URAL MATHEMATICAL JOURNAL, Vol. 8, No. 2, 2022, pp. 177–186

DOI: 10.15826/umj.2022.2.015

BIHARMONIC GREEN FUNCTION AND BISUPERMEDIAN ON INFINITE NETWORKS

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Abstract: In this article, we have discussed Biharmonic Green function on an infinite network and bimedian functions. We have proved some standard results in terms of supermedian and bimedian. Also, we have proved the Discrete Riquier problem in the setting of bimedian functions.

Keywords: Biharmonic Green function, Bimedian function, Dirichlet problem, Discrete Riquier problem, Hyperbolic networks.

1. Introduction

An electrical network X is a finite graph consisting of a finite number of nodes and branches, each branch connecting some two nodes. There is a certain resistance $r(a,b)$ on each branch $[a,b]$ connecting the nodes a and b in X; the reciprocal of the resistance is called the conductance $c(a,b)$. Thus, an electrical network X can be considered as a graph $\{X, c(a,b)\}$ with finite number of vertices (nodes) and a finite number of edges (branches); the non-negative conductance $c(a,b)$ is positive if and only if a and b are neighbors, that is $[a,b]$ is an edge in X. A vertex e in X is called a terminal vertex if e has only one neighbor in X. If a and b are neighbors, we write $b \sim a$. We assume also that if $a \sim b$, then there is only one branch $[a,b]$ connecting a and b and there is no self-loops in X; there is no edge of the form $[a,a]$ so that $c(a,b) = 0$ for all a in X. We also assume that there is always a path $\{a = a_0, a_1, a_2, \ldots, a_n = b\}$ connecting two vertices a and b in X where $a_i \sim a_{i+1}$ for $0 \leq i \leq n-1$.

An electrical current regime voltage is considered on the finite network $\{X, c(a,b)\}$ assuming the Ohm–Kirchhoff laws: if ψ is the potential function on X, when extremal currents are applied on X, then the voltage on the branch $[a,b]$ is $[\psi(b) - \psi(a)]$ and the current is $c(a,b)[\psi(b) - \psi(a)]$ so that the total current at the node a is $\sum\limits_{b\sim a} c(a,b)[\psi(a) - \psi(b)]$. Based on these basic notions, the condenser principle, the equilibrium principle, the minimum principle, etc. are proved on X in [4].

In an abstract sense, can we consider these principles on an infinite network in a meaningful manner? Is it possible to think of an infinite electrical network with Ohm–Kirchhoff laws suitably modified by Nash-Williams [10] in his remarkable paper on random walks and electrical currents in networks, where it is shown that the random walk in probability theory has features analogous to electrical networks. A random walk considered as an irreducible, reversible Markov chain will serve as a model to develop a function theory on infinite graphs analogous to that of electrical networks (Abodayeh and Anandam [1, 2], Woess [12] and Zemanian [13]). Let $\{X, p(a,b)\}$ stand for a countably infinite state X with the transition probabilities $p(a,b)$. Assume that $\{X, p(a,b)\}$ is irreducible, i.e., is it possible to move along a path from any state a to any other state b in X; it is also reversible, i.e., there is a function $\varphi(a) > 0$ on X such that $\varphi(a)p(a,b) = \varphi(b)p(b,a)$ for any two states a and b in X. Then, as an example of the analogy between random walks and electrical currents, consider two disjoint subsets A and B. Denote by $\psi(a)$ the probability that the walker starting at the state a reaches A before meeting any state in B. Then $\psi(a) = 1$ for $a \in A$ and $\psi(a) = 0$ for $a \in B$; if $a \notin A \cup B$, then $\varphi(x) = \sum_{b\sim a} p(a,b)\psi(a)$ and, since $\sum_{b\sim a} p(a,b) = 1$, we have $\sum_{b\sim a} p(a,b)[\psi(b) - \psi(a)] = 0$, which is equality analogous to the situation where the total current at the node a is 0.

