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ALPHA LABELINGS OF DISJOINT UNION OF HAIRY CYCLES

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Abstract: In this paper, we prove the following results: (1) the disjoint union of $n \geq 2$ isomorphic copies of a graph obtained by adding a pendant edge to each vertex of a cycle of order 4 admits an α-valuation; (2) the disjoint union of two isomorphic copies of a graph obtained by adding $n \geq 1$ pendant edges to each vertex of a cycle of order 4 admits an α-valuation; (3) the disjoint union of two isomorphic copies of a graph obtained by adding a pendant edge to each vertex of a cycle of order $4m$ admits an α-valuation; (4) the disjoint union of two nonisomorphic copies of a graph obtained by adding a pendant edge to each vertex of cycles of order $4m$ and $4m-2$ admits an α-valuation; (5) the disjoint union of two isomorphic copies of a graph obtained by adding a pendant edge to each vertex of a cycle of order $4m-1$ $(4m+2)$ admits a graceful valuation (an α-valuation), respectively.

Keywords: Hairy cycles, Graceful valuation, α-valuation.

1. Introduction

Notation and terminology not defined here can be found in [2]. Throughout this paper, we denote by S_n and C_n a star on $n+1$ vertices and a cycle on n vertices, respectively.

If a labeling f on a graph G with p edges is a one-to-one function from the set of vertices of G to the set $\{0, 1, \ldots, p\}$ such that, for p pairs of adjacent vertices x and y, the values $|f(x)-f(y)|$ are distinct, then f is called a *graceful valuation* (a β-labeling or a β-valuation) of G. If, in addition, there exists an integer ℓ such that, for each edge $xy \in E(G)$, one of the values $f(x)$ and $f(y)$ does not exceed ℓ and the other is strictly greater than ℓ, then the labeling f is called an α-valuation of G with *critical value* ℓ. Note that a graph with an α-valuation is necessarily bipartite. As a result, such ℓ must be smaller than the smallest of the two vertex labels that yield the edge labeled 1. Let $\{A, B\}$ be stable sets (a partition) of vertices with $x \in A$ and $y \in B$. Without loss of generality, assume that

$$A = \{x \in V(G) : f(x) \leq \ell\}, \quad B = \{y \in V(G) : f(y) > \ell\}.$$

Clearly, every α-valuation is also a graceful labeling but not conversely. Rosa pioneered in 1966 [21] the concept of graph β-labeling. He also presented certain types of vertex labeling as an important tool for decomposing the complete graph K_{2p+1} into graphs with p edges.

Theorem 1 [21]. *Let a graph G with p edges has an α-valuation. Then, for $s \in \mathbb{N}$, there exists a G-decomposition of the complete graph K_{2ps+1}.*

Specifically, β-valuations were developed to challenge Ringel's conjecture [19] that K_{2n+1} can be decomposed into $2n+1$ subgraphs that are all isomorphic to a given tree with n edges. More results about graph labeling are collected and updated regularly in the survey by Gallian [9].

The *disjoint union of graphs* $H_1 = (V_1, E_1), H_2 = (V_2, E_2), \ldots, H_n = (V_n, E_n)$ is a graph $H = H_1 \cup H_2 \cup \cdots \cup H_n$ with vertex set $V = V_1 \cup V_2 \cup \cdots \cup V_n$ and edge set $E = E_1 \cup E_2 \cup \cdots \cup E_n$, where $V_1 \cap V_2 \cap \cdots \cap V_m = \varnothing$. Lakshmi and Vangipuram [13] proved that there is an α-valuation for the quadratic graph $Q(4, 4k)$ consisting of four cycles of length $4k$, $k \geq 1$. Abrham and Kotzig [1] proved that $C_m \cup C_n$ has an α-valuation if and only if both m and n are even and $m+n \equiv 0 \pmod 4$. Eshghi and Carter [6] showed several families of graphs of the form $C_{4_{n_1}} \cup C_{4_{n_2}} \cup \cdots \cup C_{4_{n_k}}$ that have α-valuations.

The *cartesian product* $G \square H$ of two graphs G and H is the graph with the vertex set

$$V(G \square H) = V(G) \square V(H)$$

and the edge set $E(G \square H)$ satisfying the following condition:

$$(x_1, x_2)(y_1, y_2) \in E(G \square H)$$

if and only if either $x_1 = y_1$ and $x_2 y_2 \in E(H)$ or $x_2 = y_2$ and $x_1 y_1 \in E(G)$.

The *corona* [7] of two graphs H_1 and H_2, denoted by $H_1 \odot H_2$, is the graph obtained by taking one copy of H_1, which has m vertices, and m copies of H_2, and then joining the kth vertex of H_1 with an edge to every vertex in the kth copy of H_2.

A unicyclic graph H (other than a cycle) is called a *hairy cycle* if the deletion of any edge e from the cycle of H results in a caterpillar. Thus, the coronas $C_n \odot mK_1$ are examples of hairy cycles. Kumar et. al. [11, 12, 15–18] proved that the hairy cycle $C_n \odot K_1$, $n \equiv 0 \pmod 4$, and graphs obtained by joining two graceful cycles by a path admit α-valuations. They also discussed that the subdivision of a cycle and pendant edges of $C_n \square K_4$, joining two isomorphic copies of $C_n \square K_4$, $C_n \odot K_1$, $n \equiv 0 \pmod 4$, and $C_n \odot K_1$, $n \equiv 3 \pmod 4$, are graceful. Moreover, they proved that $C_n \odot rK_1$, $n \equiv 3 \pmod 4$, and $C_n \odot K_1, n \equiv 0 \pmod 4$, are k-graceful. Graf [10] established that $C_n \odot K_1$ has a graceful valuation if $n \equiv 3$ or $4 \pmod 8$.

Barrientos [3, 4] showed that if G is a graceful graph with order greater than its size, then the graphs $G \odot nK_1$ and $G + nK_1$ are graceful. He also proved that helms (graphs obtained from a wheel by attaching one pendant edge to each vertex) are graceful. Minion and Barrientos, in [5] and [14], studied the gracefulness of $G \cup P_m$ and $C_r \cup G_n$, where G_n is a caterpillar of size n. Frucht and Salinas [8] analyzed the gracefulness of $C_m \cup P_n$, $n \geq 3$. Ropp [20] showed that the graph $(C_m \square P_2) \odot K_1$ is graceful. Truszczynski [22] conjectured that all unicyclic graphs except the cycle C_n, $n \equiv 1$ or $2 \pmod 4$, are graceful.

Labeled graphs are helpful mathematical models for coding theory, such as designing optimal radar, synch-set, missile guidance, and convolution codes with high auto-correlation. They make it easier to perform optimal nonstandard integer encoding.

