(n)_2 = \sqrt{2(n+1)}, \quad n \geq 0. \tag{1.24}$$

It is useful to compare this result with the corresponding result (1.2)–(1.3) for the classical case.

1.4. Main results

The authors consider the following statements to be the main ones in this paper.

1.4.1. Limit behavior of the best constants in inequalities (1.14) and (1.16)

For the best constant in inequality (1.14), we have an analog of the above Theorem A.

Theorem 1. *The following limit relation holds for constants $M(n)_p$ in inequality (1.14) for $0 < p < \infty$:*

$$\lim_{n\to\infty} \frac{M(n)_p}{(2\pi n)^{1/p}} = A_p. \tag{1.25}$$

1.4.2. Characterization of a polynomial extremal in inequality (1.16)

Denote by $\varphi_n^* = \varphi_{n,p}^* \in \mathscr{C}_n$ a polynomial of order $n \geq 1$ with unit leading coefficient that deviates least from zero in the space $L_w^p(0, \pi/2)$ with the weight

$$w(x) = \sin^2 x \tag{1.26}$$

on the interval $(0, \pi/2)$. In other words, φ_n^* is a solution to the problem

$$\min\{\|\phi_n\|_{L_w^p(0,\pi/2)} : \phi_n \in \mathscr{C}_n^1\} = \|\varphi_n^*\|_{L_w^p(0,\pi/2)}$$

on the set $\mathscr{C}_n^1$ of polynomials (1.12) of order n with leading coefficient 1: $a_n = 1$.

Theorem 2. *For all $1 \leq p < \infty$ and $n \geq 1$, the polynomial φ_n^* of order n with unit leading coefficient that deviates least from zero in the space $L_w^p(0, \pi/2)$ with weight (1.26) is the unique extremal polynomial in inequality (1.16).*

There are statements similar to Theorem 2 in [5, Theorem 1; 6, Theorem 2; 7, Theorem 2; 9, Theorem 3; 4, Theorem 2; 3].

We will also give some estimates for the best constant $M(n)_p$ in inequality (1.14); see, in particular, Section 2.4.

2. Behavior with respect to n of the best constant in the Nikol'skii inequality for $\mathscr{C}$-polynomials

2.1. Case $p = 2$

According to (1.3) and (1.24), for $n \geq 0$, we have

$$C(n)_2 = \sqrt{2n+1},$$
$$M(n)_2 = \sqrt{2(n+1)}.$$

Thus,

$$\lim_{n\to\infty} \frac{M(n)_2}{n^{1/2}} = \lim_{n\to\infty} \frac{C(n)_2}{n^{1/2}} = \sqrt{2}.$$

Both limits exist and coincide; this fact served as an argument for the authors that this property of the quantities $M(n)_p$ will hold for all $0 < p < \infty$.

2.2. Expression of $\mathscr{C}$-polynomials in terms of the classical trigonometric polynomials

Let $\mathcal{C}_n$ be the set of even (complex) trigonometric polynomials

$$f_n(x) = \sum_{k=0}^{n} a_k \cos kx$$

of order (at most) $n \geq 1$. On this set, we consider the uniform norm

$$\|f_n\|_\infty = \|f_n\|_{C[0,\pi]} = \max\left\{|f_n(x)| : x \in [0, \pi]\right\}$$

and the integral p-norm (for $0 < p < \infty$)

$$\|f_n\|_p = \|f_n\|_{L^p(0,\pi)} = \left(\frac{1}{\pi}\int_0^\pi |f_n(x)|^p dx\right)^{1/p}.$$

It is well known (however, it is easy to show) that the best constants $C(n)_p$ in inequality (1.2) and the inequality

$$|f_n(0)| \le C(n)_p \|f_n\|_p, \quad f_n \in \mathcal{C}_n, \tag{2.1}$$

coincide.

Lemma 1. *For $f_n \in \mathcal{C}_n$, the function*

$$\phi_n(x) = f_n(2x)\cos x \tag{2.2}$$

is a $\mathscr{C}_n$-polynomial. Conversely, every $\mathscr{C}_n$-polynomial ϕ_n can be represented in the form (2.2), where $f_n \in \mathcal{C}_n$. Thus, formula (2.2) establishes one-to-one correspondence between $\mathcal{C}_n$ and $\mathscr{C}_n$.

