URAL MATHEMATICAL JOURNAL, Vol. 8, No. 2, 2022, pp. 81–93

DOI: 10.15826/umj.2022.2.007

RESTRAINED ROMAN REINFORCEMENT NUMBER IN GRAPHS

Saeed Kosari

Institute of Computing Science and Technology, Guangzhou University,
Guangzhou 510006, China
saeedkosari38@gzhu.edu.cn

Seyed Mahmoud Sheikholeslami

Department of Mathematics, Azarbaijan Shahid Madani University,
Tabriz, I.R. Iran
s.m.sheikholeslami@azaruniv.ac.ir

Mustapha Chellali

LAMDA-RO Laboratory, Department of Mathematics, University of Blida,
B.P. 270, Blida, Algeria
m_chellali@yahoo.com

Maryam Hajjari

Department of Mathematics, Azarbaijan Shahid Madani University,
Tabriz, I.R. Iran
m.hajjari260@gmail.com

Abstract: A restrained Roman dominating function (RRD-function) on a graph $G=(V,E)$ is a function f from V into $\{0,1,2\}$ satisfying: (i) every vertex u with $f(u)=0$ is adjacent to a vertex v with $f(v)=2$; (ii) the subgraph induced by the vertices assigned 0 under f has no isolated vertices. The weight of an RRD-function is the sum of its function value over the whole set of vertices, and the restrained Roman domination number is the minimum weight of an RRD-function on G. In this paper, we begin the study of the restrained Roman reinforcement number $r_{rR}(G)$ of a graph G defined as the cardinality of a smallest set of edges that we must add to the graph to decrease its restrained Roman domination number. We first show that the decision problem associated with the restrained Roman reinforcement problem is NP-hard. Then several properties as well as some sharp bounds of the restrained Roman reinforcement number are presented. In particular it is established that $r_{rR}(T)=1$ for every tree T of order at least three.

Keywords: Restrained Roman domination, Restrained Roman reinforcement.

1. Introduction

For definitions and notations not given here we refer the reader to [8]. We consider simple graphs G with vertex set $V=V(G)$ and edge set $E=E(G)$. The *order* of G is $n=n(G)=|V|$. The *open neighborhood* of a vertex v, denoted by $N(v)$ (or $N_G(v)$ to refer to G) is the set $\{u\in V(G)\mid uv\in E\}$ and its *closed neighborhood* is the set $N[v]=N_G[v]=N(v)\cup\{v\}$. The *degree* of vertex $v\in V$ is $d(v)=d_G(v)=|N(v)|$. The *maximum* and *minimum degree* in G are denoted by $\Delta=\Delta(G)$ and $\delta=\delta(G)$, respectively. A vertex of degree one is called a *leaf* and its neighbor is called a *support vertex*. As usual, the *path* (*cycle*, *complete p-partite graph*, respectively) of order n is denoted by P_n (C_n, $K_{n_1,n_2,\ldots,n_p}$, respectively). A *star* of order $n \geq 2$ is the graph $K_{1,n-1}$. For a subset $S \subseteq V$, the subgraph induced by S in G is denoted as $G[S]$.

A subset $S \subseteq V$ is a *dominating set* of G if every vertex in $V \setminus S$ has a neighbor in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G.

As an application, in the design of networks for example, it is essential to study the effect of some modifications of the graph parameters on its structure. These modifications can be deletion or addition of vertices, deletion or addition of edges. We refer the reader to chapter 7 of [8] when the graph parameter is the domination number. The *reinforcement number* $r(G)$ of a graph G is the minimum number of edges that have to be added to the graph G in order to decrease the domination number. Of course for graphs G with domination number one it was assumed that $r(G) = 0$. The concept of the reinforcement number was introduced in 1990 by Kok and Mynhardt [10], and since then it has been defined and studied for several other domination parameters, such as Roman domination [9], total Roman domination [1], quasi-total Roman domination [5], Italian domination [7], double Roman domination [4] and rainbow domination [3, 13].

In 2015, Leely Pushpam and Padmapriea [11] introduced the concept of restrained Roman domination as a new variation of Roman domination. A *restrained Roman dominating function* (RRD-function, for short) on a graph G is a function $f : V \longrightarrow \{0, 1, 2\}$ having the properties that (i) every vertex u with $f(u) = 0$ is adjacent to a vertex v with $f(v) = 2$; and (ii) the subgraph induced by the vertices assigned 0 under f has no isolated vertices. The weight of an RRD-function f is the sum

$$w(f) = \sum_{v \in V(G)} f(v)$$

and the *restrained Roman domination number* of G denoted by $\gamma_{rR}(G)$, is the minimum weight of an RRD-function on G. Any RRD-function f on G can simply be referred as $f = (V_0, V_1, V_2)$, where $V_i = \{v \in V(G) : f(v) = i\}$ for $i \in \{0, 1, 2\}$. For further studies on restrained Roman domination and its variants, see [2, 12, 14–16].

In this paper, we are interested in starting the study of the *restrained Roman reinforcement number* $r_{rR}(G)$ of a graph G defined as the cardinality of a smallest set of edges $F \subseteq E(\overline{G})$ such that $\gamma_{rR}(G + F) < \gamma_{rR}(G)$, where $\overline{G}$ denotes the complement graph of G. If there is no subset of edges F satisfying $\gamma_{rR}(G + F) < \gamma_{rI}(G)$, then we define $r_{rR}(G) = 0$. Since for any nontrivial connected graph G, $\gamma_{rR}(G) \geq 2$, we deduce that $r_{rR}(G) = 0$ for all nontrivial connected graphs with $\gamma_{rR}(G) = 2$. Moreover, a subset $E' \subseteq E(\overline{G})$ is called an $r_{rR}(G)$-set if $|E'| = r_{rR}(G)$ and $\gamma_{rR}(G + E') < \gamma_{rR}(G)$.

Further, we will prove that the decision problem associated with the Restrained Roman reinforcement is NP-hard. Then various properties of the restrained Roman reinforcement number are investigated and some sharp bounds on it are presented.

