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INEQUALITIES PERTAINING TO RATIONAL FUNCTIONS WITH PRESCRIBED POLES

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Abstract: Let $\Re_n$ be the set of all rational functions of the type $r(z)=p(z)/w(z)$, where $p(z)$ is a polynomial of degree at most n and $w(z)=\prod_{j=1}^{n}(z-a_j)$, $|a_j|>1$ for $1\leq j\leq n$. In this paper, we set up some results for rational functions with fixed poles and restricted zeros. The obtained results bring forth generalizations and refinements of some known inequalities for rational functions and in turn produce generalizations and refinements of some polynomial inequalities as well.

Keywords: Rational functions, Polynomials, Inequalities.

1. Introduction

Let P_n denote the class of all complex polynomials of degree at most n. For $a_j\in\mathbb{C}$, $j=1,2,\ldots,n$, we write

$$w(z):=\prod_{j=1}^{n}(z-a_j),\quad B(z):=\prod_{j=1}^{n}\left(\frac{1-\overline{a_j}z}{z-a_j}\right)$$

and

$$\Re_n:=\Re_n(a_1,a_2,\ldots,a_n)=\left\{\frac{p(z)}{w(z)};\ p\in P_n\right\}.$$

Then $\Re_n$ is the set of all rational functions with poles $a_j, j=1,2,\ldots,n$ at most and with finite limit at infinity. It is clear that $B(z)\in\Re_n$ and $|B(z)|=1$ for $|z|=1$. Throughout this paper, we shall assume that all the poles a_j, $j=1,2,\ldots,n$ lie in $|z|>1$.

If $p\in P_n$, then concerning the estimate of $|p'(z)|$ on the unit disk $|z|\leq 1$, we have the following famous result known as Bernstein's inequality [3].

Theorem 1 [3]. *If $p \in P_n$, then*

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|$$

with equality only for $p(z) = \lambda z^n$, $\lambda \neq 0$ being a complex number.

For polynomials having all their zeros in $|z| \leq 1$, Turàn [14] proved

Theorem 2 [14]. *If $p \in P_n$ and $p(z)$ has all its zeros in $|z| \leq 1$, then*

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \max_{|z|=1} |p(z)| \tag{1.1}$$

with equality for those polynomials, which have all their zeros on $|z| = 1$.

In literature, there exists several generalizations and refinements of inequality (1.1) (see [10–12]). V.K. Jain [6] in 1997 introduced a parameter β and proved the following result which is an interesting generalization of inequality (1.1).

Theorem 3 [6]. *If $p \in P_n$ and $p(z)$ has all its zeros in $|z| \leq 1$, then for $|\beta| \leq 1$*

$$\max_{|z|=1} \left| zp'(z) + \frac{n\beta}{2} p(z) \right| \geq \frac{n}{2} \left\{1 + \operatorname{Re}(\beta)\right\} \max_{|z|=1} |p(z)|. \tag{1.2}$$

By involving the coefficients of polynomial $p(z)$, Dubinin [4] refined inequality (1.1) and proved the following result.

Theorem 4 [4]. *If $p(z) = \sum_{j=0}^{n} \alpha_j z^j$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then*

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \left\{ 1 + \frac{1}{n} \left(\frac{|\alpha_n| - |\alpha_0|}{|\alpha_n| + |\alpha_0|} \right) \right\} \max_{|z|=1} |p(z)|.$$

As a generalization of Theorem 4, Rather et al. [9] proved the following result.

Theorem 5 [9]. *If $p(z) = \sum_{j=0}^{n} \alpha_j z^j$ is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \leq 1$, then for $|z| = 1$*

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+k} \left\{ 1 + \frac{k}{n} \left(\frac{k^n|\alpha_n| - |\alpha_0|}{k^n|\alpha_n| + |\alpha_0|} \right) \right\} \max_{|z|=1} |p(z)|. \tag{1.3}$$

Li, Mohapatra and Rodriguez [7] extended the inequality (1.1) to the rational functions $r \in \Re_n$ with prescribed poles and replace z^n by Blaschke product $B(z)$. Among other things they proved the following result.

Theorem 6 [7] . *Suppose $r \in \Re_n$, where r has exactly n poles at $a_1, a_2, \ldots, a_n$ and all the zeros of r lie in $|z| \leq 1$, then for $|z| = 1$*

$$|r'(z)| \geq \frac{1}{2} \Big\{ |B'(z)| - (n - m) \Big\} |r(z)|, \tag{1.4}$$

where m is the number of zeros of r.

