

THE VERTEX DISTANCE COMPLEMENT SPECTRUM OF SUBDIVISION VERTEX JOIN AND SUBDIVISION EDGE JOIN OF TWO REGULAR GRAPHS

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Abstract: The vertex distance complement (VDC) matrix C , of a connected graph G with vertex set consisting of n vertices, is a real symmetric matrix $[c_{ij}]$ that takes the value $n - d_{ij}$ where d_{ij} is the distance between the vertices v_i and v_j of G for $i \neq j$ and 0 otherwise. The vertex distance complement spectrum of the subdivision vertex join, $G_1 \vee G_2$ and the subdivision edge join $G_1 \underline{\vee} G_2$ of regular graphs G_1 and G_2 in terms of the adjacency spectrum are determined in this paper.

Keywords: Distance matrix, Vertex distance complement spectrum, Subdivision vertex join, Subdivision edge join.

1. Introduction

Spectral graph theory deals with the study of the eigenvalues of various matrices associated with graphs. Initially, the spectrum of the adjacency matrix of a graph was studied. Collatz and Sinogowitz initiated the exploration of this topic in 1957 [2]. Since then spectral theory of graphs is an active research area [1, 3].

In this paper, we consider the matrix derived from a type of distance matrix, viz., vertex distance complement (VDC) matrix. The VDC spectra of some classes of graphs are found in [8, 9]. The VDC matrix C of a graph G [7] is defined as follows

$$C = \begin{cases} n - d_{ij}, & i \neq j, \\ 0, & i = j, \end{cases}$$

where d_{ij} is the distance between the vertices v_i and v_j of G and n denotes the number of vertices of G .

The subdivision graph $S(G)$ of a graph G is obtained by inserting a new vertex of degree two in every edge of G . Let $V(G)$ and $I(G)$ denote respectively the existing vertex set and the set of the newly introduced vertices of the subdivision graph $S(G)$ of a graph G . The adjacency spectrum of two joins, $G_1 \dot{\vee} G_2$ and $G_1 \underline{\vee} G_2$, based on subdivision graph was determined in [4]. The distance spectrum of the same was calculated in [6].

Throughout this article we consider connected simple graphs of diameter at most two. We determine the VDC spectrum of $G_1 \dot{\vee} G_2$ and $G_1 \underline{\vee} G_2$ when G_1 and G_2 are regular graphs. The eigenvalues of $VDC(G)$ are called the *VDC-eigenvalues* of G and they form the *VDC spectrum* of G , denoted by $spec_{VDC}(G)$. We denote J and I as the all-one matrix and identity matrix, respectively, of appropriate orders.

The definitions of the subdivision graphs are as follows.

Definition 1 [4]. *The subdivision-vertex join $G_1 \dot{\vee} G_2$ of two vertex disjoint graphs G_1 and G_2 is the graph obtained from $S(G_1)$ and G_2 by joining each vertex of $V(G_1)$ with every vertex of $V(G_2)$.*

Definition 2 [4]. *The subdivision-edge join $G_1 \underline{\vee} G_2$ of two vertex disjoint graphs G_1 and G_2 is the graph obtained from $S(G_1)$ and G_2 by joining each vertex of $I(G_1)$ with every vertex of $V(G_2)$.*

The following results are very useful for computing the VDC spectrum.

Lemma 1 [3]. *Let G be an r -regular graph with adjacency matrix A and incidence matrix R . Let $A(L(G))$ denote the adjacency matrix of the line graph $L(G)$ of G . Then,*

$$RR^T = A + rI, \quad R^T R = A(L(G)) + 2I.$$

Also,

$$JR = 2J = R^T J, \quad JR^T = rJ = RJ.$$

Lemma 2 [3]. *Let G be r -regular $(n; m)$ graph with $spec(G) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$. Then*

$$spec(L(G)) = \begin{cases} 2r - 2, \\ \lambda_i + r - 2, & i = 2, 3, \dots, n, \\ -2, & m - n \text{ times.} \end{cases}$$

Also, Z is an eigenvector corresponding to the eigenvalue -2 if and only if $RZ = 0$ where R is the incidence matrix of G .