In a random walk, the Green function $G(a,b)$ represents the expected number of visits that the walker starting at a makes to reach b. The function $G(a,b)$ takes the value ∞ if $\{X, p(a,b)\}$ is recurrent, i.e., the walker starting at any state a comes back to a infinitely often; $G(a,b) < \infty$ for all pairs a, b if $\{X, p(a,b)\}$ is transient, i.e., the walker starting at a vertex a definitely wanders off from a. A situation similar to this occurs in the study of Riemann surfaces. If a Riemann surface R is parabolic, there is no Green potential on R. If R is hyperbolic, then there is a Green kernel on R.

When this analogy between random walks and functions on Riemann surfaces is properly developed, a successful application of function-theoretic methods on Riemann surfaces to solve problems in random walks on an irreducible, reversible $\{X, p(a,b)\}$ is possible. For this case, we can define the Dirichlet norm on a, and then the functional analysis methods enable us to establish a correspondence between some function-theoretic problems on a Riemann surface and problems connected with a random walk on X. For example, Lyons [9], modifying a Royden criterion on Riemann surfaces, gives a necessary and sufficient condition for a reversible Markov chain to be transient. However, these arguments establishing relations between random walks and Riemann surfaces are valid only when it is assumed that the random walk is reversible. Intending to develop a function theory on infinite networks that will be applicable even in the case of non-reversible Markov chains, we adopt here potential theoretic methods on locally compact spaces. The basic result is the solution to a generalized Dirichlet problem in infinite networks; using which we introduce the analogous of balayage, maximum principle, equilibrium principle, condenser principle, the classifications based on the notions of transient and recurrent random walks, etc. in infinite networks.

2. Preliminaries

Let N be an infinite graph that is connected and locally finite but without self-loops [4, 7]. Let $\varphi(a,b) \geq 0$ be a nonnegative number associated with each pair of vertices a and b in N such that $\varphi(a,b) > 0$ iff $b \sim a$. Then $\{N, \varphi(a,b)\}$ is called an infinite network. We do not assume that $\varphi(a,b)$ is symmetric. Given a set B in N, say that a vertex a is an interior vertex of B if a and all its neighbors are in B; denote by $\overset{\circ}{B}$ the set of all interior vertices of B, and let $\partial B = B \setminus \overset{\circ}{B}$. If $f(a)$ is a real-valued function on B, write

$$\Delta f(a) = \sum_{b\sim a} \varphi(a,b)\,[f(b) - f(a)]$$

for any $a \in \overset{\circ}{B}$. Say that $f(a)$ is superharmonic on B if $\Delta f(a) \leq 0$ for any $a \in \overset{\circ}{B}$; and $f(a)$ is said to be harmonic on B if $\Delta f(a) = 0$ for any $x \in \overset{\circ}{B}$. A function $f(a)$ on B is subharmonic if $-f(a)$ is superharmonic on B. The following statements are valid.

(1) If $\{f_n(a)\}$ is a sequence of superharmonic functions on B and $f(a) = \lim_n f_n(a)$ is real-valued on B, then $f(a)$ is superharmonic on B; consequently, if $\{g_n(a)\}$ is a sequence of superharmonic functions on B such that $g(a) = \sum_n g_n(a)$ is finite for each a in B, then $g(a)$ is superharmonic on B.

(2) **Minimum Principle:** If $s(a) \geq 0$ is superharmonic on N and $s(a_0) = 0$ for some vertex a_0, then $s \equiv 0$.

(3) **Greatest harmonic minorant:** Let f be superharmonic on B and g be subharmonic on B such that $f \geq g$ on B. Let $u(a)$ be the sequence $\mathfrak{F}$ of all subharmonic functions on B such that $u \leq f$. Let

$$\lambda(a) = \sup_{u \in \mathfrak{F}} u(a)$$

for $x \in B$. Then $\lambda(a)$ is a harmonic function on B such that if $\lambda'(a)$ is another harmonic function and $\lambda' \leq f$ on B, then $\lambda' \leq \lambda$. We call $\lambda(a)$ the greatest harmonic minorant of f on B. Similarly, we define the least harmonic majorant of g on B.

