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A QUADRUPLE INTEGRAL INVOLVING THE EXPONENTIAL LOGARITHM OF QUOTIENT RADICALS IN TERMS OF THE HURWITZ-LERCH ZETA FUNCTION[1]

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Abstract: With a possible connection to integrals used in General Relativity, we used our contour integral method to write a closed form solution for a quadruple integral involving exponential functions and logarithm of quotient radicals. Almost all Hurwitz–Lerch Zeta functions have an asymmetrical zero-distribution. All the results in this work are new.

Keywords: Quadruple integral, Hurwitz–Lerch Zeta function, Catalan's constant, Cauchy integral, Glaisher's constant.

1. Significance statement

Quadruple integrals are broadly utilized in a wide number of disciplines crossing math, science and engineering. Some interesting areas where these integrals are used are in the three-body problem and the equations of dynamics [12], integral solutions to the wave equation [9], path integrals in polymer physics [5], analytical evaluations of double integral expressions related to total variation [6], electrodynamics of moving media [8], and measurements in heat transfer [2].

The authors discovered various uses of quadruple integrals after reviewing the present literature. In some cases these integrals were separable and in some cases asymptotic expansions were used to attain a solution.The authors were unable to uncover quadruple integrals involving exponential functions and the logarithm of quotient radicals generated in terms of a closed form solution. This integral features a kernel with the product of the exponential logarithm of quotient radical functions. The log term mixes the variables so that the integral is not separable except for special values of k.

The book by Prudnikov et al. [13], is structured towards mathematicians, physicists, experts in calculus methods, instructors, graduate students, for all those concerned with integrals, higher transcendental functions and integral transforms and those keen to master the corresponding theories. This book is also of help when dealing with the modern theory of higher functions and integral transformations accessible to undergraduate and graduate students [3].

This famous book contains a vast quantity of mathematical formulae. These formulae are indefinite integrals, definite integrals, multidimensional integrals, finite and infinite sums and and multidimensional finite and infinite sums. In the book of Prudnikov et al. [13] there is a combination of integral examples expressed in terms of fundamental constants and Special functions. Since these types of integral formulae are of such high importance in science, it has encouraged us to contribute to such tables by adding definite quadruple integrals in terms of the Hurwitz–Lerch Zeta function.

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This work represents an illustration of a general approach using contour integration applied to a particular integral in the book of Prudnikov et al. [13].

2. Preliminaries

We proceed by using the contour integral method [14] and the reflection formula for the gamma function given by equation (5.5.3) in [10], applied to equation (3.1.3.9) in [13] to yield the Prudnikov quadruple contour integral representation given by:

$$\int_{\mathbb{R}_+^4} a^w w^{-k-1}(rs)^{(-m-w)/2-1}(r+s)^{(m+w+1)/2}(xy)^{(m+w)/2}(x+y)^{(-m-w-1)/2}e^{-p(r+x)-q(s+y)}dxdydrds$$
$$= -\frac{1}{2\pi i}\int_C \frac{\pi^2 a^w w^{-k-1}\csc\left(\pi(m+w)/2\right)}{pq}dw$$

where $a, k, w, m, p, q \in \mathbb{C}$, $\mathrm{Re}(m+w) > 0$, $-1 < \mathrm{Re}(m) < 0$.

3. Introduction

In this paper the main theorem derived is the quadruple definite integral given by

$$\int_{\mathbb{R}_+^4} (rs)^{-m/2-1}(r+s)^{(m+1)/2}(xy)^{m/2}(x+y)^{(-m-1)/2}e^{-p(r+x)-q(s+y)}\log^k\left(\frac{a\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)dxdydrds$$
$$= \frac{2i\pi^{k+2}e^{i\pi(k+m)/2}\Phi\left(e^{im\pi}, -k, 1/2 - i\log(a)/\pi\right)}{pq},$$

where the parameters k, a, p, q and m are general complex numbers. This integral is derived in terms of the Hurwitz–Lerch Zeta function which is a useful special function. This is a function of three complex variables which is extended by analytic continuation to the complex plane with the exception of a singularity at 1 and a branch cut between one to infinity. The Lerch function is a generalization of several important special functions namely, the geometric series, the natural logarithm, powers and exponentials, polylogarithms, the Riemann zeta function, the alternating Riemann zeta function, and the Hurwitz zeta function. One advantage of the approach in this current work is that it reveals the connection between quadruple integral formulae and classical mathematical functions.