As a generalization of inequality (1.4), Aziz and Shah [2] proved the following result.

Theorem 7 [2]. *Suppose $r \in \Re_n$, where r has exactly n poles at $a_1, a_2, \dots, a_n$ and all the zeros of r lie in $|z| \le k$, $k \le 1$, then for $|z| = 1$*

$$|r'(z)| \ge \frac{1}{2}\left\{|B'(z)| + \frac{2m - n(1+k)}{1+k}\right\}|r(z)|, \tag{1.5}$$

where m is the number of zeros of r.

Concerning the estimation of the lower bound of $\mathrm{Re}\big(zp'(z)/p(z)\big)$ on $|z| = 1$, Dubinin [4] proved the following result.

Theorem 8 [4] . *If $p(z) = \sum_{j=0}^{n} \alpha_j z^j$ is a polynomial of degree n which has all its zeros in $|z| \le 1$, then for all z on $|z| = 1$ for which $p(z) \ne 0$*

$$\mathrm{Re}\left(\frac{zp'(z)}{p(z)}\right) \ge \frac{n}{2}\left\{1 + \frac{1}{n}\left(\frac{|\alpha_n| - |\alpha_0|}{|\alpha_n| + |\alpha_0|}\right)\right\}.$$

Rather et al. [9] generalized Theorem 8 by proving the following result.

Theorem 9 [9]. *If $p(z) = \sum_{j=0}^{n} \alpha_j z^j$ is a polynomial of degree n and $p(z)$ has all its zeros in $|z| \le k, k \le 1$, then for all z on $|z| = 1$ for which $p(z) \ne 0$,*

$$\mathrm{Re}\left(\frac{zp'(z)}{p(z)}\right) \ge \frac{n}{1+k}\left\{1 + \frac{k}{n}\left(\frac{k^n|\alpha_n| - |\alpha_0|}{k^n|\alpha_n| + |\alpha_0|}\right)\right\}.$$

Concerning the estimation of the lower bound of $\mathrm{Re}\big(zr'(z)/r(z)\big)$ on $|z| = 1$, Dubinin [5] extended Theorem 8 to the rational functions and proved the following result.

Theorem 10 [5]. *Let r be a rational function of the form $r(z) = p(z)/w(z)$, where*

$$p(z) = \alpha_m z^m + \alpha_{m-1} z^{m-1} + \cdots + \alpha_1 z + \alpha_0, \quad \alpha_m \ne 0, \quad m \ge n$$

and the poles c_ν, $\nu = 1, 2, ..., n$ of r are arbitrary with $|c_\nu| \ne 1$ and let all the zeros of the function r lie in the disk $|z| \le 1$. Then, at points of the circle $|z| = 1$, other than the zeros of r, the following inequality holds

$$\mathrm{Re}\left\{\frac{zr'(z)}{r(z)}\right\} \ge \frac{1}{2}\left\{m - n + \frac{zB'(z)}{B(z)} + \frac{|\alpha_m| - |\alpha_0|}{|\alpha_m| + |\alpha_0|}\right\}. \tag{1.6}$$

For $m = n$ inequality (1.6) reduces to

$$\mathrm{Re}\left\{\frac{zr'(z)}{r(z)}\right\} \ge \frac{1}{2}\left\{\frac{zB'(z)}{B(z)} + \frac{|\alpha_m| - |\alpha_0|}{|\alpha_m| + |\alpha_0|}\right\}. \tag{1.7}$$

2. Main results

In this section, we first present the following result, which in particular furnishes a compact generalization of Theorem 10 for the case when all the poles of r lie outside the unit disk and as a consequence of this result, we get various generalizations and refinements of the above mentioned results. More precisely we prove.

Theorem 11. *Suppose $r \in \Re_n$, where r has exactly n poles and all the zeros of*

$$p(z) = \alpha_m z^m + \alpha_{m-1} z^{m-1} + \cdots + \alpha_1 z + \alpha_0, \quad \alpha_m \neq 0,$$

lie in $|z| \leq k$, $k \leq 1$. Then for all z on the circle $|z| = 1$, other than the zeros of r and $|\beta| \leq 1$

$$\begin{aligned} &\mathrm{Re}\left\{\frac{zr'(z)}{r(z)} + \frac{\beta}{1+k}|B'(z)|\right\} \\ &\geq \frac{1}{2}\left[\left\{1 + \frac{2\mathrm{Re}(\beta)}{1+k}\right\}|B'(z)| + \frac{2m - n(1+k)}{(1+k)} + \frac{2k}{1+k}\left\{\frac{k^m|\alpha_m| - |\alpha_0|}{k^m|\alpha_m| + |\alpha_0|}\right\}\right]. \end{aligned} \tag{2.1}$$

