PURSUIT-EVASION DIFFERENTIAL GAMES
WITH GRÖNWALL-TYPE CONSTRAINTS ON CONTROLS[1]

Bahrom T. Samatov

Namangan State Universiti,
316 Uychi Str., Namangan, 116019, Uzbekistan
samatov57@inbox.ru

Gafurjan Ibragimov

Universiti Putra Malaysia,
43400, UPM, Serdang, Selangor Darul Ehsan, Malaysia
ibragimov@upm.edu.my

Iroda V. Khodjibayeva

Namangan Engineering and Technology Institute,
7 Kosonsoy Str., Namangan, 160115, Uzbekistan
hodjira83@mail.ru

Abstract: A simple pursuit-evasion differential game of one pursuer and one evader is studied. The players' controls are subject to differential constraints in the form of the integral Grönwall inequality. The pursuit is considered completed if the state of the pursuer coincides with the state of the evader. The main goal of this work is to construct optimal strategies for the players and find the optimal pursuit time. A parallel approach strategy for Grönwall-type constraints is constructed and it is proved that it is the optimal strategy of the pursuer. In addition, the optimal strategy of the evader is constructed and the optimal pursuit time is obtained. The concept of a parallel pursuit strategy (Π-strategy for short) was introduced and used to solve the quality problem for "life-line" games by L.A. Petrosjan. This work develops and expands the works of Isaacs, Petrosjan, Pshenichnyi, and other researchers, including the authors.

Keywords: Differential game, Grönwall's inequality, Geometric constraint, Pursuit, Evasion, Optimal strategy, Domain of attainability, Life-line.

Introduction

According to the fundamental approaches in the theory of differential games developed by Pontryagin [27] and Krasovskii [22], a differential game is considered as a control problem from the point of view of either the pursuer or the evader. From this point of view, the game reduces either to the problem of pursuit (approach) or to the problem of evasion (escape). In this paper, we mainly focus on the pursuit problem.

The concept of "Differential Games" was initiated by Isaacs [20]. Differential games have been the object of research since 1960, and fundamental results were obtained by Pontryagin [27], Krasovskii [22], Bercovitz [4], Elliot and Kalton [9], Isaacs [20], Fleming [10], Friedman [11],

Hajek [14], Ho, Bryson, and Baron [15], Petrosjan [26], Pshenichnyi [28, 29], Subbotin [38, 39], Ushakov [41], Chikrii [7], and others.

The book of Isaacs [20] contains specific game problems that were discussed in detail and proposed for further study. One of these problems is the so-called life-line problem that was initially formulated and studied for certain special cases in [20, Problem 9.5.1]. For the case when controls of both players are subject to geometric constraints, this game has been rather comprehensively studied in the works of Petrosjan [26] based on approximating measurable controls with the most efficient piecewise constant controls that realize the parallel approach strategy. Later this approach to control in differential pursuit games was termed the Π-strategy. The strategy proposed [26] in a simple pursuit game with geometric constraints became the starting point for the development of the pursuit method in games with multiple pursuers (see, e.g., [3, 5, 12, 30–34]). Differential games where both players have admissible controls satisfying integral constraints have also been considered in several works, e.g., in [3, 32, 36, 41], although this treatment has been less comprehensive than for games with geometric constraints [3, 5, 7, 12, 30]. Also, in [35], the intercept problem was studied, when objects move in the dynamic flow field.

The constructing of optimal strategies of the players and finding the value of the game are difficult and important problems of differential games. Note that in [16–19, 21, 25, 37, 40], simple-motion differential games were studied and the existence of the value of the game was proved by constructing optimal strategies of the players.

In the theory of differential games, control functions are mainly subject to geometric, integral, or mixed constraints [8, 23]. However, differential type constraints on controls also arise in some applied problems such as ecological and technical problems [1, 24].

The present paper is also devoted to a simple pursuit-evasion differential game problem. We propose Grönwall-type constraints on the players' controls [13] for the pursuit-evasion differential game. We find the optimal pursuit time and construct optimal strategies for the players.

1. Statement of the problem

There is a huge number of works where simple-motion differential games with geometric constraints on controls of the form

$$|u| \leq \rho, \quad |v| \leq \sigma \tag{1.1}$$

were studied. The first constraint in (1.1) means that any control function $u(t)$, $t \geq 0$, satisfies the condition

$$\|u(\cdot)\|_\infty = \operatorname*{ess\,sup}_{t \geq 0} |u(t)| \leq \rho. \tag{1.2}$$

In the present paper, we propose a new set of controls of the pursuer and evader described by the following Grönwall-type constraints, respectively:

$$|u(t)|^2 \leq \rho^2 + 2k \int_0^t |u(s)|^2 ds, \quad t \geq 0, \tag{1.3}$$

and

$$|v(t)|^2 \leq \sigma^2 + 2k \int_0^t |v(s)|^2 ds, \quad t \geq 0, \tag{1.4}$$

where ρ and σ are given positive numbers and k is a given non-negative number.

