

ON THE CHARACTERIZATION OF SCALING FUNCTIONS ON NON-ARCHEMEDEAN FIELDS

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Abstract: In real life application all signals are not obtained from uniform shifts; so there is a natural question regarding analysis and decompositions of these types of signals by a stable mathematical tool. This gap was filled by Gabardo and Nashed [11] by establishing a constructive algorithm based on the theory of spectral pairs for constructing non-uniform wavelet basis in $L^2(\mathbb{R})$. In this setting, the associated translation set $\Lambda = \{0, r/N\} + 2\mathbb{Z}$ is no longer a discrete subgroup of \mathbb{R} but a spectrum associated with a certain one-dimensional spectral pair and the associated dilation is an even positive integer related to the given spectral pair. In this paper, we characterize the scaling function for non-uniform multiresolution analysis on local fields of positive characteristic (LFPC). Some properties of wavelet scaling function associated with non-uniform multiresolution analysis (NUMRA) on LFPC are also established.

Keywords: Scaling function, Fourier transform, Local field, NUMRA

1. Introduction

Multiresolution analysis (MRA) is an important mathematical tool since it provides a natural framework for understanding and constructing discrete wavelet systems. The concept of MRA provides a natural framework for understanding and constructing discrete wavelet systems. Multiresolution analysis is an increasing family of closed spaces $\{V_j : j \in \mathbb{Z}\}$ of $L^2(\mathbb{R})$ such that $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$ and $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$ which satisfies $f \in V_j$ if and only if $f(2 \cdot) \in V_{j+1}$. Moreover, there exists a function $\varphi \in V_0$ such that the collection of integer translates of the function φ , $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$, represents a complete orthonormal system for V_0 . The function φ is called scaling function or father wavelet. The concept of multiresolution analysis has been extended in various ways in recent years. These concepts are generalized to $L^2(\mathbb{R}^d)$, to lattices different from \mathbb{Z}^d , allowing the subspaces of MRA to be generated by Riesz basis instead of orthonormal basis, admitting a finite number of scaling functions, replacing the dilation factor 2 by an integer $M \geq 2$ or by an expansive matrix $A \in GL_d(\mathbb{R})$ as long as $A \subset AZ^d$. All these concepts are developed on regular lattices, that is the translation set is always a group. Recently, Gabardo and Nashed [11] considered a generalization of Mallat's [21] celebrated theory of MRA based on spectral pairs, in which the translation set acting on the scaling function associated with the MRA to generate the subspace V_0 is no longer a group, but is the union of \mathbb{Z} and a translate of \mathbb{Z} . Based on one-dimensional spectral pairs, Gabardo and Yu [12] considered sets of nonuniform wavelets in $L^2(\mathbb{R})$. In the heart of any MRA, there lies the concept of scaling functions. Cifuentes et al. [10] characterized the scaling function of MRA in a general settings. The multiresolution analysis whose scaling functions are characteristic functions some elementary properties of MRA of $L^2(\mathbb{R}^n)$ are established by Madych [20]. Zhang [26] studied scaling functions of standard MRA and wavelets. Zhang [26] characterized support of the Fourier transform of scaling functions.

The theory of wavelets, wavelet frames, multiresolution analysis, Gabor frames on local fields of positive characteristics (LFPC) are extensively studied by many researchers including Benedetto,

Behera and Jahan, Ahmed and Neyaz, Ahmad and Shah, Jiang, Li and Ji in the references [1–4, 7–9, 13, 19, 22, 24] but still more concepts required to be studied for its enhancement on LFPC. Albeverio, Kozyrev, Khrennikov, Shelkovich, Skopina and their collaborators also established the theory of MRA and wavelets on the p -adic field \mathbb{Q}_p in a series of papers [5, 6, 14–18], where \mathbb{Q}_p is a local field of characteristic 0. Recently, Shah and Abdullah [23] have generalized the concept of multiresolution analysis on Euclidean spaces \mathbb{R}^n to nonuniform multiresolution analysis on local fields of positive characteristic, in which the translation set acting on the scaling function associated with the multiresolution analysis to generate the subspace V_0 is no longer a group, but is the union of \mathcal{Z} and a translate of \mathcal{Z} , where $\mathcal{Z} = \{u(n) : n \in \mathbb{N}_0\}$ is a complete list of (distinct) coset representation of the unit disc \mathfrak{D} in the locally compact Abelian group \mathbb{K}^+ . More precisely, this set is of the form $\Lambda = \{0, r/N\} + \mathcal{Z}$, where $N \geq 1$ is an integer and r is an odd integer such that r and N are relatively prime. They call this a *nonuniform multiresolution analysis* on local fields of positive characteristic. Inspired by the work of Shah and Abdullah [23], we in this paper establish the characterization of scaling function for nonuniform multiresolution on local fields of positive characteristic. Some properties of wavelet scaling functions associated with NUMRA on LFPC are established.

The remainder of the paper is structured as follows. In Section 2, we discuss preliminary results on local fields as well as some definitions and auxiliary results. Section 3 is devoted to the characterization of scaling function associated with nonuniform multiresolution analysis on LFPC.

2. Preliminaries on local fields

2.1. Local fields

A local field \mathbb{K} is a locally compact, non-discrete and totally disconnected field. If it is of characteristic zero, then it is a field of p -adic numbers \mathbb{Q}_p or its finite extension. If \mathbb{K} is of positive characteristic, then \mathbb{K} is a field of formal Laurent series over a finite field $GF(p^c)$. If $c = 1$, it is a p -series field, while for $c \neq 1$, it is an algebraic extension of degree c of a p -series field. Let \mathbb{K} be a fixed local field with the ring of integers

$$\mathfrak{D} = \{x \in K : |x| \leq 1\}.$$

Since K^+ is a locally compact Abelian group, we choose a Haar measure dx for K^+ . The field K is locally compact, non-trivial, totally disconnected and complete topological field endowed with non-Archimedean norm $|\cdot| : \mathbb{K} \rightarrow \mathbb{R}^+$ satisfying

- (a) $|x| = 0$ if and only if $x = 0$;
- (b) $|xy| = |x||y|$ for all $x, y \in \mathbb{K}$;
- (c) $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in \mathbb{K}$.