(4) **Generalised Dirichlet Solution:** Let F be an arbitrary set in N and $B \subset \overset{\circ}{F}$. Suppose that $u(a)$ is a real-valued function on $F \setminus B$ such that there exist a superharonic function f and a subharmonic function g on F such that $f \geq g$ on F and $f \geq u \geq g$ on $F \setminus B$. Then there exists a function λ on F such that $\lambda = u$ on $F \setminus B$ and $\Delta\lambda(a) = 0$ for $a \in B$. This generalised Dirichlet solution λ on F is uniquely determined if F is a finite set.

3. Biharmonic Green Function

Definition 1 (Potential). *A nonnegative superharmonic function p defined on a subset B is said to be potential if and only if the greatest harmonic minorant of p on B is* 0.

Definition 2 (Bipotential). *A potential u in (N, p) is said to be a bipotential if and only if $(-\Delta)u = p$, where p is a potential in N. We say that N is a bipotential network if there exists a positive bipotential on N.*

Definition 3 (Biharmonic Green function). *For a fixed vertex z in N, a potential $u_z(a)$ in (N, p) is said to be the biharmonic Green function with biharmonic support $\{z\}$ if and only if $(-\Delta)u_z(a) = G_z(a)$, where $G_z(a)$ is the harmonic Green function with harmonic support z.*

Proposition 1. *The biharmonic Green function exists on (N, p) if and only if there is a positive bipotential on (N, p).*

P r o o f. Clearly, the binarmonic Green function is a bipotential. Conversely, let u be a positive bipotential, $(-\Delta)u = p$. Then,

$$u(a) = \sum_b G(a, b)p(b).$$

Let z be a fixed vertex. Then, for some $\lambda > 0$, $G_z(b) \leq \lambda p(b)$ for any $b \in N$ (Domination Principle). Hence,

$$Q_z(a) = \sum_b G(a, b)G_z(b)$$

is a well-defined potential such that $(-\Delta)u_z(a) = G_z(a)$. □

Theorem 1. *Let (N,p) be a bipotential infinite network, and let $u_y(a)$ be the biharmonic Green potential on (N,p). If $\sum_b f(b)u_b(a)$ is finite at some vertex a_0 for some $f > 0$, then*

$$u(a) = \sum_b f(b)u_b(a)$$

is a bipotential on (N,p). Conversely, every bipotential $u(a)$ can be represented as

$$u(a) = \sum_b f(b)u_b(a),$$

where $f(a) = (-\Delta)^2 u(a)$.

P r o o f. Let $(-\Delta)u = p$ on (N,p). For a finite set E in (N,p), write

$$s(a) = u(a) - \sum_{b\in E}(-\Delta)p(b)u_b(a).$$

Then,

$$(-\Delta)s(a) = p(a) - \sum_{b\in E}(-\Delta_q)p(b)G_b(a) = \sum_{b\notin E}(-\Delta)p(b)G_b(a) \geq 0.$$

Hence, s is superharmonic on (N,p), and since

$$-s(a) \leq \sum_{b\in E}(-\Delta)p(b)u_b(a),$$

we conclude that $-s \leq 0$. Hence,

$$u(a) \geq \sum_{b\in E}(-\Delta)p(b)u_b(a).$$

Allow E to grow into (N,p), to conclude that

$$u(a) \geq \sum_{b\in E}(-\Delta)p(b)u_b(a).$$

Write

$$\varphi(a) = q(a) - \sum_{b\in(N,p)}(-\Delta)p(b)u_b(a).$$

Then,

$$(-\Delta)\varphi(a) = p(a) - \sum_{b\in(N,p)}(-\Delta)p(b)G_b(a) = 0.$$

Hence, $\varphi(a)$ is a nonnegative harmonic function majorized by the potential $u(a)$. Hence, $\varphi = 0$ so that

$$u(a) = \sum_{b\in(N,p)}(-\Delta)p(b)u_b(a)$$

for any $a \in (N,p)$. If $f(a) = (-\Delta)p(a)$, then $f \geq 0$ and $f(a) = (-\Delta)^2 u(a)$. Conversely, suppose that

$$u(a) = \sum_b f(b)u_b(a),$$

which is a convergent sum of potentials if $u(a_0)$ is finite at some vertex a_0. Then, $u(a)$ is a potential in (N, p) and

$$(-\Delta)u(a) = \sum_b f(b)(-\Delta)u_b(a) = \sum_b f(b)G_b(a),$$

being finite at each a, defines a potential $p(a)$. Thus, $(-\Delta)u(a) = p(a)$. □