This definite integral will be used to derive special cases in terms of special functions and fundamental constants and we summarize most of the evaluations in Table 7 for easy reading. The derivations follow the method used by us in [14]. This method involves using a form of the generalized Cauchy's integral formula given by

$$\frac{y^k}{\Gamma(k+1)} = \frac{1}{2\pi i}\int_C \frac{e^{wy}}{w^{k+1}}dw, \tag{3.1}$$

where C is in general an open contour in the complex plane where the bilinear concomitant has the same value at the end points of the contour. We then multiply both sides by a function of x, y, z and t, then take a definite quadruple integral of both sides. This yields a definite integral in terms of a contour integral. Then we multiply both sides of equation (3.1) by another function of x, y, r and s and take the infinite sums of both sides such that the contour integral of both equations are the same.

4. Definite integral of the contour integral

We use the method in [14]. The variable of integration in the contour integral is $\alpha = w + m$. The cut and contour are in the first quadrant of the complex α-plane. The cut approaches the origin from the interior of the first quadrant and the contour goes round the origin with zero radius and is on opposite sides of the cut. Using a generalization of Cauchy's integral formula we form the quadruple integral by replacing y by

$$\log\left(\frac{a\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)$$

and multiplying by

$$(rs)^{-m/2-1}(r+s)^{(m+1)/2}(xy)^{m/2}(x+y)^{(-m-1)/2}e^{-p(r+x)-q(s+y)}$$

then taking the definite integral with respect to $x \in [0,\infty)$, $y \in [0,\infty)$, $r \in [0,\infty)$ and $s \in [0,\infty)$ to obtain

$$\begin{aligned}
&\frac{1}{\Gamma(k+1)}\int_{\mathbb{R}_+^4}(rs)^{-m/2-1}(r+s)^{(m+1)/2}(xy)^{m/2}(x+y)^{(-m-1)/2}e^{-p(r+x)-q(s+y)}\\
&\qquad\times\log^k\left(\frac{a\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)dxdydrds\\
&=\frac{1}{2\pi i}\int_{\mathbb{R}_+^4}\int_C a^w w^{-k-1}(rs)^{(-m-w)/2-1}(r+s)^{(m+w+1)/2}(xy)^{(m+w)/2}(x+y)^{(-m-w-1)/2}\\
&\qquad\times e^{-p(r+x)-q(s+y)}dwdxdydrds\\
&=\frac{1}{2\pi i}\int_C\int_{\mathbb{R}_+^4} a^w w^{-k-1}(rs)^{(-m-w)/2-1}(r+s)^{(m+w+1)/2}(xy)^{(m+w)/2}(x+y)^{(-m-w-1)/2}\\
&\qquad\times e^{-p(r+x)-q(s+y)}dxdydrdsdw\\
&=-\frac{1}{2\pi i}\int_C\frac{\pi^2 a^w w^{-k-1}\csc(\pi(m+w)/2)}{pq}dw
\end{aligned}\tag{4.2}$$

from equation (3.1.3.9) in [13] where

$$\operatorname{Re}(w+m)>0,\quad \operatorname{Re}(p)>0,\quad \operatorname{Re}(q)>0,\quad -1<\operatorname{Re}(m)<0$$

and using the reflection formula (8.334.3) in [4] for the Gamma function. We are able to switch the order of integration over α, x, y, r and s using Fubini's theorem since the integrand is of bounded measure over the space $\mathbb{C}\times[0,\infty)\times[0,\infty)\times[0,\infty)\times[0,\infty)$.

5. The Hurwitz–Lerch zeta function and infinite sum of the contour integral

5.1. The Hurwitz–Lerch zeta function

The Hurwitz–Lerch Zeta function (see Section 1.11 in [1]) has a series representation given by

$$\Phi(z,s,v)=\sum_{n=0}^{\infty}(v+n)^{-s}z^n,$$

where $|z|<1, v\neq 0,-1,\ldots$ and is continued analytically by its integral representation given by

$$\Phi(z,s,v)=\frac{1}{\Gamma(s)}\int_0^\infty\frac{t^{s-1}e^{-vt}}{1-ze^{-t}}dt=\frac{1}{\Gamma(s)}\int_0^\infty\frac{t^{s-1}e^{-(v-1)t}}{e^t-z}dt$$

where $\operatorname{Re}(v)>0$, and either $|z|\leq 1$, $z\neq 1$, $\operatorname{Re}(s)>0$, or $z=1$, $\operatorname{Re}(s)>1$.