We finish this section by observing that any $r_{rR}(G)$-set of a connected graph G with $\gamma_{rR}(G) \geq 3$ can decrease the restrained Roman domination number of G by at most two.

Proposition 1. *Let G be a connected graph with $\gamma_{rR}(G) \geq 3$. If F is an $r_{rR}(G)$-set, then*

$$\gamma_{rR}(G) - 2 \leq \gamma_{rR}(G + F) \leq \gamma_{rR}(G) - 1.$$

Both bounds are sharp.

P r o o f. By assumption, $\gamma_{rR}(G + F) < \gamma_{rR}(G)$, whence the upper bound follows. To show the lower bound, let us assume that

$$\gamma_{rR}(G + F) \leq \gamma_{rR}(G) - 3.$$

Let f be a $\gamma_{rR}(G+F)$-function and let $uv \in F$ such that $0 \in \{f(u), f(v)\}$. If such an edge does not exist, then f is an RRD-function of G leading to the contradiction

$$\gamma_{rR}(G) \leq \gamma_{rR}(G+F).$$

Hence we suppose that uv exists, and let $F' = F - \{uv\}$. Without loss of generality, suppose that $f(u) = 0$. If $f(v) = 1$, then f is an RRD-function of G leading to the contradiction

$$\gamma_{rR}(G) \leq \gamma_{rR}(G+F)$$

too. Hence assume that $f(v) \neq 1$.

First let $f(v) = 2$. If u has a neighbor w in $G+F'$ with $f(w) \geq 1$, then the function g defined by $g(w) = 2$ and $g(x) = f(x)$ otherwise, is an RRD-function of $G+F'$ yielding as above to the contradiction $\gamma_{rR}(G+F') < \gamma_{rR}(G)$. Hence we assume that each neighbor u in $G+F'$ is assigned 0 under f. Let $x_1, \ldots, x_k$ be the neighbors of u in $G+F'$. If $k = 1$ and x_1 has a neighbor assigned 0 other than u, then the function $g(u) = 1$ and $g(x) = f(x)$ otherwise, is an RRD-function of $G+F'$ yielding

$$\gamma_{rR}(G+F') \leq \gamma_{rR}(G+F) + 1 < \gamma_{rR}(G),$$

this is a contradiction. If $k = 1$ and x_1 has no neighbor assigned 0 other than u, then the function $g(u) = g(x_1) = 1$ and $g(x) = f(x)$ otherwise, is an RRD-function of $G+F'$ and thus

$$\gamma_{rR}(G+F') \leq \gamma_{rR}(G+F) + 2 < \gamma_{rR}(G),$$

it is a contradiction too. Hence assume that $k \geq 2$. If some x_i has no neighbor assigned 0 other than u, then the function $g(x_i) = 2$ and $g(x) = f(x)$ otherwise, is an RRD-function of $G+F'$ yielding again $\gamma_{rR}(G+F') < \gamma_{rR}(G)$. Hence we assume that for each i, x_i has at least two neighbors assigned 0 under f. In this case, we have $g(u) = 1$ and $g(x) = f(x)$ otherwise, it is an RRD-function of $G+F'$ and thus

$$\gamma_{rR}(G+F') < \gamma_{rR}(G).$$

Finally, assume that $f(v) = 0$. Since F is an $r_{rR}(G)$-set, we can suppose, without loss of generality, that all neighbors of u in $G+F'$ have positive labels under f. Now, if v has a neighbor with weight 0 in $G+F'$, then the function $g(u) = 1$ and $g(x) = f(x)$ otherwise, it is an RRD-function of $G+F'$ while if v has no neighbor with weight 0 in $G+F'$, then the function $g(u) = g(v) = 1$ and $g(x) = f(x)$ otherwise, is an RRD-function of $G+F'$. Both situations yield the contradiction $\gamma_{rR}(G+F') < \gamma_{rR}(G)$. Consequently,

$$\gamma_{rR}(G+F) \geq \gamma_{rR}(G) - 2.$$

The upper bound of Proposition 1 is attained for the cycle C_4, while the lower bound is attained for the cycle C_6. □

2. NP-hardness result

The aim of this section, is to show that the decision problem associated with the Restrained Roman reinforcement is NP-hard. Consider the following decision problem.

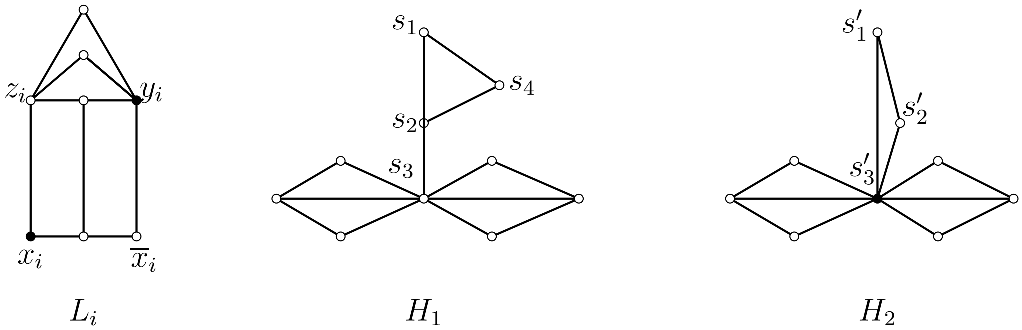

Figure 1. The graphs L_i and $H = H_1 \cup H_2$.

Restrained Roman reinforcement problem (RR-reinforcement)

Instance: *A nonempty graph G and a positive integer k.*
Question: *Is $r_{rR}(G) \leq k$?*

We show that the NP-hardness of the RR-reinforcement problem by transforming the well-known 3-SAT problem to it in polynomial time. Recall that the 3-SAT problem specified below was proven to be NP-complete in [6].

3-SAT problem

Instance: A collection $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$ of clauses over a finite set X of variables such that $|C_j| = 3$ for every $j \in \{1, 2, \dots, m\}$.
Question: Is there a truth assignment for X that satisfies all the clauses in $\mathcal{C}$?