The result is best possible in the case $\beta = 0$, and equality holds for

$$r(z) = \frac{(z+k)^m}{(z-a)^n} \quad \text{and} \quad B(z) = \left(\frac{1-az}{z-a}\right)^n, \quad \text{at} \quad z = 1, \quad a > 1 \quad \text{and} \quad \beta = 0.$$

Remark 1. Taking $\beta = 0$, and using the fact that

$$|B'(z)| = \frac{zB'(z)}{B(z)}$$

on $|z| = 1$, inequality (2.1) reduces to the following inequality

$$\mathrm{Re}\left\{\frac{zr'(z)}{r(z)}\right\} \geq \frac{1}{2}\left[\frac{zB'(z)}{B(z)} + \frac{2m - n(1+k)}{(1+k)} + \frac{2k}{1+k}\left\{\frac{k^m|\alpha_m| - |\alpha_0|}{k^m|\alpha_m| + |\alpha_0|}\right\}\right]. \tag{2.2}$$

One can easily note that for $\beta = 0$, Theorem 11 is an extension of Theorem 9 to the rational functions. On the other hand if we take $k = 1$ and $m = n$ in inequality (2.2), we shall obtain inequality (1.7).

Remark 2. Now for the points on the circle $|z| = 1$, other than the zeros of r and $|\beta| \leq 1$, one can easily prove that

$$\left|\frac{zr'(z)}{r(z)} + \frac{\beta}{1+k}|B'(z)|\right| \geq \mathrm{Re}\left\{\frac{zr'(z)}{r(z)} + \frac{\beta}{1+k}|B'(z)|\right\}.$$

In view of this, Theorem 11 reduces to the following result, which contributes a generalization and refinement of inequality (1.5).

Corollary 1. *Suppose $r \in \Re_n$, where r has exactly n poles and all the zeros of r lie in $|z| \leq k$, $k \leq 1$, that is $r(z) = p(z)/w(z)$ with*

$$p(z) = \alpha_m z^m + \alpha_{m-1} z^{m-1} + \cdots + \alpha_1 z + \alpha_0, \quad \alpha_m \neq 0.$$

Then for all z on $|z| = 1$ other than the zeros of r and $|\beta| \leq 1$

$$\left|\frac{zr'(z)}{r(z)} + \frac{\beta}{1+k}|B'(z)|\right| \geq \frac{1}{2}\left[\left\{1 + \frac{2\mathrm{Re}\,(\beta)}{1+k}\right\}|B'(z)| + \frac{2m - n(1+k)}{(1+k)} + \frac{2k}{1+k}\left\{\frac{k^m|\alpha_m| - |\alpha_0|}{k^m|\alpha_m| + |\alpha_0|}\right\}\right].$$

The result is best possible in the case $\beta = 0$, and equality holds for

$$r(z) = \frac{(z+k)^m}{(z-a)^n} \quad \text{and} \quad B(z) = \left(\frac{1-az}{z-a}\right)^n, \quad \text{at} \quad z = 1, \quad a > 1 \quad \text{and} \quad \beta = 0.$$

Remark 3. For $k=1$, Corollary 1 reduces to the following result, which yields a generalization as well as refinement of inequality (1.4).

Corollary 2. *Suppose $r \in \Re_n$, where r has exactly n poles and all the zeros of r lie in $|z| \le 1$, that is $r(z) = p(z)/w(z)$ with*

$$p(z) = \alpha_m z^m + \alpha_{m-1} z^{m-1} + \cdots + \alpha_1 z + \alpha_0, \quad \alpha_m \neq 0.$$

Then for all z on $|z|=1$ other than the zeros of r and $|\beta| \le 1$

$$\left| \frac{zr'(z)}{r(z)} + \frac{\beta}{2}|B'(z)| \right| \ge \frac{1}{2}\left[\{1 + \mathrm{Re}\,(\beta)\}\,|B'(z)| - (n-m) + \left\{ \frac{|\alpha_m| - |\alpha_0|}{|\alpha_m| + |\alpha_0|} \right\} \right]. \tag{2.3}$$