This study focuses on graceful and α-valuation of some disconnected graphs. The concept of graceful and α-valuation in graph theory has attracted attention from many researchers during the past three decades. The earlier studies motivated us to research the problem that *the disjoint union of various hairy cycles* $C_{m_1}^{S_n} \cup C_{m_2}^{S_n} \cup \cdots \cup C_{m_k}^{S_n}$ *admits an α-valuation*, which we partially solve in the present paper.

2. Results

Theorem 2. *Let G be the graph obtained by the disjoint union of n isomorphic copies of the hairy cycle $C_4^{S_1}$. Then, G admits an α-valuation with exactly one missing number $\rho = 4n - 2$ and the critical value $\sigma = 4n$.*

P r o o f. Let $n \in \mathbb{N}$, and let G^k, $1 \leq k \leq n$, be the kth part of G. Let w_i^k and x_i^k, where $i = 1, 2, 3, 4$ and $k = 1, 2, \ldots, n$, denote vertices of the cycle and leaves of the kth part of G, respectively. Clearly, $|V(G)| = |E(G)| = 8n$.

To define $\Im : V(G) \to \{0, 1, 2, \ldots, 8n\}$, we label the vertices of G^1 as follows:

$$\Im(w_1^1) = 0, \quad \Im(w_2^1) = 8n-1, \quad \Im(w_3^1) = 3, \quad \Im(w_4^1) = 8n-3,$$
$$\Im(x_1^1) = 8n, \quad \Im(x_2^1) = 1, \quad \Im(x_3^1) = 8n-2, \quad \Im(x_4^1) = 4.$$

Next, we label the vertices of the remaining parts of G, $2 \le k \le n$, as follows:

$$\Im(w_i^k) = \begin{cases} 4k + 3(i-3)/2 & \text{if} \quad i = 1,3, \\ 4(2n-k+1) - i/2 & \text{if} \quad i = 2,4, \end{cases}$$
$$\Im(x_i^k) = \begin{cases} 4(k+1) - 5i/2 & \text{if} \quad i = 2,4, \\ 8n - 4k + (11-3i)/2 & \text{if} \quad i = 1,3. \end{cases}$$

Define the edge labeling $f^\star$ on $E(G^1)$ by

$$f^\star(wx) = |\Im(w) - \Im(x)|$$

for $wx \in E(G)$ as follows:

$$f^\star(w_1^1w_2^1) = 8n-1, \quad f^\star(w_2^1w_3^1) = 8n-4, \quad f^\star(w_3^1w_4^1) = 8n-6, \quad f^\star(w_4^1w_1^1) = 8n-3,$$
$$f^\star(w_1^1x_1^1) = 8n, \quad f^\star(w_2^1x_2^1) = 8n-2, \quad f^\star(w_3^1x_3^1) = 8n-5, \quad f^\star(w_4^1x_4^1) = 8n-7.$$

We label the remaining edges of G as follows:

$$f^\star(w_i^kw_{i+1}^k) = 8(n-k+1) - 2i \quad \text{for} \quad i = 1,3,$$
$$f^\star(w_2^kw_3^k) = 8(n-k) + 3,$$
$$f^\star(w_4^kw_1^k) = 8(n-k) + 5,$$
$$f^\star(w_i^kx_i^k) = 8(n-k) + 10 - 3i \quad \text{for} \quad i = 1,2,3,$$
$$f^\star(w_4^kx_4^k) = 8(n-k+1).$$

It is clear that all the vertex and edge labels are distinct. Therefore, the graph G is graceful. Next, we prove that the graceful function $\Im$ is an α-valuation with the missing number $\rho = 4n-2$ and the critical value $\sigma = 4n$. Since the vertex set V of G is partitioned into two sets, $V = A \cup B$, we have

$$A = \{0, 1, 3, 4, 5, 7, 8, 2, 9, 11, 12, 6, \ldots, 4n-3, 4n-1, 4n, 4n-6\},$$
$$B = \{8n, 8n-1, 8n-2, 8n-3, \ldots, 4n+1\}.$$

Clearly, A and B are independent sets. The number $\sigma = 4n$ satisfies $f^\star(w) \le \sigma < f^\star(x)$ for every ordered pair $(w, x) \in A \times B$. Therefore, $\Im$ is an α-valuation of G (see $C_4^{S_1} \cup C_4^{S_1} \cup C_4^{S_1}$ in Fig. 1). □

Theorem 3. *The disjoint union of two isomorphic copies of $C_4^{S_n}$ admits an α-valuation with exactly one missing number $\rho = 4(n+1)$ and the critical value $\sigma = 4n+5$.*

P r o o f. Let $i = 1, 2, 3, 4$ and $j = 1, 2, \ldots, n$. Denote by u_i (v_i) and u_{ij} (v_{ij}) the vertices of the cycle and leaves, respectively, in the first and second copies of $C_4^{S_n}$, respectively. Clearly,

$$|V(C_4^{S_n} \cup C_4^{S_n})| = |E(C_4^{S_n} \cup C_4^{S_n})| = 8(n+1).$$

Define $\vartheta : V(C_4^{S_n} \cup C_4^{S_n}) \to \{0, 1, 2, \ldots, 8(n+1)\}$ as follows: we label the vertices of the cycle of the first copy of $C_4^{S_n}$ by

$$\vartheta(u_1) = 0, \quad \vartheta(u_2) = 7n+8, \quad \vartheta(u_3) = n+2, \quad \vartheta(u_4) = 6n+7$$

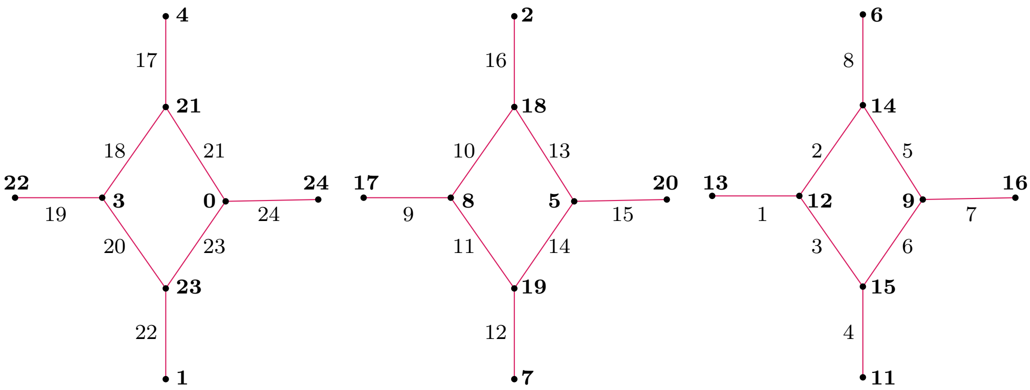

Figure 1. An α-valuation of $C_4^{S_1} \cup C_4^{S_1} \cup C_4^{S_1}$.

