URAL MATHEMATICAL JOURNAL, Vol. 8, No. 2, 2022, pp. 71–80

DOI: 10.15826/umj.2022.2.006

PERIODIC SOLUTIONS OF A CLASS OF SECOND ORDER NEUTRAL DIFFERENTIAL EQUATIONS WITH MULTIPLE DIFFERENT DELAYS

Rabah Khemis

Laboratory of Applied Mathematics and History and Didactics of Mathematics (LAMAHIS),
University of 20 August 1955,
B.P.26 route d'El-Hadaiek, Skikda, 21000 Algeria

kbra28@yahoo.fr

Abdelouaheb Ardjouni

Department of Mathematics and Informatics, Souk Ahras University,
Souk Ahras, 41000 Algeria

abd_ardjouni@yahoo.fr

Ahlème Bouakkaz

Laboratory of Applied Mathematics and History and Didactics of Mathematics (LAMAHIS),
University of 20 August 1955,
B.P.26 route d'El-Hadaiek, Skikda, 21000 Algeria

ahlemkholode@yahoo.com

Abstract: The present work mainly probes into the existence and uniqueness of periodic solutions for a class of second-order neutral differential equations with multiple delays. Our approach is based on using Banach and Krasnoselskii's fixed point theorems as well as the Green's function method. Besides, two examples are exhibited to validate the effectiveness of our findings which complement and extend some relevant ones in the literature.

Keywords: Fixed point theorem, Green's function, Neutral differential equation, Periodic solutions.

1. Introduction

We frequently encounter neutral delay differential equations in the modeling of many phenomena in various domains such as physics, biology, population dynamics, medicine, epidemiology, economics, etc.

The investigation on such equations has been one of the most attracting topics in the literature. Recently, these equations have received a considerable attention and many researchers have sought to study them. For some related works, we refer the interested reader to some of them [1, 2, 4, 6, 8–10, 12, 13] and the references cited therein.

Stimulated by the aforementioned publications, we propose the following class of second order neutral differential equations

$$\frac{d^2}{dt^2}x\left(t\right)+p\left(t\right)\frac{d}{dt}x\left(t\right)+q\left(t\right)x\left(t\right)+\frac{d^2}{dt^2}\Big[k\left(t\right)x\left(t\right)-\sum_{\ell=1}^{n}c_{\ell}\left(t\right)x\left(t-\tau_{\ell}\left(t\right)\right)\Big]=e(t), \tag{1.1}$$

where $p, q \in \mathcal{C}(\mathbb{R}, (0, \infty))$, $k, c_\ell, \tau_\ell \in \mathcal{C}^2(\mathbb{R}, (0, \infty))$, $\ell = \overline{1, n}$ and $e \in \mathcal{C}(\mathbb{R}, [0, \infty))$ are T-periodic functions.

In the current work, the authors aim is to establish sufficient conditions under which Banach and Krasnoselskii's fixed point theorems are guaranteed to work and hence the existence and uniqueness of periodic solutions of the equation (1.1) are proved. The general idea of our technique is to convert the equation (1.1) into an equivalent integral one in order to pave the way for the application of Banach and Krasnoselskii's fixed point theorems. Indeed, this last one with the help of Arzelà-Ascoli theorem and some properties of the obtained Green's kernel, is a proper means for achieving our desired goals.

The key contributions of this work can be summarized as follows.

(i) New sufficient conditions that ensure the existence of periodic solutions of the equation (1.1) are established.

(ii) The studied problems in [1, 3–5, 7, 9, 12] are with globally Lipschitz source terms while this condition is not required here.

The basic frame of this paper is as follows. Section 2, provides some preliminary results and prerequisites that will be used in the sequel. Section 3 is dedicated to the statements and the proofs of our main results. In Section 4, we present two examples to which our main findings can be applied. The conclusion is included in the last section.

2. Preliminaries

Let

$$P_T = \left\{x \in \mathcal{C}(\mathbb{R}, \mathbb{R}t),\ x(t+T) = x(t)\right\}, \quad T > 0,$$

endowed with the supremum norm

$$\|x\| = \sup_{t \in \mathbb{R}} |x(t)| = \sup_{t \in [0,T]} |x(t)|,$$

be a Banach space.