P r o o f. Let $f_n \in \mathcal{C}_n$ be a trigonometric polynomial. Let

$$\phi(x) = f_n(2x)\cos x = \sum_{k=0}^{n} a_k \cos 2kx \cos x.$$

Applying again formula (1.23), we have

$$2\cos 2kx \cos x = \cos(2k+1)x + \cos(2k-1)x.$$

Using this relation, we find

$$2\phi(x) = 2f_n(2x)\cos x = 2\sum_{k=0}^{n} a_k \cos 2kx \cos x =$$

$$= \sum_{k=0}^{n} a_k \cos(2k+1)x + \sum_{k=0}^{n} a_k \cos(2k-1)x = \sum_{k=0}^{n} a_k \cos(2k+1)x + \sum_{k=-1}^{n-1} a_{k+1}\cos(2k+1)x.$$

As a result, we obtain the representation

$$2\phi(x) = \sum_{k=-1}^{n} (a_k + a_{k+1})\cos(2k+1)x = (2a_0 + a_1)\cos x + \sum_{k=1}^{n}(a_k + a_{k+1})\cos(2k+1)x, \tag{2.3}$$

where $a_{-1} = 0$ and $a_{n+1} = 0$. The function (2.3) is a $\mathscr{C}_n$-polynomial.

Let us prove the inverse statement, i.e., let us prove that an arbitrary $\mathscr{C}_n$-polynomial

$$\phi(x) = \sum_{k=0}^{n} \lambda_k \cos(2k+1)x$$

can be represented in the form (2.2). Rewrite this polynomial in the form

$$\phi(x) = \sum_{k=-1}^{n} \lambda'_k \cos(2k+1)x, \tag{2.4}$$

where $\lambda'_{-1} = \lambda'_0 = \lambda_0/2$ and $\lambda'_k = \lambda_k$, $1 \le k \le n$.

It suffices to represent the polynomial 2ϕ in the form (2.3). The latter means that the coefficients $\{\lambda'_k\}_{k=-1}^n$ of the polynomial (2.4) can be represented as

$$\lambda'_k = a_k + a_{k+1}, \quad -1 \le k \le n; \quad a_{-1} = 0, \quad a_{n+1} = 0. \tag{2.5}$$

It is easy to see that the formulas

$$a_k = \sum_{\ell=0}^{n-k} (-1)^\ell \lambda'_{k+\ell} = \lambda'_k - a_{k+1}, \quad k = n, n-1, \dots, 0,$$

give a solution to system (2.5). Lemma 1 is proved. $\square$

Representation (2.2) implies that inequality (1.16) is equivalent to the inequality

$$|f_n(0)| \le C(n,\sigma)_p \|f_n\|_{L^p_\sigma(0,\pi)}, \quad f_n \in \mathcal{C}_n, \tag{2.6}$$

on the set $\mathcal{C}_n$ with weight $\sigma(t) = \cos^p(t/2)$. More exactly, the following assertion holds.

Lemma 2. *For $0 < p < \infty$ and $n \ge 1$, inequality (1.16) on the set $\mathscr{C}_n$ and inequality (2.6) on $\mathcal{C}_n$ are equivalent; specifically:*

(1) *the best constants in inequalities (2.6) and (1.16) are related by the equality*

$$C(n,\sigma)_p = M(n,0)_p; \tag{2.7}$$

(2) *extremal polynomials in these inequalities are related by (2.2).*

P r o o f. Using relation (2.2), we find that, for an arbitrary polynomial $\phi_n \in \mathscr{C}_n$,

$$|f_n(0)| = |\phi_n(0)| \le M(n,0)_p \|\phi_n\|_{L^p(0,\pi/2)},$$

where

$$\|\phi_n\|_{L^p(0,\pi/2)} = \left(\frac{2}{\pi}\int_0^{\pi/2} |f_n(2x)\cos x|^p dx\right)^{1/p} = \left(\frac{1}{\pi}\int_0^{\pi} |f_n(t)\cos(t/2)|^p dt\right)^{1/p} = \|f_n\|_{L^p_\sigma(0,\pi)}.$$

Lemma 2 is proved. $\square$

The following statement contains a quantitative relation between the constants in inequalities (1.14), (1.16), and (1.2).

Lemma 3. *For $0 < p < \infty$ and $n \ge 1$, the best constants in inequalities (1.14), (1.16), and (2.1) (or, equivalently, (1.2)) are related as follows:*

$$C(n)_p \le M(n,0)_p, \tag{2.8}$$
$$M(n)_p \le C(n+1)_p. \tag{2.9}$$

P r o o f. For polynomials $f_n \in \mathcal{C}_n$, we have $\|f_n\|_{L^p_\sigma(0,\pi)} \le \|f_n\|_{L^p(0,\pi)}$. Therefore, the best constants in (2.1) and (2.6) are related by the inequality $C(n)_p \le C(n,\sigma)_p$. This and (2.7) imply (2.8).