and the vertices of the cycle of the second copy of $C_4^{S_n}$ by

$$\vartheta(v_1) = 6(n+1), \quad \vartheta(v_2) = 3(n+1), \quad \vartheta(v_3) = 5(n+1), \quad \vartheta(v_4) = 4n+5,$$

respectively. Label the remaining vertices of the leaves in the graph $C_4^{S_n} \cup C_4^{S_n}$ as follows:

$$\vartheta(u_{ik}) = \begin{cases} 8(n+1) - (k-1) - \dfrac{(n+1)(i-1)}{2} & \text{if} \quad i = 1, 3, \\ k + \dfrac{(n+2)(i-2)}{2} & \text{if} \quad i = 2, 4, \end{cases}$$

$$\vartheta(v_{1k}) = 2(n+1), \quad \vartheta(v_{2k}) = 6(n+1) - k$$

for $1 \leq k \leq n$, and

$$\vartheta(v_{3k}) = 3n + k + 4, \quad \vartheta(v_{4k}) = 5(n+1) - k \quad \text{for} \quad 1 \leq k < n,$$

$$\vartheta(v_{3n}) = n + 1, \quad \vartheta(v_{4n}) = 3n + 4.$$

It can be verified that all vertices of the graph are labeled and the labels are distinct. Now, we construct labels for the edge set E of the graph. Define a labeling f on $E(C_4^{S_n} \cup C_4^{S_n})$ by $f(uv) = |\vartheta(u) - \vartheta(v)|$ for $uv \in E$. We label the edges of the cycle of the first copy of $C_4^{S_n}$ by

$$f(u_1u_2) = 7n+8, \quad f(u_2u_3) = 6(n+1), \quad f(u_3u_4) = 5(n+1), \quad f(u_4u_1) = 6n+7$$

and the edges of the cycle of the second copy of $C_4^{S_n}$ by

$$f(v_1v_2) = 3(n+1), \quad f(v_2v_3) = 2(n+1), \quad f(v_3v_4) = n, \quad f(v_4v_1) = 2n+1.$$

Label the remaining edges of the leaves in the graph $C_4^{S_n} \cup C_4^{S_n}$ as follows:

$$f(u_iu_{ik}) = \begin{cases} 8(n+1) - (k-1) - (n+1)(i-1) & \text{if} \quad i = 1, 2, \\ 8(n+1) - k - (n+1)(i-1) & \text{if} \quad i = 3, 4, \end{cases}$$

for $1 \leq k \leq n$,

$$f(v_iv_{ik}) = \begin{cases} 4(n+1) - k - (n+1)(i-1) & \text{if} \quad i = 1, 2, \\ 4(n+1) - k - 1 - (n+1)(i-1) & \text{if} \quad i = 3, 4, \end{cases}$$

for $1 \le k < n$, and

$$f(v_3 v_{3n}) = 4(n+1), \quad f(v_4 v_{4n}) = n+1.$$

It is quite clear that all the vertex and edge labels are distinct. Therefore, the graph $C_4^{S_n} \cup C_4^{S_n}$ is graceful. Next, we prove that this graceful function ϑ is an α-valuation with the missing number $\rho = 4(n+1)$ and the critical value $\sigma = 4n+5$. Since the vertex set V of $C_4^{S_n} \cup C_4^{S_n}$ is partitioned into two sets, $V = R \cup S$, we have

$$R = \{0, 1, 2, \ldots, n, n+2, n+3, \ldots, 2(n+1), 6(n+1), 6n+5, 6n+4, \\ \ldots, 5n+6, 2(2n+3), 4n+5, \ldots, 5n+4, 3n+4\}$$

and

$$S = \{8(n+1), 8n+7, 8n+6, \ldots, 7n+9, 7n+8, 7(n+1), \ldots, 2(3n+4), 2n+3, 2n+4, \\ \ldots, 3n+2, 3(n+1), 3n+5, 3n+6, \ldots, 4n+3, n+1\}.$$

Clearly, R and S are independent sets. The number $\sigma = 4n+5$ satisfies $f(u) \le \sigma < f(v)$ for every ordered pair $(u, v) \in R \times S$. Therefore, ϑ is an α-valuation of $C_4^{S_n} \cup C_4^{S_n}$ (see $C_4^{S_3} \cup C_4^{S_3}$ in Fig. 2). $\square$

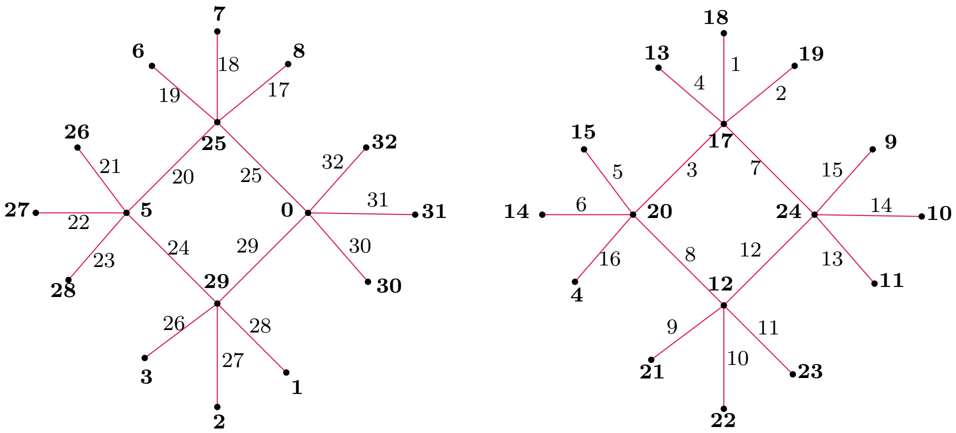

Figure 2. An α-valuation of $C_4^{S_3} \cup C_4^{S_3}$.

Theorem 4. *Let $C_{4m}^{S_1} \cup C_n^{S_1}$, $n \in \{4m, 4m-1, 4m-2\}$, be the disjoint union of hairy cycles. If there is a function*

$$\phi : V(C_{4m}^{S_1} \cup C_n^{S_1}) \to \{0, 1, 2, \ldots, 2(4m+n)\},$$

then

(i) *the graph $C_{4m}^{S_1} \cup C_n^{S_1}$, $n \in \{4m, 4m-2\}$, admits an α-valuation;*

(ii) *the graph $C_{4m}^{S_1} \cup C_{4m-1}^{S_1}$, $m \ge 1$, admits a graceful valuation.*

P r o o f. Let $C_{4m}^{S_1}$ be the graph (a hairy cycle) obtained by adding a pendant vertex to each vertex of the cycle of order $4m$. To prove this theorem, we need to prove the following claims.