Property (c) is called the ultrametric inequality. Let $\mathfrak{B} = \{x \in \mathbb{K} : |x| < 1\}$ be the prime ideal of the ring of integers \mathfrak{D} in \mathbb{K} . Then, the residue space $\mathfrak{D}/\mathfrak{B}$ is isomorphic to a finite field $GF(q)$, where $q = p^c$ for some prime p and $c \in \mathbb{N}$. Since K is totally disconnected and \mathfrak{B} is both prime and principal ideal, so there exist a prime element \mathfrak{p} of \mathbb{K} such that $\mathfrak{B} = \langle \mathfrak{p} \rangle = \mathfrak{p}\mathfrak{D}$.

Let

$$\mathfrak{D}^* = \mathfrak{D} \setminus \mathfrak{B} = \{x \in \mathbb{K} : |x| = 1\}.$$

Clearly, \mathfrak{D}^* is a group of units in \mathbb{K}^* and if $x \neq 0$, then can write $x = \mathfrak{p}^n y$, $y \in \mathfrak{D}^*$. Moreover, if $\mathcal{U} = \{a_m : m = 0, 1, \dots, q-1\}$ denotes the fixed full set of coset representatives of \mathfrak{B} in \mathfrak{D} , then

every element $x \in K$ can be expressed uniquely as

$$x = \sum_{\ell=k}^{\infty} c_{\ell} \mathfrak{p}^{\ell}, \quad \text{with } c_{\ell} \in \mathcal{U}.$$

Recall that \mathfrak{B} is compact and open, so each fractional ideal

$$\mathfrak{B}^k = \mathfrak{p}^k \mathfrak{D} = \{x \in K : |x| < q^{-k}\}$$

is also compact and open and is a subgroup of K^+ . We use the notation in Taibleson's book [25]. In the rest of this paper, we use the symbols \mathbb{N}, \mathbb{N}_0 and \mathbb{Z} to denote the sets of natural, non-negative integers and integers, respectively.

Let χ be a fixed character on K^+ that is trivial on \mathfrak{D} but non-trivial on \mathfrak{B}^{-1} . Therefore, χ is constant on cosets of \mathfrak{D} so if $y \in \mathfrak{B}^k$, then $\chi_y(x) = \chi(y, x), x \in K$. Suppose that χ_u is any character on K^+ , then the restriction $\chi_u|_{\mathfrak{D}}$ is a character on \mathfrak{D} . Moreover, as characters on $\mathfrak{D}, \chi_u = \chi_v$ if and only if $u - v \in \mathfrak{D}$. Hence, if $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representative of \mathfrak{D} in K^+ , then, as it was proved in [25], the set $\{\chi_{u(n)} : n \in \mathbb{N}_0\}$ of distinct characters on \mathfrak{D} is a complete orthonormal system on \mathfrak{D} .

We now impose a natural order on the sequence $\{u(n)\}_{n=0}^{\infty}$. We have $\mathfrak{D}/\mathfrak{B} \cong GF(q)$ where $GF(q)$ is a c -dimensional vector space over the field $GF(p)$. We choose a set

$$\{1 = \zeta_0, \zeta_1, \zeta_2, \dots, \zeta_{c-1}\} \subset \mathfrak{D}^*$$

such that $\text{span} \{\zeta_j\}_{j=0}^{c-1} \cong GF(q)$. For $n \in \mathbb{N}_0$ satisfying

$$0 \leq n < q, \quad n = a_0 + a_1 p + \dots + a_{c-1} p^{c-1}, \quad 0 \leq a_k < p, \quad k = 0, 1, \dots, c-1,$$

we define

$$u(n) = (a_0 + a_1 \zeta_1 + \dots + a_{c-1} \zeta_{c-1}) \mathfrak{p}^{-1}.$$

Also, for

$$n = b_0 + b_1 q + b_2 q^2 + \dots + b_s q^s, \quad n \in \mathbb{N}_0, \quad 0 \leq b_k < q, \quad k = 0, 1, 2, \dots, s,$$

we set

$$u(n) = u(b_0) + u(b_1) \mathfrak{p}^{-1} + \dots + u(b_s) \mathfrak{p}^{-s}.$$

This defines $u(n)$ for all $n \in \mathbb{N}_0$. In general, it is not true that $u(m+n) = u(m) + u(n)$. But, if $r, k \in \mathbb{N}_0$ and $0 \leq s < q^k$, then

$$u(rq^k + s) = u(r) \mathfrak{p}^{-k} + u(s).$$

Further, it is also easy to verify that $u(n) = 0$ if and only if $n = 0$ and

$$\{u(\ell) + u(k) : k \in \mathbb{N}_0\} = \{u(k) : k \in \mathbb{N}_0\}$$

for a fixed $\ell \in \mathbb{N}_0$. Hereafter we use the notation $\chi_n = \chi_{u(n)}, n \geq 0$.

Let the local field \mathbb{K} be of characteristic $p > 0$ and $\zeta_0, \zeta_1, \zeta_2, \dots, \zeta_{c-1}$ be as above. We define a character χ on K as follows:

$$\chi(\zeta_{\mu} \mathfrak{p}^{-j}) = \begin{cases} \exp(2\pi i/p), & \mu = 0 \quad \text{and} \quad j = 1, \\ 1, & \mu = 1, \dots, c-1 \quad \text{or} \quad j \neq 1. \end{cases}$$

2.2. Fourier transforms on local fields

The Fourier transform of $f \in L^1(K)$ is denoted by $\hat{f}(\xi)$ and defined by

$$\mathcal{F}\{f(x)\} = \hat{f}(\xi) = \int_K f(x) \overline{\chi_\xi(x)} dx.$$

It is noted that

$$\hat{f}(\xi) = \int_K f(x) \overline{\chi_\xi(x)} dx = \int_K f(x) \chi(-\xi x) dx.$$

The properties of Fourier transforms on local field \mathbb{K} are much similar to those of on the classical field \mathbb{R} . In fact, the Fourier transform on local fields of positive characteristic have the following properties:

- The map $f \rightarrow \hat{f}$ is a bounded linear transformation of $L^1(\mathbb{K})$ into $L^\infty(\mathbb{K})$, and $\|\hat{f}\|_\infty \leq \|f\|_1$.
- If $f \in L^1(\mathbb{K})$, then \hat{f} is uniformly continuous.
- If $f \in L^1(\mathbb{K}) \cap L^2(\mathbb{K})$, then $\|\hat{f}\|_2 = \|f\|_2$.