Inequality (2.3) is sharp in the case $\beta = 0$ and equality holds for

$$r(z) = \frac{(z+1)^m}{(z-a)^n} \quad \text{and} \quad B(z) = \left(\frac{1-az}{z-a} \right)^n, \quad \text{at} \quad z = 1, \quad a > 1 \quad \text{and} \quad \beta = 0.$$

Remark 4. Taking $w(z) = (z-\alpha)^n$, $|\alpha| > 1$, so that

$$B(z) = \left(\frac{1-\alpha z}{z-\alpha} \right)^n$$

with $m = n$ in Corollary 1, we get

$$\begin{aligned} &\left| z\left(\frac{p'(z)}{p(z)} + \frac{n}{z-\alpha} \right) + \frac{\beta}{1+k}|B'(z)| \right| \\ \ge \frac{1}{2}&\left[\left\{ 1 + \frac{2\mathrm{Re}\,(\beta)}{1+k} \right\} |B'(z)| + \frac{n(1-k)}{1+k} + \frac{2k}{1+k}\left(\frac{k^n|\alpha_n| - |\alpha_0|}{k^n|\alpha_n| + |\alpha_0|} \right) \right]. \end{aligned} \tag{2.4}$$

Letting $|\alpha| \to \infty$ in inequality (2.4) and noting that $|B'(z)| \to n|z|^{n-1} = n$ for $|z|=1$, we get the following result.

Corollary 3. *If $p(z) = \sum_{j=0}^{n} \alpha_j z^j$ is a polynomial of degree n, having all its zeros in $|z| \le k$, $k \le 1$, then for $|\beta| \le 1$ and $|z| = 1$*

$$\left| zp'(z) + \frac{n\beta}{1+k}p(z) \right| \ge \frac{n}{1+k}\left\{ 1 + \mathrm{Re}(\beta) + \frac{k}{n}\left(\frac{k^n|\alpha_n| - |\alpha_0|}{k^n|\alpha_n| + |\alpha_0|} \right) \right\} |p(z)|. \tag{2.5}$$

Since $k^n|\alpha_n| \ge |\alpha_0|$, therefore Corollary 3 refines as well as generalizes the well known polynomial inequality (1.2) due to Jain [6].

Remark 5. For $\beta = 0$, inequality (2.5) reduces to inequality (1.3).

Next, we prove the following refinement of Corollary 3.

Theorem 12. *If $p(z) = \sum_{j=0}^{n} \alpha_j z^j$ is a polynomial of degree n, having all its zeros in $|z| \le k$, $k \le 1$, then for $|\beta| \le 1$ and $|z| = 1$*

$$\begin{aligned} \left| zp'(z) + \frac{n\beta}{1+k}p(z) \right| \ge& \frac{n}{1+k}\left\{ 1 + \mathrm{Re}\,(\beta) + \frac{k}{n}\left(\frac{k^n|\alpha_n| - m^* - |\alpha_0|}{k^n|\alpha_n| + m^* + |\alpha_0|} \right) \right\} |p(z)| \\ &+ \frac{nm^*}{1+k}\left\{ \left| 1 + \mathrm{Re}\,(\beta) + \frac{k}{n}\left(\frac{k^n|\alpha_n| - m^* - |\alpha_0|}{k^n|\alpha_n| + m^* + |\alpha_0|} \right) - |\beta| \right| \right\}, \end{aligned}$$

where $m^* = \min_{|z|=k} |p(z)|$.

Taking $\beta = 0$ in Theorem 12, we get the following result.

Corollary 4. *If* $p(z) = \sum_{j=0}^{n} \alpha_j z^j$ *is a polynomial of degree* n*, having all its zeros in* $|z| \leq k$*,* $k \leq 1$*, then for* $|z| = 1$

$$|p'(z)| \geq \frac{n}{1+k}\left\{1 + \frac{k}{n}\left(\frac{k^n|\alpha_n| - m^* - |\alpha_0|}{k^n|\alpha_n| + m^* + |\alpha_0|}\right)\right\}|p(z)| + \frac{nm^*}{1+k}\left\{\left|1 + \frac{k}{n}\left(\frac{k^n|\alpha_n| - m^* - |\alpha_0|}{k^n|\alpha_n| + m^* + |\alpha_0|}\right)\right|\right\},$$

where $m^* = \min_{|z|=k}|p(z)|$.

Remark 6. Since $m^* \geq 0$, hence Corollary 4 is a refinement Theorem 5.