Throughout this paper we will assume that the following hypothesis are fulfilled. Here p, q, k, e, c_ℓ and τ_ℓ are T-periodic real-valued functions such that

$$\begin{aligned} p(t+T) = p(t), \quad q(t+T) = q(t), \quad k(t+T) = k(t), \\ e(t+T) = e(t), \quad c_\ell(t+T) = c_\ell(t), \quad \tau_\ell(t+T) = \tau_\ell(t), \quad \ell = \overline{1, n}, \end{aligned} \tag{2.1}$$

and

$$\int_0^T p(s)ds > 0, \quad \int_0^T q(s)ds > 0, \quad \tau_\ell(t) \geq \tau_\ell^* > 0, \quad \ell = \overline{1, n}. \tag{2.2}$$

Lemma 1 [10]. *If (2.1) and (2.2) hold and*

$$\frac{R_1 \left[\exp\left(\int_0^T p(u)\,du\right) - 1\right]}{Q_1 T} \geq 1, \tag{2.3}$$

where

$$R_1 = \max_{t \in [0,T]} \left| \int_t^{t+T} \frac{\exp\left(\int_t^s p(u)\,du\right)}{\exp\left(\int_0^T p(u)\,du\right) - 1} q(s)ds \right|,$$

and

$$Q_1 = \left(1 + \exp\left(\int_0^T p(u)\,du\right)\right)^2 R_1^2,$$

then there are continuous and T-periodic functions a and b such that

$$b(t) > 0, \quad \int_0^T a(u)du > 0, \quad a(t) + b(t) = p(t),$$

and

$$\frac{d}{dt}b(t) + a(t)b(t) = q(t),$$

for all $t \in \mathbb{R}$. Furthermore, if $\phi \in P_T$ then the equation

$$x''(t) + p(t)x'(t) + q(t)x(t) = \phi(t)$$

has a T-periodic solution. Moreover, the periodic solution can be expressed as

$$x(t) = \int_t^{t+T} G(t,s)\phi(s)ds,$$

where

$$\begin{aligned} G(t,s) = &\frac{\int_t^s \exp\left[\int_t^u b(v)\,dv + \int_u^s a(v)\,dv\right] du}{\left[\exp\left(\int_0^T a(u)\,du\right) - 1\right]\left[\exp\left(\int_0^T b(u)\,du\right) - 1\right]} \\ &+ \frac{\int_s^{t+T} \exp\left[\int_t^u b(v)\,dv + \int_u^{s+T} a(v)\,dv\right] du}{\left[\exp\left(\int_0^T a(u)\,du\right) - 1\right]\left[\exp\left(\int_0^T b(u)\,du\right) - 1\right]}. \end{aligned} \tag{2.4}$$

Corollary 1 [12]. *If G is the Green's function given by (2.4), then G satisfies*

$$G(t,t+T) = G(t,t), \; G(t+T,s+T) = G(t,s),$$

$$\frac{\partial}{\partial s}G(t,s) = a(s)G(t,s) - \frac{\exp\left(\int_t^s b(v)\,dv\right)}{\exp\left(\int_0^T b(v)\,dv\right) - 1},$$

$$\frac{\partial}{\partial t}G(t,s) = -b(t)G(t,s) + \frac{\exp\left(\int_t^s a(v)\,dv\right)}{\exp\left(\int_0^T b(v)\,dv\right) - 1},$$

$$\frac{\partial^2}{\partial s^2}G(t,s) = \left(a(s) + a'(s)\right)G(t,s) - (a(s) + b(s))\frac{\exp\left(\int_t^s b(v)\,dv\right)}{\exp\left(\int_0^T b(v)\,dv\right) - 1}.$$