Let us prove inequality (2.9). A polynomial $\phi_n \in \mathscr{C}_n$ has the form

$$\phi_n(x) = \sum_{k=0}^{n} a_k \cos(2k+1)x.$$

Writing it in exponential form, we find

$$2\phi_n(x) = \sum_{k=0}^{n} a_k \left(e^{i(2k+1)x} + e^{-i(2k+1)x}\right) = \sum_{k=0}^{n} a_k e^{i(2k+1)x} + \sum_{k=0}^{n} a_k e^{-i(2k+1)x} =$$

$$= e^{-ix} \sum_{k=0}^{n} a_k e^{i2(k+1)x} + e^{-ix} \sum_{k=0}^{n} a_k e^{-i2kx} = e^{-ix} \left(\sum_{k=1}^{n+1} a_{k-1} e^{i2kx} + \sum_{k=0}^{n-1} a_{k+1} e^{-i2kx} \right).$$

The function

$$g_{n+1}(x) = \sum_{k=1}^{n+1} a_{k-1} e^{ikx} + \sum_{k=0}^{n-1} a_{k+1} e^{-ikx}$$

is a trigonometric polynomial of order $n+1$. The functions $2\phi_n$ and g_{n+1} are related as $2\phi_n(x) = e^{-ix} g_{n+1}(2x)$, $x \in \mathbb{R}$, or, equivalently, as

$$2\phi_n(x/2) = e^{-ix/2} g_{n+1}(x), \quad x \in \mathbb{R}. \tag{2.10}$$

The following inequality holds for the polynomial g_{n+1} (cf. (1.2)):

$$\|g_{n+1}\|_{C_{2\pi}} \le C(n+1)_p \|g_{n+1}\|_{\mathcal{L}^p_{2\pi}}. \tag{2.11}$$

As a consequence of (2.10), we have $\|g_{n+1}\|_{C_{2\pi}} = 2\|\phi_n\|_{C[0,\pi/2]}$ and

$$\|g_{n+1}\|_{\mathcal{L}^p_{2\pi}} = \left(\frac{1}{2\pi}\int_0^{2\pi} |g_{n+1}(x)|^p dx\right)^{1/p} = [x/2 = t] = 2\left(\frac{1}{\pi}\int_0^{\pi} |\phi_n(t)|^p dt\right)^{1/p} =$$

$$= 2\left(\frac{2}{\pi}\int_0^{\pi/2} |\phi_n(t)|^p dt\right)^{1/p} = 2\|\phi_n\|_{L^p(0,\pi/2)}.$$

Consequently, inequality (2.11) is equivalent to the inequality

$$\|\phi_n\|_{C[0,\pi/2]} \le C(n+1)_p \|\phi_n\|_{L^p(0,\pi/2)}.$$

Comparing this inequality with (1.14), we conclude that inequality (2.9) holds. Lemma 3 is proved completely. □

2.3. Proof of Theorem 1

By Lemma 3, for all $0 < p < \infty$ and $n \ge 1$, we have the inequalities

$$C(n)_p \le M(n)_p \le C(n+1)_p.$$

As a consequence, we have

$$\frac{C(n)_p}{(2\pi n)^{1/p}} \le \frac{M(n)_p}{(2\pi n)^{1/p}} \le \frac{C(n+1)_p}{(2\pi(n+1))^{1/p}} \left(\frac{n}{n+1}\right)^{1/p}.$$

Passing here to the limit as $n \to \infty$ and using the result (1.10) of Theorem A, we obtain (1.25). Theorem 1 is proved. □

2.4. Estimates

2.4.1. Monotonicity of a constant in (1.14) in p

Let $0 < p_1 < p_2 < \infty$. For an arbitrary $\phi_n \in \mathscr{C}_n$, we have

$$\|\phi\|_{p_2} = \left(\frac{2}{\pi}\int_0^{\pi/2} |\phi(t)|^{p_2} dt\right)^{1/p_2} = \left(\frac{2}{\pi}\int_0^{\pi/2} |\phi(t)|^{p_1} |\phi(t)|^{p_2-p_1} dt\right)^{1/p_2} \le$$

$$\le \left(\frac{2}{\pi}\int_0^{\pi/2} |\phi(t)|^{p_1} dt\right)^{1/p_2} \left(\|\phi\|_{C[0,\pi/2]}\right)^{(p_2-p_1)/p_2} = \left(\|\phi\|_{p_1}\right)^{p_1/p_2} \left(\|\phi\|_{C[0,\pi/2]}\right)^{(p_2-p_1)/p_2}.$$

Consider inequality (1.14) with the parameter p_2:

$$\|\phi\|_{C[0,\pi/2]} \le M(n)_{p_2} \|\phi\|_{p_2} \le M(n)_{p_2} \left(\|\phi\|_{p_1}\right)^{p_1/p_2} \left(\|\phi\|_{C[0,\pi/2]}\right)^{(p_2-p_1)/p_2}.$$

