

# THE MINIMAL DOMINATING SETS IN A DIRECTED GRAPH AND THE KEY INDICATORS SET OF SOCIO-ECONOMIC SYSTEM<sup>1</sup>

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**Abstract:** The paper deals with a digraph with non-negative vertex weights. A subset  $W$  of the set of vertices is called dominating if any vertex that not belongs to it is reachable from the set  $W$  within precisely one step. A dominating set is called minimal if it ceases to be dominating when removing any vertex from it. The paper investigates the problem of searching for a minimal dominating set of maximum weight in a vertex-weighted digraph. An integer linear programming model is proposed for this problem. The model is tested on random instances and the real problem of choosing a family of key indicators in a specific socio-economic system. The paper compares this model with the problem of choosing a dominating set with a fixed number of vertices.

**Keywords:** Combinatorial Optimization, Boolean programming, Minimal dominating set, Key indicators.

## 1. Introduction

Let  $G$  be a directed graph with a vertex set  $V$  and an arc set  $E$ ,  $|V| = n$ . Denote by  $ij$  the arc from a vertex  $i$  to a vertex  $j$  of the digraph  $G$ . A set  $W \subseteq V$  is called dominating if, for every vertex  $j \in V \setminus W$ , there is a vertex  $i \in W$  such that  $ij \in E$ . A dominating subset  $W \subseteq V$  is called minimal dominating if none of its proper subsets is dominating. The problem of finding a minimal dominant set of maximum cardinality (Upper Domination) is widely known. In [3], its  $NP$ -hardness for the case of an undirected graph was proven. This quickly leads to the  $NP$ -hardness in the case of a directed graph. Some classes of graphs on which Upper Domination is polynomially solvable are known. This concerns those graphs on which the maximum cardinality of the minimal dominating set coincides with the independence number (for example, bipartite graphs), and the independence number for graphs of these classes can be computed in polynomial time. In addition to bipartite graphs [4], Upper Domination is polynomially solvable for chordal graphs [7], generalized series-parallel graphs [6], and graphs with bounded clique-width [5]. Much attention has recently been paid to the approximation properties of this problem. For example, in [2], it is shown that Upper Domination does not allow for  $n^{1-\epsilon}$  approximation for any  $\epsilon > 0$  unless  $P = NP$ . This makes Upper Domination significantly harder than the problem of the dominating set (without the condition of minimality) with the cardinality, which is bounded from above. We consider a weighted Upper Domination on a directed graph with positive vertex weights, which we will call the Weight MinDom Problem (WMDP). We are interested in using an integer linear program to find an exact solution to the problem. Let us denote the families of all dominating and

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minimal dominating sets in the digraph  $G$  by  $I(G)$  and  $I_m(G)$ , respectively. Let  $c_i > 0$ ,  $i \in V$ , be weights of vertices  $P$  of the graph  $G$ . By the weight of the subset  $S \subseteq V$ , we mean the number  $c(S) = \sum_{i \in S} c_i$ . In this notation, the combinatorial setting of the WMDP takes the form: find  $W^* \in I_m(G)$  such that  $c(W^*) \geq c(W)$  for any  $W \in I_m(G)$ . We will associate the vertex set  $V$  of the digraph  $G$  with the Euclidean space  $\mathbb{R}^V$  by the one-to-one correspondence between the elements of the set  $V$  and the coordinate axes of the space  $\mathbb{R}^V$ . In other words,  $\mathbb{R}^V$  is the space of column vectors whose components are indexed by elements of the set  $V$ . For every  $S \subseteq V$  we consider its incidence vector  $x^S \in \mathbb{R}^V$  with the coordinates  $x_i^S = 1$  for  $i \in S$  and  $x_i^S = 0$  for  $i \notin S$ . We define the polytopes of dominating sets and minimal dominating sets as