The Fourier transform of a function $f \in L^2(\mathbb{K})$ is defined by

$$\hat{f}(\xi) = \lim_{k \rightarrow \infty} \hat{f}_k(\xi) = \lim_{k \rightarrow \infty} \int_{|x| \leq q^k} f(x) \overline{\chi_\xi(x)} dx,$$

where $f_k = f \Phi_{-k}$ and Φ_k is the characteristic function of \mathfrak{B}^k . Furthermore, if $f \in L^2(\mathfrak{D})$, then we define the Fourier coefficients of f as

$$\hat{f}(u(n)) = \int_{\mathfrak{D}} f(x) \overline{\chi_{u(n)}(x)} dx.$$

The series

$$\sum_{n \in \mathbb{N}_0} \hat{f}(u(n)) \chi_{u(n)}(x)$$

is called the Fourier series of f . From the standard L^2 -theory for compact Abelian groups, we conclude that the Fourier series of f converges to f in $L^2(\mathfrak{D})$ and Parseval's identity holds:

$$\|f\|_2^2 = \int_{\mathfrak{D}} |f(x)|^2 dx = \sum_{n \in \mathbb{N}_0} |\hat{f}(u(n))|^2.$$

3. Nonuniform MRA on local fields

Definition 1. For an integer $N \geq 1$ and an odd integer r with $1 \leq r \leq qN - 1$ such that r and N are relatively prime, we define

$$\Lambda = \left\{ 0, \frac{u(r)}{N} \right\} + \mathcal{Z}$$

and

$$\Delta_N = \{u(m)N + \mathfrak{p}u(j) : m \in \mathbb{Z}, 0 \leq j \leq N - 1\},$$

where

$$\mathcal{Z} = \{u(n) : n \in \mathbb{N}_0\}.$$

It is easy to verify that Λ is not a group on local field \mathbb{K} , but is the union of \mathcal{Z} and a translate of \mathcal{Z} .

Following is the definition of nonuniform multiresolution analysis (NUMRA) on local fields of positive characteristic given by Shah and Abdullah [23].

Definition 2. For an integer $N \geq 1$ and an odd integer r with $1 \leq r \leq qN - 1$ such that r and N are relatively prime, an associated NUMRA on local field \mathbb{K} of positive characteristic is a sequence of closed subspaces $\{V_j : j \in \mathbb{Z}\}$ of $L^2(\mathbb{K})$ such that the following properties hold:

- (a) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;
- (b) $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(\mathbb{K})$;
- (c) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$;
- (d) $f(\cdot) \in V_j$ if and only if $f(\mathfrak{p}^{-1}N\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$;
- (e) There exists a function φ in V_0 such that $\{\varphi(\cdot - \lambda) : \lambda \in \Lambda\}$, is a complete orthonormal basis for V_0 .

It is worth noticing that, when $N = 1$, one recovers the definition of an MRA on local fields of positive characteristic $p > 0$. When, $N > 1$, the dilation is induced by $\mathfrak{p}^{-1}N$ and $|\mathfrak{p}^{-1}| = q$ ensures that $qN\Lambda \subset \mathcal{Z} \subset \Lambda$. For every $j \in \mathbb{Z}$, define W_j to be the orthogonal complement of V_j in V_{j+1} .

Then we have

$$V_{j+1} = V_j \oplus W_j \quad \text{and} \quad W_\ell \perp W_{\ell'} \quad \text{if} \quad \ell \neq \ell'.$$

It follows that for $j > J$,

$$V_j = V_J \oplus \bigoplus_{\ell=0}^{j-J-1} W_{j-\ell},$$

where all these subspaces are orthogonal. By virtue of condition (b) in the Definition 2, this implies

$$L^2(\mathbb{K}) = \bigoplus_{j \in \mathbb{Z}} W_j,$$

a decomposition of $L^2(\mathbb{K})$ into mutually orthogonal subspaces.

As in the standard scheme, one expects the existence of $qN - 1$ number of functions so that their translation by elements of Λ and dilations by the integral powers of $\mathfrak{p}^{-1}N$ form an orthonormal basis for $L^2(\mathbb{K})$.

Let a and b be any two fixed elements in \mathbb{K} . Then, for any prime \mathfrak{p} and $m, n \in \mathbb{N}_0$, let $D_{\mathfrak{p}}, T_{u(n)a}$ and $E_{u(m)b}$ be the unitary operators acting on $f \in L^2(\mathbb{K})$ defined by:

$$\begin{aligned} T_{u(n)a}f(x) &= f(x - u(n)a), & (\text{Translation}), \\ E_{u(m)b}f(x) &= \chi(u(m)bx)f(x), & (\text{Modulation}), \\ D_{\mathfrak{p}}f(x) &= \sqrt{qN}f(\mathfrak{p}^{-1}Nx), & (\text{Dilation}). \end{aligned}$$

Then for any $f \in L^2(K)$, the following results can easily be verified:

$$\begin{aligned} \mathcal{F}\{T_{u(n)a}f(x)\} &= E_{-u(n)a}\mathcal{F}\{f(x)\}, \\ \mathcal{F}\{E_{u(m)b}f(x)\} &= T_{u(m)b}\mathcal{F}\{f(x)\}, \\ \mathcal{F}\{D_{\mathfrak{p}^j}f(x)\} &= D_{\mathfrak{p}^{-j}}\mathcal{F}\{f(x)\}, \\ D_{\mathfrak{p}^j}T_{u(n)a} &= T_{(qN)^{-j}u(n)a}D_{\mathfrak{p}^j}. \end{aligned}$$

We state the following lemmas which will be very useful in establishing the results and whose proof can be found in [23].