Theorem 1. *Problem RR-reinforcement is NP-hard for an arbitrary graph.*

P r o o f. Let $X = \{x_1, x_2, \dots, x_n\}$ and $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$ be an arbitrary instance of 3-SAT problem. We will build a graph G and a positive integer k such that $r_{rR}(G) \leq k$ if and only if $\mathcal{C}$ is satisfiable.

For each $i \in \{1, 2, \dots, n\}$, we associate to the variable $x_i \in X$ a copy of the graph L_i as depicted in Figure 1, and for each $j \in \{1, 2, \dots, m\}$, we associate to the clause $C_j = \{u_j, v_j, w_j\} \in \mathcal{C}$ a vertex c_j by adding the edge-set $E_j = \{c_j u_j, c_j v_j, c_j w_j\}$. Finally, we enclose the graph H illustrated in Figure 1 by connecting vertices s_1, s'_1 to every vertex c_j. Clearly, the resulting graph G is of order $8n+m+19$ and size $11n+5m+27$ and hence G can be built in polynomial time. Set $k = 1$. Figure 2 provides an example of the resulting graph when $X = \{x_1, x_2, x_3, x_4\}$ and $\mathcal{C} = \{C_1, C_2, C_3\}$, where $C_1 = \{x_1, x_2, \bar{x}_3\}$, $C_2 = \{\bar{x}_1, x_2, x_4\}$ and $C_3 = \{\bar{x}_2, x_3, x_4\}$.

It is easy to verify that for any $\gamma_{rR}(G)$-function g we must have

$$\sum_{v \in V(L_j)} g(v) \geq 4$$

for each $j \in \{1, 2, \dots, n\}$. Moreover, to restrained Roman dominate all vertices of $V(H)$, we need that

$$\sum_{i=1}^{m} g(c_i) + g(V(H)) \geq 6.$$

Therefore

$$\gamma_{rR}(G) = w(g) \geq 4n + 6.$$

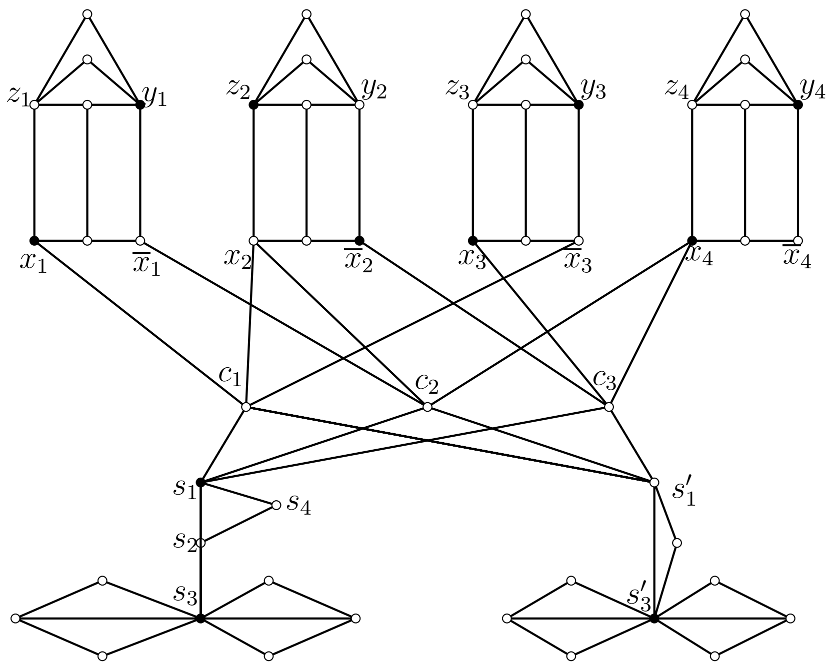

Figure 2. An instance of the restrained Roman reinforcement number problem resulting from an instance of 3-SAT. Here $k = 1$ and $\gamma_{rR}(G) = 22$, where the black vertex p means there is a RRDF f with $f(p) = 2$.

Basing on the assignment given to the graph in Figure 2, one can easily define an RRD-function of G with weight $4n + 6$, which consequently leads to $\gamma_{rR}(G) = 4n + 6$.

In the following, we show that $\mathcal{C}$ is satisfiable if and only if $r_{rR}(G) = 1$. Let $\mathcal{C}$ be satisfiable and $t : X \to \{T, F\}$ a satisfying function for $\mathcal{C}$. We build a subset S of vertices of G as follows. If $t(x_i) = T$, then put the vertices x_i and y_i in S; while if $t(x_i) = F$, then put the vertices $\overline{x_i}$ and z_i in S. So $|S| = 2n$. Define the function h on $V(G)$ by $h(x) = 2$ for every $x \in S$, $h(s_1) = 1$, $h(s_3) = h(s'_3) = 2$ and $h(y) = 0$ for the remaining vertices. It is easy to verify that h is an RRD-function of $G + s_4s_3$ of weight

$$4n + 5 < \gamma_{rR}(G) = 4n + 6,$$

and hence $r_{rR}(G) = 1$.

Conversely, let $r_{rR}(G) = 1$. Then, there is an edge $e = uv \in E(\overline{G})$ for which

$$\gamma_{rR}(G + e) < 4n + 6.$$

Let $g = (V_0, V_1, V_2)$ be a $\gamma_{rR}(G + e)$-function. Since whatever the added edge e, we have $g(V(L_i)) \geq 4$, and thus vertices u and v cannot both belong to $V(L_i)$ (for otherwise $\gamma_{rR}(G + e) \geq 4n + 6$). On the other hand, since $r_{rR}(G) = 1$ and $g(V(L_i)) \geq 4$, we must have

$$\sum_{j=1}^{m} g(c_j) + g(V(H)) < 6.$$

Since also whatever the added edge e, we have $g(V(H)) \geq 5$, we conclude that $g(V(H)) = 5$. In particular, this is only possible if $g(s'_1) = 0$, $g(s_1) \leq 1$ and

$$\sum_{j=1}^{m} g(c_j) = 0.$$

In addition, we note that if $\{x_i, \bar{x_i}\} \subseteq V_2$ or $\{x_i, \bar{x_i}\} \cap V_1 \neq \emptyset$ for some i, then $g(V(L_i)) \geq 5$ which results in the contradiction

$$\gamma_{rR}(G+e) \geq 4n+6.$$

Thus, $|\{x_i, \bar{x_i}\} \cap V_2| \leq 1$ and $\{x_i, \bar{x_i}\} \cap V_1 = \emptyset$ for every $i \in \{1, \ldots, n\}$. Therefore each vertex c_j must have a neighbor in $\{x_i, \bar{x_i}\}$ for some i which is assigned a 2. In this case, define the mapping $t : X \to \{T, F\}$ by

$$t(x_i) = \begin{cases} T & \text{if} \quad f(x_i) = 2, \\ F & \text{otherwise} \end{cases} \tag{2.1}$$

for $i \in \{1, \ldots, n\}$.