Furthermore, by putting

$$
\begin{gathered}
A=\int_0^T p(u)\,du, \quad B=T^2\exp\left(\frac{1}{T}\int_0^T \ln(q(u))\,du\right),\\
M_1=\frac{1}{2}\left(A-\sqrt{A^2-4b}\right), \quad M_2=\frac{1}{2}\left(A+\sqrt{A^2+4b}\right),\\
\alpha_1=\frac{T}{\left(e^{M_2}-1\right)^2}, \quad \alpha_2=\frac{T\exp\left(\int_0^T p(u)\,du\right)}{\left(e^{M_1}-1\right)^2},\\
H(t,s)=\frac{\exp\left(\int_t^s b(v)\,dv\right)}{\exp\left(\int_0^T b(v)\,dv\right)-1}, \quad \beta=\frac{\exp\left(\int_0^T b(v)\,dv\right)}{\exp\left(\int_0^T b(v)\,dv\right)-1},\\
H^*(t,s)=\frac{\exp\left(\int_t^s a(v)\,dv\right)}{\exp\left(\int_0^T b(v)\,dv\right)-1}, \quad \beta^*=\frac{\exp\left(\int_0^T a(v)\,dv\right)}{\exp\left(\int_0^T b(v)\,dv\right)-1},
\end{gathered}
$$

and if $A^2\geq 4B$, *then we have*

$$0<\alpha_1\leq G(t,s)\leq\alpha_2, \quad |H(t,s)|\leq\beta, \quad |H^*(t,s)|\leq\beta^*.$$

3. Existence and uniqueness of periodic solutions

Lemma 2. *Suppose that* (2.1)–(2.3) *hold. If* $x\in P_T\cap\mathcal{C}^2(\mathbb{R},\mathbb{R})$, *then* x *is a solution of* (1.1) *if and only if* x *is a solution of the following equation*

$$
\begin{aligned}
x(t)&=\frac{1}{1+k(t)}\sum_{\ell=1}^n c_\ell(t)\,x(t-\tau_\ell(t))+\frac{1}{1+k(t)}\int_t^{t+T} e(s)\,G(t,s)\,ds\\
&+\int_t^{t+T}\frac{a(s)+b(s)}{1+k(t)}\left[k(s)\,x(s)-\sum_{\ell=1}^n c_\ell(s)\,x(s-\tau_\ell(s))\right]H(t,s)\,ds\\
&-\int_t^{t+T}\frac{a(s)+a'(s)}{1+k(t)}\left[k(s)\,x(s)-\sum_{\ell=1}^n c_\ell(s)\,x(s-\tau_\ell(s))\right]G(t,s)ds.
\end{aligned}
\tag{3.1}
$$

P r o o f. Let $x\in P_T\cap\mathcal{C}^2(\mathbb{R},\mathbb{R})$. From Lemma 1, we get

$$
\begin{aligned}
x(t)&=\int_t^{t+T}\left\{\frac{\partial}{\partial s}\left[k(s)\,x(s)-\sum_{\ell=1}^n c_\ell(s)\,x(s-\tau_\ell(s))\right]\right\}\frac{\partial}{\partial s}G(t,s)\,ds+\int_t^{t+T} e(s)\,G(t,s)\,ds\\
&=\left[k(s)\,x(s)-\sum_{\ell=1}^n c_\ell(s)\,x(s-\tau_\ell(s))\right]\frac{\partial}{\partial s}G(t,s)\bigg|_t^{t+T}\\
&-\int_t^{t+T}\left[k(s)\,x(s)-\sum_{\ell=1}^n c_\ell(s)\,x(s-\tau_\ell(s))\right]\frac{\partial^2}{\partial s^2}G(t,s)\,ds+\int_t^{t+T} e(s)\,G(t,s)\,ds.
\end{aligned}
$$