Let the dynamics of the pursuer x and the evader y be described by the following equations:

$$\dot{x} = u, \quad x(0) = x_0,$$
$$\dot{y} = v, \quad y(0) = y_0,$$

(1.5)

where $x, y, x_0, y_0, u, v \in \mathbb{R}^n$, $n \geq 1$, and $x_0 \neq y_0$.

Definition 1. *Functions $u(\cdot) = \big(u_1(\cdot), u_2(\cdot), \ldots, u_n(\cdot)\big)$ and $v(\cdot) = \big(v_1(\cdot), v_2(\cdot), \ldots, v_n(\cdot)\big)$ satisfying conditions* (1.3) *and* (1.4) *are called the controls of the pursuer and evader, respectively.*

Denote by $\mathbb{U}$ and $\mathbb{V}$ the sets of all controls of the pursuer and evader, respectively. Pairs $(x_0, u(\cdot))$, $u(\cdot) \in \mathbb{U}$, and $(y_0, v(\cdot))$, $v(\cdot) \in \mathbb{V}$, generate the following trajectories:

$$x(t) = x_0 + \int_0^t u(s)ds, \quad y(t) = y_0 + \int_0^t v(s)ds$$

of the pursuer and evader, respectively.

We use the following statement.

Lemma 1 (Grönwall [13]). *If*

$$|\omega(t)|^2 \leq \alpha^2 + 2k \int_0^t |\omega(s)|^2 ds,$$

then $|\omega(t)| \leq \alpha e^{kt}$, where $\omega(t)$, $t \geq 0$, is a measurable function and α and k are non-negative numbers.

By Lemma 1, if $u(\cdot) \in \mathbb{U}$ and $v(\cdot) \in \mathbb{V}$, then

$$|u(t)| \leq \rho e^{kt}, \quad |v(t)| \leq \sigma e^{kt}, \quad t \geq 0.$$

(1.6)

It can be easily checked that the converse is not true, that is, inequalities (1.6) do not imply inequalities (1.3) and (1.4). To define the notions of optimal strategies of the players and the optimal pursuit time, we consider two games.

1.1. The minimax payoff of the game

Denote by $B(x, r)$ the ball of radius r centered at a point x.

Definition 2. *A continuous function*

$$U(x_0, y_0, t, v), \quad U : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times B(O, \sigma e^{kt}) \to B(O, \rho e^{kt}),$$

where O stands for the origin, is called a strategy of the pursuer.

Hence, at the current time t, the pursuer is allowed to know the initial states x_0, y_0, the current time t, and the value of the evader's control $v(t)$.

Definition 3. *We say that a strategy $U = U(x_0, y_0, t, v)$ guarantees the completion of the pursuit by time $T(U)$ if, for any control of the evader $v(t)$, $t \geq 0$, we have $x(\tau) = y(\tau)$ at some time $\tau \in [0, T(U)]$, where $(x(\cdot), y(\cdot))$ is the solution of the initial value problem*

$$\dot{x} = U(x_0, y_0, t, v(t)), \quad x(0) = x_0,$$
$$\dot{y} = v, \quad y(0) = y_0.$$

We say that $T(U)$ is a guaranteed pursuit time. Note that any number T', $T' \geq T(U)$, is also a guaranteed pursuit time corresponding to the strategy U. Denote by $T^*(U)$ the exact lower bound of the guaranteed pursuit times $T(U)$ corresponding to the strategy U.

The pursuer tries to minimize the number $T^*(U)$ by choosing their strategy U while the evader tries to maximize $T^*(U)$ by choosing their control $v(\cdot)$.

Definition 4. *A strategy U_0 is called an optimal strategy of the pursuer if $T^*(U) \geq T^*(U_0)$ for any strategy U of the pursuer. The number $T^*(U_0)$ is called the minimax payoff of the game.*

1.2. The maximin payoff of the game

Definition 5. *A continuous function*

$$V(x_0, y_0, t, x, y), \quad V : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to B(O, \sigma e^{kt}),$$

is called a strategy of the evader if the following initial value problem

$$
\begin{aligned}
\dot{x} &= u, & x(0) &= x_0, \\
\dot{y} &= V(x_0, y_0, t, x, y), & y(0) &= y_0,
\end{aligned}
\tag{1.7}
$$

has a unique solution $(x(t), y(t))$, $t \geq 0$.

Definition 6. *We say that a strategy V guarantees the evasion on the time interval $[0, T(V))$ if, for any control $u(t)$ of the pursuer, $t \geq 0$, the condition $x(t) \neq y(t)$ holds for all $t \in [0, T(V))$, where $(x(t), y(t))$ is the solution of (1.7). The number $T(V)$ is called a guaranteed evasion time.*

Denote by $T_*(V)$ the exact upper bound of numbers $T(V)$ corresponding to the strategy V. The evader tries to maximize $T_*(V)$ by choosing their strategy V while the pursuer tries to minimize it by choosing their control $u(\cdot)$. If $T_*(V) = \infty$, we say that the evasion is possible.