Lemma 1. For an integer $N \geq 1$ and an odd integer r with $1 \leq r \leq qN - 1$ such that r and N are relatively prime. Let $\varphi \in L^2(\mathbb{K})$ with $\|\varphi\|^2 = 1$, then

(i) the family $\{\varphi(\xi - \lambda) : \lambda \in \Lambda\}$ is an orthonormal system for fixed r if and only if

$$\sum_{k \in \mathbb{N}_0} |\widehat{\varphi}(\xi + \mathfrak{p}u(k))|^2 = q \quad \text{a.e. } \xi \in \mathbb{K}$$

and

$$\sum_{k \in \mathbb{N}_0} \chi\left(\frac{u(r)}{N}u(k)\right) \overline{|\widehat{\varphi}(\xi + \mathfrak{p}u(k))|^2} = 0 \quad \text{a.e. } \xi \in \mathbb{K};$$

(ii) the family $\{\varphi(\xi - \lambda) : \lambda \in \Delta_N\}$ is an orthonormal system for every odd integer r if and only if

$$|\widehat{\varphi}(\xi - \gamma)|^2 = 1, \quad \text{a.e. } \xi \in \mathbb{K}.$$

Lemma 2. Let (V_j, φ) be non-uniform multiresolution analysis, where

$$V_0 = \overline{\text{span}} \{\varphi(x - \lambda) : \lambda \in \Lambda\}.$$

Then the necessary and sufficient condition for the existence of associated wavelets is

$$\sum_{\gamma \in \Delta_N} |\widehat{\varphi}(\xi - \gamma)|^2 = 1 \quad \text{a.e. } \xi \in \mathbb{K}.$$

Lemma 3. Let $\mathcal{S} \subset \mathbb{K}$ be measurable and $\Lambda_0 = \{0, u(a)\} + \mathcal{Z}$. Then (\mathcal{S}, Λ_0) is a spectral pair if and only if there exist an integer $N \geq 1$ and an odd integer r with $1 \leq r \leq qN - 1$, such that N and r are relatively prime, $a = r/N$ and

$$\sum_{j=0}^{N-1} \delta_{j/2} \star \sum_{n \in \mathbb{N}_0} \delta_{nN} \star \Phi_{\mathcal{S}} = 1.$$

4. Characterization of scaling functions on LFPC

In this section, we establish the characterization of scaling functions associated with nonuniform multiresolution analysis on LFPC. We also provide the sufficient condition for the frequency band of the scaling function on LFPC.

Theorem 1. A nonzero function $\varphi \in L^2(\mathbb{K})$ is a scaling function for wavelet NUMRA if and only if the following conditions are satisfied

(i) $\sum_{\gamma \in \Delta_N} |\widehat{\varphi}(\xi - \gamma)|^2 = 1 \quad \text{a.e. } \xi \in \mathbb{K};$

(ii) $\lim_{j \rightarrow \infty} |\widehat{\varphi}(\mathfrak{p}^{-1}N)^j \xi|^2 = 1 \quad \text{a.e. } \xi \in q^2\mathcal{D};$

(iii) there exist functions $m^1(\xi), m^2(\xi)$ locally integrable, q -periodic functions such that

$$\widehat{\varphi}(\mathfrak{p}^{-1}N\xi) = m(\xi)\widehat{\varphi}(\xi) \quad \text{a.e. } \xi \in \mathbb{K},$$

where

$$m(\xi) = m^1(\xi) + \chi\left(\frac{u(r)}{N}\xi\right) \overline{m^2(\xi)}.$$

P r o o f. Suppose $\psi \in L^2(\mathbb{K})$ is a scaling function for wavelet NUMRA, say $\{V_j, \varphi\}_{j \in \mathbb{Z}}$. Then by Lemma 2, we must have

$$\sum_{\gamma \in \Delta_N} |\widehat{\varphi}(\xi - \gamma)|^2 = 1 \quad \text{a.e. } \xi \in \mathbb{K}. \quad (4.1)$$

This gives (i). Since $\varphi \in V_0$, we have $D_{\mathfrak{p}^{-1}}\varphi \in V_{-1} \subseteq V_0$. Thus we can write

$$D_{\mathfrak{p}^{-1}}\varphi = \sum_{\lambda \in \Lambda} a_\lambda T_\lambda \varphi.$$

Taking the Fourier transform of both sides, we get

$$D_{\mathfrak{p}}\widehat{\varphi} = \sum_{\lambda \in \Lambda} a_\lambda E_{-\lambda} \widehat{\varphi}.$$

So we can write,

$$\widehat{\varphi}(\mathfrak{p}^{-1}N\gamma) = m(\gamma)\widehat{\varphi}(\gamma),$$

where

$$m(\gamma) = m^1(\gamma) + \overline{\chi\left(\frac{u(r)}{N}\gamma\right)m^2(\gamma)}$$

and m^1, m^2 are q -periodic and locally integrable functions. This proves (iii).

Next we show that (ii) holds. Let $f \in L^2(\mathbb{K})$ be such that $\widehat{f}(\gamma) = \Phi_{q^2\mathfrak{D}}(\gamma)$. Then

$$\|f\|^2 = \|\widehat{f}\|^2 = q.$$

As (V_j, φ) is NUMRA so if P_j is orthogonal projection onto V_j , we must have

$$\|f - P_j f\|^2 \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

That is

$$\|P_j f\| \rightarrow \|f\| \quad \text{as } j \rightarrow \infty.$$

Since $\{T_\lambda \varphi\}_{\lambda \in \Lambda}$ is an orthonormal bases for V_0 so $\{D_{\mathfrak{p}^j} T_\lambda \varphi\}_{\lambda \in \Lambda}$ is an orthonormal basis for V_j . Thus