5.2. Derivation of the contour integral

Using equation (3.1) and replacing y by

$$\log(a)+\frac{1}{2}i\pi(2y+1)$$

then multiplying both sides by

$$\frac{2i\pi^2 e^{i\pi m(2y+1)/2}}{pq}$$

taking the infinite sum over $y\in[0,\infty)$ and simplifying in terms of the Hurwitz–Lerch Zeta function we obtain

$$\begin{aligned}\frac{2i\pi^{k+2}e^{i\pi(k+m)/2}\Phi\left(e^{im\pi},-k,1/2-i\log(a)/\pi\right)}{pq\Gamma(k+1)}&=\frac{1}{2\pi i}\sum_{y=0}^{\infty}\int_C\frac{2i\pi^2a^w w^{-k-1}e^{i\pi(2y+1)(m+w)/2}}{pq}dw\\ =\frac{1}{2\pi i}\int_C\sum_{y=0}^{\infty}\frac{2i\pi^2a^w w^{-k-1}e^{i\pi(2y+1)(m+w)/2}}{pq}dw&=-\frac{1}{2\pi i}\int_C\frac{\pi^2a^w w^{-k-1}\csc\left(\pi(m+w)/2\right)}{pq}dw\end{aligned}\tag{5.1}$$

from equation (1.232.2) in [4] where $\mathrm{Im}(w+m)>0$ in order for the sum to converge.

Theorem 1. *For $k,a,p,q,m\in\mathbb{C}$,*

$$\begin{aligned}\int_{\mathbb{R}_+^4}&(rs)^{-m/2-1}(r+s)^{(m+1)/2}(xy)^{m/2}(x+y)^{(-m-1)/2}e^{-p(r+x)-q(s+y)}\\ &\times\log^k\left(\frac{a\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)dxdydrds\\ &=\frac{2i\pi^{k+2}e^{i\pi(k+m)/2}\Phi\left(e^{im\pi},-k,1/2-i\log(a)/\pi\right)}{pq}.\end{aligned}\tag{5.2}$$

P r o o f. Observe the right-hand sides of (4.2) and (5.1) are the same so we can simplify the gamma function and equate the left-hand sides to yield the stated result. □

6. Main results

In the proceeding section we will evaluate equation (5.2) in terms of special functions and fundamental constants, Hurwitz zeta function $\zeta(s,a)$, given in Section 25.11 in [10], Catalan's constant C, given by equation (25.11.40) in [10], Riemann zeta function $\zeta(s)$, given in Section 25.2 in [10], Glaisher's constant A, given by equation (5.17.6) in [10] and equation (2.2.1.2.7) in [7], and Euler's constant γ, given by equation (5.2.3) in [10].

Example 1.

$$\int_{\mathbb{R}_+^4}\frac{\sqrt[4]{r+s}e^{-3r-2s-3x-2y}\left(\pi^2-4\log^2\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)\right)}{(rs)^{3/4}\sqrt[4]{xy}\sqrt[4]{x+y}\left(4\log^2\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)+\pi^2\right)^2}dxdydrds=\frac{48C+\pi^2}{576\sqrt{2}}$$

and

$$\int_{\mathbb{R}_+^4}\frac{\sqrt[4]{rs}\sqrt[4]{r+s}(xy)^{3/4}e^{-3r-2s-3x-2y}\log\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)}{rsxy\sqrt[4]{x+y}\left(4\log^2\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)+\pi^2\right)^2}dxdydrds=\frac{1}{16\pi}\left(\frac{C}{3\sqrt{2}}-\frac{\pi^2}{144\sqrt{2}}\right).$$

P r o o f. Use equation (5.2) and set $k=-2$, $a=i$, $m=-1/2$, $p=3$, $q=2$, rationalize the denominator and compare real and imaginary parts and simplify in using entry (2) in table (64:12:7) in [11]. □

Example 2.

$$\int_{\mathbb{R}_+^4} \frac{\sqrt{rs}\sqrt{xy}e^{-3r-2s-3x-2y}}{rsxy\left(\log^2\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)+\pi^2\right)}dxdydrds=\frac{4-\pi}{6}$$

and

$$\int_{\mathbb{R}_+^4} \frac{\sqrt{rs}\sqrt{xy}e^{-3r-2s-3x-2y}\log\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)}{rsxy\left(\log^2\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)+\pi^2\right)}dxdydrds=0.$$