Claim 1. The graph $C_{4m}^{S_1} \cup C_{4n}^{S_1}$, $m = n$ and $m \ge 1$, admits an α-valuation.

Claim 2. The graph $C_{4m}^{S_1} \cup C_{4m-2}^{S_1}$ admits an α-valuation.

Claim 3. The graph $C_{4m}^{S_1} \cup C_{4m-1}^{S_1}$, $m \geq 1$, admits a graceful valuation.

Before proving the claims, we fix a labeling of the hairy cycle $C_{4m}^{S_1}$ in $C_{4m}^{S_1} \cup C_n^{S_1}$. Because, throughout the proof, the labeling of the first part $C_{4m}^{S_1}$ is the same. For each $t \in \{4m, 4m-1, 4m-2\}$, we need to define a labeling ϕ on $C_{4m}^{S_1} \cup C_t^{S_1}$ and prove that this is an α-labeling. So, first, we define the labeling of the first part of the union as follows.

Let u_i and v_i, $i = 1, 2, \ldots, 4m$, be the vertices of the cycle and leaves of $C_{4m}^{S_1}$, respectively. Then,

$$\phi(u_i) = \begin{cases} i-1 & \text{if } \; i \leq 2m \; \text{ and } \; i \; \text{ is odd,} \\ i & \text{if } \; i > 2m \; \text{ and } \; i \; \text{ is odd,} \\ 2(4m+t)-(i-1) & \text{if } \; i \; \text{ is even,} \end{cases}$$

$$\phi(v_i) = \begin{cases} 2(4m+t)-(i-1) & \text{if } \; i \; \text{ is odd,} \\ i-1 & \text{if } \; i \leq 2m \; \text{ and } \; i \; \text{ is even,} \\ i & \text{if } \; i > 2m \; \text{ and } \; i \; \text{ is even.} \end{cases}$$

Next, we define the edge labeling g on the edges of $C_{4m}^{S_1}$ by

$$g(uv) = |\phi(u) - \phi(v)|$$

for $uv \in E$ as follows:

$$g(u_i u_{i+1}) = \begin{cases} 2(4m+t)-2i+1 & \text{if } \; i < 2m, \\ 2(4m+t)-2i & \text{if } \; 2m \leq i < 4m, \end{cases}$$

$$g(u_{4m}, u_1) = 4m + 2t + 1,$$

$$g(u_i v_i) = \begin{cases} 2(4m+t-i+1) & \text{if } \; i \leq 2m, \\ 2(4m+t)-2(i-1)-1 & \text{if } \; i > 2m. \end{cases}$$

Proof of Claim 1. This claim holds for only $m = n$.

Let x_i and y_i, $i = 1, 2, \ldots, 4n$, be the vertices of the cycle and leaves of $C_{4n}^{S_1}$ (the second part). Clearly,

$$|V(C_{4m}^{S_1} \cup C_{4m}^{S_1})| = |E(C_{4m}^{S_1} \cup C_{4m}^{S_1})| = 16m.$$

We now define the labeling of $C_{4m}^{S_1}$ as follows.

Case 1: *m is even.* We label the vertices by $\phi(x_{4m}) = 10m$,

$$\phi(x_i) = \begin{cases} 4m+i & \text{if } \; i \; \text{ is odd,} \\ 12m+1-i & \text{if } \; i \leq m \; \text{ and } \; i \; \text{ is even,} \\ 12m-i & \text{if } \; m < i < 2m \; \text{ and } \; i \; \text{ is even,} \\ 12m-1-i & \text{if } \; 2m \leq i \leq 4m-2 \; \text{ and } \; i \; \text{ is even,} \end{cases}$$

$$\phi(y_{3m-1}) = 11m, \quad \phi(y_{4m}) = 2m,$$

$$\phi(y_i) = \begin{cases} 12m-(i-1) & \text{if } \; i < m \; \text{ and } \; i \; \text{ is odd,} \\ 12m-i & \text{if } \; m \leq i < 2m \; \text{ and } \; i \; \text{ is odd,} \\ 12m-1-i & \text{if } \; 2m < i < 3m-1, \quad 3m-1 < i \leq 4m-1, \quad \text{ and } \; i \; \text{ is odd,} \\ 4m+i & \text{if } \; i \leq 4m-2 \; \text{ and } \; i \; \text{ is even.} \end{cases}$$

It can be verified that all vertices of the graph are labeled and all labels are distinct. Label the set

E of edges in the graph as follows:

$$g(x_i x_{i+1}) = \begin{cases} 2(4m-i) & \text{if} \quad i \le m, \\ 2(4m-i)-1 & \text{if} \quad m < i < 2m-1, \\ 2(4m-i)-2 & \text{if} \quad 2m-1 \le i \le 4m-2, \end{cases}$$

$$g(x_{4m-1}x_{4m}) = 2m+1, \quad g(x_{4m}x_1) = 6m-1,$$

$$g(x_i y_i) = \begin{cases} 2(4m-i)+1 & \text{if} \quad i \le m, \\ 2(4m-i) & \text{if} \quad m < i < 2m, \\ 2(4m-i)-1 & \text{if} \quad 2m \le i < 3m-1, \quad 3m-1 < i \le 4m-1. \end{cases}$$

$$g(x_{3m-1}y_{3m-1}) = 4m+1, \quad g(x_{4m}y_{4m}) = 8m.$$

Suppose that $m = n$ and m is even. Then, the labeling of $C_{4m}^{S_1} \cup C_{4n}^{S_1}$ is a graceful valuation. Moreover, the labeling of $C_{4m}^{S_1} \cup C_{4m}^{S_1}$ is actually an α-valuation with the critical value $2m-1$, and the number $9m/4$ is not assigned to any vertex of $C_{4m}^{S_1} \cup C_{4m}^{S_1}$.