$$\|P_j f\|^2 = \sum_{\lambda \in \Lambda} |\langle f, D_{\mathfrak{p}^j} T_\lambda \varphi \rangle|^2 \rightarrow \frac{1}{q} \quad \text{a.e. } j \rightarrow \infty \quad (4.2)$$

$$\begin{aligned} \sum_{\lambda \in \Lambda} |\langle f, D_{\mathfrak{p}^j} T_\lambda \varphi \rangle|^2 &= \sum_{\lambda \in \mathcal{Z}} |\langle f, D_{\mathfrak{p}^j} T_\lambda \varphi \rangle|^2 + \sum_{\lambda \in (u(r)/N + \mathcal{Z})} |\langle f, D_{\mathfrak{p}^j} T_\lambda \varphi \rangle|^2 \\ &= \sum_{\lambda \in \mathcal{Z}} |\widehat{f}, \widehat{D_{\mathfrak{p}^j} T_\lambda \varphi}|^2 + \sum_{\lambda \in (u(r)/N + \mathcal{Z})} |\widehat{f}, \widehat{D_{\mathfrak{p}^j} T_\lambda \varphi}|^2 \\ &= \sum_{k \in \mathbb{N}_0} \left| \int_{\mathbb{K}} (qN)^{-j/2} \widehat{f}(\gamma) \chi_{u(k)} \left(\frac{\gamma}{(\mathfrak{p}^{-1}N)^j} \right) \overline{\widehat{\varphi} \left(\frac{\gamma}{(\mathfrak{p}^{-1}N)^j} \right)} d\gamma \right|^2 \\ &+ \sum_{k \in \mathbb{N}_0} \left| \int_{\mathbb{K}} (qN)^{-j/2} \widehat{f}(\gamma) \chi_{u(k)} \left(\frac{\gamma}{(\mathfrak{p}^{-1}N)^j} \left(\frac{u(r)}{N} + \mathfrak{p}u(k) \right) \right) \overline{\widehat{\varphi} \left(\frac{\gamma}{(\mathfrak{p}^{-1}N)^j} \right)} d\gamma \right|^2 \end{aligned}$$

$$\begin{aligned}
&= \sum_{k \in \mathbb{N}_0} \left| \int_{q^2 \mathfrak{D}} (qN)^{-j/2} \chi_{u(k)} \left(\frac{\gamma}{(\mathfrak{p}^{-1}N)^j} \right) \overline{\widehat{\varphi} \left(\frac{\gamma}{(\mathfrak{p}^{-1}N)^j} \right)} d\gamma \right|^2 \\
&= \sum_{k \in \mathbb{N}_0} \left| \int_{q^2 \mathfrak{D}} (qN)^{-j/2} \chi_{u(k)} \left(\frac{\gamma}{(\mathfrak{p}^{-1}N)^j} \left(\frac{u(r)}{N} + \mathfrak{p}u(k) \right) \right) \overline{\widehat{\varphi} \left(\frac{\gamma}{(\mathfrak{p}^{-1}N)^j} \right)} d\gamma \right|^2
\end{aligned}$$

Putting

$$\frac{\gamma}{(qN)^j} = \eta,$$

we obtain

$$\begin{aligned}
\sum_{\lambda \in \Lambda} |\langle f, D_{\mathfrak{p}^j} T_\lambda \varphi \rangle|^2 &= \frac{(qN)^j}{2} \left\{ \sum_{k \in \mathbb{N}_0} \left| \int_{(\mathfrak{p}^{-1}N)^{-j} \mathfrak{D}} \sqrt{q} \chi_{u(k)}(\mathfrak{p}^{-1}\eta) \overline{\widehat{\varphi}(\eta)} d\eta \right|^2 \right. \\
&\quad \left. + \sum_{k \in \mathbb{N}_0} \left| \int_{(\mathfrak{p}^{-1}N)^{-j} \mathfrak{D}} \sqrt{q} \chi_{u(k)}(\mathfrak{p}^{-1}\eta) \overline{\widehat{\varphi}(\eta)} d\eta \right|^2 \right\} \\
&= \frac{(qN)^j}{2} \left\{ \sum_{k \in \mathbb{N}_0} \left| \int_{q^2 \mathfrak{D}} \Phi_{(\mathfrak{p}^{-1}N)^{-j} \mathfrak{D}} \sqrt{q} \chi_{u(k)}(\mathfrak{p}^{-1}\eta) \overline{\widehat{\varphi}(\eta)} d\eta \right|^2 \right. \\
&\quad \left. + \sum_{k \in \mathbb{N}_0} \left| \int_{q^2 \mathfrak{D}} \Phi_{(\mathfrak{p}^{-1}N)^{-j} \mathfrak{D}} \chi \left(\frac{u(r)}{N} \eta \right) \sqrt{q} \chi_{u(k)}(\mathfrak{p}^{-1}\eta) \overline{\widehat{\varphi}(\eta)} d\eta \right|^2 \right\},
\end{aligned}$$

because $(\mathfrak{p}^{-1}N)^{-j} \mathfrak{D} \subseteq q^2 \mathfrak{D}$, for any $j \geq 0$. Therefore from (4.2) and from the fact that $\{\sqrt{q} \chi_{u(k)}(\mathfrak{p}^{-1}\eta)\}$ is an orthonormal basis for $L^2(q\mathfrak{D})$, we get

$$\sum_{\lambda \in \Lambda} |\langle f, D_{\mathfrak{p}^j} T_\lambda \varphi \rangle|^2 = (qN)^j \int_{(\mathfrak{p}^{-1}N)^{-j} \mathfrak{D}} |\widehat{\varphi}(\eta)|^2 d\eta \rightarrow \frac{1}{q} \quad j \rightarrow \infty.$$

Putting $\mu = (qN)^j \eta$, we get

$$\int_{q^2 \mathfrak{D}} |\widehat{\varphi}(\mathfrak{p}^{-1}N)^j \mu|^2 d\mu \rightarrow \frac{1}{q} \quad \text{as } j \rightarrow \infty. \quad (4.3)$$

Let

$$h(\xi) = \lim_{j \rightarrow \infty} |\widehat{\varphi}(\mathfrak{p}^{-1}N)^j \xi|^2.$$