Dividing by the latter factor, we obtain the inequality

$$\left(\|\phi\|_{C[0,\pi/2]}\right)^{p_1/p_2} \le M(n)_{p_2} \left(\|\phi\|_{p_1}\right)^{p_1/p_2}.$$

Raise it to the power p_2/p_1:

$$\|\phi\|_{C[0,\pi/2]} \le \left(M(n)_{p_2}\right)^{p_2/p_1} \|\phi\|_{p_1}.$$

This inequality holds for any polynomial $\phi \in \mathscr{C}_n$; hence,

$$M(n)_{p_1} \le \left(M(n)_{p_2}\right)^{p_2/p_1},$$

or

$$\left(M(n)_{p_1}\right)^{p_1} \le \left(M(n)_{p_2}\right)^{p_2}, \quad 0 < p_1 < p_2 < \infty; \tag{2.12}$$

this is the required property of monotonicity.

2.4.2. Estimates for a constant in (1.14)

As a special case of (2.12), we have the inequality

$$M(n)_p \ge \left(M(n)_1\right)^{1/p}, \quad 1 < p < \infty.$$

Using now inequalities (2.8) and (1.6), we obtain the lower estimate

$$M(n)_p \ge (n+1)^{1/p}, \quad 1 \le p < \infty.$$

Using (2.12) and (1.24), we obtain the estimates

$$M(n)_p \le (2(n+1))^{1/p}, \quad 0 < p \le 2,$$
$$M(n)_p \ge (2(n+1))^{1/p}, \quad p \ge 2.$$

Finally, inequalities (2.9) and (1.5) imply the estimate

$$M(n)_p \le 4(n+1)^{1/p}, \quad 0 < p < \infty.$$

3. Characterization of an extremal $\mathscr{C}$-polynomial in the Nikol'skii inequality

The primary purpose of this section is to prove Theorem 2, which characterizes the extremal $\mathscr{C}_n$-polynomial in the Nikol'skii inequality (1.16). As mentioned above, statements similar to Theorem 2 can be found in the authors' papers, personal and with co-authors: [5–7, 9]. The ideas contained in these statements have been summarized in [3]. The fact that we need [3] holds for inequalities similar to inequality (1.16) for algebraic polynomials in spaces L^p with weight. Because of this, we first rewrite inequality (1.16) in terms of algebraic polynomials, use Theorem 4 from [3], and then make a conclusion related to inequality (1.16).

3.1. Equivalent to (1.16) inequality for algebraic polynomials on an interval

We can associate inequalities (1.14) and (1.16) on the set $\mathscr{C}_n$ with equivalent inequalities on the set of algebraic polynomials. For this, we describe $\mathscr{C}$-polynomials in terms of algebraic polynomials. Denote by $\mathcal{P}_n$ the set of algebraic polynomials

$$\rho_n(t) = \sum_{\ell=0}^{n} c_\ell t^\ell \tag{3.1}$$

of degree (at most) n with complex coefficients. In further consideration, we will need some properties of the Chebyshev polynomials

$$T_\nu(t) = \cos(\nu x), \quad x = \arccos t, \quad t \in [-1, 1], \tag{3.2}$$

of the first kind of degree $\nu \geq 0$; these properties can be found, for example, in [21, Ch. 2, Sect. 2.2]). A polynomial T_ν, $\nu \geq 1$, can be represented as

$$T_\nu(t) = \frac{\nu}{2} \sum_{k=0}^{[\nu/2]} \frac{(-1)^k}{\nu - k} C^k_{\nu-k} (2t)^{\nu-k} = 2^{\nu-1} t^\nu - \nu 2^{\nu-3} t^{\nu-2} + \cdots . \tag{3.3}$$

A polynomial T_ν is even or odd in accordance with the evenness of the number ν, and its leading coefficient (for $\nu \geq 1$) is $2^{\nu-1}$.

Lemma 4. *For $n \geq 1$, the relation*

$$\phi_n(x) = t\rho_n(t^2), \quad t = \cos x, \quad x \in [0, \pi], \quad t \in [-1, 1], \tag{3.4}$$

establishes a bijection between the set of polynomials $\phi_n \in \mathscr{C}_n$ of order n and the set $\mathcal{P}_n$ of algebraic polynomials ρ_n of degree n. Under this correspondence, the leading coefficient $a_n(\phi_n)$ of a $\mathscr{C}$-polynomial ϕ_n and the leading coefficient $c_n(\rho_n)$ of the polynomial ρ_n are related by the formula

$$c_n(\rho_n) = 2^{2n} a_n(\phi_n); \tag{3.5}$$

in addition, the following equalities hold:

$$\phi_n(0) = \rho_n(1), \tag{3.6}$$

$$\|\phi_n\|_{L^p(0,\pi/2)} = \|\rho_n\|_{L^p_\upsilon(0,1)}, \quad \upsilon(u) = \upsilon(u)_p = \frac{u^{(p-1)/2}}{\pi\sqrt{1-u}}. \tag{3.7}$$