Since

$$\left[k(s)\,x(s) - \sum_{\ell=1}^{n} c_\ell(s)\,x(s-\tau_\ell(s))\right] \frac{\partial}{\partial s} G(t,s)\Big|_t^{t+T}$$
$$= -k(t)\,x(t) + \sum_{\ell=1}^{n} c_\ell(t)\,x(t-\tau_\ell(t)),$$

and

$$\frac{\partial^2}{\partial s^2} G(t,s) = \big(a(s)+a'(s)\big)\,G(t,s) - (a(s)+b(s))\,H(t,s),$$

then

$$(1+k(t))\,x(t) = \sum_{\ell=1}^{n} c_\ell(t)\,x(t-\tau_\ell(t)) + \int_t^{t+T} e(s)\,G(t,s)\,ds$$
$$+ \int_t^{t+T} (a(s)+b(s)) \left[k(s)\,x(s) - \sum_{\ell=1}^{n} c_\ell(s)\,x(s-\tau_\ell(s))\right] H(t,s)\,ds$$
$$- \int_t^{t+T} \big(a(s)+a'(s)\big) \left[k(s)\,x(s) - \sum_{\ell=1}^{n} c_\ell(s)\,x(s-\tau_\ell(s))\right] G(t,s)ds.$$

Dividing both sides of the above equation by $1+k(t)$, we obtain (3.1). The converse implication can be obtained by the derivation of (3.1). □

Fore ease of exposition, we will use the following notations

$$\lambda_1 = \max_{t\in[0,T]} |a(t)|, \quad \lambda_1^* = \max_{t\in[0,T]} \left|a'(t)\right|, \quad \sigma = \max_{t\in[0,T]} |e(t)|,$$
$$\mu_1 = \max_{t\in[0,T]} |b(t)|, \quad \delta_\ell = \max_{t\in[0,T]} |c_\ell(t)|, \quad \ell = \overline{1,n},$$
$$\rho_0 = \min_{t\in[0,T]} |k(t)|, \quad \rho_1 = \max_{t\in[0,T]} |k(t)|, \quad \rho_1^* = \max_{t\in[0,T]} \left|k'(t)\right|.$$

Furthermore, we suppose that

$$\Gamma_1 = \frac{1}{1+\rho_0} \sum_{\ell=1}^{n} \delta_\ell < 1, \tag{3.2}$$

and there exists $L > 0$ which satisfies the following estimate

$$\Gamma_2 = \frac{T\alpha_2\sigma}{1+\rho_0} + \Gamma_3 L \le L, \tag{3.3}$$

where

$$\Gamma_3 = \frac{1}{1+\rho_0} \left(T\Big(\rho_1 + \sum_{\ell=1}^{n} \delta_\ell\Big)\Big(\beta(\lambda_1+\mu_1) + \alpha_2(\lambda_1+\lambda_1^*)\Big) + \sum_{\ell=1}^{n} \delta_\ell \right).$$

For employing Krasnoselskii's fixed point theorem, we need to define an operator that can be expressed as a sum of two operators, one of which is continuous and compact and the other is a contraction.

Indeed, from Lemma 2, we can define an operator $\mathcal{S} : P_T \longrightarrow P_T$ as follows

$$(\mathcal{S}\varphi)(t) = (\mathcal{S}_1\varphi)(t) + (\mathcal{S}_2\varphi)(t),$$

where

$$(\mathcal{S}_1\varphi)(t) = \frac{1}{1+k(t)}\sum_{\ell=1}^{n} c_\ell(t)\varphi(t-\tau_\ell(t)) + \frac{1}{1+k(t)}\int_t^{t+T} e(s)G(t,s)ds,$$

and

$$(\mathcal{S}_2\varphi)(t) = \int_t^{t+T} \frac{a(s)+b(s)}{1+k(t)}\left[k(s)\varphi(s) - \sum_{\ell=1}^{n} c_\ell(s)\varphi(s-\tau_\ell(s))\right] H(t,s)\,ds$$
$$-\int_t^{t+T} \frac{a(s)+a'(s)}{1+k(t)}\left[k(s)\varphi(s) - \sum_{\ell=1}^{n} c_\ell(s)\varphi(s-\tau_\ell(s))\right] G(t,s)ds.$$

Clearly, $(\mathcal{S}_i\varphi)(t+T) = (\mathcal{S}_i\varphi)(t)\,,\ i=1,2$ which shows that operators $\mathcal{S}_i$ are well defined.

To reach our target, it suffices to prove the existence of at least one fixed point of the operator $\mathcal{S}_1+\mathcal{S}_2$. This is due to the fact that the sought solution of equation (1.1) is just a fixed point of $\mathcal{S}_1+\mathcal{S}_2$ and vice versa.