3. Lemmas

For the proof of our results, we need the following lemmas. The first lemma is due to A. Aziz and B.A. Zargar [1].

Lemma 1 [1]. *If* $|z| = 1$*, then*

$$\mathrm{Re}\left(\frac{zw'(z)}{w(z)}\right) = \frac{n - |B'(z)|}{2},$$

where $w(z) = \prod_{j=1}^{n}(z - a_j)$.

The following lemma is due to Rather et al. [9].

Lemma 2 [9]. *If* $\langle\zeta_j\rangle_{j=1}^{m}$ *be a finite collection of real numbers such that* $0 \leq \zeta_j \leq 1$*,* $j = 1, 2, \ldots, m$*, then*

$$\sum_{j=1}^{m}\frac{1-\zeta_j}{1+\zeta_j} \geq \frac{1 - \prod_{j=1}^{m}\zeta_j}{1 + \prod_{j=1}^{m}\zeta_j}.$$

The next lemma is due to Mezerji et al. [13].

Lemma 3 [13]. *If* $p(z)$ *is a polynomial of degree* n*, having all its zeros in* $|z| \leq k$*,* $k \leq 1$*, then for any* β *with* $|\beta| \leq 1$*,*

$$\min_{|z|=1}\left|zp'(z) + \frac{n\beta}{1+k}p(z)\right| \geq \frac{nm^*}{k^n}\left|1 + \frac{\beta}{1+k}\right|,$$

where $m^* = \min_{|z|=k}|p(z)|$.

4. Proof of Theorem 11

P r o o f. Since $r \in \Re_n$ and all the zeros of $r(z)$ lie in $|z| \leq k$, $k \leq 1$, that is $r(z) = p(z)/w(z)$ with

$$p(z) = \alpha_m \prod_{j=1}^{m}(z - b_j) = \alpha_m z^m + \alpha_{m-1}z^{m-1} + \cdots + \alpha_1 z + \alpha_0,$$
$$\alpha_m \neq 0, \quad |b_j| \leq k \leq 1, \quad j = 1, 2, 3, ..., m.$$

Then for $|\beta| \le 1$ and for all z on $|z| = 1$, where $r(z) \neq 0$, we have

$$\begin{aligned}\mathrm{Re}\left\{\frac{zr'(z)}{r(z)} + \frac{\beta}{1+k}|B'(z)|\right\} &= \mathrm{Re}\left\{\frac{zr'(z)}{r(z)}\right\} + \frac{|B'(z)|}{1+k}\mathrm{Re}\{\beta\}\\ &= \mathrm{Re}\left\{\frac{zp'(z)}{p(z)} - \frac{zw'(z)}{w(z)}\right\} + \frac{|B'(z)|}{1+k}\mathrm{Re}\{\beta\}\\ &= \mathrm{Re}\left\{\frac{zp'(z)}{p(z)}\right\} - \mathrm{Re}\left\{\frac{zw'(z)}{w(z)}\right\} + \frac{|B'(z)|}{1+k}\mathrm{Re}\{\beta\}.\end{aligned}$$

Using Lemma 1, we have for $|\beta| \le 1$ and for all z on $|z| = 1$, where $r(z) \neq 0$,

$$\begin{aligned}\mathrm{Re}\left\{\frac{zr'(z)}{r(z)} + \frac{\beta}{1+k}|B'(z)|\right\} &= \mathrm{Re}\sum_{j=1}^{m}\left\{\frac{z}{z-b_j}\right\} - \left\{\frac{n-|B'(z)|}{2}\right\} + \frac{|B'(z)|}{1+k}\mathrm{Re}\{\beta\}\\ &= \sum_{j=1}^{m}\mathrm{Re}\left\{\frac{z}{z-b_j}\right\} - \frac{n}{2} + \frac{1}{2}\left\{1 + \frac{2\mathrm{Re}(\beta)}{1+k}\right\}|B'(z)|.\end{aligned} \tag{4.1}$$

Now it can be easily verified that for $|z| = 1$ and $|b_j| \le k \le 1$, we have

$$\mathrm{Re}\left\{\frac{z}{z-b_j}\right\} \ge \left\{\frac{1}{1+|b_j|}\right\}.$$