$$P(G) = \text{conv}\{x^W \in \mathbb{R}^V \mid W \in I(G)\}$$

and

$$P_m(G) = \text{conv}\{x^W \in \mathbb{R}^V \mid W \in I_m(G)\},$$

respectively. Here  $\text{conv}$  means the convex hull. Note that if  $c \in \mathbb{R}^V$  is a column vector with components  $c_i$ , then the following equalities are true for all  $S \subseteq V$ :

$$\sum_{i \in S} c_i = \sum_{i \in V} c_i x_i^S = c^T x^S,$$

where  $T$  is the transpose sign. Thus, the WMDP can be considered in the polyhedral setting as the problem of maximizing the linear function  $c^T x$  on the vertex set of the polytope  $P_m(G)$ .

In this paper, we obtain the following results. First, the nonlinear characterization of minimal dominating sets is extended from the case of an undirected graph to a directed graph. Second, a linear integer programming model is constructed for the problem under consideration. This model requires the introduction of additional variables different from the variables of the space  $\mathbb{R}^V$ . This increases the dimension of the problem but allows us to formalize the condition of minimality of dominating sets in terms of linear inequalities. Third, we propose an ILP model for the approximate solution to the problem. This model is defined by replacing the minimality condition of the dominating set with the fixation of its cardinality. Since the power of the dominating set does not exceed  $n$ , we can use enumeration by cardinality. This increases the number of ILP problems to be solved. But, as the computational experiment shows, the time to solve each of them is significantly shorter than that of the original exact model.

The paper is structured as follows: Section 2 contains a nonlinear model in the space  $\mathbb{R}^V$  that describes the set of incidence vectors of minimal dominating sets; Section 3 contains a Boolean linear programming model for the WMDP; Section 4 contains a description of the applied problem of the key indicators system, a brief overview of the results of the research conducted by economists, and the ways to formalize this problem in terms of WMDP; Section 5 presents the results of a numerical experiment.

## 2. Nonlinear characterization of minimal dominating sets

For every vertex  $i \in V$ , define

$$N_-[i] = \{j \in V \mid ji \in E\} \cup \{i\},$$

$$N_+[i] = \{j \in V \mid ij \in E\} \cup \{i\}.$$

In this notation, the dominating set in a graph  $G$  is a set  $W \subseteq V$  such that the condition  $N_-[i] \cap W \neq \emptyset$  is satisfied for each vertex  $i \in V$ .

**Lemma 1.** *The dominating set  $W$  in a digraph  $G$  is minimal if and only if, for every vertex  $i \in W$ , there exists a vertex  $j \in N_+[i]$  such that  $N_-[j] \cap W = \{i\}$ .*

*P r o o f.* Necessity. Let  $W$  be the minimal dominating set. Let us suppose that there exists a vertex  $k \in W$  such that  $N_-[j] \cap W \neq \{k\}$  for all  $j \in N_+[k]$ . Since the vertex  $k$  obviously belongs to both sets  $N_-[j]$  and  $W$ , this assumption means that their intersection contains at least one more vertex in addition to the vertex  $k$ . Consequently, for every vertex  $j \in N_+[k]$ , the set  $(N_-[j] \cap W) \setminus \{k\}$  is nonempty. Let us prove that the set  $W' = W \setminus \{k\}$  is dominating. It must be shown that the condition  $N_-[j] \cap W' \neq \emptyset$  holds for all  $j \in V \setminus W'$ . It is clear that  $V \setminus W' = (V \setminus W) \cup \{k\}$ . If  $j = k$ , then in view of that  $k \in N_+[k]$  and the assumption made, we have  $N_-[k] \cap W' \neq \emptyset$ . Assume that  $j \in V \setminus W$ . If  $j \in N_+[k]$ , then, as it is shown above,  $N_-[j] \cap W' = (N_-[j] \cap W) \setminus \{k\} \neq \emptyset$ . Finally, according to the definition of the dominating set, if  $j \notin N_+[k]$ , then it is reachable from a vertex of the set  $W$  different from  $k$ . Thus, the set  $W' = W \setminus \{k\}$  is dominating, which contradicts the minimality of the dominating set  $W$ .