We show that t satisfies the truth assignment for $\mathcal{C}$. It is enough to show that every clause in $\mathcal{C}$ is satisfied by t. Consider an arbitrary clause $C_j \in \mathcal{C}$ for some $j \in \{1, \ldots, m\}$. If c_j is dominated by x_i, then $g(x_i) = 2$ and so $t(x_i) = T$. If c_j is dominated by $\bar{x_i}$, then $g(\bar{x_i}) = 2$ and hence $t(x_i) = F$ and $t(\bar{x_i}) = T$. Therefore, in either case the clause C_j is satisfied. The arbitrariness of j shows that all clauses in $\mathcal{C}$ are satisfied by t, that is, $\mathcal{C}$ is satisfiable. This completes the proof of the theorem. □

3. Exact values

In this section, we determine the restrained Roman reinforcement number of some classes of graphs including paths, cycles and complete p-partite graphs for any integer $p \geq 2$. As observed in [11], for every connected graph G of order $n \geq 2$, we have $2 \leq \gamma_{rR}(G) \leq n$. A characterization of all connected graphs of order n with $\gamma_{rR}(G) \in \{2, 3, n\}$ was provided in [11, 14] as follows.

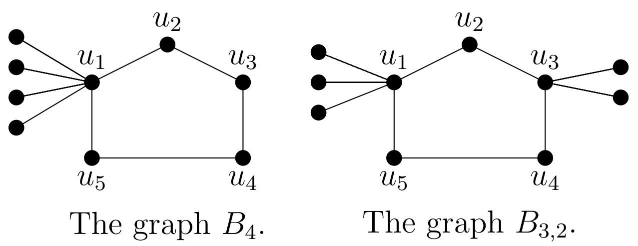

Figure 3. Graphs B_4 and $B_{3,2}$.

Let $C := (u_1u_2u_3u_4u_5)$ be a cycle of length 5 and let B_p be the graph obtained from C by adding $p \geq 1$ new vertices attached by edges at u_1 and let $B_{p,q}$ be the graph obtained from C by adding $p \geq 1$ new vertices attached by edges at u_1 and $q \geq 1$ other new vertices attached by edges at u_3 (see Fig. 3). Recall that the *diameter,* $\operatorname{diam}(G)$, of a graph G is the maximum distance between the pair of vertices.

Proposition 2 [11]. *Let G be a connected graph of order $n \geq 2$. Then*

(a) $\gamma_{rR}(G) = 2$ *if and only if* $n = 2$ *or* $\Delta(G) = n - 1$ *and* $\delta(G) \geq 2$;

(b) $\gamma_{rR}(G) = n$ *if and only if* $G \simeq C_4, C_5, B_p, B_{p,q}$ *or* G *is a tree with* $\operatorname{diam}(G) \leq 5$.

Proposition 3 [14]. *Let G be a connected graph of order $n \geq 4$. Then $\gamma_{rR}(G) = 3$ if and only if G satisfies one of the following conditions*:

(i) $\Delta(G) = n - 1$ *and* G *has exactly one leaf*;

(ii) $\Delta(G) = n - 2$ *and* G *has a vertex* u *of degree* $n - 2$ *such that the induced subgraph* $G[N(u)]$ *has no isolated vertex.*

On the other hand, the exact values of the restrained Roman domination number have been established in [11] for paths, cycles and complete p-partite graphs.

Proposition 4 [11]. *The following conditions holds*:

(a) $\gamma_{rR}(P_n) = n$ *for* $1 \leq n \leq 6$ *and* $\gamma_{rR}(P_n) = \lceil (2n+1)/3 \rceil + 1$ *for* $n \geq 7$;

(b) $\gamma_{rR}(C_n) = 2\lceil n/3 \rceil$ *when* $n \not\equiv 2 \pmod 3$ *and* $\gamma_{rR}(C_n) = 2\lceil n/3 \rceil + 1$ *otherwise*;

(c) $\gamma_{rR}(K_{m,n}) = 4$ *for* $m, n \geq 2$;

(d) *if* $K_{n_1,n_2,\ldots,n_p}$ *is the complete* p*-partite graph such that* $p \geq 3$ *and* $n_1 \leq n_2 \leq \ldots \leq n_p$, *then* $\gamma_{rR}(K_{1,n_2,\ldots,n_p}) = 2$, $\gamma_{rR}(K_{2,n_2,\ldots,n_p}) = 3$ *and* $\gamma_{rR}(K_{n_1,n_2,\ldots,n_p}) = 4$ *for* $n_1 \geq 3$.

Now we are ready to find the restrained Roman reinforcement number for paths, cycles and complete p-partite graphs, $p \geq 2$.

Proposition 5. *For* $n \geq 3$, $r_{rR}(P_n) = 1$.