Theorem 1. *Suppose that conditions (2.1)–(2.3), (3.2) and (3.3) hold. Then equation (1.1) admits at least one periodic solution $x \in P_T$ which satisfies $\|x\| \le L$.*

P r o o f. For establishing the existence of periodic solutions, we use Krasnoselskii's fixed point theorem ([11]). The proof will be made in three steps.

Step 1. We show that $\mathcal{S}_1$ is a contraction mapping.

Let $\varphi_1, \varphi_2 \in P_T$, we have

$$|(\mathcal{S}_1\varphi_1)(t) - (\mathcal{S}_1\varphi_2)(t)| \le \sum_{\ell=1}^{n}\frac{c_\ell(t)}{1+k(t)}\left|\varphi_1(t-\tau_\ell(t)) - \varphi_2(t-\tau_\ell(t))\right| \le \Gamma_1\|\varphi_1-\varphi_2\|.$$

From (3.2), we deduce that $\mathcal{S}_1$ is a contraction mapping.

Step 2. We show that $\mathcal{S}_2$ is continuous and compact mapping.

Let $\varphi_1, \varphi_2 \in P_T$. For $\varepsilon > 0$ and $\eta = \Lambda\varepsilon$, where

$$\Lambda = \frac{1+\rho_0}{T\left(\rho_1 + \sum\limits_{\ell=1}^{n}\delta_\ell\right)\left(\beta(\lambda_1+\mu_1) + \alpha_2(\lambda_1+\lambda_1^*)\right)},$$

we obtain

$$\|\varphi_1-\varphi_2\| \le \eta \Longrightarrow \|\mathcal{S}_2\varphi_1 - \mathcal{S}_2\varphi_2\| < \varepsilon,$$

which shows the continuity of $\mathcal{S}_2$.

On the other hand, let $\hbar > 0$, $\mathbb{K} = \{\varphi \in P_T,\ \|\varphi\| \le \hbar\}$ and $\{\varphi_n\}_{n\in\mathbb{N}}$ be a sequence from $\mathbb{K}$. We have

$$\|\mathcal{S}_2\varphi_n\| \le \frac{T\hbar\left(\rho_1 + \sum\limits_{\ell=1}^{n}\delta_\ell\right)}{1+\rho_0}\left(\beta(\lambda_1+\mu_1) + \alpha_2(\lambda_1+\lambda_1^*)\right), \tag{3.4}$$

and

$$\frac{d}{dt}\left(\mathcal{S}_2\varphi_n\right)(t) = \frac{a(t)+b(t)}{1+k\left(t\right)}\left[k\left(t\right)\varphi_n\left(t\right) - \sum_{\ell=1}^{n} c_\ell\left(t\right)\varphi_n\left(t-\tau_\ell\left(t\right)\right)\right]$$

$$-\frac{b\left(t\right)\left(1+k\left(t\right)\right)+k'\left(t\right)}{\left(1+k\left(t\right)\right)^2}\int_t^{t+T}\left(a(s)+b(s)\right)\left[k\left(s\right)\varphi_n\left(s\right) - \sum_{\ell=1}^{n} c_\ell\left(s\right)\varphi_n\left(s-\tau_\ell\left(s\right)\right)\right]H\left(t,s\right)ds$$

$$+\frac{b(t)\left(1+k\left(t\right)\right)+k'\left(t\right)}{\left(1+k\left(t\right)\right)^2}\int_t^{t+T}\left(a(s)+a'(s)\right)\left[k\left(s\right)\varphi_n\left(s\right) - \sum_{\ell=1}^{n} c_\ell\left(s\right)\varphi_n\left(s-\tau_\ell\left(s\right)\right)\right]G(t,s)ds$$

$$-\frac{1}{1+k\left(t\right)}\int_t^{t+T}\left(a(s)+a'(s)\right)\left[k\left(s\right)\varphi_n\left(s\right) - \sum_{\ell=1}^{n} c_\ell\left(s\right)\varphi_n\left(s-\tau_\ell\left(s\right)\right)\right]H^*\left(t,s\right)ds.$$