Sufficiency. We will show that no single vertex of the set  $W$  can be discarded without losing the dominance property. Take an arbitrary vertex  $k \in W$  and consider the set  $W' = W \setminus \{k\}$ . Since  $k \in W$ , by the assumption, there is a vertex  $j \in N_+[k]$  such that  $N_-[j] \cap W = \{k\}$ . Then  $N_-[j] \cap W' = (N_-[j] \cap W) \setminus \{k\}$ , which means that the set  $W'$  is not dominating. Hence, due to the arbitrariness of the vertex  $k \in W$ , the set  $W$  is a minimal dominating one.

The lemma is proved. □

It is easy to see that an integer vector  $x \in \mathbb{R}^V$  is an incidence vector of a dominating set (without the minimality condition) if and only if it satisfies the constraints

$$\sum_{j \in N_-[i]} x_j \geq 1, \quad i \in V, \tag{2.1}$$

$$0 \leq x_i \leq 1, \quad i \in V. \tag{2.2}$$

The following theorem immediately follows from Lemma 1.

**Theorem 1.** *An integer vector  $x \in \mathbb{R}^V$  satisfying conditions (2.1)–(2.2) is an incidence vector of a minimal dominating set if and only if it satisfies the system of constraints*

$$x_i \cdot \prod_{j \in N_+[i]} (1 - \sum_{k \in N_-[j]} x_k) = 0, \quad i \in V.$$

In [3], a similar characterization of minimal dominating sets without the condition of the directivity of the original graph was described.

### 3. Linear integer model

To use integer linear programming, first of all, it is necessary to have good polyhedral relaxation of the polytope  $P_m(G)$ . Under “good” polyhedral relaxation, we mean a system of linear equations and inequalities in the space  $\mathbb{R}^V$  whose integer solutions are no other but all vertices of the polytope  $P_m(G)$ . It is easy to see that, for the polytope  $P(G)$  of dominating sets without the minimality condition, the polyhedral relaxation satisfying these requirements is the polyhedron  $M(G)$  defined by the constraints (2.1)–(2.2). For this polyhedron, we will also use the matrix notation

$$M(G) = \{x \in \mathbb{R}^V \mid Ax \geq 1, 0 \leq x \leq 1\},$$

where  $A$  is an  $(n \times n)$ -matrix with coefficients  $a_{ji} = 1$  for  $j \in N_-[i]$  and  $a_{ji} = 0$  in other cases. Since  $P_m(G)$  is obviously a subset of the polyhedron  $M(G)$ , we can consider the polyhedron  $M(G)$  to be a relaxation of minimal dominating sets polytope. However,  $M(G)$  contains incidence vectors of all dominating sets, including those that do not have the minimality property. We will construct a system of linear equations and inequalities that will allow us to formalize the WMDP as a Boolean linear programming problem. The model will use additional variables  $y_{ij} \in \{0, 1\}$ ,  $i, j = 1, 2, \dots, n$ .

**Theorem 2.** *The system of inequalities with respect to variables  $x_k, y_{ik} \in \{0, 1\}$ ,  $i, k = 1, 2, \dots, n$ ,*

$$\sum_{j=1}^n a_{ij}x_j \geq 1, \quad i = 1, 2, \dots, n, \quad (3.1)$$

$$\sum_{i=1}^n y_{ik} - x_k \geq 0, \quad i = 1, 2, \dots, n, \quad (3.2)$$

$$\sum_{j=1}^n a_{ij}x_j - a_{ik}x_k \leq n(1 - y_{ik}), \quad i, k = 1, 2, \dots, n, \quad (3.3)$$

$$x_k, y_{ik} \in \{0, 1\}, \quad i, k = 1, 2, \dots, n, \quad (3.4)$$

has a solution if and only if the  $(0, 1)$ -vector  $x$  is the incidence vector of the minimal dominating set of the graph  $G$ .