Theorem 1 (Perron–Frobenius). *If all entries of an $n \times n$ matrix are positive, then it has a unique maximal eigenvalue. Its eigenvector has positive entries.*

2. The VDC spectrum of $G_1 \dot{\vee} G_2$

Theorem 2. *Let G_i be an r_i regular graph with n_i vertices and m_i edges, for $i = 1, 2$. If $\{\lambda_{i1}, \lambda_{i2}, \dots, \lambda_{in_i}\}$ denotes the adjacency spectrum corresponding to the adjacency matrix A_i of G_i , the $spec_{VDC}(G_1 \dot{\vee} G_2)$ consists of*

$$(i) \quad 2\lambda_{1i} + 2r_1 - n + 2, \quad \text{for } i = 2, 3, \dots, n_1;$$

(ii) $-n$, repeated $m_1 - 1$ times;

(iii) $\lambda_{2i} - n + 2$, for $i = 2, 3, \dots, n_2$;

(iv) the 3 roots of the equation

$$\begin{aligned}
 & x^3 - (n_1n - 2n_1 + n_2n - 2n_2 + m_1n^2 - 4m_1 + 4r_1 + r_2 - 3n + 4)x^2 \\
 & - (2n_1n_2n - 3n_1n_2 + n_1m_1 - 2n_1r_1n + 2n_1r_1 - n_1r_2n + 2n_1r_2 + 2n_1n^2 \\
 & - 6n_1n + 4n_1 + 2n_2m_1n - 4n_2m_1 - 4n_2r_1n + 8n_2r_1 + 2n_2n^2 - 6n_2n + 4n_2 - m_1r_2n \\
 & + 4m_1r_2 + 2m_1n^2 - 8m_1n + 4m_1 - 4r_1r_2 + 8r_1n - 8r_1 + 2r_2n - 2r_2 - 3n^2 + 8n - 4)x \\
 & - (2n_1n_2m_1 - 4n_1n_2r_1n + 4n_1n_2r_1 + 2n_1n_2n^2 - 3n_1n_2n - n_1m_1r_2 \\
 & + n_1m_1n - 2n_1m_1 + 2n_1r_1r_2n - 2n_1r_1r_2 - 2n_1r_1n^2 + 6n_1r_1n - 4n_1r_1 - n_1r_2n^2 \\
 & + 2n_1r_2n + n_1n^3 - 4n_1n^2 + 4n_1n + 2n_2m_1n^2 - 8n_2m_1 - 4n_2r_1n^2 + 8n_2r_1n + n_2n^3 \\
 & - 4n_2n^2 + 4n_2n - m_1r_2n^2 + 2m_1r_2n + 4m_1r_2 + m_1n^3 - 4m_1n^2 + 8m_1 - 4r_1r_2n \\
 & + 4r_1n^2 - 8r_1n + r_2n^2 - 2r_2n - n^3 + 4n^2 - 4n) = 0,
 \end{aligned}$$

where $n = n_1 + m_1 + n_2$.

P r o o f. Given that G_1 and G_2 are regular graphs with regularity r_1 and r_2 respectively. Let R be the incidence matrix of G_1 and $A(L(G_1))$ be the adjacency matrix of the line graph of G_1 . The distance matrix of a graph with diameter at most two and adjacency matrix A can be rewritten as $A + 2\bar{A}$ or $2(J - I) - A$ [5].

The subdivision-vertex join $G_1 \dot{\vee} G_2$ has $n = n_1 + m_1 + n_2$ vertices. With the proper labeling of vertices, the VDC matrix of $G_1 \dot{\vee} G_2$ is a square matrix of order n given by

$$C = \begin{pmatrix} (n-2)(J-I) & (n-3)J+2R & (n-1)J \\ (n-3)J+2R^T & (n-4)(J-I)+2A(L(G_1)) & (n-2)J \\ (n-1)J & (n-2)J & (n-2)(J-I)+A_2 \end{pmatrix}.$$

Let X be an eigenvector corresponding to the eigenvalue $\lambda_{1i} \neq r_1$ of A_1 . Using Lemma 1, we note that

$$A(L(G_1))R^T X = (\lambda_{1i} + r_1 - 2)R^T X.$$

Hence, $\lambda_{1i} + r_1 - 2$ are the eigenvalues of $A(L(G_1))$ with an eigenvector $R^T X$.

By Perron–Frobenius theorem, X and $R^T X$ are orthogonal to the all-one vector J .