Using this in inequality (4.1), we get for $|\beta| \le 1$ and for all z on $|z| = 1$, where $r(z) \neq 0$,

$$\begin{aligned}\mathrm{Re}\left\{\frac{zr'(z)}{r(z)} + \frac{\beta}{1+k}|B'(z)|\right\} &\ge \sum_{j=1}^{m}\left\{\frac{1}{1+|b_j|}\right\} - \frac{n}{2} + \frac{1}{2}\left\{1 + \frac{2\mathrm{Re}(\beta)}{1+k}\right\}|B'(z)|\\ &= \frac{1}{2}\left\{1 + \frac{2\mathrm{Re}(\beta)}{1+k}\right\}|B'(z)| + \sum_{j=1}^{m}\left\{\frac{1}{1+|b_j|} - \frac{1}{1+k}\right\} + \frac{m}{1+k} - \frac{n}{2}\\ &= \frac{1}{2}\left\{1 + \frac{2\mathrm{Re}(\beta)}{1+k}\right\}|B'(z)| + \frac{2m-n(1+k)}{2(1+k)} + \frac{k}{1+k}\sum_{j=1}^{m}\left\{\frac{k-|b_j|}{k+k|b_j|}\right\}\\ &\ge \frac{1}{2}\left\{1 + \frac{2\mathrm{Re}(\beta)}{1+k}\right\}|B'(z)| + \frac{2m-n(1+k)}{2(1+k)} + \frac{k}{1+k}\sum_{j=1}^{m}\left\{\frac{k-|b_j|}{k+|b_j|}\right\}\\ &= \frac{1}{2}\left\{1 + \frac{2\mathrm{Re}(\beta)}{1+k}\right\}|B'(z)| + \frac{2m-n(1+k)}{2(1+k)} + \frac{k}{1+k}\sum_{j=1}^{m}\left\{\frac{1-|b_j|/k}{1+|b_j|/k}\right\}.\end{aligned} \tag{4.2}$$

Since $|b_j|/k \le 1$, therefore by invoking Lemma 2, we conclude from inequality (4.2) that for $|\beta| \le 1$ and for all z on $|z| = 1$, where $r(z) \neq 0$,

$$\begin{aligned}&\mathrm{Re}\left\{\frac{zr'(z)}{r(z)} + \frac{\beta}{1+k}|B'(z)|\right\}\\ &\ge \frac{1}{2}\left\{1 + \frac{2\mathrm{Re}(\beta)}{1+k}\right\}|B'(z)| + \frac{2m-n(1+k)}{2(1+k)} + \frac{k}{1+k}\left\{\frac{1-\prod_{j=1}^{m} c|b_j|/k}{1+\prod_{j=1}^{m}|b_j|/k}\right\}\\ &= \frac{1}{2}\left\{1 + \frac{2\mathrm{Re}(\beta)}{1+k}\right\}|B'(z)| + \frac{2m-n(1+k)}{2(1+k)} + \frac{k}{1+k}\left\{\frac{k^m|\alpha_m| - |\alpha_0|}{k^m|\alpha_m| + |\alpha_0|}\right\}.\end{aligned}$$

$\square$

5. Proof of Theorem 12

Proof. If $p(z)$ has a zero on $|z|=k$, then the result follows from Corollary 3. We assume that all the zeros of $p(z)$ lie in $|z|<k$, $k\le 1$, so that $m^*>0$ and we have $m^*\le |p(z)|$ for $|z|=k$. By Rouche's theorem for every λ with $|\lambda|<1$, the polynomial $h(z)=p(z)-\lambda m^*$ has all its zeros in $|z|<k$, $k\le 1$. Applying Corollary 3 to the polynomial $h(z)$, we get for $\lambda,\beta\in\mathbb{C}$ with $|\lambda|<1$, $|\beta|\le 1$ and $|z|=1$,

$$\begin{aligned}&\left|zp'(z)+\frac{n\beta}{1+k}\{p(z)-\lambda m^*\}\right|\\ &\ge \frac{n}{2}\left\{1+\frac{2\mathrm{Re}(\beta)}{1+k}+\frac{1-k}{1+k}+\frac{2k}{n(1+k)}\left(\frac{k^n|\alpha_n|-|\lambda m^*-\alpha_0|}{k^n|\alpha_n|+|\lambda m^*-\alpha_0|}\right)\right\}|p(z)-\lambda m^*|.\end{aligned}$$

or

$$\begin{aligned}&\left|zp'(z)+\frac{n\beta}{1+k}p(z)-\frac{n\beta}{1+k}\lambda m^*\right|\\ &\ge \frac{n}{2}\left\{1+\frac{2\mathrm{Re}(\beta)}{1+k}+\frac{1-k}{1+k}+\frac{2k}{n(1+k)}\left(\frac{k^n|\alpha_n|-|\lambda|m^*-|\alpha_0|}{k^n|\alpha_n|+|\lambda|m^*+|\alpha_0|}\right)\right\}|p(z)-\lambda m^*|.\end{aligned}\tag{5.1}$$