P r o o f. Use equation (5.2) and set $k=-1$, $a=-1$, $m=-1$, $p=3$, $q=2$, rationalize the denominator and compare real and imaginary parts and simplify in using entry (1) in table (64:12:7) in [11]. □

Example 3.

$$\begin{aligned}\int_{\mathbb{R}_+^4} &\frac{e^{-2r-3s-2x-3y}\left((r+s)^{3/8}\sqrt[4]{xy}-\sqrt[4]{rs}\sqrt[8]{r+s}\sqrt[4]{x+y}\right)}{(rs)^{7/8}(xy)^{3/8}(x+y)^{3/8}\log\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)}dxdydrds\\ &=\frac{2}{3}\pi\tanh^{-1}\left(\cos\left(\frac{\pi}{8}\right)-\sin\left(\frac{\pi}{8}\right)\right).\end{aligned}$$

P r o o f. Use equation (5.2) and form a second equation by replacing $m\to n$ and take their difference. Next we set $k=-1$, $a=1$, $m=-3/4$, $n=-1/4$, $p=2$, $q=3$ and simplify using equation (9.559) in [4] and entry (3) in table (64:12:7) in [11]. □

Example 4.

$$\begin{aligned}\int_{\mathbb{R}_+^4} &\frac{e^{-r-s-x-y}\left(\sqrt[6]{r+s}\sqrt[24]{xy}-\sqrt[24]{rs}\sqrt[8]{r+s}\sqrt[24]{x+y}\right)}{(rs)^{2/3}(xy)^{3/8}\sqrt[6]{x+y}\log\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)}dxdydrds\\ &=2\pi\log\left(\sqrt{3}\tan\left(\frac{3\pi}{16}\right)\right).\end{aligned}$$

P r o o f. Use equation (5.2) and form a second equation by replacing $m\to n$ and take their difference. Next we set $k=-1$, $a=1$, $m=-3/4$, $n=-2/3$, $p=1$, $q=1$ and simplify using equation (9.559) in [4] and entry (3) in table (64:12:7) in [11]. □

Example 5.

$$\int_{\mathbb{R}_+^4} \frac{e^{-r-2(s+y)-x}}{\sqrt{rs}\sqrt{xy}\left(\log\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)+i\pi\right)^2}dxdydrds=4(C-1).$$

P r o o f. Use equation (5.2) and set $a = e^{ai}$, $m = -1$, $p = 1$, $q = 2$ and simplify in terms of the Hurwitz zeta function using entry (3) in table (64:12:7) in [11]. Next apply l'Hopitals' rule as $k \to -1$ and simplify in terms of the digamma function $\psi^{(0)}(a)$ given by equation (5.15.1) in [10]. Next take the first partial derivative with respect to a and set $a = \pi$ and simplify in terms of Catalan's constant C. □

Example 6.

$$\int_{\mathbb{R}_+^4} \frac{e^{-r-2(s+y)-x}}{\sqrt{rs}\sqrt{xy}\left(\log\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)+i\pi\right)^3} dxdydrds = -\frac{i\left(\pi^3-32\right)}{4\pi}.$$

P r o o f. Use equation (5.2) and set $a = e^{ai}, m = -1, p = 1, q = 2$ and simplify in terms of the Hurwitz zeta function using entry (3) in table (64:12:7) in [11]. Next apply l'Hopitals' rule as $k \to -1$ and simplify in terms of the digamma function $\psi^{(0)}(a)$. Next take the second partial derivative with respect to a and set $a = \pi$ and simplify in terms of π. □

Proposition 1. *For all $a, k, p, q \in \mathbb{C}$ the equality is true*

$$\begin{aligned} &\int_{\mathbb{R}_+^4} \frac{e^{-p(r+x)-q(s+y)} \log^k\left(\frac{a\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)}{\sqrt{rs}\sqrt{xy}} dxdydrds \\ &= \frac{2ie^{i\pi(k-1)/2}\pi^{k+2}\left(2^k\zeta\left(-k, 1/2\cdot(1/2 - i\log(a)/\pi)\right) - 2^k\zeta\left(-k, 1/2\cdot(3/2 - i\log(a)/\pi)\right)\right)}{pq}. \end{aligned} \tag{6.1}$$

P r o o f. Use equation (5.2) and set $m = -1$ and simplify using entry (4) in table (64:12:7) in [11]. □