Let

$$\Upsilon = \begin{pmatrix} X \\ R^T X \\ 0 \end{pmatrix}.$$

Then,

$$2\lambda_{1i} + 2r_1 - n + 2, \quad i = 2, 3, \dots, n_1$$

is an eigenvalue of the VDC matrix of $G_1 \dot{\vee} G_2$ corresponding to the eigenvector Υ . This is because

$$\begin{aligned} & \begin{pmatrix} (n-2)(J-I) & (n-3)J+2R & (n-1)J \\ (n-3)J+2R^T & (n-4)(J-I)+2A(L(G)) & (n-2)J \\ (n-1)J & (n-2)J & (n-2)(J-I)+A_2 \end{pmatrix} \begin{pmatrix} X \\ R^T X \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -(n-2)X+2(A_1+r_1I)X \\ 2R^T X-(n-4)R^T X+2A(L(G_1))R^T X \\ 0 \end{pmatrix} = \begin{pmatrix} (2\lambda_{1i}+2r_1-n+2)X \\ (2\lambda_{1i}+2r_1-n+2)R^T X \\ 0 \end{pmatrix} \\ &= (2\lambda_{1i}+2r_1-n+2) \begin{pmatrix} X \\ R^T X \\ 0 \end{pmatrix}. \end{aligned}$$

By a similar reasoning, if Y is an eigenvector of $A(L(G_1))$ corresponding to the eigenvalue $\lambda_{1i}+r_1-2$, for $i=2,3,\dots,n_1$,

$$\Phi = \begin{pmatrix} RY \\ -Y \\ 0 \end{pmatrix}$$

is an eigenvector of VDC matrix of $G_1 \dot{\vee} G_2$ corresponding to the eigenvalue $-n$. (Note that the line graph of a regular graph is also regular).

Hence, $-n$ is an eigenvalue of $G_1 \dot{\vee} G_2$ repeated n_1-1 times.

Now, -2 is an eigenvalue of $A(L(G_1))$ with multiplicity m_1-n_1 . Let Z be an eigenvector of $A(L(G_1))$ corresponding to the eigenvalue -2 . Then, by Lemma 2, $RZ=0$ and by Perron-Frobenius theorem, $JZ=0$.

Let

$$\Omega = \begin{pmatrix} 0 \\ Z \\ 0 \end{pmatrix}.$$

Then $-n$ is an eigenvalue of the VDC matrix of $G_1 \dot{\vee} G_2$ repeated m_1-n_1 times with an eigenvector Ω . This is because

$$\begin{aligned} & \begin{pmatrix} (n-2)(J-I) & (n-3)J+2R & (n-1)J \\ (n-3)J+2R^T & (n-4)(J-I)+2A(L(G)) & (n-2)J \\ (n-1)J & (n-2)J & (n-2)(J-I)+A_2 \end{pmatrix} \begin{pmatrix} 0 \\ Z \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ -(n-4)Z+2A(L(G_1))Z \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -nZ \\ 0 \end{pmatrix}. \end{aligned}$$

In total, $-n$ is an eigenvalue of $G_1 \dot{\vee} G_2$ repeated m_1-1 times.

Now, let $\lambda_{2i} \neq r_2$ be an eigenvalue of G_2 with an eigenvector W . Since G_2 is regular, $JW=0$. Hence

$$\Psi = \begin{pmatrix} 0 \\ 0 \\ W \end{pmatrix}$$

is an eigenvector of the VDC matrix of $G_1 \dot{\vee} G_2$ corresponding to the eigenvalue $\lambda_{2i}-n+2$, for $i=2,3,\dots,n_2$. Thus, we have obtained $n_1+m_1+n_2-3$ eigenvalues.

The remaining three eigenvalues are to be determined. We note that all the eigenvectors constructed so far, are orthogonal to

$$\begin{pmatrix} J \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ J \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ J \end{pmatrix}.$$

The remaining three eigenvectors are spanned by these three vectors and is of the form

$$\Theta = \begin{pmatrix} \alpha J \\ \beta J \\ \gamma J \end{pmatrix}.$$

for some $(\alpha, \beta, \gamma) \neq (0, 0, 0)$.

Thus, if ρ is an eigenvalue of the VDC matrix with an eigenvector Θ , then from $C\Theta = \rho\Theta$, we can see that the remaining three eigenvalues are obtained from the matrix

$$\begin{pmatrix} (n-2)(n_1-1) & (n-3)m_1+2r_1 & (n-1)n_2 \\ (n-3)n_1+4 & n(m_1-1)-4(m_1-r_1) & (n-2)n_2 \\ (n-1)n_1 & (n-2)m_1 & (n-2)(n_2-1)+r_2 \end{pmatrix}.$$

Thus we determine the VDC spectrum of $G_1 \dot{\vee} G_2$. □

3. The VDC spectrum of $G_1 \underline{\vee} G_2$

In this section we present the VDC spectrum of $G_1 \underline{\vee} G_2$.