*P r o o f.* Let  $(x, y)$  be a solution of system (3.1)–(3.4). By constraints (3.1), the vector  $x$  is an incidence vector of a set  $W \in I(G)$ . Assume that  $W \notin I_m(G)$ . Then there is  $k \in W$  such that  $W \setminus \{k\} \in I(G)$ . This means that constraints (3.1) are also satisfied for the vector  $x^{W \setminus \{k\}}$ . This and constraints (3.3) result in that

$$n(1 - y_{ik}) \geq \sum_{j=1}^n a_{ij}x_j^W - a_{ik}x_k^W = \sum_{j=1}^W a_{ij}x_j^{W \setminus \{k\}} \geq 1,$$

for all  $i = 1, 2, \dots, n$ . Hence,  $y_{ik} = 0$ . However, (3.2) implies

$$\sum_{i=1}^n y_{ik} \geq x_k^W = 1,$$

a contradiction. Now let  $W \in I_m(G)$ . Define the following values of the variables  $y_{ik}$  for  $i, k = 1, 2, \dots, n$ :

$$y_{ik} = \begin{cases} 0 & \text{if } k \notin W; \\ 0 & \text{if } k \in W \text{ and } \sum_{j=1}^n a_{ij}x_j^W - a_{ik}x_k^W \geq 1; \\ 1 & \text{if } k \in W \text{ and } \sum_{j=1}^n a_{ij}x_j^W - a_{ik}x_k^W < 1. \end{cases}$$

We will show that the pair  $(x^W, y)$  is a solution to system (3.1)–(3.4). Inequalities (3.1) hold because  $W \in I_m(G)$ . Inequality (3.3) is always true for  $y_{ik} = 0$  and holds for  $y_{ik} = 1$  by construction. Inequalities (3.2) are obviously true for all  $x_k^W = 0$ . Let us show that they are also true for  $x_k^W = 1$ . Since  $W \in I_m(G)$ , we have  $x^{W \setminus \{k\}} \notin I_m(G)$ . Therefore, for all  $k \in W$ , there exists  $l \in \{1, 2, \dots, n\}$  such that

$$\sum_{j=1}^n a_{lj}x_j^W - a_{lk}x_k^W = \sum_{j=1}^n a_{lj}x_j^{W \setminus \{k\}} = 0.$$

Then  $y_{lk} = 1$  by construction. Therefore, for all  $x_k^W = 1$ , we have

$$\sum_{i=1}^n y_{ik} - x_k^W \geq y_{lk} - x_k^W = 0.$$

The proof is complete.  $\square$

Thus, the integer formulation of the WMDP is to maximize the function  $c(x) = c^T x$  under conditions (3.1)–(3.4).

#### 4. Practical application for key indicators problem

Recently, in data analysis, an approach based on selecting a specific subset from a large number of indicators has been used. Using this subset of indicators, we can conclude the state of the whole system. This subset, as a rule, should fully reflect the state of the system, have a high sensitivity to the changes in the situation, and interact with other indicators to a sufficiently strong degree. In other words, we distinguish some “key” indicators from the list of indicators. In modern economic science, when analyzing the state of the regional economy, much attention is paid to creating an indicative evaluation system. This approach is based on the selection of a system of key indicators and the analysis of their values. The number of approaches to the formation of key indicator systems grows with each new publication (see, for example, [8–10, 12], etc.). However, the ambiguity of the proposed formulations, the methodological inconsistency of concepts, and especially the lack of a sound methodology for calculating the numerical characteristics make it difficult, in our opinion, to build an objective concept of an indicative assessment of the economic situation. In the literature on economics, the criteria for selecting these limited indicators set are not clear. In this case, when analyzing the existing systems of key indicators, we again come across the opinions of the experts, the number of which often prevail over the number of indicators themselves. We propose to consider the minimal dominating sets of a special graph associated with the economic system as key indicator systems. We are far from thinking that this approach is the best. It should be considered as a means applicable for a certain range of situations, as an element of a hybrid analysis of not only the economic condition but also a wider range of tasks. In [1, 11] there is a comparison of different key indicator systems of economic security, among which there is a system obtained in the framework of our approach.