P r o o f. First, let us show that an arbitrary $\mathscr{C}$-polynomial

$$\phi_n(x) = \sum_{k=0}^{n} a_k \cos(2k+1)x \tag{3.8}$$

of order $n \geq 0$ can be presented in the form (3.4), where ρ_n is an algebraic polynomial (3.1) of degree n. Replace $\cos(2k+1)x$ with $T_{2k+1}(t)$, $t = \cos x$, $t \in [-1,1]$, in (3.8). A polynomial T_{2k+1} is odd, and consequently, can be represented in the form $T_{2k+1}(t) = tR_k(t^2)$, where R_k is a real algebraic polynomial of degree k. The leading coefficient of the polynomial R_k is 2^{2k} and $R_k(1) = 1$. Therefore, polynomial (3.8) can be represented as (3.4), where ρ_n is some algebraic polynomial (3.1) of degree n; the leading coefficients of the polynomials ϕ_n and ρ_n satisfy the relation (3.5).

Conversely, let $\rho_n \in \mathcal{P}_n$. Consider the function $f(x) = \cos x\, \rho_n(\cos^2 x)$. Based on the representation (3.1), we have

$$f(x) = \cos x\, \rho_n(\cos^2 x) = \sum_{\ell=0}^{n} c_\ell \cos^{2\ell+1} x. \tag{3.9}$$

The function $\cos^{2\ell+1} x$ is an even trigonometric polynomial of order $2\ell+1$:

$$\cos^{2\ell+1} x = \sum_{k=0}^{\ell} \rho_{\ell,k} \cos(2k+1)x. \tag{3.10}$$

This fact is, of course, known. However, it is easy to obtain (by induction on ℓ) starting from the representations (3.2) and (3.3). In particular, it follows from (3.3) that the leading coefficient of the representation (3.10) is $\rho_{\ell,\ell} = 2^{-2\ell}$. Thus, function (3.9) is a $\mathscr{C}$-polynomial of the form (3.8).

Relation (3.4) establishes a bijection between $\mathscr{C}_n$ and $\mathcal{P}_n$. Formula (3.6) is obvious. Let us verify (3.7):

$$\|\phi_n\|_{L^p(0,\pi/2)} = \left(\frac{2}{\pi}\int_0^{\pi/2} |\phi_n(x)|^p dx\right)^{1/p} = \left(\frac{2}{\pi}\int_0^{\pi/2} |\cos x \rho_n(\cos^2 x)|^p dx\right)^{1/p} = [\cos x = t] =$$
$$= \left(\frac{2}{\pi}\int_0^{1} |t\rho_n(t^2)|^p \frac{dt}{\sqrt{1-t^2}}\right)^{1/p} = \left[t^2 = u\right] = \left(\frac{1}{\pi}\int_0^{1} |\rho_n(u)|^p \frac{u^{(p-1)/2}du}{\sqrt{1-u}}\right)^{1/p} =$$
$$= \left(\int_0^1 |\rho_n(u)|^p \upsilon(u)du\right)^{1/p}, \quad \upsilon(u) = \upsilon(u)_p = \frac{u^{(p-1)/2}}{\pi\sqrt{1-u}}.$$

Thus, we obtain equality (3.7). Lemma 4 is proved completely. □

Corollary 1. *The representation* (3.4) *implies that for* $n \geq 1$ *the polynomial* $\phi \in \mathscr{C}_n$, $\phi \not\equiv 0$, *can have on* $[0,\pi/2)$ *at most* n *zeros, taking into account their multiplicities, i.e.,* $\mathscr{C}_n$ *is the Chebyshev system on* $[0,\pi/2)$.

Corollary 2. *Lemma 4 implies that inequalities* (1.14) *and* (1.16) *can be associated with equivalent sharp inequalities on the set of algebraic polynomials*:

$$\|\rho_n\|_\infty \leq \mathcal{M}(n)_p \|\rho_n\|_{L^p_\upsilon(0,1)}, \quad \rho_n \in \mathcal{P}_n, \tag{3.11}$$

with constant $\mathcal{M}(n)_p = M(n)_p$ *and*

$$|\rho_n(1)| \leq \mathcal{M}(n,1)_p \|\rho_n\|_{L^p_\upsilon(0,1)}, \quad \rho_n \in \mathcal{P}_n, \tag{3.12}$$

with constant $\mathcal{M}(n,1)_p = M(n,0)_p$. *Moreover, a polynomial* $\varrho_n \in \mathcal{P}_n$ *is extremal in inequalities* (3.11) *and* (3.12) *if and only if the polynomial* $\varphi_n \in \mathscr{C}_n$ *related to* ϱ_n *by formula* (3.4) *is extremal in inequalities* (1.14) *and* (1.16).