Theorem 3. *Let G_i be r_i regular graph with n_i vertices and m_i edges, for $i = 1, 2$. If $\{\lambda_{i1}, \lambda_{i2}, \dots, \lambda_{in_i}\}$ denotes the adjacency spectrum corresponding to the adjacency matrix A_i of G_i , then, the $\text{spec}_{VDC}(G_1 \underline{\vee} G_2)$ consists of*

- (i) $\lambda_{1i} + 3 \pm \sqrt{(\lambda_{1i} + 1)^2 + 4(\lambda_{1i} + r_1)} - n$, for $i = 2, 3, \dots, n_1$;
- (ii) $-n + 2$, repeated $m_1 - n_1$ times;
- (iii) $\lambda_{2i} - n + 2$, for $i = 2, 3, \dots, n_2$;
- (iv) the 3 roots of the equation

$$\begin{aligned} & x^3 - (n_1n - 4n_1 + n_2n - 2n_2 + m_1n - 2m_1 + 2r_1 + r_2 - 3n + 8)x^2 \\ & - (2n_1n_2n - 4n_1n_2 + n_1m_1 + 2n_1r_1n - 6n_1r_1 - n_1r_2n + 4n_1r_2 + 2n_1n^2 - 12n_1n + 16n_1 \\ & + 2n_2m_1n - 3n_2m_1 - 2n_2r_1n + 4n_2r_1 + 2n_2n^2 - 10n_2n + 12n_2 - 2m_1r_1n + 4m_1r_1 + 2m_1n^2 \\ & - 6m_1n - m_1r_2n - 2r_1r_2 + 4r_1n + 2m_1r_2 + 2r_2n - 6r_2 - 3n^2 + 16n - 20)x \\ & - (2n_1n_2m_1 + 4n_1n_2r_1n - 8n_1n_2r_1 + 8n_1n_2 + 2n_1r_1n^2 - 8n_1r_2 - 16n_1 \\ & - 4n_2m_1r_1n + 6n_2m_1r_1 - 3n_2m_1n - 16n_2 + 2m_1r_1r_2n - 4m_1r_1r_2 - 2m_1r_1n^2 \\ & + 8m_1r_1n - 8m_1r_1 - 4r_1r_2 + 2n_1n_2n^2 - 8n_1n_2n - n_1m_1r_2 + n_1m_1n - 2n_1m_1 \\ & - 2n_1r_1r_2n + 6n_1r_1r_2 + 2n_1r_1n^2 - 10n_1r_1n + 12n_1r_1 - n_1r_2n^2 + 6n_1r_2n \\ & + n_1n^3 - 8n_1n^2 + 20n_1n + 2n_2m_1n^2 - 4n_2m_1 - 2n_2r_1n^2 + 8n_2r_1 + n_2n^3 \\ & - 8n_2n^2 + 20n_2n - m_1r_2n^2 + 2m_1r_2n + 4m_1r_2 + m_1n^3 - 4m_1n^2 + 8m_1 - 2r_1r_2n \\ & + 2r_1n^2 - 8r_1 + r_2n^2 - 6r_2n - n^3 + 8n^2 - 20n + 8r_2 + 16) = 0. \end{aligned}$$

where $n = n_1 + m_1 + n_2$.

P r o o f. Given that G_1 and G_2 are regular graphs with regularity r_1 and r_2 respectively. Let R be the incidence matrix of G_1 . $G_1 \underline{\vee} G_2$ has $n = n_1 + m_1 + n_2$ vertices. With the proper labeling of vertices, the VDC matrix of $G_1 \underline{\vee} G_2$ of order n is given by

$$C = \begin{pmatrix} (n-4)(J-I) + 2A_1 & (n-3)J + 2R & (n-2)J \\ (n-3)J + 2R^T & (n-2)(J-I) & (n-1)J \\ (n-2)J & (n-1)J & (n-2)(J-I) + A_2 \end{pmatrix}.$$

Let $\lambda_{1i} \neq r_1$ be an eigenvalue of A_1 with an eigenvector X . By Perron–Frobenius theorem, X is orthogonal to the all-one vector J .