So, let  $V = \{1, 2, \dots, n\}$  be a particular set of indicators given a priori, for each of which there are a technique for calculating its value at each given moment in time and a set of these values for a certain period (several years, months, days, etc.). We believe that the statistics available allow us to calculate the sample correlation coefficients  $k(i, j)$  between any pair of indicators  $i, j \in V$ . We will introduce two additional considerations to the correlation matrix. First, from a practical point of view, the value of the correlation coefficient may turn out to be so small that the dependence on the corresponding indicators can be neglected. In this regard, as a control parameter of the model, we introduce the value  $\alpha$  — the dependence threshold — thus assuming that if  $|k(i, j)| < \alpha$ , then  $|k(i, j)| = 0$ . Second, if there is a relationship between the two indicators in terms of a causal relationship, the following alternative is important: “ $i$  determines  $j$ ” or “ $j$  determines  $i$ .” For example, if the random variable  $i$  is the number of fires in the village, and  $j$  is the number of fire brigade visits during the same period, then it is clear that “ $i$  determines  $j$ ” and not vice versa. Such a causal relationship is denoted by  $i \rightarrow j$ . Nevertheless, in some cases, a situation is possible where the primary and secondary indicators cannot be determined, or they are interdependent. In such cases, we will write  $i \leftrightarrow j$ , which, basically, is equivalent to a pair of conditions  $i \rightarrow j$  and  $j \rightarrow i$ . As a result, for any chosen dependence threshold  $\alpha$ , we will fix the correlation matrix  $K(\alpha)$ .

Now we will associate our set of exponents  $V$  and the correlation matrix  $K(\alpha)$  with the directed graph  $G_\alpha$ , the vertex set of which is the set of indicators  $V$ , and the set of arcs  $E$  is defined as follows: from  $i$  to  $j$  there is an arc if and only if  $|k(i, j)| \geq \alpha$  and  $i \rightarrow j$ . The graph  $G_\alpha$  is called the  $\alpha$ -correlation graph of the set of indicators under consideration. A subset  $W \subseteq V$  is called a key indicators system if  $W$  is a minimal dominating set in the graph  $G_\alpha$ . In this approach, an important feature of a key indicators system is that the indicators belonging to the system are key ones only in the aggregate. This consideration becomes more noticeable because, in general, there are many minimal dominating sets in a graph. We will define the weight (degree of influence) of the indicator  $i$  as

$$c_i = \sum_{j \in V | ij \in E} |k(i, j)|$$

and the weight of the subset  $S \subseteq V$  as

$$c(S) = \sum_{i \in S} c_i.$$

## 5. Numerical experiment

Using the constructed WMDP model in numerical experiments on random data, we noticed that, for a relatively large dimension ( $n > 60$ ), it takes too much time to solve the problem (more than 3 hours). We used a computer Intel(R) Celeron(R) CPU N2830 2.16GHz and the commercial package IBM ILOG CPLEX Optimization Studio 12.10. Initially, the interest in the problem of the minimal dominating set of maximum weight was caused by the problem of finding a system of key indicators among some a priori given set of indicators of economic security discussed in the previous section. These indicators and their values were withdrawn from the website of the Federal State Statistics Service of the Russian Federation. Their number is estimated at several hundred. In this regard, a question arose of using additional considerations that would allow us to find acceptable solutions to practical problems in a reasonable time.