Let us reformulate the problem of studying inequality (3.12) as an approximation problem. Consider the set

$$\mathcal{P}_n[1] = \big\{\rho_n \in \mathcal{P}_n \colon \rho_n(1) = 1\big\} \tag{3.13}$$

of polynomials with a fixed unit value at unity: $\rho_n(1) = 1$. On this set, we define the value

$$e_n[1]_p = \inf\big\{\|\rho_n\|_{L^p_\upsilon(0,1)} \colon \rho_n \in \mathcal{P}_n[1]\big\} \tag{3.14}$$

of the least deviation from zero in the space $L^p_\upsilon(0,1)$ of the class of polynomials (3.13). It is clear that $e_n[1]_p = 1/\mathcal{M}(n,1)_p$. Moreover, polynomials extremal in problem (3.14) and in inequality (3.12) coincide (up to a multiplicative constant). Thus, the problem on the sharp inequality (3.12) is equivalent to problem (3.14) on the least deviation from zero of the class (3.13).

Lemma 5. *For $1 \le p < \infty$ and $n \ge 1$, the following statements hold for a polynomial $\varrho_n \in \mathcal{P}_n$ extremal in inequality* (3.12) *and such that $\varrho_n(1) = 1$.*

(1) *The polynomial ϱ_n is characterized by the property*

$$\int_0^1 \rho_{n-1}(u)(1-u)\upsilon(u)|\varrho_n(u)|^{p-1}\operatorname{sign}\varrho_n(u)du = 0, \quad \rho_{n-1} \in \mathcal{P}_{n-1}. \tag{3.15}$$

(2) *The polynomial ϱ_n has degree n, all its n roots are real, simple, and lie on the interval $(0,1)$; in this sense, the polynomial ϱ_n is real.*

(3) *The polynomial ϱ_n is unique.*

P r o o f. Most of the results of this lemma are contained in [3]. We find it difficult in some cases to make precise references to this paper. Therefore, we have to repeat some arguments from [3] with clarifications and explanations.

In Theorem 4 of [3], in particular, the following properties of a polynomial ϱ_n extremal in the inequality (3.12) were proved.

($1'$) The polynomial ϱ_n is real; more exactly, it has real coefficients and, hence, takes real values on the real axis.

($2'$) The polynomial ϱ_n is characterized by the property (3.15).

It follows that the polynomial ϱ_n has n sign changes on the interval $(0,1)$. Otherwise, the polynomial with simple zeros at the sign change points of the polynomial ϱ_n would have order at most $n-1$ and the property (3.15) would not hold on it.

Finally, let us verify that an extremal polynomial ϱ_n is unique for all $1 \le p < \infty$. In fact, let ϱ_n and ζ_n be two polynomials that solve problem (3.14). By the inequality $\|\varrho_n + \zeta_n\|_{L^p_\upsilon} \le \|\varrho_n\|_{L^p_\upsilon} + \|\zeta_n\|_{L^p_\upsilon}$, their half-sum $(\varrho_n + \zeta_n)/2$ has the same property; hence, we have the equality $\|\varrho_n + \zeta_n\|_{L^p_\upsilon} = \|\varrho_n\|_{L^p_\upsilon} + \|\zeta_n\|_{L^p_\upsilon}$. For $1 < p < \infty$, since the space $L^p_\upsilon(0,1)$ is strictly normalized, it immediately follows that $\zeta_n = \varrho_n$. If $p = 1$, then we can only assert so far that the signs of the polynomials ζ_n and ϱ_n coincide almost everywhere on $[0,1]$. The zeros of these polynomials are simple and lie on the interval $(0,1)$, so the polynomials ζ_n and ϱ_n have the same set of zeros and the same value at the point $u = 1$: $\zeta_n(1) = \varrho_n(1) = 1$, hence, these polynomials coincide. Thus, the extremal polynomial is unique for $p = 1$ too.

Lemma 5 is proved completely. □

Based on the weight υ defined in (3.7), let us define the following weight on the interval $(0,1)$:

$$\varpi(x) = (1-u)\,\upsilon(u) = \frac{u^{(p-1)/2}\sqrt{1-u}}{\pi}. \tag{3.16}$$

Consider the problem on the least deviation from zero

$$u(\mathcal{P}_n^1)_{L^p_\varpi(0,1)} = \min\{\|\rho_n\|_{L^p_\varpi(0,1)} : \rho_n \in \mathcal{P}_n^1\} \tag{3.17}$$

in the space $L^p_\varpi(0,1)$ of the set $\mathcal{P}_n^1$ of algebraic polynomials of degree n whose leading coefficient is 1. Denote by $\varrho_n^* = \varrho_{n,\varpi,p}^*$ a polynomial solving this problem:

$$u(\mathcal{P}_n^1)_{L^p_\varpi(0,1)} = \|\varrho_n^*\|_{L^p_\varpi(0,1)};$$

it is called a polynomial of degree n with unit leading coefficient that deviates least from zero in the space $L^p_\varpi(0,1)$.