Proposition 2. *Let (N, p) be a bipotential infinite network. For $z \in (N, p)$, if $u_z(a)$ and $G_z(a)$ are the biharmonic and harmonic Green potentials, then $u_z(a) > G_z(a)$ for any $a \in (N, p)$.*

P r o o f. Since $(-\Delta)u_z(a) = G_z(a)$, $(-\Delta)G_z(a) = \delta_z(a)$, and $G_z(z) \geq G_z(a)$ for all a (Domination principle), we have

$$(-\Delta)\left[\frac{u_z(a)}{G_z(z)}\right] = \frac{G_z(a)}{G_z(z)} \geq \delta_z(a) = (-\Delta)G_z(a).$$

Hence,

$$u(a) = \frac{u_z(a)}{G_z(z)} - G_z(a)$$

is a superharmonic function such that $-u(a) \leq G_z(a)$; hence, $-u \leq 0$ on (N, p). Consequently,

$$u_z(a) \geq G_z(z)G_z(a) > G_z(a)$$

since $G_z(z) > 1$. □

Proposition 3. *Let u be a potential in (N, p), $(-\Delta)u = p$. Suppose that $0 \leq f \leq p$. Then, there exists a potential v, $v \leq u$, such that $(-\Delta)v = f$ on (N, p).*

P r o o f. Let

$$u(a) = \sum_b G(a, b)p(b) \geq \sum_b G(a, b)f(b) = v(a),$$

then $v(a)$ is a potential, $v \leq u$ and $(-\Delta)v(a) = f(a)$ for all $a \in (N, p)$. □

Corollary 1. *Let (N, p) be a bipotential infinite network. If u is a potentials with finite harmonic support in (N, p), then there exist is a bipotential v on (N, p) such that $(-\Delta)v = u$ on (N, p).*

P r o o f. By hypothesis, there are postive potentials p and q such that $(-\Delta)u = p$ on (N, p). Since u has finite harmonic support, $u \leq \lambda p$ on (N, p) for some $\lambda > 0$ (Domination Principle). Hence use the above Proposition , there is a potential $v \leq \lambda q$ such that $(-\Delta)v = u$ on (N, p). □

Lemma 1. *Let F be a finite subset in any infinite network (N, p). Let $E \subset \overset{\circ}{F}$ and $f \geq 0$ be a real-valued function on E. Then there exist a potential u on F such that $(-\Delta)u(a) = f(a)$ for any $a \in E$.*

P r o o f. Assume that f is defined on F by giving it values 0 in $F \setminus E$. Let $G_b^F(a)$ be the Green function in F with point harmonic support $b \in \overset{\circ}{F}$ such that $G_b^F(a) = 0$ if $a \in \partial F$. Let

$$u(a) = \sum_{b \in F} f(b)G_b^F(a).$$

Then, u is a potential in F such that $(-\Delta)u(a) = f(a)$ if $a \in E$. □

4. Transient and hyperbolic networks

Let $\{N, p(a,b)\}$ be a countable set of state space N with transition probabilities $\{p(a,b)\}$. Let E be a fixed vertex in N. If a walker starting at e comes back to the state e infinitely often, i.e., with probability 1, then $\{N, p(a,b)\}$ is said to be recurrent; otherwise, it is transient (see [5, 6]).

In classical potential theory (Brelot [8] and Al-Gwaiz M.A., Anandam V. [3]), a superharmonic function $s(a) \geq 0$ is called a potential if its greatest harmonic minorant is 0. In the discrete case, we can show that the superharmoinc function $s(a) \geq 0$ on an infinite network $\{N, \varphi(a,b)\}$ is a potential if its greatest harmonic minorant is 0. If there exists a potential $p(a) > 0$ on N, then we say that $\{N, \varphi(a,b)\}$ is a hyperbolic network; otherwise, it is called a parabolic network.