P r o o f. Let $P_n := w_1 w_2 \ldots w_n$. If $n \equiv 0 \pmod 3$, then the function g defined by

$$g(w_{3i+1}) = 2$$

for $0 \leq i \leq (n-3)/3$ and $g(w) = 0$ otherwise, is an RRD-function of $P_n + w_1 w_n$ of weight $2n/3$. If $n \equiv 2 \pmod 3$, then the function g defined by $g(w_n) = 2$, $g(w_{3i+1}) = 2$ for $0 \leq i \leq (n-5)/3$ and $g(x) = 0$ otherwise, is an RRD-function of $P_n + w_1 w_{n-2}$ of weight $(2n+2)/3$. Finally, if $n \equiv 1 \pmod 3$, then the function g defined by $g(w_n) = 1$, $g(w_{3i+1}) = 2$ for $0 \leq i \leq (n-4)/3$ and $g(w) = 0$ otherwise, is an RRD-function of $P_n + w_1 w_{n-1}$ of weight $(2n+1)/3$. All considered cases show that $r_{rR}(P_n) = 1$. □

Proposition 6. For $n \geq 4$,

$$r_{rR}(C_n) = \begin{cases} 2 & \text{if } n \equiv 0 \pmod 3, \\ 1 & \text{otherwise.} \end{cases}$$

P r o o f. Assume that $C_n := (w_1 w_2 \ldots w_n)$ be a cycle on n vertices. If $n \equiv r \pmod 3$ with $r \in \{1, 2\}$, then by a similar argument to that used in the proof of Proposition 5, we can see that $r_{rR}(C_n) = 1$. Hence we assume that $n \equiv 0 \pmod 3$. First, since the function g defined by $g(w_{n-2}) = 1$, $g(w_{3i+1}) = 2$ for $0 \leq i \leq (n-6)/3$ and $g(x) = 0$ otherwise, is an RRD-function of $C_n + \{w_1 w_{n-1}, w_1 w_{n-3}\}$ of weight $(2n-3)/3 = \gamma_{rR}(C_n) - 1$ (Proposition 4-(b)), we deduce that $r_{rR}(C_n) \leq 2$.

Now we prove the inverse inequality. For this purpose, we need only to show that adding an arbitrary edge e cannot decrease $\gamma_{rR}(C_n)$. Observe that for any edge $e \in \overline{C_n}$,

$$\gamma_{rR}(C_n + e) \leq \gamma_{rR}(C_n).$$

Let e be an arbitrary edge in $\overline{C_n}$ and let f be a $\gamma_{rR}(C_n + e)$-function. Suppose first that there are three consecutive vertices w_i, w_{i+1}, w_{i+2} such that $f(w_i) = f(w_{i+1}) = f(w_{i+2}) = 0$, say for $i = 1$.

Then the edge e must join w_2 to some vertex assigned 2, say w_k, with $k \notin \{1, 3\}$. Also, to restrained Roman dominate w_1 and w_3, we must also have $f(w_4) = f(w_n) = 2$.

Consider the cycles $C' := (w_2w_3 \ldots w_k)$ of order $k-1$ and $C'' := (w_2w_k \ldots w_nw_1)$ of order $n-k+3$. Let $k-1 \equiv s_1 \pmod 3$ and $n-k+3 \equiv s_2 \pmod 3$. Notice that $s_1 = 0$ and $s_2 = 2$; $s_1 = 2$ and $s_2 = 0$ or $s_1 = s_2 = 1$. Assume that $k-1 \equiv 0 \pmod 3$ (the case $n-k+3 \equiv 0 \pmod 3$ is similar). Then $n-k+3 \equiv 2 \pmod 3$, and since the restrictions of f on $V(C')$ and $V(C'')$ are RRD-functions, we deduce from Proposition 4-(b), that

$$\begin{aligned}\gamma_{rR}(C_n + e) &= f(V(C')) + f(V(C'')) - 2 \\ \geq \gamma_{rR}(C') + \gamma_{rR}(C'') - 2 &= \frac{2(k-1)}{3} + \frac{2(n-k+3)+3+2}{3} - 2 = \frac{2n+3}{3} > \gamma_{rR}(C_n).\end{aligned}$$

Assume now that $s_1 = s_2 = 1$. Then, as above, it follows from Proposition 4-(b) that

$$\begin{aligned}\gamma_{rR}(C_n + e) &= f(V(C')) + f(V(C'')) - 2 \\ \geq \gamma_{rR}(C') + \gamma_{rR}(C'') - 2 &= \frac{2(k-1)+3+1}{3} + \frac{2(n-k+3)+3+1}{3} - 2 = \frac{2n+6}{3} > \gamma_{rR}(C_n).\end{aligned}$$

Thus in either case we obtain a contradiction. Next suppose there are three consecutive vertices w_i, w_{i+1}, w_{i+2} such that $f(w_i) + f(w_{i+1}) + f(w_{i+2}) = 1$, say for $i = 1$.

If $f(w_2) = 1$, then $f(w_1) = f(w_3) = 0$ and each of w_1 and w_3 must be adjacent a vertex assigned 2 as well as to a vertex assigned 0. This possible only if $e = w_1w_3$ and so

$$H = (C_n + e) - w_2$$

is a cycle on $n-1$ vertices, where the restriction of f to H is an RRD-function. It follows that

$$\gamma_{rR}(C_n + e) = f(V(H)) + 1 \geq \gamma_{rR}(H) + 1,$$

and by Proposition 4-(b), we obtain

$$\gamma_{rR}(C_n + e) \geq \frac{2(n-1)+3+2}{3} + 1 > \gamma_{rR}(C_n)$$

which is a contradiction. Hence we can assume that $f(w_2) = 0$. Without loss of generality, let $f(w_1) = 1$ and $f(w_3) = 0$. To restrained Roman dominate w_2, the edge e must join w_2 to a vertex with label 2, say w_k. Likewise for w_3 we must have $f(w_4) = 2$. Now, consider the cycles $C' := (w_2w_3 \ldots w_k)$ of order $k-1$ and the path $P' := w_k \ldots w_nw_1$ of order $n-k+2$.

Let $k-1 \equiv s_1 \pmod 3$ and $n-k+2 \equiv s_2 \pmod 3$. Notice that $s_1 = 0$ and $s_2 = 1$; $s_1 = 1$ and $s_2 = 0$ or $s_1 = s_2 = 2$. Notice also that the restrictions of f on $V(C')$ and $V(P')$ are RRD-functions, and thus

$$\gamma_{rR}(C_n + e) = f(V(C')) + f(V(P')) - 2 \geq \gamma_{rR}(C') + \gamma_{rR}(P') - 2.$$

Now using Propositions 4-(a,b), we get a contradiction as before.