Now for every $\beta\in\mathbb{C}$ with $|\beta|\le 1$ and $k>0$,

$$k|\beta|\le |1+k+\beta|.$$

or,

$$\left|1+\frac{\beta}{1+k}\right|\ge\frac{k}{1+k}|\beta|,\quad\text{for}\quad |\beta|\le 1\quad\text{and}\quad k>0.$$

Using this in Lemma 3, we have for $|z|=1$, $|\beta|\le 1$ and $k\le 1$,

$$\left|zp'(z)+\frac{n\beta}{1+k}p(z)\right|\ge\frac{nm^*}{k^n}\left|1+\frac{\beta}{1+k}\right|\ge\frac{nm^*}{k^{n-1}}\frac{|\beta|}{1+k}\ge\left|\frac{n\beta}{1+k}\lambda m^*\right|\quad\text{for}\quad|\lambda|<1.$$

In view of this, choosing argument of λ in left hand side of (5.1) such that

$$\left|zp'(z)+\frac{n\beta}{1+k}p(z)-\frac{n\beta}{1+k}\lambda m^*\right|=\left|zp'(z)+\frac{n\beta}{1+k}p(z)\right|-\frac{n|\beta|}{1+k}|\lambda|m^*,$$

we obtain from inequality (5.1), for $|\beta|\le 1$ and $|z|=1$,

$$\begin{aligned}&\left|zp'(z)+\frac{n\beta}{1+k}p(z)\right|-\frac{n|\beta|}{1+k}|\lambda|m^*\\ &\ge \frac{n}{2}\left\{1+\frac{2\mathrm{Re}(\beta)}{1+k}+\frac{1-k}{1+k}+\frac{2k}{n(1+k)}\left(\frac{k^n|\alpha_n|-|\lambda|m^*-|\alpha_0|}{k^n|\alpha_n|+|\lambda|m^*+|\alpha_0|}\right)\right\}\{|p(z)|-|\lambda|m^*\}.\end{aligned}$$

or

$$\begin{aligned}&\left|zp'(z)+\frac{n\beta}{1+k}p(z)\right|\\ &\ge \frac{n}{2}\left\{1+\frac{2\mathrm{Re}(\beta)}{1+k}+\frac{1-k}{1+k}+\frac{2k}{n(1+k)}\left(\frac{k^n|\alpha_n|-|\lambda|m^*-|\alpha_0|}{k^n|\alpha_n|+|\lambda|m^*+|\alpha_0|}\right)\right\}|p(z)|\\ &+\frac{nm^*}{2}|\lambda|\left[\frac{2|\beta|}{1+k}-\left\{1+\frac{2\mathrm{Re}(\beta)}{1+k}+\frac{1-k}{1+k}+\frac{2k}{n(1+k)}\left(\frac{k^n|\alpha_n|-|\lambda|m^*-|\alpha_0|}{k^n|\alpha_n|+|\lambda|m^*+|\alpha_0|}\right)\right\}\right].\end{aligned}\tag{5.2}$$

Again by inequality (5.1), we have for $|\lambda| < 1$, $|\beta| \leq 1$ and $|z| = 1$,

$$\begin{aligned}&\left|zp'(z) + \frac{n\beta}{1+k}p(z)\right| + \left|\frac{n\beta}{1+k}\lambda m^*\right|\\ \geq \frac{n}{2}&\left\{1 + \frac{2\mathrm{Re}(\beta)}{1+k} + \frac{1-k}{1+k} + \frac{2k}{n(1+k)}\left(\frac{k^n|\alpha_n| - |\lambda|m^* - |\alpha_0|}{k^n|\alpha_n| + |\lambda|m^* + |\alpha_0|}\right)\right\}\{|p(z)| + |\lambda|m^*\}.\end{aligned}$$

or

$$\begin{aligned}&\left|zp'(z) + \frac{n\beta}{1+k}p(z)\right|\\ &\geq \frac{n}{2}\left\{1 + \frac{2\mathrm{Re}(\beta)}{1+k} + \frac{1-k}{1+k} + \frac{2k}{n(1+k)}\left(\frac{k^n|\alpha_n| - |\lambda|m^* - |\alpha_0|}{k^n|\alpha_n| + |\lambda|m^* + |\alpha_0|}\right)\right\}|p(z)|\\ &+\frac{nm^*}{2}|\lambda|\left\{1 + \frac{2\mathrm{Re}(\beta)}{1+k} + \frac{1-k}{1+k} + \frac{2k}{n(1+k)}\left(\frac{k^n|\alpha_n| - |\lambda|m^* - |\alpha_0|}{k^n|\alpha_n| + |\lambda|m^* + |\alpha_0|}\right) - \frac{2|\beta|}{1+k}\right\}.\end{aligned} \tag{5.3}$$