Let $\{N, \varphi(a,b)\}$ be an infinite network. Write

$$\varphi(a) = \sum_{b \sim a} \varphi(a,b).$$

Then $0 < \varphi(a) < \infty$. Write

$$p(a,b) = \frac{\varphi(a,b)}{\varphi(a)}.$$

Then $\{N, p(a,b)\}$ becomes a probability space, which need not be reversible. Therefore, we can say that $\{N, \varphi(a,b)\}$ is transient when the associated probability space $\{N, p(a,b)\}$ is transient.

Theorem 2. *The infinite network $\{N, \varphi(a,b)\}$ is transient if and only if it is hyperbolic.*

P r o o f. Let e be a fixed vertex in N. Consider a sequence of finite subsets $\{F_n\}$ such that $e \in \overset{\circ}{F}_1$, $F_n \subset \overset{\circ}{F}_{n+1}$, and $N = \bigcup_n F_n$. For a vertex a in N, let $\psi_n(a)$ denote the probability that the walker starting at a reaches the vertex e before contacting any vertex in F_n^C. Then $\psi_n(e) = 1$, $\psi_n(a) = 0$ for $a \notin F_n$, and

$$\psi_n(a) = \sum_b p(a,b)\psi_n(b)$$

for $a \notin \{e\} \cup \{F_n^C\}$. Since

$$\sum_b p(a,b) = 1$$

for all a, we have

$$\sum_b p(a,b)[\psi_n(b) - \psi_n(a)] = 0;$$

that is $\Delta\psi_n(a) = 0$ if $a \notin \{e\} \cup \{F_n^C\}$. Since $\{\psi_n(a)\}$ is an increasing sequence, $\psi(a) = \lim_n \psi_n(a)$ exists and $0 \leq \psi(a) \leq 1$ for all a in N. Clearly, $\psi(a)$ denotes the probability that the walker starting at $\{e\}$ returns to $\{e\}$. Consequently, $\psi \equiv 1$ if and only if N is recurrent. Hence, $\{N, \psi(a,b)\}$ is transient if and only if ψ is not the constant 1.

Now, another interpretation of $\psi_n(a)$ is that it is the Dirichlet solution with boundary values $\psi_n(e) = 1$ and $\psi_n(a) = 0$ if $a \notin \overset{\circ}{F}_n$. Hence, if we extend ψ_n to the whole space N assuming it equal to 0 on F_n, then $\psi_n(a)$ becomes subharmonic at each vertex other than e, harmonic at each vertex in $\overset{\circ}{F}_n \setminus \{e\}$, and superharmonic at e. Hence, in the limit, we find that $\psi(a)$ is a nonnegative superharmonic function on N that is harmonic outside the vertex e. Consequently, if ψ is not the constant 1, then ψ is a positive superharmonic function that is not harmonic on N. Let $h(a)$ be the greatest harmonic minorant of $\psi(a)$ on N. Then, $p(a) = \psi(a) - h(a)$ is a positive superharmonic function that is a potential on N. That is, $\{N, \psi(a,b)\}$ is hyperbolic. Then, the following statements are equivalent:

(1) the function φ is not the constant 1;

(2) the probability space $\{N, p(a,b)\}$ is transient;

(3) the infinite network $\{N, \varphi(a,b)\}$ is hyperbolic.

□

5. Bimedian functions on infinite networks

In this section, we assume that $\{N, \varphi(a,b)\}$ is an infinite network that is a tree without terminal vertices. Write

$$Au(a) = u(a) - \sum_{b \sim a} \varphi(a,b)u(b)$$

for a real-valued function $u(a)$ on N. Note that A is the Lapalcian operator $-\Delta$ if

$$\varphi(a) = \sum_{b \sim a} \varphi(a,b) = 1$$

for all a in N.

Definition 4. *A real-valued function $u(a)$ on N is said to be supermedian if*

$$u(a) \geq \sum_{b \sim a} \varphi(a,b)u(b)$$

for all a in N; $u(a)$ is said to be median if

$$u(a) = \sum_{b \sim a} \varphi(a,b)u(b)$$

for all a in N.