Finally, let

$$f(w_i) + f(w_{i+1}) + f(w_{i+2}) \geq 2$$

for each $1 \leq i \leq n$, where the sum in indices is taken modulo n. Then we have

$$\gamma_{rR}(C_n + e) = \frac{1}{3}\sum_{i=1}^{n}(f(w_i) + f(w_{i+1}) + f(w_{i+2})) \geq \frac{2n}{3} = \gamma_{rR}(C_n),$$

and therefore, $\gamma_{rR}(C_n + e) = \gamma_{rR}(G)$. Consequently, $r_{rR}(C_n) = 2$ as desired. $\square$

Proposition 7. For integers $1 \leq r \leq s$ with $r+s \geq 3$,

$$r_{rR}(K_{r,s}) = \begin{cases} 1 & \text{if} \quad r = 1,2,3, \\ r-2 & \text{if} \quad r \geq 4. \end{cases}$$

P r o o f. Let $X = \{x_1, x_2, \ldots, x_r\}$ and $Y = \{y_1, y_2, \ldots, y_s\}$ be the partite sets of $K_{r,s}$.

If $r = 1$, then the function g defined by $g(x_1) = 2$, $g(y_1) = g(y_2) = 0$ and $g(x) = 1$ otherwise, is an RRD-function of $K_{1,s} + y_1y_2$ of weight $n-1$ and it follows from Proposition 2-(b) that $r_{rR}(K_{1,s}) = 1$.

If $r = 2$, then the function g defined by $g(x_1) = 2$ and $g(x) = 0$ otherwise, is an RRD-function of $K_{2,s} + x_1x_2$ of weight 2 and we get from Proposition 2-(a) that $r_{rR}(K_{2,s}) = 1$.

If $r = 3$, then the function g defined by $g(x_1) = 2, g(x_2) = 1$ and $g(x) = 0$ otherwise, is an RRD-function of $K_{3,s} + x_1x_3$ of weight 3 and by Proposition 4-(c), we have $r_{rR}(K_{3,s}) = 1$.

Let $r \geq 4$. First we observe that the function g defined by $g(x_1) = 2$, $g(x_2) = 1$ and $g(x) = 0$ otherwise, is an RRD-function of $K_{r,s} + \{x_1x_i \mid 3 \leq i \leq r\}$ of weight 3 and thus by Proposition 4-(c), $r_{rR}(K_{r,s}) \leq r-2$.

To show that $r_{rR}(K_{r,s}) \geq r-2$, let F be an $r_{rR}(K_{r,s})$-set. Then

$$2 \leq \gamma_{rR}(K_{r,s} + F) \leq 3.$$

By Propositions 2-(a) and 3 we must have $\Delta(K_{r,s} + F) \geq r+s-2$ and this implies that $|F| \geq r-2$. Therefore $r_{rR}(K_{r,s}) = r-2$ and the proof is complete. □

Proposition 8. *Let $K_{n_1,n_2,\ldots,n_p}$ be the complete p-partite graph such that $p \geq 3$ and $3 \leq n_1 \leq n_2 \leq \ldots \leq n_p$. Then $r_{rR}(K_{n_1,n_2,\ldots,n_p}) = n_1 - 2$.*

P r o o f. Let $G = K_{n_1,n_2,\ldots,n_p}$ and $X_1 = \{x_1, \ldots, x_{n_1}\}$, $X_2 = \{y_1, \ldots, y_{n_2}\}, \ldots, X_p$ be the partite sets of G. Let F be an $r_{rR}(G)$-set. By Proposition 4-(d) we deduce that $\gamma_{rR}(G+F) \in \{2,3\}$, and by Propositions 2-(a) and 3 we must have

$$\Delta(G+F) \geq n_1 + \cdots + n_p - 2$$

implying that $|F| \geq n_1 - 2$. On the other hand, the function g defined by $f(x_1) = 2$, $f(x_2) = 1$ and $f(x) = 0$ otherwise, is an RRD-function of $G + \{x_ix_1 \mid 3 \leq i \leq n_1\}$ yielding $r_{rR}(G) \leq |F| = n_1 - 2$. Consequently, $r_{rR}(G) = n_1 - 2$. □

4. Graphs with small restrained Roman reinforcement number

In this section, we study graphs with small restrained Roman reinforcement number. We begin with the following lemma.

Lemma 1. *If G is a connected graph of order $n \geq 3$ with $\gamma_{rR}(G) = n$, then $r_{rR}(G) = 1$.*

P r o o f. By Proposition 2, $G \simeq C_4, C_5, B_p, B_{p,q}$ or G is a tree with $\mathrm{diam}(G) \leq 5$. If $G \in \{C_4, C_5\}$, then the desired result follows from Proposition 6. If $G \in \{B_p, B_{p,q}\}$, then the function g defined by $g(u_1) = 2$, $g(u_2) = g(u_5) = 0$ and $g(x) = 1$ otherwise, is an RRD-function of $G + u_2u_5$ and hence $r_{rR}(G) = 1$. Hence, we assume that G is a tree with diameter at most 5.

Let $v_1v_2 \ldots v_k$ $(k \geq 3)$ be a diametral path in G. Define the function f by $f(v_1) = f(v_3) = 0$, $f(v_2) = 2$ and $f(x) = 1$ for the remaining vertices. Clearly, f is an RRD-function of $G + v_1v_3$ and hence $r_{rR}(G) = 1$. □

Proposition 9. *Let G be a connected graph of order $n \geq 4$ with $\gamma_{rI}(G) \geq 3$. If $f = (V_0, V_1, V_2)$ is a $\gamma_{rR}(G)$-function with $V_1 \neq \emptyset$, then*

$$r_{rR}(G) = 1.$$

P r o o f. Let $f = (V_0, V_1, V_2)$ be a $\gamma_{rR}(G)$-function such that $V_1 \neq \emptyset$. If $\gamma_{rR}(G) = n$, then the desired result comes from Lemma 1.