Hence

$$\left|\frac{d}{dt}\left(\mathcal{S}_2\varphi_n\right)(t)\right| \le \Gamma_4, \tag{3.5}$$

where

$$\Gamma_4 = \hbar\Big(\rho_1 + \sum_{\ell=1}^{n}\delta_\ell\Big)\left(\frac{\left(\lambda_1+\mu_1\right)+T\beta^*\left(\lambda_1+\lambda_1^*\right)}{1+\rho_0}\right.$$
$$\left.+T\frac{\left(\mu_1\left(1+\rho_1\right)+\rho_1^*\right)\left(\beta\left(\lambda_1+\mu_1\right)+\alpha_2\left(\lambda_1+\lambda_1^*\right)\right)}{\left(1+\rho_0\right)^2}\right).$$

It follows from (3.4), (3.5) and the Arzelà-Ascoli theorem [14] that $\mathcal{S}_2$ is a compact operator.

Step 3. If L is defined as in (3.3), let

$$\mathbb{M} = \left\{\varphi \in P_T,\ \left\|\varphi\right\| \le L\right\}.$$

In view of (3.3), if $\varphi_1, \varphi_2 \in \mathbb{M}$, then

$$\left\|\mathcal{S}_1\varphi_1 + \mathcal{S}_2\varphi_2\right\| \le \Gamma_2 \le L,$$

which proves that

$$\mathcal{S}_1\varphi_1 + \mathcal{S}_2\varphi_2 \in \mathbb{M}, \quad \forall\varphi_1, \varphi_2 \in \mathbb{M}.$$

From these three steps, we conclude that the operator $\mathcal{S}_2 + \mathcal{S}_2$ has at least one fixed point $x \in P_T$ with $\|x\| \le L$. Consequently, the equation (1.1) has at least one periodic solution in $\mathbb{M}$. □

Theorem 2. *Suppose that conditions* (2.1)–(2.3) *and* (3.2) *hold. If* $\Gamma_3 < 1$*, then the equation* (1.1) *has a unique periodic solution* $x \in P_T$*.*

P r o o f. Let $\varphi_1, \varphi_2 \in P_T$, we have

$$\left|\left(\mathcal{S}\varphi_1\right)(t) - \left(\mathcal{S}\varphi_2\right)(t)\right| \le \Gamma_3\left\|\varphi_1 - \varphi_2\right\|.$$

Since $\Gamma_3 < 1$, the Banach fixed point theorem [11] guarantees that the operator $\mathcal{S}$ has a unique fixed point which is the unique periodic solution of the equation (1.1). □

4. Examples

Example 1. Let $L = 3\pi$. We consider the following equation

$$\begin{aligned} x''(t) + \frac{5}{12}x'(t) + \frac{1}{24}x(t) + \Bigg(\frac{1}{100}x(t) - \left(\frac{1}{120}\sin^2 2\pi t\right) x\left(t - \pi \sin^2 2\pi t\right) \\ - \left(\frac{1}{150}\cos^2 2\pi t\right) x\left(t - 2\pi \cos^4 2\pi t\right)\Bigg)'' = \frac{1}{10}\sin^4 2\pi t. \end{aligned} \tag{4.1}$$

Here

$$p(t) = \frac{5}{12}, \quad p(t) = \frac{1}{24}, \quad k(t) = \frac{1}{100}, \quad c_1(t) = \frac{1}{120}\sin^2 2\pi t,$$
$$c_2(t) = \frac{1}{150}\cos^2 2\pi t, \quad \tau_1(t) = \pi \sin^2 2\pi t, \quad \tau_2(t) = 2\pi\cos^4 2\pi t,$$
$$e(t) = \frac{100}{1010}\sin^4 2\pi t, \quad T = 1,$$

which implies

$$A = \frac{5}{12}, \quad B = \frac{1}{24}, \quad A^2 = \frac{25}{144} > 4B^2 = \frac{1}{6}, \quad R_1 = \frac{1}{10},$$
$$Q_1 = \frac{1}{100}\left(e^{5/12}+1\right)^2, \quad \frac{R_1\left[\exp\left(\int_0^T p(u)\,du\right) - 1\right]}{Q_1 T} \simeq 22.367 > 1,$$
$$M_1 = \frac{1}{6}, \quad M_2 = \frac{1}{4}, \quad \alpha_2 \simeq 46.118, \quad \beta \simeq 6.5139, \quad \Gamma_1 = \frac{3}{202} < 1,$$
$$\Gamma_2 \simeq 8.0746 < L = 3\pi, \quad \Gamma_3 \simeq 0.36742 < 1.$$