Remark 1.

(1) A supermedian function is the same as superharmonic if and only if $\varphi(a) = \sum_{b \sim a} \varphi(a,b) = 1$ for all a in N.

(2) A solution to the Schrödinger equation corresponds to a median function if and only if $\varphi(a) \leq 1$ for all a in N and $\varphi(a_0) < 1$ for at least one vertex a_0 in N.
We can develop a theory of supermedian functions exactly in the same way as the theory of discrete superharmonic functions. For example, we have the following.

   (a) If $u(a)$ is supermedian and $v(a)$ is submedian such that $u(a) \geq v(a)$ on N, then there exists a median function $h(a)$ on N such that $u(a) \geq h(a) \geq v(a)$; and if $h^{'}(a)$ is another median function such that $u(a) \geq h^{'} \geq v(a)$, then $h^{'}(a) \geq h(a)$.

   (b) If $u(a) \geq 0$ is supermedian, then there exists a unique decomposition $u(a) = p(a)+h(a)$, where $p(a)$ is a superpotential (i.e., a nonnegative supermedian function whose greatest median minorant is 0) and $h(a) \geq 0$ is a median function. Recall that a finite or infinite graph is known as a tree if there is no closed path of the form $\{a_0, a_1, \ldots, a_n = a_0\}$ with more than 2 distinct vertices.

Corollary 2. *Let $\{N, \varphi(a,b)\}$ be an infinite tree without terminal vertex. Then, for any vertex e in N, there exists a supermedian function $\varphi_e(a)$ on N such that $\varphi_e(a)$ is a median function at each vertex in $N \setminus \{e\}$, i.e., $\varphi_e(a)$ is not median at e.*

P r o o f. Let F be the set consisting of $\{e\}$ and all its neighbors. Define a function $u(a)$ on F such that $u(a) = 1$ and $u(a) = 0$ at each neighbor of e. Then extend $v(a)$ to N as in the above theorem to get the function $v(a)$ which equal to $u(a)$ on F and is a median function at each vertex $a \neq e$. Note that $u(a)$, and hence $v(a)$, is superharmonic at e but not median. Denote the function $v(x)$ by $\varphi_e(a)$ to prove the statement in the corollary. $\square$

Remark 2. This function $\varphi_e(a)$ is an analog of the Newtonian potential function $1/|x|$ in $\mathbb{R}^3$ if there is a positive superpotential on N; otherwise, $\varphi_e(a)$ is an analog of the logarithmic function $\log(1/|x|)$ in $\mathbb{R}^2$.

Theorem 3. *Let $\{N, \varphi(a,b)\}$ be an infinite tree without terminal vertices. Let F be a connected subset of N, and let $u(a)$ be a real-valued function on F. Then, there exists a real-valued function $v(a)$ on N such that $v(a) = u(a)$ if $a \in F$ and $v(a)$ is a median function at each vertex not in $\overset{\circ}{F}$.*

P r o o f. Let $a_0 \in \partial F$. Let $\{a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_m\}$ be the neighbors of a_0, where $\{a_1, \dots, a_k\}$ are in F and $\{b_1, \dots, b_m\}$ are outside F. Note that the latter subset $\{b_1, b_2, \dots, b_m\}$ is non-empty since $a_0 \in \partial F$. Choose a constant λ and define a function $v(a)$ on $F \cup \{$all neighbors of $a_0\}$ such that

$$v(a) = \begin{cases} u(a) & \text{for} \quad a \in F, \\ \lambda & \text{for} \quad a \notin F. \end{cases}$$

Now, if the constant λ is chosen so that

$$v(a_0) = \sum_{i=1}^{k} \varphi(a_0, a_i) u(a_i) + \lambda \sum_{j=1}^{m} t(a_0, b_j),$$

then $v(a)$ is a median function at the vertex a_0.

This procedure can be adopted with respect to each vertex on ∂F. Denoting this extended function also by $v(a)$, we get a function $v(a)$ defined on $Nbr(F)$, which consists of F and all neighbors of each vertex in F such that $v(x) = u(x)$ if $a \in F$ and $v(a)$ is a median function at each vertex in ∂F.