To achieve this goal, we consider one more model for choosing a system of key indicators. We discard the minimality condition of the dominating sets and replace this condition with a search for the maximum weight of the dominating set of a given cardinality. From an economic point of view, this approach also makes sense. It is true that when distinguishing a system of key indicators at a practical level, two basic considerations are important. Firstly, there should be relatively few key indicators and, secondly, they should have a significant impact on the situation. For this, in principle, the dominance condition is sufficient. The number of key indicators can be considered as an external condition. Under this assumption, the problem is formalized as a Boolean programming problem of the following form

$$\max \left\{ c^T x \mid Ax \geq 1, \sum_{i=1}^n x_i = h, x \in \{0, 1\}^V \right\}. \quad (5.1)$$

We will denote such a problem by  $WDP(h)$ . In this approach, we can solve a series of  $WDP(h)$  problems by decreasing the value of the  $h$  parameter until the problem ceases to have a solution. It is clear that, in this case, we also get the minimal dominating set of the least cardinality. However, it remains an open question, whether it will be the minimum dominating set of maximum weight. Since the dimension of  $WDP(h)$  is less than the dimension of WMDP, this approach may be computationally less time-consuming than using the WMDP model directly. In this regard, our computational experiment has the following objectives.

1. Evaluation of solutions obtained using the WDP( $h$ ) procedure. The comparison is carried out both in terms of the weight and power of the obtained obtained.
2. The behavior of the WDP( $h$ ) procedure concerning the structural properties of the digraph  $G$ .

We will use the following notation:

WDP( $h$ ) — the procedure based on a monotonic decrease in the values of the parameter  $h$  in the model (5.1). The procedure ends when WDP( $h$ ) has a solution, and WDP( $h - 1$ ) does not have one;

WMDP — the model (3.1)–(3.4) with the objective function  $f(x) = c^T x$ ;

$f1, h1$  — the optimal value of the objective function in the WDP( $h$ ) procedure and the cardinality of the optimal solution, respectively;

$f2, h2$  — the optimal value of the objective function in the WMDP and the cardinality of the optimal solution, respectively;

$\Delta = \frac{f2 - f1}{f2}$  — the relative error of the WDP( $h$ ) procedure;

$\frac{h1}{h2}$  — the ratio of cardinalities of optimal solutions in WDP( $h$ ) and WMDP;

$time1, time2$  — the time to solve the problem by the WDP( $h$ ) and WMDP algorithms, respectively (in minutes);

$\rho = \frac{|E|}{n^2 - n}$  — the density of the digraph  $G$ ;

$\alpha$  — the dependence threshold (see Section 4).

The comparison was carried out on ten problems with random data. For each problem, the input correlation matrix was  $K(0)$  for  $n = 40$ . From the matrix  $K(0)$ ,  $K(\alpha)$  problems were obtained for  $\alpha = 0.2, 0.4, 0.5, 0.7, 0.8$ . The objective function was formed by the formula

$$c_i = \sum_{j \in V | ij \in E} |k(i, j)|,$$

where  $k(i, j) \in [-1, 1]$  are the coefficients of the corresponding matrix  $K(\alpha)$ . As a result, for each value of  $\alpha$ , we got ten problems with random input data. As mentioned above, to solve the integer linear programming problems arising in the experiment, we used the IBM ILOG CPLEX Optimization Studio 12.10 package.

Table 1 shows the average values (for ten instances) of the above parameters for each  $\alpha$ .

As we can see, with an increase in the dependence threshold, the characteristics  $\Delta$  and  $(h1/h2)$  of WDP( $h$ ) procedure improve. However, it should be noted that, with the growth of  $\alpha$ , the weights of optimal solutions to WMDP problems decrease. This is important for the key indicators problem since the weight of the found dominating set characterizes the degree of its influence on the set of indicators as a whole. In this regard, it is advisable to choose  $\alpha$  equal to 0.7 or 0.8.