Then

$$0 \leq h(\xi) \leq 1 \quad \text{a.e. } \xi \in q^2 \mathfrak{D}.$$

Indeed for any fixed $j \in \mathbb{Z}$ by using (4.1), we have

$$0 \leq |\widehat{\varphi}(\mathfrak{p}^{-1}N)^j \xi|^2 \leq 1 \quad \text{a.e. } \xi \in q^2 \mathfrak{D}.$$

This gives

$$0 \leq h(\xi) = \lim_{j \rightarrow \infty} |\widehat{\varphi}(\mathfrak{p}^{-1}N)^j \xi|^2 \leq 1 \quad \text{a.e. } \xi \in q^2 \mathfrak{D}.$$

Now invoking the Lebesgue-dominated convergence theorem, we obtain

$$\lim_{j \rightarrow \infty} \int_{q^2 \mathfrak{D}} |\widehat{\varphi}((\mathfrak{p}^{-1}N)^j \mu)|^2 d\mu = \int_{q^2 \mathfrak{D}} \lim_{j \rightarrow \infty} |\widehat{\varphi}((\mathfrak{p}^{-1}N)^j \mu)|^2 d\mu = \frac{1}{q}.$$

Thus

$$\int_{q^2 \mathfrak{D}} h(\xi) d\xi = \frac{1}{q} = \int_{q^2 \mathfrak{D}} 1 d\xi.$$

That is

$$\int_{q^2\mathfrak{D}} (1 - h(\xi))d\xi = 0,$$

so by using

$$0 \leq h(\xi) \leq 1 \quad \text{a.e.} \quad \xi \in q^2\mathfrak{D},$$

we get $h(\xi) = 1$ a.e. $\xi \in q^2\mathfrak{D}$. Hence (ii) is proved.

Conversely, let $\varphi \in L^2(\mathbb{K})$ satisfying (i)–(iii). We define closed subspaces V_j of $L^2(\mathbb{K})$ in the following way.

For $j = 0$ let $V_j = \overline{\text{span}} \{ \varphi(\xi - \lambda) : \lambda \in \Lambda \}$ and for $j \neq 0$ let $V_j = \{ f : f((\mathfrak{p}^{-1}N)^{-j}\xi) \in V_0 \}$. We will show (V_j, φ) forms wavelet NUMRA. Using Lemma 1, the sequence $\{T_\lambda\varphi\}_{\lambda \in \Lambda}$ is an orthonormal basis for V_0 .

By definition of V_j , it can be easily shown that $f(\gamma) \in V_j$ if and only if

$$f((\mathfrak{p}^{-1}N)\gamma) \in V_{j+1},$$

which clearly implies $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$. To prove $V_j \subseteq V_{j+1}$, it is sufficient to show that $V_0 \subseteq V_1$. First we show that

$$V_j = \left\{ f \in L^2(\mathbb{K}) : \widehat{f}((\mathfrak{p}^{-1}N)^j\gamma) = (m_j^1(\gamma) + \chi \left(\frac{u(r)}{N}\gamma \right) \overline{m_j^2(\gamma)}) \widehat{\varphi}(\gamma) \right\}, \quad (4.4)$$

where m_j^1, m_j^2 are locally integrable, q -periodic functions. Let $f \in V_j$, then

$$\frac{1}{(qN)^{j/2}} D_{\mathfrak{p}^{-j}} f(\gamma) \in V_0,$$

as $\{T_\lambda\varphi\}_{\lambda \in \Lambda}$ is an orthonormal basis for V_0 , so there exist $\{c_\lambda^j\} \in \ell^2(\mathbb{N}_0)$ such that

$$\frac{1}{(qN)^{j/2}} D_{\mathfrak{p}^{-j}} f(\gamma) = \sum_{\lambda \in \Lambda} c_\lambda^j T_\lambda \varphi.$$

On taking Fourier transform of both sides, we obtain

$$\widehat{f}((\mathfrak{p}^{-1}N)^j\gamma) = \sum_{\lambda \in \Lambda} c_\lambda^j \overline{\chi_\lambda(\mathfrak{p}^{-1}\gamma)} \widehat{\varphi}(\gamma) = \left\{ m_j^1(\gamma) + \chi \left(\frac{u(r)}{N}\gamma \right) \overline{m_j^2(\gamma)} \right\} \widehat{\varphi}(\gamma),$$

where m_j^1 and m_j^2 are locally integrable and q -periodic functions. If $f \in L^2(\mathbb{K})$ satisfies

$$\widehat{f}((\mathfrak{p}^{-1}N)^j\gamma) = \left\{ m_j^1(\gamma) + \chi \left(\frac{u(r)}{N}\gamma \right) \overline{m_j^2(\gamma)} \right\} \widehat{\varphi}(\gamma)$$

for some m_j^1 and m_j^2 are locally integrable and q -periodic functions, then we can write

$$\widehat{f}((\mathfrak{p}^{-1}N)^j\xi) = \left\{ \sum_{k \in \mathbb{Z}} c_k^j \overline{\chi_{u(k)}(\mathfrak{p}^{-1}\gamma)} + \chi \left(\frac{u(r)}{N}\gamma \right) \sum_{k \in \mathbb{N}_0} d_k^j \overline{\chi_{u(k)}(\mathfrak{p}^{-1}\gamma)} \right\} \widehat{\varphi}(\gamma)$$

for some scalars $\{c_k^j\}$ and $\{d_k^j\}_{k \in \mathbb{N}_0} \in \ell^2(\mathbb{N}_0)$. Therefore

$$\frac{D_{\mathfrak{p}^j} \widehat{f}(\gamma)}{(qN)^{j/2}} = \sum_{\lambda \in \Lambda} l_\lambda^j \overline{\chi_{u(k)}(\mathfrak{p}^{-1}\gamma)} \widehat{\varphi}(\gamma)$$

for some $\{l_\lambda^j\}_{\lambda \in \Lambda} \in \ell^2(\mathbb{N}_0)$. By taking inverse Fourier transform on both sides, we obtain

$$\widehat{f}((\mathfrak{p}^{-1}N)^j\gamma) = \sum_{\lambda \in \Lambda} p_\lambda^j T_\lambda \varphi,$$

where

$$\sum_{\lambda \in \Lambda} |p_\lambda^j|^2 < \infty$$

which shows $f(\gamma) \in V_j$. Hence $V_j (j \in \mathbb{Z})$ are given by (4.4).