Let us test the condition under which

$$\Upsilon = \begin{pmatrix} tX \\ R^T X \\ 0 \end{pmatrix}$$

is an eigenvector of the given VDC matrix.

If Υ is an eigenvector of the VDC matrix of $G_1 \underline{\vee} G_2$ corresponding to the eigenvalue η , then $C\Upsilon = \eta\Upsilon$ implies

$$\begin{pmatrix} (n-4)(J-I) + 2A_1 & (n-3)J + 2R & (n-2)J \\ (n-3)J + 2R^T & (n-2)(J-I) & (n-1)J \\ (n-2)J & (n-1)J & (n-2)(J-I) + A_2 \end{pmatrix} \begin{pmatrix} tX \\ R^T X \\ 0 \end{pmatrix} = \eta \begin{pmatrix} tX \\ R^T X \\ 0 \end{pmatrix}$$

i. e.,

$$-(n-4)t + 2t\lambda_{1i} + 2\lambda_{1i} + 2r_1 = \eta t \tag{3.1}$$

and

$$2t - (n-2) = \eta. \tag{3.2}$$

Substituting the value of η from equation (3.2) in equation (3.1), we get a quadratic equation in t as

$$t^2 - (1 + \lambda_{1i})t - (\lambda_{1i} + r_1) = 0$$

Hence

$$t = \frac{(1 + \lambda_{1i}) \pm \sqrt{(1 + \lambda_{1i})^2 + 4(\lambda_{1i} + r_1)}}{2}.$$

Thus corresponding to each eigenvalue $\lambda_{1i} \neq r_1$ of A_1 , we get two VDC eigenvalues $\eta = 2t + 2 - n$ of $G_1 \underline{\vee} G_2$ and hence a total of $2(n_1 - 1)$ VDC eigenvalues are obtained.

Now, -2 is an eigenvalue of $A(L(G_1))$ with multiplicity $m_1 - n_1$. Let Z be an eigenvector of $A(L(G_1))$ with eigenvalue -2 . Then, by Lemma 2, $RZ = 0$.

However,

$$\Omega = \begin{pmatrix} 0 \\ Z \\ 0 \end{pmatrix}$$

is an eigenvector of the VDC matrix of $G_1 \underline{\vee} G_2$ corresponding to the eigenvalue $-n + 2$.

Let $\lambda_{2i} \neq r_2$ be an eigenvalue of G_2 with an eigenvector W . Then,

$$\Psi = \begin{pmatrix} 0 \\ 0 \\ W \end{pmatrix}$$

is an eigenvector of the VDC matrix of $G_1 \underline{\vee} G_2$ corresponding to the eigenvalue $\lambda_{2i} - n + 2$, for $i = 2, 3, \dots, n_2$.

Thus, we have obtained $n_1 + m_1 + n_2 - 3$ eigenvalues.

Next, we will determine the remaining three eigenvalues. We note that all the eigenvectors constructed are orthogonal to

$$\begin{pmatrix} J \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ J \\ 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ J \end{pmatrix}.$$

The remaining three eigenvectors are spanned by these three vectors and is of the form

$$\Theta = \begin{pmatrix} \alpha J \\ \beta J \\ \gamma J \end{pmatrix}$$

for some $(\alpha, \beta, \gamma) \neq (0, 0, 0)$. Thus, if ρ is an eigenvalue of C with an eigenvector Θ then from $C\Theta = \rho\Theta$, we can see that the remaining three eigenvalues are obtained from the matrix

$$\begin{pmatrix} (n-4)(n_1-1) + 2r_1 & (n-3)m_1 + 2r_1 & (n-2)n_2 \\ (n-3)n_1 + 4 & (n-2)(m_1-1) & (n-1)n_2 \\ (n-2)n_1 & (n-1)m_1 & (n-2)(n_2-1) + r_2 \end{pmatrix}.$$

□

4. Conclusion

In this paper we have computed the Vertex Distance Complement Spectrum of Subdivision Vertex Join, $G_1 \dot{\vee} G_2$, and Subdivision Edge Join, $G_1 \underline{\vee} G_2$ of regular graphs G_1 and G_2 . The work can be extended to graphs with diameter greater than two, graphs that are not regular etc. It is worth exploring the nature of the spectrum of graphs with arbitrary subdivisions.

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