Proposition 2. *For all $k \in \mathbb{C}$ then,*

$$\int_{\mathbb{R}_+^4} \frac{e^{-r-s-x-y} \log^k\left(\frac{i\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)}{\sqrt{rs}\sqrt{xy}} dxdydrds = -2\left(2^{k+1}-1\right)e^{i\pi k/2}\pi^{k+2}\zeta(-k). \tag{6.2}$$

P r o o f. Use equation (6.2) and set $a = i$, $p = q = 1$ and simplify using entry (2) in table (64:12:7) in [11]. □

Example 7.

$$\int_{\mathbb{R}_+^4} \frac{e^{-r-s-x-y}\sqrt{\log\left(\frac{i\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)}}{\sqrt{rs}\sqrt{xy}} dxdydrds = -2\left(2\sqrt{2}-1\right)e^{i\pi/4}\pi^{5/2}\zeta\left(-\frac{1}{2}\right).$$

P r o o f. Use equation (6.2) and set $k = 1/2$ and simplify. □

Example 8.

$$\int_{\mathbb{R}_+^4}\frac{e^{-3(r+x)-4(s+y)}}{\sqrt{rs}\sqrt{xy}\log\left(\frac{i\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)}dxdydrds=-\frac{1}{6}i\pi\log(2).$$

P r o o f. Use equation (6.1) set $a=i$ and apply l'Hopital's rule as $k\to-1$ and set $q=3$, $q=4$ and simplify. □

Example 9.

$$\int_{\mathbb{R}_+^4}\frac{e^{-r-s-x-y}\log\left(\log\left(\frac{i\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)\right)}{\sqrt{rs}\sqrt{xy}}dxdydrds=\frac{1}{2}\pi^2(\log(4)+i\pi).$$

P r o o f. Use equation (6.1) set $a=i$ and take the first partial derivative with respect to k and set $k=0$, $p=q=1$ and simplify. □

Example 10.

$$\int_{\mathbb{R}_+^4}\frac{e^{-r-s-x-y}\log\left(\log\left(\frac{i\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)\right)}{\sqrt{rs}\sqrt{xy}\log\left(\frac{i\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)}dxdydrds=\pi\log(2)(2i\gamma+\pi-i(\log(2)+2\log(\pi))).$$

P r o o f. Use equation (6.1) set $a=i$ and take the first partial derivative with respect to k then apply l'Hopital's rule as $k\to-1$ and set $p=q=1$ and simplify. □

Example 11.

$$\int_{\mathbb{R}_+^4}\frac{e^{-r-s-x-y}\log\left(\log\left(\frac{i\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)\right)}{\sqrt{rs}\sqrt{xy}\log^2\left(\frac{i\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)}dxdydrds=\frac{1}{12}\pi^2(-24\log(A)+2\gamma-i\pi+\log(16)).$$

P r o o f. Use equation (6.1) set $a=i$ and take the first partial derivative with respect to k and set $k=-2$, $p=q=1$ and simplify. □

Proposition 3. *For all $a,p,q\in\mathbb{C}$, $\mathrm{Re}(a)>0$ then,*

$$\int_{\mathbb{R}_+^4}\frac{\log\left(\log\left(\frac{a\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)\right)e^{-p(r+x)-q(s+y)}}{\sqrt{rs}\sqrt{xy}}dxdydrds=\frac{\pi^2\left(4\log\left(\frac{\sqrt{2\pi}\Gamma(3/4-i\log(a)/2\pi)}{\Gamma((\pi-2i\log(a))/4\pi)}\right)+i\pi\right)}{2pq}.$$

P r o o f. Use equation (6.1) and take the first partial derivative with respect to k and set $k=0$ and simplify using equation (25.11.18) in [10] □