Theorem 3. *For all $1 \le p < \infty$ and $n \ge 1$, the polynomial ϱ_n^* of degree n with unit leading coefficient that deviates least from zero in the space $L^p_\varpi(0,1)$ with weight (3.16) is the unique extremal polynomial in inequality (3.12).*

P r o o f. The polynomial ϱ_n^* is characterized by the property that the function $|\varrho_n^*|^{p-1}\mathrm{sign}\,\varrho_n^*$ is orthogonal to the space $\mathcal{P}_{n-1}$ (see, for example, [20, Ch. 3, Sect. 3.3, Theorems 3.3.1 and 3.3.2]):

$$\int_0^1 \varpi(x)\,\rho_{n-1}(x)|\varrho_n^*(x)|^{p-1}\mathrm{sign}\,\varrho_n^*(x)\,dx = 0, \quad \rho_{n-1} \in \mathcal{P}_{n-1}.$$

This property is the same as property (3.15). Therefore, the polynomials ϱ_n and ϱ_n^* can differ only by a multiplicative constant. Theorem 3 is proved. □

3.2. Characterization of a $\mathscr{C}$-polynomial extremal in inequality (1.16)

Let us apply the results of the previous section to describing the characteristic properties of $\mathscr{C}_n$-polynomials extremal in inequalities (1.14) and (1.16).

3.2.1. An analog of Lemma 5 in the set of $\mathscr{C}$-polynomials

Let us reformulate Lemma 5 for the extremal $\mathscr{C}_n$-polynomial of inequality (1.16).

Lemma 6. *For $1 \le p < \infty$ and $n \ge 1$, the following statements hold for polynomials $\varphi_n \in \mathscr{C}_n$ extremal in inequality (1.16) and such that $\varphi_n(0) = 1$.*

(1) *The polynomial φ_n is characterized by the property*

$$\int_0^{\pi/2} \phi_{n-1}(t)(\sin^2 t)\,|\varphi_n(t)|^{p-1}\mathrm{sign}\,\varphi_n(t)dt = 0, \quad \phi_{n-1} \in \mathscr{C}_{n-1}. \tag{3.18}$$

(2) *The polynomial φ_n has order n. The polynomial φ_n has n simple roots on the interval $(0,\pi/2)$.*

(3) *The polynomial φ_n is unique.*

P r o o f. Let us employ the statements of Lemmas 4 and 5. Let $\varrho_n \in \mathcal{P}_n$ be an extremal (algebraic) polynomial in inequality (1.16) with the property $\varrho_n(1) = 1$. The polynomial ϱ_n and the polynomial φ_n are related by (3.4).

Let us check that the relation (3.18) coincides with (is equivalent to) (3.15). For this, we transform the functional on the left-hand side of (3.15). For a polynomial $\rho_{n-1} \in \mathcal{P}_{n-1}$, we define a polynomial $\phi_{n-1} \in \mathscr{C}_{n-1}$ that is expressed in terms of ρ_{n-1} by formula (3.4). Based on the left-hand side of (3.15), we find

$$\int_0^1 \rho_{n-1}(u)(1-u)v(u)|\varrho_n(u)|^{p-1}\text{sign}\,\varrho_n(u)du =$$
$$= \frac{1}{\pi}\int_0^1 \rho_{n-1}(u)(1-u)\frac{1}{\sqrt{1-u}}|\sqrt{u}\varrho_n(u)|^{p-1}\text{sign}\,\varrho_n(u)du =$$
$$= \frac{1}{\pi}\int_0^1 \rho_{n-1}(u)\sqrt{1-u}|\sqrt{u}\varrho_n(u)|^{p-1}\text{sign}\,\varrho_n(u)du = \left[u=t^2\right] =$$
$$= \frac{2}{\pi}\int_0^1 t\rho_{n-1}(t^2)\sqrt{1-t^2}|t\varrho_n(t^2)|^{p-1}\text{sign}\,\varrho_n(t^2)dt = [t=\cos x] =$$
$$= \frac{2}{\pi}\int_0^{\pi/2} \phi_{n-1}(x)\sin^2 x\,|\varphi_n(x)|^{p-1}\text{sign}\,\varphi_n(x)dt.$$