Case 2: *m is odd.* If $m = 1$, the labeling follows from Theorem 2. If $m \ge 2$, the labeling is defined as follows:

$$\phi(w_{3m}) = 7m+2, \quad \phi(w_{4m}) = 10m,$$

$$\phi(w_i) = \begin{cases} 4m+i & \text{if} \quad i \le m \quad \text{and} \quad i \quad \text{is odd}, \\ 4m+1+i & \text{if} \quad m < i < 3m, \quad 3m < i \le 4m-1, \quad \text{and} \quad i \quad \text{is odd}, \\ 12m+1-i & \text{if} \quad i \le 2m \quad \text{and} \quad i \quad \text{is even}, \\ 12m-i & \text{if} \quad 2m < i \le 4m-2 \quad \text{and} \quad i \quad \text{is even}, \end{cases}$$

$$\phi(z_{3m+1}) = 5m+1, \quad \phi(z_{4m}) = 2m,$$

$$\phi(z_i) = \begin{cases} 12m+1-i & \text{if} \quad i < 2m \quad \text{and} \quad i \quad \text{is odd}, \\ 12m-i & \text{if} \quad 2m < i \le 4m-1 \quad \text{and} \quad i \quad \text{is odd}, \\ 4m+i & \text{if} \quad i < m \quad \text{and} \quad i \quad \text{is even}, \\ 4m+1+i & \text{if} \quad m < i < 3m, \quad 3m+1 < i \le 4m-2, \quad \text{and} \quad i \quad \text{is even}. \end{cases}$$

It can be verified that all the vertices of the graph are labeled and the labels are distinct. We now construct labels for the set E of edges in the graph as follows:

$$g(w_{3m-1}w_{3m}) = 2m-1, \quad g(w_{3m}w_{3m+1}) = 2m-3, \quad g(w_{4m-1}w_{4m}) = 2m, \quad g(w_{4m}w_1) = 6m-1,$$

$$g(w_i w_{i+1}) = \begin{cases} 2(4m-i) & \text{if} \quad i \le m, \\ 2(4m-i)-1 & \text{if} \quad m < i \le 2m, \\ 2(4m-i)-2 & \text{if} \quad 2m < i < 3m-1, \quad 3m < i \le 4m-2, \end{cases}$$

$$g(w_{3m}z_{3m}) = 2(m-1), \quad g(w_{3m+1}z_{3m+1}) = 4m-2, \quad g(w_{4m}z_{4m}) = 8m,$$

$$g(w_i z_i) = \begin{cases} 2(4m-i)+1 & \text{if} \quad i \le m, \\ 2(4m-i) & \text{if} \quad m < i \le 2m, \\ 2(4m-i)-1 & \text{if} \quad 2m < i < 3m, \quad 3m+1 < i \le 4m-1. \end{cases}$$

Through the close examination of the above function ϕ, it can be seen that the induced edge labeling is bijective. It is clear that all the vertex labels are distinct. The edge labels are computed from these vertex labels and are also found to be distinct from 1 to $16m$. Therefore, $C_{4m}^{S_1} \cup C_{4n}^{S_1}$, where m is odd and $m = n$, is a graceful valuation. Moreover, the labeling of $C_{4m}^{S_1} \cup C_{4m}^{S_1}$ is actually an α-valuation with the critical value $2m$, and the number $7m+4/4$ is not assigned to any vertex of $C_{4m}^{S_1} \cup C_{4m}^{S_1}$. This completes the proof of Claim 1.

Proof of Claim 2. Let a_i and b_i, $i = 1, 2, \ldots, 4m-2$, be the vertices of the cycle and leaves of $C_{4m-2}^{S_1}$. Define the labeling of $C_{4m-2}^{S_1}$ as follows:

$$\phi(a_{2m}) = 6m + 1,$$

$$\phi(a_i) = \begin{cases} 3(4m-1) - i & \text{if } \ i \ \text{ is odd,} \\ 4m + i & \text{if } \ 1 < i \le 2m-1, \quad 2m < i \le 4m-2, \quad \text{and} \quad i \quad \text{is even,} \end{cases}$$

$$\phi(b_{2m+1}) = 2m,$$

$$\phi(b_i) = \begin{cases} 4m + i & \text{if } \ 1 \le i \le 2m-1, \quad 2m+1 < i < 4m-2, \quad \text{and} \quad i \quad \text{is odd,} \\ 3(4m-1) - i & \text{if } \ i \ \text{ is even.} \end{cases}$$

Moreover, this produces the edge labels of $C_{4m-2}^{S_1}$:

$$g(a_{4m-2}a_1) = 4m-2, \quad g(a_{2m}b_{2m}) = 4(m-1), \quad g(a_{2m+1}b_{2m+1}) = 4m-2,$$

$$g(a_i a_{i+1}) = \begin{cases} 2(4m-2-i) & \text{if } \ i < 2m-1, \quad 2m < i < 4m-2, \\ 8m-5-2i & \text{if } \ 2m-1 \le i \le 2m, \end{cases}$$

$$g(a_i b_i) = 8m - 3 - 2i \quad \text{if } \ i \le 2m-1, \quad i > 2m+1.$$

Through these combined labelings of the hairy cycles $C_{4m}^{S_1}$ (defined before Claim 1) and $C_{4m-2}^{S_1}$ bring out the labeling of $C_{4m}^{S_1} \cup C_{4m-2}^{S_1}$, and its induced edge labeling is bijective. It is clear that all the vertex labels are distinct. The edge labels are computed from these vertex labels and are also found to be distinct from 1 to $16m-4$. Therefore, $C_{4m}^{S_1} \cup C_{4m-2}^{S_1}$ is a graceful valuation. Moreover, the labeling of $C_{4m}^{S_1} \cup C_{4m-2}^{S_1}$ is actually an α-valuation with critical value $5m-2$, and the number $3m/2$ is not a label of $C_{4m}^{S_1} \cup C_{4m-2}^{S_1}$ (see $C_8^{S_1} \cup C_6^{S_1}$ in Fig. 3). This completes the proof of Claim 2.