Hence assume that $\gamma_{rR}(G) < n$. Then $V_0 \neq \emptyset$ and so $V_2 \neq \emptyset$. Since G is connected and $V_1 \neq \emptyset$, there exists a vertex $w \in V_1$ such that w is dominated by $V_0 \cup V_2$. Note that if w has a neighbor in V_0 and another one in V_2, then reassigning w provides an RRD-function with weight $\gamma_{rR}(G) - 1$, a contradiction.

Now, if w is adjacent to a vertex in V_2, then the function g defined by $g(w) = 0$ and $g(x) = f(x)$ otherwise, is an RRD-function of $G + wz$ where $z \in V_0$, of weight less than $\gamma_{rR}(G)$, and thus $r_{rR}(G) = 1$. If w is adjacent to a vertex in V_0, then the function g defined by $g(w) = 0$ and $g(x) = f(x)$ otherwise, is an RRD-function of $G + wu$ where $u \in V_2$, of weight less than $\gamma_{rR}(G)$ and so $r_{rR}(G) = 1$. This completes the proof. $\square$

Proposition 10. *Let G be a connected graph of order n with $\gamma_{rR}(G) \geq 3$. Then $r_{rR}(G) = 1$ if and only if $\gamma_{rR}(G) = n$ or G has a function $f = (V_0, V_1, V_2)$ of weight less than $\gamma_{rR}(G)$ such that one of the following conditions holds*:

(i) $G[V_0]$ *has at most two isolated vertices and V_2 dominates V_0;*

(ii) $G[V_0]$ *has no isolated vertices and there is exactly one vertex $v \in V_0$ which is not dominated by V_2.*

P r o o f. If $\gamma_{rR}(G) = n$, then by Lemma 1 we have $r_{rR}(G) = 1$. Hence suppose that $\gamma_{rR}(G) < n$, and let $f = (V_0, V_1, V_2)$ be a function on G with weight less than $\gamma_{rR}(G)$ satisfying (i) or (ii). Since $\omega(f) < \gamma_{rR}(G) \leq n - 2$, we have $|V_0| \geq 2$. In the case V_2 is non-empty, let $u \in V_2$. Now, if (ii) holds, then $V_2 \neq \emptyset$ and f is an RRD-function of $G + uv$.

Assume now that (i) holds. If $G[V_0]$ has two isolated vertices w, v, then f is an RRD-function of $G + \{wv\}$ and if $G[V_0]$ has exactly one isolated vertex, say w, then f is an RRD-function of $G + \{wz\}$, where z is any vertex in $V_0 - \{w\}$. Hence in either case $r_{rR}(G) = 1$.

Conversely, let $r_{rR}(G) = 1$ and suppose that $\{uv\}$ is an $r_{rR}(G)$-set. If $\gamma_{rR}(G) = n$, then we are done. Hence suppose that $\gamma_{rR}(G) \leq n - 1$ and let f be a $\gamma_{rR}(G + \{uv\})$-function. Notice that vertices u and v cannot be assigned both positive values under f (otherwise f is an RRD-function of G). Without loss of generality, assume that $f(u) = 0$. If $f(v) = 0$, then f is a function satisfying (i). Hence assume that $f(v) \geq 1$. If u is adjacent to a vertex with label 2 other than v, then f is an RRD-function of G. Hence u is not dominated by V_2 in G and so f is a function satisfying (ii). This completes the proof. $\square$

Proposition 11. *Let G be a connected graph of order n with $\gamma_{rR}(G) \geq 3$. If $\delta(G) = 1$, then $r_{rR}(G) = 1$.*

P r o o f. First note that $n \geq 3$, since $\gamma_{rR}(G) \geq 3$. If $\gamma_{rR}(G) = n$, then the result comes from Lemma 1. Hence we assume that $\gamma_{rR}(G) < n$, and let $f = (V_0, V_1, V_2)$ be a $\gamma_{rR}(G)$-function. We have $V_0 \neq \emptyset$ (because $\gamma_{rR}(G) < n$) and thus $V_2 \neq \emptyset$.

Let u be a support vertex of G and u_1 a leaf neighbor of u. By definition we have $f(u_1) \geq 1$. If $f(u_1) = 1$ or $f(u) = 1$, then the desired result comes from Proposition 9.

Hence we assume that $f(u_1) = 2$ and $f(u) \neq 1$. The minimality of f implies that $f(u) = 0$. Note that u_1 is the only neighbor of u which assigned a 2, for otherwise u_1 can be reassigned the value 1 instead of 2. Let w be a neighbor of u with label 0. To Roman dominate w, there is a vertex v such that $f(v) = 2$. Then the function g defined on $G + uv$ by $g(u_1) = 1$ and $g(x) = f(x)$ otherwise, is an RRD-function of $G + \{uv\}$ with weight $\omega(f) - 1$. Consequently, $r_{rR}(G) = 1$. □

Corollary 1. *For any tree T of order $n \geq 3$, $r_{rR}(T) = 1$.*

5. Bounds on $r_{rR}(G)$

In this section, we present some sharp upper bounds on the restrained Roman reinforcement number of a graph. Given a set $S \subseteq V$ of vertices in a graph G and a vertex $v \in S$, the *external private neighborhood* of v with respect to S in the set

$$\operatorname{epn}(v, S) = \big\{u \in V - S \mid N(u) \cap S = \{v\}\big\}.$$

Proposition 12. *Let G be a connected graph with $\gamma_{rI}(G) \geq 3$. If $f = (V_0, V_1, V_2)$ is a $\gamma_{rR}(G)$-function with $V_2 \neq \emptyset$, then*

$$r_{rR}(G) \leq \min\big\{|\operatorname{epn}(v, V_2) \cap V_0| \;:\; v \in V_2\big\}.$$