Now you can see that the conditions (3.18) and (3.15) hold or do not hold simultaneously. Lemma 6 is proved. □

Consider a problem similar (3.17) on the value

$$U(\mathscr{C}_n^1)_{L_\sigma^p(0,\pi/2)} = \min\{\|\phi_n\|_{L_\sigma^p(0,\pi/2)} \colon \phi_n \in \mathscr{C}_n^1\} \tag{3.19}$$

of the least deviation from zero in the space $L_\sigma^p(0,\pi/2)$ with weight $\sigma(x) = \sin^2 x$ of the set $\mathscr{C}_n^1$ of algebraic polynomials of degree n whose leading coefficient is 1. Denote by $\varphi_n^* = \varphi_{n,\sigma,p}^*$ the polynomial of degree n with unit leading coefficient that deviates least from zero in the space $L_\sigma^p(0,\pi/2)$, i.e., the polynomial that solves problem (3.19):

$$U(\mathscr{C}_n^1)_{L_\sigma^p(0,\pi/2)} = \|\varphi_n^*\|_{L_\sigma^p(0,\pi/2)}.$$

Lemma 7. *For $1 \le p < \infty$ and $n \ge 1$, the following statements hold for problems (3.17) and (3.19).*

(1) *The values of the problems are related by the equality*

$$U(\mathscr{C}_n^1)_{L_\sigma^p(0,\pi/2)} = 2^{2n}\,u(\mathcal{P}_n^1)_{L_\varpi^p(0,1)}.$$

(2) *A polynomial $\varrho_n^* \in \mathcal{P}_n^1$ extremal in problem (3.17) and a polynomial $\varphi_n^* \in \mathscr{C}_n^1$ extremal in problem (3.19) are related by the equality*

$$\varphi_n^*(x) = 2^{2n}t\,\varrho_n^*(t^2), \quad t=\cos x, \quad x\in[0,\pi/2], \quad t\in[0,1].$$

P r o o f. Let $\rho_n \in \mathcal{P}_n$, and let $\phi_n \in \mathscr{C}_n$ be the polynomial expressed in terms of ρ_n by formula (3.4). We have

$$\|\rho_n\|^p_{L_\varpi^p(0,1)} = \frac{1}{\pi}\int_0^1 u^{(p-1)/2}\sqrt{1-u}|\rho_n(u)|^p du = [u=t^2] =$$
$$= \frac{2}{\pi}\int_0^1 \sqrt{1-t^2}|t\rho_n(t^2)|^p dt = [t=\cos x] = \frac{2}{\pi}\int_0^{\pi/2}\sin^2 x|\phi_n(x)|^p dx = \|\phi_n\|^p_{L_\sigma^p(0,\pi/2)}.$$

Consequently, the norms satisfy the equality

$$\|\rho_n\|_{L_\varpi^p(0,1)} = \|\phi_n\|_{L_\sigma^p(0,\pi/2)}.$$

According to (3.5), we have

$$\phi_n \in \mathscr{C}_n^1 \quad \Longleftrightarrow \quad 2^{-2n}\rho_n \in \mathcal{P}_n^1.$$

From here, all the assertions of Lemma 7 follow. Lemma 7 is proven. □

3.2.2. The proof of Theorem 2

For an algebraic polynomial $\rho \in \mathcal{P}_n$ and a $\mathscr{C}_n$-polynomial $\phi \in \mathscr{C}_n$ related by (3.4), we will say that they are (3.4)-related.

(1) According to Corollary 2, the polynomials $\varrho \in \mathcal{P}_n$ and $\phi \in \mathscr{C}_n$ extremal in the Nikol'skii inequalities (1.14) and (1.16) and inequalities (3.11) and (3.12), respectively, are (3.4)-related. According to Lemmas 5 and 6, these polynomials are unique up to numerical factors.

(2) Let $\varrho^* \in \mathcal{P}_n^1$ and $\varphi_n^* \in \mathscr{C}_n^1$ be extremal polynomials, i.e., polynomials that deviate least from zero in problems (3.17) and (3.19), respectively. According to Lemma 7, the polynomials ρ_n^* and $2^{2n}\varphi^*$ are also (3.4)-related.

(3) According to Theorem 3, the polynomial ϱ^* is extremal in inequalities (3.11) and (3.12). Consequently, φ^* is also extremal in inequalities (1.14) and (1.16).

Thus, Theorem 2 is proved. □

4. Conclusions

The system of functions (1.20) is in some sense a "quarter" of the classical trigonometric system. However, as it turned out (see Theorem 1 and Lemma 3), the best constants in inequalities (1.2) and (1.14) are very close. The reason for this is not clear to the authors. What will be the situation with other extremal problems in these systems, the authors also do not know.

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