7. Summary table of quadruple integrals involving

$f(x,y,r,s)$	$\int_{\mathbb{R}_+^4} f(x,y,r,s)dudx$
$\frac{\sqrt[4]{r+s}e^{-3r-2s-3x-2y}\left(\pi^2-4\log^2\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)\right)}{(rs)^{3/4}\sqrt[4]{xy}\sqrt[4]{x+y}\left(4\log^2\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)+\pi^2\right)^2}$	$\frac{48C+\pi^2}{576\sqrt{2}}$
$\frac{\sqrt[4]{rs}\sqrt[4]{r+s}(xy)^{3/4}e^{-3r-2s-3x-2y}\log\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)}{rsxy\sqrt[4]{x+y}\left(4\log^2\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)+\pi^2\right)^2}$	$\frac{1}{16\pi}\left(\frac{C}{3\sqrt{2}}-\frac{\pi^2}{144\sqrt{2}}\right)$
$\frac{\sqrt{rs}\sqrt{xy}e^{-3r-2s-3x-2y}}{rsxy\left(\log^2\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)+\pi^2\right)}$	$\frac{4-\pi}{6}$
$\frac{\sqrt{rs}\sqrt{xy}e^{-3r-2s-3x-2y}\log\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)}{rsxy\left(\log^2\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)+\pi^2\right)}$	0
$\frac{e^{-2r-3s-2x-3y}\left((r+s)^{3/8}\sqrt[4]{xy}-\sqrt[4]{rs}\sqrt[8]{r+s}\sqrt[4]{x+y}\right)}{(rs)^{7/8}(xy)^{3/8}(x+y)^{3/8}\log\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)}$	$\frac{2}{3}\pi\tanh^{-1}\left(\cos\left(\frac{\pi}{8}\right)-\sin\left(\frac{\pi}{8}\right)\right)$
$\frac{e^{-r-s-x-y}\left(\sqrt[6]{r+s}\sqrt[24]{xy}-\sqrt[24]{rs}\sqrt[8]{r+s}\sqrt[24]{x+y}\right)}{(rs)^{2/3}(xy)^{3/8}\sqrt[6]{x+y}\log\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)}$	$2\pi\log\left(\sqrt{3}\tan\left(\frac{3\pi}{16}\right)\right)$
$\frac{e^{-r-2(s+y)-x}}{\sqrt{rs}\sqrt{xy}\left(\log\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)+i\pi\right)^2}$	$4(C-1)$
$\frac{e^{-r-2(s+y)-x}}{\sqrt{rs}\sqrt{xy}\left(\log\left(\frac{\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)+i\pi\right)^3}$	$-\frac{i\left(\pi^3-32\right)}{4\pi}$
$\frac{e^{-r-s-x-y}\log^k\left(\frac{i\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)}{\sqrt{rs}\sqrt{xy}}$	$-2\left(2^{k+1}-1\right)e^{i\pi k/2}\pi^{k+2}\zeta(-k)$
$\frac{e^{-r-s-x-y}\sqrt{\log\left(\frac{i\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)}}{\sqrt{rs}\sqrt{xy}}$	$-2\left(2\sqrt{2}-1\right)e^{i\pi/4}\pi^{5/2}\zeta\left(-\frac{1}{2}\right)$
$\frac{e^{-3(r+x)-4(s+y)}}{\sqrt{rs}\sqrt{xy}\log\left(\frac{i\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)}$	$-\frac{1}{6}i\pi\log(2)$
$\frac{e^{-r-s-x-y}\log\left(\log\left(\frac{i\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)\right)}{\sqrt{rs}\sqrt{xy}}$	$\frac{1}{2}\pi^2(\log(4)+i\pi)$
$\frac{e^{-r-s-x-y}\log\left(\log\left(\frac{i\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)\right)}{\sqrt{rs}\sqrt{xy}\log\left(\frac{i\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)}$	$\pi\log(2)(2i\gamma+\pi-i(\log(2)+2\log(\pi)))$
$\frac{\log\left(\log\left(\frac{a\sqrt{r+s}\sqrt{xy}}{\sqrt{rs}\sqrt{x+y}}\right)\right)e^{-p(r+x)-q(s+y)}}{\sqrt{rs}\sqrt{xy}}$	$\frac{\pi^2\left(4\log\left(\frac{\sqrt{2\pi}\Gamma(3/4-i\log(a)/2\pi)}{\Gamma((\pi-2i\log(a))/4\pi)}\right)+i\pi\right)}{2pq}$

8. Discussion

In this work we used our contour integral method to derive a quadruple integral involving the logarithm of quotient radicals in terms of the Hurwitz–Lerch Zeta transcendent. The integrals derived are not easy to numerically evaluate as we suspect the presence of singularities and the integrand maybe highly oscillatory. The importance of this work is that we are able to write down a closed form solution for this integral. This is advantageous as we now have the Hurwitz–Lerch Zeta function with analytic continuation to use in order to evaluate this quadruple integral. We also employed Wolfram Mathematica to assist with numerical computation where needed. We will use our contour method to derive other multiple integrals for future work.

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