Now we are ready to show that $V_0 \subseteq V_1$. Let $f(\gamma) \in V_0$. Then by (4.4), we can write

$$f(\gamma) = \left\{ m_0^1(\gamma) + \overline{\chi\left(\frac{u(r)}{N}\gamma\right)} m_0^2(\gamma) \widehat{\varphi}(\gamma) \right\},$$

where m_0^1 and m_0^2 are locally integrable, q -periodic functions. Therefore,

$$\widehat{f}(\gamma) = G(\gamma)m(\gamma)\widehat{\varphi}(\gamma), \quad (4.5)$$

where

$$G(\gamma) = m_0^1(\mathfrak{p}^{-1}N\gamma) + \overline{\chi_{u(r)}(\mathfrak{p}^{-1}\gamma)} m_0^2(\mathfrak{p}^{-1}N\gamma)$$

and

$$m(\gamma) = m^1(\gamma) + \overline{\chi\left(\frac{u(r)}{N}\gamma\right)} m^2(\gamma).$$

This gives

$$G(\gamma)m(\gamma) = G(\gamma) \left\{ m^1(\gamma) + \overline{\chi\left(\frac{u(r)}{N}\gamma\right)} m^2(\gamma) \right\} = G(\gamma)m^1(\gamma) + \overline{\chi\left(\frac{u(r)}{N}\gamma\right)} G(\gamma)m^2(\gamma). \quad (4.6)$$

Using the conditions (i) and (iii), it can be easily shown that functions $m^1(\gamma)$ and $m^2(\gamma)$ are bounded. Also since $m^1(\gamma)$, $m^2(\gamma)$ and $G(\gamma)$ are q -periodic, therefore the functions $G(\gamma)m^1(\gamma)$ and $G(\gamma)m^2(\gamma)$ are q -periodic and

$$\int_{\mathfrak{D}} |G(\gamma)m^1(\gamma)|^2 d\gamma, \quad \int_{\mathfrak{D}} |G(\gamma)m^2(\gamma)|^2 d\gamma < \infty.$$

Thus by using (4.4)–(4.6), we infer that $f(\gamma) \in V_1$. Hence $V_0 \subseteq V_1$.

To prove that $\bigcup_{j \in \mathbb{Z}} V_j = L^2(\mathbb{K})$, it sufficient to show that, for any $f \in L^2(\mathbb{K})$, we have

$$\|P_j f - f\|^2 = \|f\|^2 - \|P_{4.12_j} f\|^2 \rightarrow 0 \quad \text{as } j \rightarrow \infty,$$

where P_j is the orthonormal projection onto V_j . Let $f \in L^2(\mathbb{K})$ be such that $\widehat{f} \in C_c(\mathbb{K})$. Now we have

$$\begin{aligned} \|P_j f\|^2 &= \sum_{\lambda \in \Lambda} |\langle f, D_{\mathfrak{p}^j} T_\lambda \varphi \rangle|^2 = \sum_{\lambda \in \Lambda} |\langle f, \widehat{D_{\mathfrak{p}^j} T_\lambda \varphi} \rangle|^2 \\ &= \sum_{k \in \mathbb{N}_0} \left| \int_{\mathbb{K}} (qN)^{-1/2} \widehat{f}(\gamma) \chi_{u(k)} \left(\frac{\gamma}{(\mathfrak{p}^{-1}N)^j} \left(\frac{u(r)}{N} + \mathfrak{p}u(k) \right) \right) \overline{\widehat{\varphi} \left(\frac{\gamma}{(\mathfrak{p}^{-1}N)^j} \right)} d\gamma \right|^2 \\ &\quad + \sum_{k \in \mathbb{N}_0} \left| \int_{\mathbb{K}} (qN)^{-1/2} \widehat{f}(\gamma) \chi_{u(k)} \left(\frac{\gamma}{(\mathfrak{p}^{-1}N)^j} \left(\frac{u(r)}{N} + \mathfrak{p}u(k) \right) \right) \overline{\widehat{\varphi} \left(\frac{\gamma}{(2N)^j} \right)} d\gamma \right|^2 \\ &= \sum_{k \in \mathbb{N}_0} \left| \int_{\mathbb{K}} (qN)^{j/2} \widehat{f}((\mathfrak{p}^{-1}N)^j \xi) \chi_{u(k)}(\mathfrak{p}^{-1}\xi) \overline{\widehat{\varphi}(\xi)} d\xi \right|^2 \\ &\quad + \sum_{k \in \mathbb{N}_0} \left| \int_{\mathbb{K}} (qN)^{j/2} \widehat{f}((\mathfrak{p}^{-1}N)^j \xi) \chi_{u(k)}(\mathfrak{p}^{-1}\xi) \overline{\widehat{\varphi}(\xi)} d\xi \right|^2. \end{aligned} \quad (4.7)$$

Since \widehat{f} has compact support, we can choose j so large that

$$\text{supp } \widehat{f}((\mathfrak{p}^{-1}N)^j\xi) \subseteq q^2\mathfrak{D}.$$