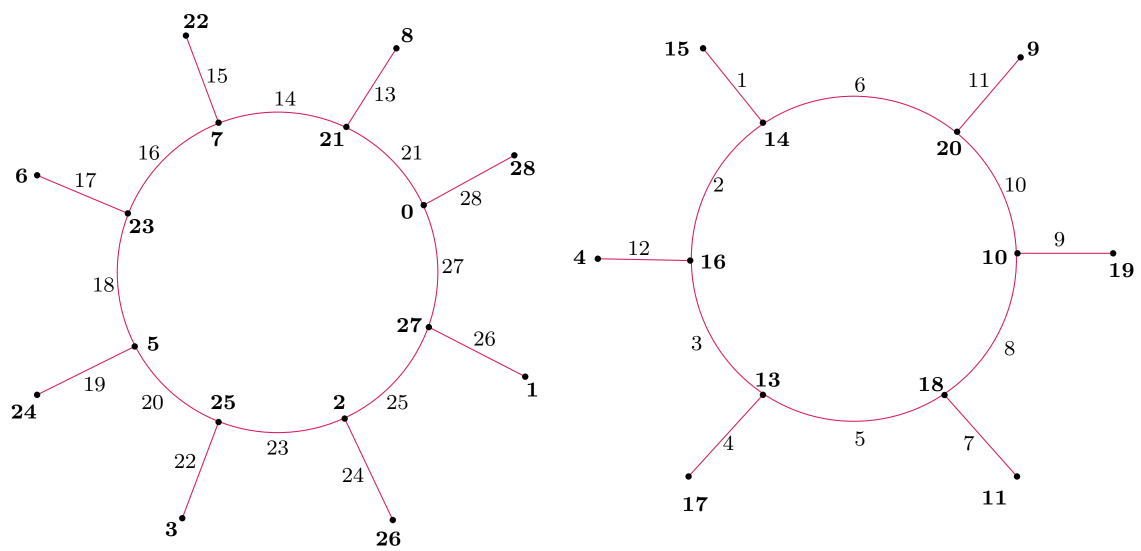

Figure 3. An α-valuation of $C_8^{S_1} \cup C_6^{S_1}$.

Proof of Claim 3. Let w_i and z_i, $i = 1, 2, \ldots, 4m-1$, be the vertices of the cycles and leaves of $C_{4m-1}^{S_1}$. Clearly,

$$|V(C_{4m}^{S_1} \cup C_{4m-1}^{S_1})| = |E(C_{4m}^{S_1} \cup C_{4m-1}^{S_1})| = 16m - 2.$$

Since the labeling of $C_{4m}^{S_1}$ is defined at the beginning of the proof, we only need to specify a labeling of $C_{4m-1}^{S_1}$ and $C_{4m-1}^{S_1}$, and we do this as follows.

Case 1: *m is even.* Define

$$\phi(x_{4m-1}) = 2(5m-1),$$

$$\phi(x_i) = \begin{cases} 4m+i & \text{if} \quad i \le 4m-3 \quad \text{and} \quad i \quad \text{is odd}, \\ 12m-1-i & \text{if} \quad i \le m \quad \text{and} \quad i \quad \text{is even}, \\ 2(6m-1)-i & \text{if} \quad m < i < 2m \quad \text{and} \quad i \quad \text{is even}, \\ 3(4m-1)-i & \text{if} \quad i \ge 2m \quad \text{and} \quad i \quad \text{is even}, \end{cases}$$

$$\phi(y_{3m-1}) = 11m-2, \quad \phi(x_{4m-1}) = 2m,$$

$$\phi(y_i) = \begin{cases} 12m-1-i & \text{if} \quad i < m \quad \text{and} \quad i \quad \text{is odd}, \\ 2(6m-1)-i & \text{if} \quad m < i < 2m \quad \text{and} \quad i \quad \text{is odd}, \\ 3(4m-1)-i & \text{if} \quad 2m < i < 3m-1, \quad 3m-1 < i < 4m-1, \quad \text{and} \quad i \quad \text{is odd}, \\ 4m+i & \text{if} \quad i \quad \text{is even}. \end{cases}$$

It can be verified that all the vertices of the graph are labeled and the labels are distinct. We now construct the set E of edge labels in the graph as follows:

$$g(x_{4m-2}x_{4m-1}) = 2m-1, \quad g(x_{4m-1}x_1) = 3(2m-1),$$

$$g(x_i x_{i+1}) = \begin{cases} 2(4m-1-i) & \text{if} \quad i \le m, \\ 2(4m-i)-3 & \text{if} \quad m < i < 2m-1, \\ 2(4m-2-i) & \text{if} \quad 2m-1 \le i \le 4m-3, \end{cases}$$

$$g(x_{3m-1}y_{3m-1}) = 4m-1, \quad g(x_{4m-1}y_{4m-1}) = 2(4m-1),$$

$$g(x_i y_i) = \begin{cases} 2(4m-i)-1 & \text{if} \quad i \le m, \\ 2(4m-1-i) & \text{if} \quad m < i \le 2m-1, \\ 2(4m-i)-3 & \text{if} \quad 2m-1 < i < 3m-1, \quad 3m-1 < i \le 4m-2. \end{cases}$$

Case 2: *m is odd and* $m \ge 1$. Figure 4 shows a graceful valuation for $m = 1$.

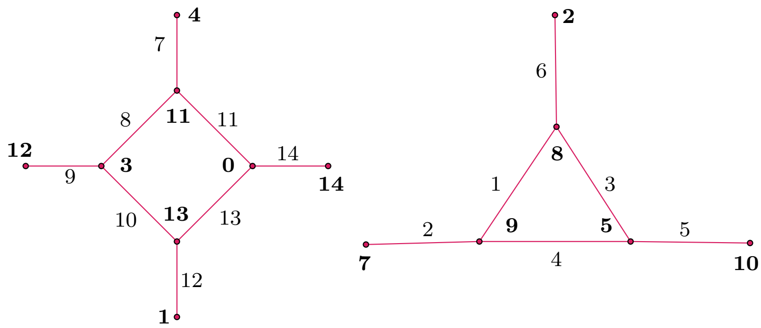

Figure 4. A graceful valuation of $C_4^{S_1} \cup C_3^{S_1}$.

If $m > 1$, then we define

$$\phi(w_{3m}) = 7m+2, \quad \phi(w_{4m-1}) = 10m-2,$$

$$\phi(w_i) = \begin{cases} 4m+i & \text{if} \quad i \le m \quad \text{and} \quad i \quad \text{is odd}, \\ 4m+1+i & \text{if} \quad m < i < 3m, \quad 3m < i < 4m-1, \quad \text{and} \quad i \quad \text{is odd}, \\ 12m-1-i & \text{if} \quad i \le 2m \quad \text{and} \quad i \quad \text{is even}, \\ 2(6m-1)-i & \text{if} \quad i > 2m \quad \text{and} \quad i \quad \text{is even}, \end{cases}$$

$$\phi(z_{4m-1}) = 2m, \quad \phi(z_{3m+1}) = 5m+1,$$

$$\phi(z_i) = \begin{cases} 12m-1-i & \text{if} \quad i < 2m \quad \text{and} \quad i \quad \text{is odd}, \\ 2(6m-1)-i & \text{if} \quad 2m < i \leq 4m-3 \quad \text{and} \quad i \quad \text{is odd}, \\ 4m+i & \text{if} \quad i < m \quad \text{and} \quad i \quad \text{is even}, \\ 4m+i+1 & \text{if} \quad m < i < 3m+1, \quad 3m+1 < i \leq 4m-1, \quad \text{and} \quad i \quad \text{is even}. \end{cases}$$