Now from inequality (5.2) and inequality (5.3), we get for $|\beta| \leq 1$ and $|z| = 1$,

$$\begin{aligned}\left|zp'(z) + \frac{n\beta}{1+k}p(z)\right| &\geq \frac{n}{2}\left\{1 + \frac{2\mathrm{Re}(\beta)}{1+k} + \frac{1-k}{1+k} + \frac{2k}{n(1+k)}\left(\frac{k^n|\alpha_n| - |\lambda|m^* - |\alpha_0|}{k^n|\alpha_n| + |\lambda|m^* + |\alpha_0|}\right)\right\}|p(z)|\\ &+\frac{nm^*}{2}|\lambda|\left\{\left|1 + \frac{2\mathrm{Re}(\beta)}{1+k} + \frac{1-k}{1+k} + \frac{2k}{n(1+k)}\left(\frac{k^n|\alpha_n| - |\lambda|m^* - |\alpha_0|}{k^n|\alpha_n| + |\lambda|m^* + |\alpha_0|}\right) - \frac{2|\beta|}{1+k}\right|\right\}.\end{aligned}$$

Letting $|\lambda| \to 1$, we obtain for $|z| = 1$,

$$\begin{aligned}\left|zp'(z) + \frac{n\beta}{1+k}p(z)\right| &\geq \frac{n}{1+k}\left\{1 + \mathrm{Re}(\beta) + \frac{k}{n}\left(\frac{k^n|\alpha_n| - m^* - |\alpha_0|}{k^n|\alpha_n| + m^* + |\alpha_0|}\right)\right\}|p(z)|\\ &+\frac{nm^*}{1+k}\left\{\left|1 + \mathrm{Re}(\beta) + \frac{k}{n}\left(\frac{k^n|\alpha_n| - m^* - |\alpha_0|}{k^n|\alpha_n| + m^* + |\alpha_0|}\right) - |\beta|\right|\right\},\end{aligned}$$

which proves Theorem 12. □

6. A remark on a recent result concerning rational functions

Recently Idrees Qasim [8] claimed to have proved various results regarding Bernstein-type inequalities for rational functions with prescribed poles and restricted zeros. Among other things he claimed to have proved the following result.

Theorem 13 [8]. *If $r(z) = p(z)/w(z) \in \Re_n$, where $p(z) = \sum_{j=0}^{n} \alpha_j z^j$, $|b|.|\alpha_n| \leq |\alpha_0|$, r has exactly n poles at $a_1, a_2, \ldots, a_n$, and $r(z) \neq 0$ in $|z| > 1$, then for $|z| = 1$,*

$$|r'(z)| \geq \frac{1}{2}\left[|B'(z)| + \frac{\sqrt{|\alpha_n|} - \sqrt{|\alpha_0|}}{\sqrt{|\alpha_n|}}\right](|r(z)| + m^{**}),$$

where $m^{**} = \min_{|z|=1}|r(z)|$ and $b = a_1 a_2 \ldots a_n$.

Since it is assumed throughout the paper that all the poles $(a_1, a_2, ..., a_n)$ of rational function r lie outside unit disk, therefore,

$$|b| = |a_1 \times a_2 \times ... \times a_n| > 1. \tag{6.1}$$

On the other hand, it is also assumed that all the zeros $(z_1, z_2, ..., z_n)$ of r lie in the disc $|z| \leq 1$, implies

$$\frac{|\alpha_0|}{|\alpha_n|} = |z_1 \times z_2 \times ... \times z_n| \leq 1. \tag{6.2}$$

From (6.1) and (6.2), it follows that $|b|.|\alpha_n| > |\alpha_0|$, which is contrary to the hypothesis $|b|.|\alpha_n| \leq |\alpha_0|$ given in the statement of the Theorem 13. Hence the statement of the Theorem 13 is self-contradicting, as such Theorem 13 and its consequences are never applicable.

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