Repeat this procedure with respect to $v(x)$ as $Nbr(f)$. Since N is a connected network,

$$N = \dots Nbr[Nbr[Nbr(f)]]$$

so that $v(a)$ is a function defined on N such that $v(a) = u(a)$ if $a \in F$ and $v(a)$ is a median function at each vertex not in $\overset{\circ}{F}$.

Theorem 4. *Let $f(a)$ be a real-valued function on N. Then there exists a function $u(a)$ on N such that $Au(a) = f(a)$ for every a in N.*

P r o o f. From Theorem 3, we have a function $\varphi_e(a)$ such that $A\varphi_e(a) = \lambda\delta_e(a)$, where $\lambda > 0$ is a constant and $\delta_e(a)$ is the Dirac function. Write $q_e(a) = 1/\lambda \cdot \varphi_e(a)$. Thus, we conclude that given any vertex e in N, there exists a real-valued function $q_e(a)$ on N such that $Aq_e(a) = \delta_e(a)$.

Take a finite exhaustion $\{E_n\}$ of N, i.e., E_n is a non-empty finite set, $E_n \subset \overset{\circ}{E}_{n+1}$, and $N = \cup E_n$. For $n > 1$, let

$$u_n(a) = \sum_{e \in E_{n+1} \setminus E_n} q_e(a) f(e).$$

Then, $Au_n(a) = 0$ for $x \notin E_{n+1} \setminus E_n$ and $Au_n(a) = f(a)$ for $x \in E_{n+1} \setminus E_n$. Define

$$u(a) = \sum_{n=1}^{\infty} u_n(a),$$

where

$$u_1(a) = \sum_{e \in E_1} q_e(a) f(e)$$

is such that $Au_1(a) = 0$ for $x \notin E_1$ and $Au_1(a) = f(a)$ for $a \in E_1$. Note that the infinite sum is well-defined. For, if a_0 is any vertex in N, then $a_0 \in \overset{\circ}{E}_m$ for some m and $\sum_{n=m}^{\infty} u_n(a)$ is a convergent series consisting of functions that are median at the vertex a_0. Consequently, $u(a)$ is a well-defined function on N such that $Au(a) = f(a)$ for all $a \in N$.

Definition 5 (Bimedian) [11]. *A real-valued function $v(a)$ on N is said to be bimedian if there exists a median function $u(a)$ on N such that $Av(a) = u(a)$ for all $a \in N$. If A is the Laplacian operator, then $v(a)$ is called a biharmonic function on N.*

Theorem 5 (Discrete Riquier problem). *Let E be a finite subset of N. Let f and g be two real-valued functions on ∂E. Then, there exists a unique bimedian function v on E such that $Av(a) = f(a)$ and $v(a) = g(a)$ for $a \in \partial E$.*

P r o o f. Let $h_1(a)$ be the unique Dirichlet solution on E such that $Ah_1(a) = 0$ for $a \in \overset{\circ}{E}$ and $h_1(a) = f(a)$ for $a \in \partial E$. By Theorem 4, we can choose a function $s(a)$ on E such that $As(a) = h_1(a)$ on E.

Let $h_2(a)$ be the unique Dirichlete solution on E such that $Ah_2(a) = 0$ for $a \in \overset{\circ}{E}$ and $h_2(a) = g(a) - s(a)$ on ∂E. Take $v(a) = s(a) + h_2(a)$. Then, $v(a) = g(a)$ on ∂E and $Av(a) = As(a) = h_1(a)$ for $a \in \overset{\circ}{E}$, so that $A[Av(a)] = Ah_1(a) = 0$ for $a \in \overset{\circ}{E}$; further, $Av(a) = f(a)$ for $a \in \partial E$. Thus, $v(a)$ is the unique bimedian function on E such that $Av(a) = f(a)$ and $v(a) = g(a)$ for $a \in \partial E$. □

Acknowledgement

We thank the referee for very useful comments. The first author acknowledges the support given by the Vellore Institute of Technology through the Teaching cum Research Associate fellowship (VIT/HR/2019/5944 dated 18th September 2019).

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