In addition to these problems on 40 vertices, ten instances on 50 vertices with  $\alpha = 0.7$  were considered. The relative error and the ratio of cardinalities of optimal solutions turned out to be comparable with a similar situation for  $n = 40$ . However, the average time to solve these ten instances using WMDP was 46 minutes. Problems with  $n > 70$  and  $\alpha = 0.7$  were not resolved in 3 hours.

And finally, the third observation. As mentioned above, we withdrew 63 indicators of economic security characterizing the socio-economic situation in the Omsk region of Russia from the website of the Federal State Statistics Service of the Russian Federation. The correlation coefficients were calculated for the period of 2010–2017. These indicators and the rationale for their choice are

Table 1. Comparison of the WDP(h) and WMDP procedures in terms of time, relative error, and the ratio of cardinalities of optimal solutions for  $n = 40$ .

$\alpha$	$\rho_{med}$	$(time2)_{med}$	$(time1)_{med}$	$\Delta_{med}$	$(h1/h2)_{med}$
0.2	0.43	3.56	0.02	0.44	0.48
0.4	0.32	3.81	0.03	0.5	0.44
0.5	0.27	3.99	0.04	0.47	0.46
0.7	0.16	2.46	0.03	0.39	0.53
0.8	0.11	0.99	0.02	0.31	0.61

Table 2. The results of solving the problems with real data for  $n = 63$  and  $n = 50$ .

$n$	$\alpha$	$\rho$	$time2$	$f2$	$h2$	$time1$	$f1$	$h1$	$\Delta$	$h1/h2$
63	0.7	0.18	0.06	182.5	15	0.04	127.5	11	0.30	0.73
63	0.8	0.12	0.08	126.4	19	0.06	123.7	18	0.02	0.95
50(63)	0.7	0.18	0.08			0.05			0.27	0.79

described in detail in [11]. The problem was solved using the WMDP and WDP( $h$ ) models. In the same work, the found key system of indicators was compared with other approaches proposed by various researchers and organizations, including the Economic Security Strategy of the Russian Federation until 2030. As it turned out, the problem with real data on 63 indicators with the use of the WMDP model was solved quite quickly (see Table 2). In addition, to prove the algorithm on real data, 50 indicators (10 samples) were randomly chosen from the given 63 indicators. The average values of the results for these ten samples are also shown in Table 2 (the row titled “50(63)”) <sup>2</sup>.

Noteworthy that the problems on real data, in contrast to random instances, are quickly solved by using the WMDP model. This observation makes reasonable the further investigation of the minimal dominating set problem.

## 6. Conclusion

In this paper, we propose an integer linear model of formalizing the problem of a weighted minimal dominating set in a directed graph. This is a generalization of the Upper Domination problem about the minimal dominating set of maximum cardinality in an ordinary graph, which is widely discussed in the works today. A model for an approximate solution to the problem is proposed. This model is based on discarding the minimality condition of the dominant set. An experimental analysis of this approximation is carried out. The average values of relative error estimates are obtained both for the weight and the cardinality of optimal solutions. It is noted that the relative error of the approximate solution decreases with lesser graph density.

As an application of the results obtained, a formalization of a key indicators system concept is proposed. This concept is widely used within the indicator approach to the analysis of socio-economic systems. We define the set of key indicators as a subset of the original set of indicators that has the most significant impact on the situation. Unfortunately, this approach does not exclude the participation of the expert community in selecting key indicators since our algorithms provide more than one solution. We believe that our approach should be considered as an apparatus applicable for a certain range of situations as an element of a hybrid analysis not only for

<sup>2</sup>Some empty cells in the third row of Table 2 are because this row contains the average values of the results.



economic security but also for a wider range of economic and social problems. The calculations based on real data characterizing the economic security of the Omsk region of the Russian Federation are made. These calculations should be considered primarily as an example of our approach being applied. Implementation-wise, the foundation for the proposed approach is the thesis that the same key indicators system cannot be universal for different socio-economic entities since the deep interconnections between the indicators can have a different nature determined by the territory specifics.

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