Then, using the fact that $\{\sqrt{q}\chi_{u(k)}(\xi)\}$ is an orthonormal basis for $L^2(q^2\mathfrak{D})$ and by (4.7), we get

$$\begin{aligned} \|P_j f\|^2 &= \frac{(qN)^j}{2} \left\{ \sum_{k \in \mathbb{N}_0} \left| \int_{q^2\mathfrak{D}} \widehat{f}((\mathfrak{p}^{-1}N)^j\xi) \sqrt{q}\chi_{u(k)}(\mathfrak{p}^{-1}\xi) \overline{\widehat{\varphi}(\xi)} d\xi \right|^2 \right. \\ &\quad \left. + \sum_{k \in \mathbb{N}_0} \left| \int_{q^2\mathfrak{D}} \widehat{f}((\mathfrak{p}^{-1}N)^j\xi) \sqrt{q}e^{2\pi i \frac{r}{N}\xi} \chi_{u(k)}(\mathfrak{p}^{-1}\eta) \overline{\widehat{\varphi}(\eta)} d\xi \right|^2 \right\} \\ &= (qN)^j \int_{q^2\mathfrak{D}} |\widehat{f}((\mathfrak{p}^{-1}N)^j\xi) \overline{\widehat{\varphi}(\xi)}|^2 d\xi \end{aligned} \quad (4.8)$$

Putting $(\mathfrak{p}^{-1}N)^j\xi = \eta$ in (4.8) and invoking the Lebesgue-dominated convergence theorem, we get

$$\|P_j f\|^2 = \int_{(\mathfrak{p}^{-1}N)^{-j}\mathfrak{D}} |\widehat{f}(\eta) \overline{\widehat{\varphi}(\mathfrak{p}^{-1}N)^{-j}\eta}|^2 d\eta \rightarrow \|f\|^2 \quad \text{as } j \rightarrow \infty.$$

Thus the proof is complete. \square

In the context of Fourier domain, the following theorem gives necessary condition for scaling function of wavelet NUMRA on LFPC.

Theorem 2. *If φ be a scaling function of wavelet NUMRA and $\widehat{\varphi}$ is continuous then $|\widehat{\varphi}(0)| = 1$ and $\widehat{\varphi}(u(m)N - u(j)) = 0$, where $m \in \mathbb{N}_0$, $0 \leq j \leq N-1$. In particular $\widehat{\varphi}(u(m)N) = 0$ for $m \in \mathbb{N}_0$ and $\widehat{\varphi}(-\mathfrak{p}u(j)) = 0$, $0 \leq j \leq N-1$.*

P r o o f. By (4.3), we have

$$\lim_{j \rightarrow \infty} \int_{q^2\mathfrak{D}} |\widehat{\varphi}(\mathfrak{p}^{-1}N)^{-j}\mu|^2 d\mu = \frac{1}{q}$$

as $|\widehat{\varphi}|$ is continuous. By virtue of Lebesgue dominated convergence theorem, we obtain $|\widehat{\varphi}(0)| = 1$. Since φ is a scaling function for wavelet NUMRA, we have

$$\sum_{\gamma \in \Delta_N} |\widehat{\varphi}(\xi - \gamma)|^2 = 1 \quad \text{a.e. } \xi \in \mathbb{K}. \quad (4.9)$$

Suppose

$$\widehat{\varphi}(u(m)N - \mathfrak{p}u(j)) = a \neq 0$$

for some m, j not both zero together. Then

$$|\widehat{\varphi}(\xi)| + |\widehat{\varphi}(\xi + u(m)N - \mathfrak{p}u(j))|^2 > 1 + a^2, \quad \text{when } \xi \in \mathfrak{p}^\epsilon\mathfrak{D}$$

for some $\epsilon > 0$ which contradicts (4.9). \square

The following theorem gives the sufficient conditions for the frequency band of the scaling function of wavelet NUMRA on LFPC.

Theorem 3. *Let \mathfrak{U} be a compact subset of \mathbb{K} such that*

- (i) $\mathcal{U} \subseteq (\mathfrak{p}^{-1}N)\mathcal{U}$;
- (ii) $\bigcup_{m \in \mathbb{N}_0} (\mathfrak{p}^{-1}N)^j \mathcal{U} = \mathbb{K}$;
- (iii) $\sum_{j=0}^{N-1} \delta_{j/2} \star \sum_{m \in \mathbb{N}_0} \delta_{mN} \star \Phi_{\mathcal{U}} = 1$.

Then \mathcal{U} is the frequency band function for some wavelet NUMRA.

P r o o f. Let

$$V_j = \left\{ f \in L^2(\mathbb{K}) : \text{supp } \widehat{f} \subset (\mathfrak{p}^{-1}N)^j \mathcal{U}, \quad j \in \mathbb{Z} \right\}$$

and $\psi \in L^2(\mathbb{K})$ be such that $\widehat{\psi} = \Phi_{\mathcal{U}}$. Using hypothesis (i) and the definition of V_j , we have $V_j \subseteq V_{j+1}$ and $f((\mathfrak{p}^{-1}N)^j \gamma) \in V_j$ if and only if $f((\mathfrak{p}^{-1}N)^{j+1} \gamma) \in V_{j+1}$. By hypothesis (ii) and the definition of V_j , we get $\bigcup_{j \in \mathbb{Z}} V_j = L^2(\mathbb{K})$. By using Lemma 3 and hypothesis (iii), we get that (\mathcal{U}, Λ) is a spectral pair. Now we have

$$\widehat{T_{\lambda} \varphi}(\xi) = \overline{\chi_{\lambda}(\xi)} \widehat{\varphi}(\xi) = \overline{\chi_{\lambda}(\xi)} \Phi_{\mathcal{U}}(\xi)$$

and the Fourier transform is the unitary operator. Thus $\{T_{\lambda} \varphi\}_{\lambda \in \Lambda}$ is an orthonormal basis for V_0 . By virtue of Lemma 3, we infer that $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$. Hence \mathcal{U} is frequency band for wavelet NUMRA (V_j, φ) . \square

5. Conclusion

In the present paper, we have given a complete characterization of the scaling function for the non-uniform multiresolution analysis on local fields of positive characteristic. Theorem 1 characterizes the nonzero square integrable functions on $L^2(\mathbb{K})$ to be a scaling functions for the wavelet NUMRA by means of three simple conditions. Furthermore Theorem 3 expresses a compact subset of \mathbb{K} to be the band scaling function of wavelet NUMRA on LFPC by means of three conditions. The present study can be extended in fractional settings and in the context of Multiresolution Analysis associated with Linear Canonical Transform.

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