It follows from Theorem 2 that the equation (4.1) has a unique solution $x \in P_T$ which satisfies $\|x\| \leq 3\pi$.

The following example shows the usefulness of Theorem 1 when the Banach fixed point theorem cannot be applied.

Example 2. We consider the following equation

$$\begin{aligned} x''(t) + \frac{5}{12}x'(t) + \frac{1}{24}x(t) + \Bigg(\left(6\frac{\left(e^{1/6}-1\right)^2}{5e^{1/3}-5e^{1/6}+3e^{5/12}}x(t)\right) \\ - \left(2\frac{\left(e^{1/6}-1\right)^2}{5e^{1/3}-5e^{1/6}+3e^{5/12}}\sin^2 2\pi t\right) x\left(t - \pi\sin^2 2\pi t\right) \\ - \left(4\frac{\left(e^{1/6}-1\right)^2}{5e^{1/3}-5e^{1/6}+3e^{5/12}}\sin^2 2\pi t\right) x\left(t - 2\pi\cos^4 2\pi t\right)\Bigg)'' = 0. \end{aligned} \tag{4.2}$$

Here

$$p(t) = \frac{5}{12}, \quad p(t) = \frac{1}{24}, \quad k(t) = 6\frac{\left(e^{1/6}-1\right)^2}{5e^{1/3}-5e^{1/6}+3e^{5/12}},$$
$$c_1(t) = 2\frac{\left(e^{1/6}-1\right)^2}{5e^{1/3}-5e^{1/6}+3e^{5/12}}\sin^2 2\pi t, \quad c_2(t) = 4\frac{\left(e^{1/6}-1\right)^2}{5e^{1/3}-5e^{1/6}+3e^{5/12}}\cos^2 2\pi t,$$
$$\tau_1(t) = \pi\sin^2 2\pi t, \quad \tau_2(t) = 2\pi\cos^4 2\pi t, \quad e(t) = 0, \quad T = 1,$$

which implies

$$A = \frac{5}{12}, \quad B = \frac{1}{24}, \quad A^2 = \frac{25}{144} > 4B^2 = \frac{1}{6}, \quad R_1 = \frac{1}{10},$$
$$Q_1 = \frac{1}{100}\left(e^{5/12}+1\right)^2, \quad \frac{R_1\left[\exp\left(\int_0^T p(u)\,du\right)-1\right]}{Q_1 T} \simeq 22.367 > 1,$$
$$M_1 = \frac{1}{6}, \quad M_2 = \frac{1}{4}, \quad \alpha_2 \simeq 46.118, \quad \beta \simeq 6.5139, \quad \Gamma_1 = 0.03391 < 1,$$
$$\Gamma_2 = L \le L, \quad \forall L > 0, \quad \Gamma_3 = 1.$$

Since $\Gamma_3 = 1$, we can not use Theorem 2, but $\Gamma_2 = L \le L$, so we can apply Theorem 1 to prove that the equation (4.2) has at least one periodic solution $x \in P_T$ which satisfies $\|x\| \le L$.

5. Conclusion

In this paper, by utilizing both the Banach and Krasnoselskii's fixed point theorems and the Green's functions method, a class of second-order neutral differential equations with multiple delays has been investigated. To be more precise, we have discussed the existence and uniqueness of periodic solutions by transforming the equation (1.1) into an equivalent integral one and then by using the Banach and Krasnoselskii's fixed point theorems.

Acknowledgements

The authors are grateful to the referees for their valuable comments which have led to improvement of the presentation.

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