It can be verified that all the vertices of the graph are labeled and the labels are distinct. We now construct labels for the set E of edges in the graph as follows:

$$g(w_{3m-1}w_{3m}) = 2m-3, \quad g(w_{3m}w_{3m+1}) = 2m-5,$$
$$g(w_{4m-2}w_{4m-1}) = 2(m-1), \quad g(w_{4m-1}w_1) = 3(2m-1),$$
$$g(w_i w_{i+1}) = \begin{cases} 2(4m-1-i) & \text{if} \quad i \leq m, \\ 2(4m-i)-3 & \text{if} \quad m < i \leq 2m, \\ 2(4m-2-i) & \text{if} \quad 2m < i < 3m-1, \quad 3m < i < 4m-2, \end{cases}$$
$$g(w_{3m}z_{3m}) = 2(m-2), \quad g(w_{3m+1}z_{3m+1}) = 4(m-1), \quad g(w_{4m-1}z_{4m-1}) = 2(4m-1),$$
$$g(w_i z_i) = \begin{cases} 2(4m-i)-1 & \text{if} \quad i \leq m, \\ 2(4m-1-i) & \text{if} \quad m < i \leq 2m, \\ 2(4m-i)-3 & \text{if} \quad 2m < i < 3m, \quad 3m+1 < i < 4m-1. \end{cases}$$

We see that the labels of the edges of $C_{4m}^{S_1} \cup C_{4m-1}^{S_1}$ are distinct. Therefore, it can be easily shown that the graph $C_{4m}^{S_1} \cup C_{4m-1}^{S_1}$ has graceful valuations (see $C_8^{S_1} \cup C_7^{S_1}$ in Fig. 5). This completes the proof of Claim 3. □

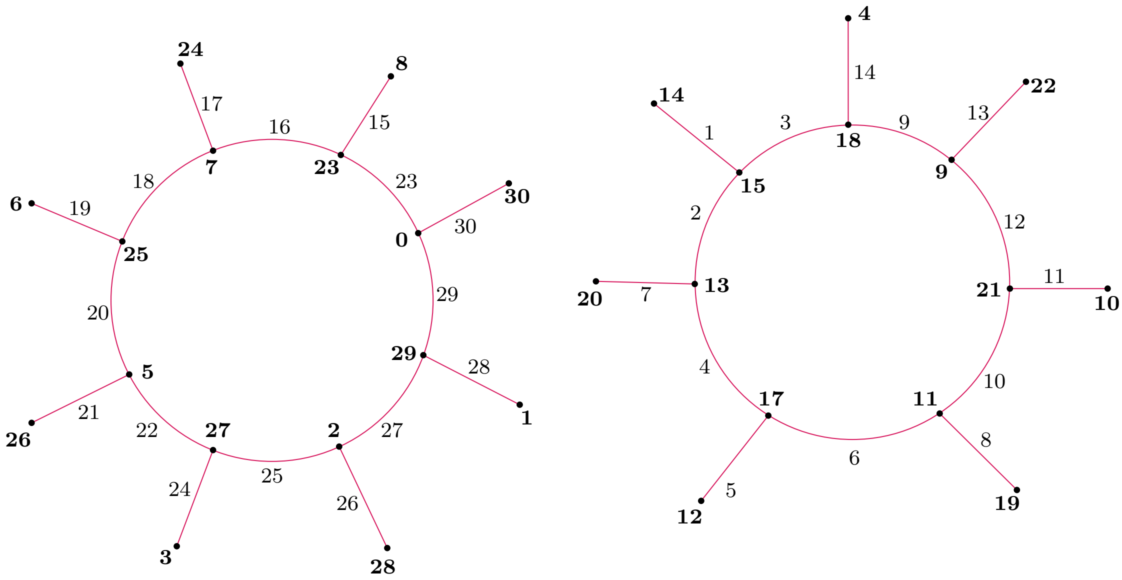

Figure 5. A graceful valuation of $C_8^{S_1} \cup C_7^{S_1}$.

Theorem 5. *The graph obtained by the disjoint union of two isomorphic copies of all hairy cycles $C_{4m+2}^{S_1}$ admits an α-valuation with exactly one missing number $\rho = 4(2m+1)$ and the critical value $\sigma = 2(3m+2)$.*

P r o o f. Let $i = 1, 2, 3, \ldots, 4m+2$, and let p_i (r_i) and q_i (s_i) denote the vertices of the cycle and leaves, respectively, in the first and second copies of $C_{4m+2}^{S_1}$, respectively. Clearly,

$$|V(C_{4m+2}^{S_1} \cup C_{4m+2}^{S_1})| = |E(C_{4m+2}^{S_1} \cup C_{4m+2}^{S_1})| \ = \ 8(2m+1).$$

Define a function $\xi : V(C_{4m+2}^{S_1} \cup C_{4m+2}^{S_1}) \to \{0, 1, 2, \ldots, 8(2m+1)\}$ as follows. Figure 6 shows an α-valuation for $m = 1$.

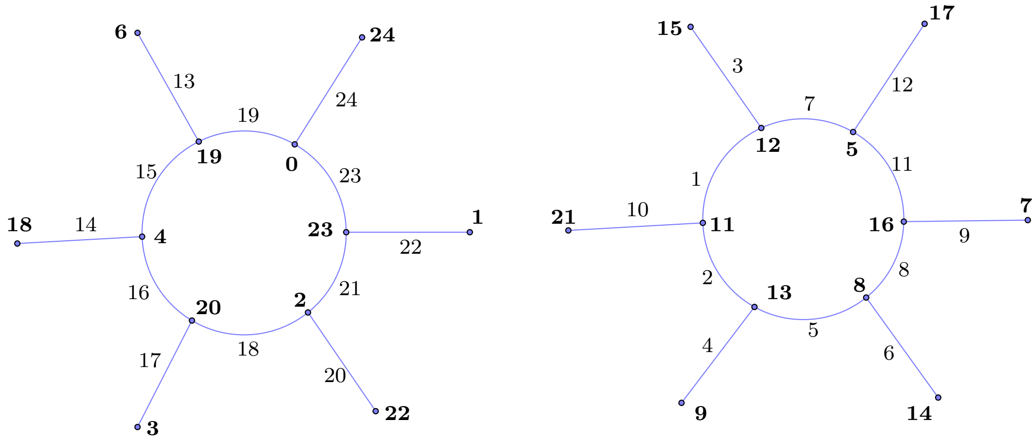

Figure 6. An α-valuation of $C_6^{S_1} \cup C_6^{S_1}$.