P r o o f. Let $f = (V_0, V_1, V_2)$ be a $\gamma_{rR}(G)$-function with $V_2 \neq \emptyset$. If $|\operatorname{epn}(v, V_2) \cap V_0| = 0$ for some vertex $v \in V_2$, then reassigning v the value 1 instead of 2 provides an RRD-function of weight less than $\gamma_{rR}(G)$ leading to a contradiction. Hence $|\operatorname{epn}(v, V_2) \cap V_0| \geq 1$ for every $v \in V_2$. Let u be a vertex in V_2 such that

$$|\operatorname{epn}(u, V_2) \cap V_0| = \min\{|\operatorname{epn}(v, V_2) \cap V_0| :\; v \in V_2\}$$

and let $\operatorname{epn}(u, V_2) \cap V_0 = \{u_1, \ldots, u_\epsilon\}$. If $|V_2| \geq 2$ and $w \in V_2 - \{u\}$, then the function g defined by $g(u) = 1$ and $g(x) = f(x)$ otherwise, is an RRD-function of $G + \{wx \mid x \in \operatorname{epn}(v, V_2) \cap V_0\}$ of weight less than $\gamma_{rR}(G)$ and so

$$r_{rR}(G) \leq \min\{|\operatorname{epn}(v, V_2) \cap V_0| :\; v \in V_2\}.$$

Hence assume that $V_2 = \{u\}$. Then u dominates all vertices in V_0. Since $\gamma_{rR}(G) \geq 3$, we have $V_1 \neq \emptyset$ and the desired result follows from Proposition 9. □

We observe that for any $\gamma_{rR}(G)$-function $f = (V_0, V_1, V_2)$, every vertex u of V_2 can have at most $d_G(u)$ neighbors in V_0. Whence we have the following corollary.

Corollary 2. *Let G be a connected graph with $\gamma_{rR}(G) \geq 3$ and $f = (V_0, V_1, V_2)$ a $\gamma_{rR}(G)$-function with $|V_2| \geq 1$. Then $r_{rR}(G) \leq \Delta$.*

Corollary 3. *Let G be a connected graph with $\gamma_{rR}(G) \geq 3$ containing a path $v_1v_2v_3v_4v_5$ in which $d_G(v_i) = 2$ for $i \in \{2, 3, 4\}$. Then $r_{rR}(G) \leq 2$.*

P r o o f. If $\gamma_{rR}(G) = n$, then the result is immediate from Lemma 1. Hence we assume that $\gamma_{rR}(G) < n$, and let $f = (V_0, V_1, V_2)$ be a $\gamma_{rR}(G)$-function. By Proposition 9, we may assume that $V_1 = \emptyset$. Then we must have $2 \in \{f(v_2), f(v_3), f(v_4)\}$ and the result follows from Proposition 12. □

Using Propositions 9 and 12 we obtain the next result.

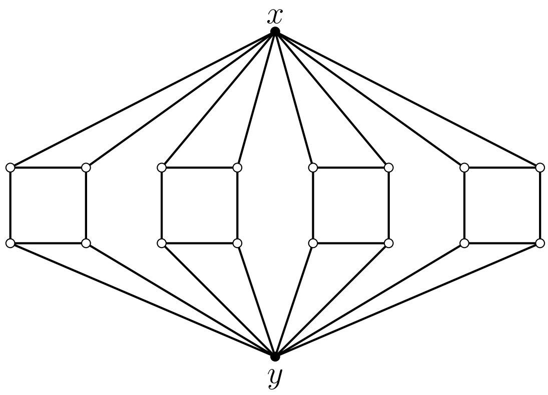

Figure 4. A graph G of order 18 and $r_{rR}(G) = 4$.

Theorem 2. *For any graph G of order $n \geq 3$, we have*

$$r_{rR}(G) \leq \max\{1, (2n - \gamma_{rR}(G))/\gamma_{rR}(G)\}.$$

Moreover, the bound is sharp.

P r o o f. If $\gamma_{rR}(G) = 2$, then $r_{rR}(G) = 0$ and the result is true.

If $\gamma_{rR}(G) = n$, then by Lemma 1, $r_{rR}(G) = 1$ and the desired result follows.

Hence we assume that $3 \leq \gamma_{rR}(G) < n$, and let $f = (V_0, V_1, V_2)$ be a $\gamma_{rR}(G)$-function. If $V_1 \neq \emptyset$, then the result follows from Proposition 9. Thus suppose that $V_1 = \emptyset$. Then $\gamma_{rR}(G)/2 = |V_2| \geq 2$ and clearly

$$|\mathrm{epn}(u, V_2) \cap V_0| \leq (2n - \gamma_{rR}(G))/\gamma_{rR}(G)$$

for some $u \in V_2$. Now, the result is immediate by Proposition 9.

To show the sharpness, consider the graph G illustrated in Figure 4. It is easy to see that $\gamma_{rR}(G) = 4$ and the function f on G defined by $f(x) = f(y) = 2$ and $f(z) = 0$ otherwise, is the unique $\gamma_{rR}(G)$-function. Then

$$r_{rR}(G) \leq (2n - \gamma_{rR}(G))/\gamma_{rR}(G) = 8.$$

Now let F be an $r_{rR}(G)$-set. Then $\gamma_{rR}(G+F) \leq 3$ and so $\Delta(G+F) \geq n-2$ (see Propositions 2-(a) and 3). This implies that $|F| \geq 8$, and consequently,

$$r_{rR}(G) = 8 = (2n - \gamma_{rR}(G))/\gamma_{rR}(G).$$

□

6. Conclusion

The main objective of this paper was to start the study of the restrained Roman reinforcement number $r_{rR}(G)$ of a graph G. We first showed that the decision problem associated with the restrained Roman reinforcement problem is NP-hard, and then various properties as well as some sharp bounds of the restrained Roman reinforcement number have been established. In particular we showed that $r_{rR}(T) = 1$ for every tree T of order at least three and that $r_{rR}(G) \leq \Delta(G)$ for any connected graph G with $\gamma_{rR}(G) \geq 3$. As a future work, one can focus on the problem of characterizing all connected graphs G such that $r_{rR}(G) = \Delta(G)$.

Acknowledgements

This research was financially supported by fund from Hubei Province Key Laboratory of Intelligent Information Processing and Real-time Industrial System (Wuhan University of Science and Technology).

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