For $m > 1$, we label the vertices of $C_{4m+2}^{S_1} \cup C_{4m+2}^{S_1}$ as follows:

$$\xi(p_{4m+2}) = 12m + 7,$$

$$\xi(p_i) = \begin{cases} 16m + 9 - i & \text{if} \quad i < 2m+1 \quad \text{and} \quad i \quad \text{is even}, \\ 16m + 8 - i & \text{if} \quad 2m+1 \le i \le 4m \quad \text{and} \quad i \quad \text{is even}, \\ i - 1 & \text{if} \quad i \quad \text{is odd}, \end{cases}$$

$$\xi(q_{4m+1}) = 12m + 6, \quad \xi(q_{4m+2}) = 4m + 2,$$

$$\xi(q_i) = \begin{cases} 16m + 9 - i & \text{if} \quad i \le 2m+1 \quad \text{and} \quad i \quad \text{is odd}, \\ 16m + 8 - i & \text{if} \quad 2m+1 < i \le 4m-1 \quad \text{and} \quad i \quad \text{is odd}, \\ i - 1 & \text{if} \quad i \le 4m \quad \text{and} \quad i \quad \text{is even}, \end{cases}$$

$$\xi(r_1) = 4m + 1, \quad \xi(r_{2m+3}) = 6m + 5, \quad \xi(r_{2m+2}) = 10m + 3,$$

$$\xi(r_i) = \begin{cases} 4m + 1 + i & \text{if} \quad 1 < i \le 2m+1, \quad 2m+3 < i \le 4m+1, \quad \text{and} \quad i \quad \text{is odd}, \\ 12m + 6 - i & \text{if} i < 2m+2, \quad 2m+2 < i \le 4m+2, \quad \text{and} \quad i \quad \text{is even}, \end{cases}$$

$$\xi(s_{2m+1}) = 10m + 4, \quad \xi(s_{2m+3}) = 14m + 7, \quad \xi(s_{2m+5}) = 10m + 5,$$

$$\xi(s_i) = \begin{cases} 12m + 6 - i & \text{if} \quad i < 2m+1 \quad \text{and} \quad i \quad \text{is odd}, \\ 12m + 8 - i & \text{if} \quad 2m+5 < i < 4m+2 \quad \text{and} \quad i \quad \text{is odd}, \\ 4m + 1 + i & \text{if} \quad 1 < i \le 2m+2 \quad \text{and} \quad i \quad \text{is even}, \\ 4m + 3 + i & \text{if} \quad 2m+2 < i \le 4m+2 \quad \text{and} \quad i \quad \text{is even}. \end{cases}$$

Clearly, ξ is injective. Now, we prove that the induced labeling

$$\ell : E(C_{4m+2}^{S_1} \cup C_{4m+2}^{S_1}) \to \{1, 2, \ldots, 8(2m+1)\}$$

defined as $\ell(xy) = |\xi(x) - \xi(y)|$ for $xy \in E(C_{4m+2}^{S_1} \cup C_{4m+2}^{S_1})$ is bijective.

The induced edge labeling ℓ has the following values:

$$\ell(p_{4m+2}p_1) = 12m+7, \quad \ell(p_{4m+1}p_{4m+2}) = 8m+7,$$

$$\ell(p_ip_{i+1}) = \begin{cases} 16m+9-2i & \text{if} \quad i < 2m+1, \\ 16m+8-2i & \text{if} \quad 2m+1 \le i \le 4m, \end{cases}$$

$$\ell(p_{4m+1}q_{4m+1}) = 8m+6, \quad \ell(p_{4m+2}q_{4m+2}) = 8m+5,$$

$$\ell(p_iq_i) = \begin{cases} 16m+10-2i & \text{if} \quad i \le 2m+1, \\ 16m+9-2i & \text{if} \quad 2m+1 < i \le 4m, \end{cases}$$

$$\ell(r_1r_2) = 8m+3, \quad \ell(r_{2m+2}r_{2m+3}) = 4m-2, \quad \ell(r_{4m+2}r_1) = 4m+3,$$

$$\ell(r_{2m+1}r_{2m+2}) = 4m+1, \quad \ell(r_{2m+3}r_{2m+4}) = 4m-3,$$

$$\ell(r_ir_{i+1}) = 8m+4-2i, \quad \text{for} \quad 1 < i < 2m+1, \quad 2m+4 \le i \le 4m+1,$$

$$\ell(r_1s_1) = 8m+4, \quad \ell(r_{2m+3}r_{2m+3}) = 8m+2, \quad \ell(r_{2m+5}s_{2m+5}) = 4m-1,$$

$$\ell(r_{4m+2}s_{4m+2}) = 1, \quad \ell(r_{2m+1}s_{2m+1}) = 4m+2, \quad \ell(r_{2m+2}s_{2m+2}) = 4m,$$

$$\ell(r_is_i) = \begin{cases} 8m+5-2i & \text{if} \quad 1 < i < 2m+1, \\ 8m-2i+3 & \text{if} \quad 2m+4 \le i < 4m+2 \quad \text{and} \quad i \quad \text{is even}, \\ 8m-2i+7 & \text{if} \quad i > 2m+5 \quad \text{and} \quad i \quad \text{is odd}. \end{cases}$$

It is clear that all the vertex and edge labels are distinct. Therefore, the graph $C_{4m+2}^{S_1} \cup C_{4m+2}^{S_1}$ is graceful. Next, we prove that the above graceful function ξ is an α-valuation with the missing number $\rho = 4(2m+1)$ and the critical value $\sigma = 2(3m+2)$. Since the vertex set V of $C_{4m+2}^{S_1} \cup C_{4m+2}^{S_1}$ is partitioned into two sets, $V = X \cup Y$, we have

$$X=\{0,1,2,\dots,4m,4m+2,4m+1,4m+3,4m+4,\dots,6m+3,6m+5,6m+7,6m+6,\dots,8m+5\}$$

and

$$Y=\{8(2m+1),16m+7,16m+6,\dots,14m+8,14m+6,\dots,12m+6,12m+7,12m+5,12m+4,$$
$$\dots,10m+6,10m+4,\dots,14m+7,10m+2,10m+5,\dots,8m+4\}.$$

Clearly, X and Y are independent sets. The number $\sigma = 2(3m+2)$ satisfies $f(x) \le \sigma < f(y)$ for every ordered pair $(x,y) \in X \times Y$. Therefore, ξ is an α-valuation of $C_{4m+2}^{S_1} \cup C_{4m+2}^{S_1}$. □

3. Conclusion

This paper discussed graceful and α-valuations of certain disconnected graphs. Finding general characterizations for such graphs is an open problem. We also propose the following problem.

Problem 3.1. The disjoint union of various hairy cycles $C_{m_1}^{S_n} \cup C_{m_2}^{S_n} \cup \dots \cup C_{m_k}^{S_n}$ admits an α-labeling.

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