Definition 7. *A strategy V_0 of the evader is called optimal if the inequality $T_*(V) \leq T_*(V_0)$ holds for any strategy V of the evader. The number $T_*(V_0)$ is called the maximin payoff of the game. If $T^*(U_0) = T_*(V_0)$, then this number is called the optimal pursuit time.*

This paper is devoted to solving the following problems under Grönwall-type constraints on the controls.

Problem 1. *Construct optimal strategies of the pursuer and evader, and find the optimal pursuit time in the game.*

Problem 2. *Solve a "life-line" differential game.*

2.   The main result

In this section, we construct optimal strategies for the players and give a formula for the optimal pursuit time.

2.1. Construction of the Π_{Gr}-strategy

To construct a strategy for the pursuer, we first assume that the pursuer knows t, $x(t)$, $y(t)$, and $v(t)$ at the current time t. After constructing the strategy, we abandon the information about the current players' positions $x(t)$ and $y(t)$.

Let $x(t) \neq y(t)$, $\xi = \xi(t) = z(t)/|z(t)|$, and $z(t) = x(t) - y(t)$. Based on the classical method for deriving a Π-strategy (see, for example, [2, 20, 26, 28]), we assume that, for a constant vector $v \in \mathbb{R}^n$, the velocity $u \in \mathbb{R}^n$ is chosen so that the following relations hold:

$$u = v - \lambda\xi, \tag{2.8}$$

$$|u|^2 = |v|^2 + \delta e^{2kt}, \tag{2.9}$$

where λ is a non-negative parameter and $\delta = \rho^2 - \sigma^2$. Substituting (2.8) into (2.9), we obtain the following equation for λ:

$$\lambda^2 - 2\lambda\langle v, \xi\rangle - \delta e^{2kt} = 0,$$

where $\langle v, \xi\rangle$ denotes the inner product of vectors v and ξ in $\mathbb{R}^n$. To construct the strategy of the pursuer, we use the following root:

$$\lambda(t, v, z) = \langle v, \xi\rangle + \sqrt{\langle v, \xi\rangle^2 + \delta e^{2kt}}. \tag{2.10}$$

Note that $\lambda(t, v, z)$ is not necessarily positive for all v and z. We call the root (2.10) the resolving function (see [7],[29]) and present some of its important properties.

Property 1. *If $\delta \geq 0$, then the function $\lambda(t, v, z)$ is continuous and non-negative for all $(t, v, z) \in [0, \infty) \times \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$.*

Now, substituting the resolving function (2.10) into (2.8), we obtain

$$\mathbf{u}(t, v, z) = v - \lambda(t, v, z)\xi \tag{2.11}$$

that satisfies (2.9). Let $z_0 = x_0 - y_0$, and let $v(\cdot) \in \mathbb{V}$ be an arbitrary control of the evader. If the pursuer applies strategy (2.11), then, by (1.5) and (2.11), the dynamics of the vector z is described by the following initial value problem:

$$\dot{z} = \dot{x} - \dot{y} = -\lambda(t, v(t), z)\frac{z}{|z|}, \quad z(0) = z_0. \tag{2.12}$$

Obviously, for the initial value problem (2.12), the hypotheses of the Caratheodory existence theorem are satisfied if $z \neq 0$, and therefore it has a unique absolutely continuous solution $(t, z(t))$, which starts from the point $(0, z_0)$ since $z_0 \neq 0$. The following statement justifies the term of "parallel approach" for the strategy (2.11).

Lemma 2. *For every z_0, $z_0 \neq 0$, and $v(\cdot) \in \mathbb{V}$, there exists a scalar function $\Lambda(\cdot)$ such that $z(t) = z_0\Lambda(t, v(\cdot), z(\cdot))$.*

P r o o f. We obtain from (2.12) that

$$\dot{z}_i = -\frac{\lambda(t, v(t), z)}{|z|}z_i, \quad z_i(0) = z_{i0},$$

where $i = 1, 2, \ldots, n$ and z_i is a scalar coordinate of the vector $z \in \mathbb{R}^n$. Then the latter differential equation can be transformed to the form

$$z_i(t) = z_{i0}\Lambda(t, v(\cdot), z(\cdot)), \quad \Lambda(t, v(\cdot), z(\cdot)) = \exp\left\{-\int_0^t \frac{1}{|z(s)|}\lambda(s, v(s), z(s))ds\right\}.$$

and the proof of Lemma 2 is complete.   □

Lemma 3. *If $\rho \geq \sigma$, then the following equation holds for every z_0, $z_0 \neq 0$ and $v(\cdot) \in \mathbb{V}$ on some time interval $[0, t^*)$:*

$$\boldsymbol{u}(t, v(t), z(t)) = \boldsymbol{u}(t, v(t), z_0). \tag{2.13}$$

P r o o f. The function $\lambda(t, v, z)$ defined by (2.10) is homogeneous in z. Therefore, $\mathbf{u}(t, v, z)$ is homogeneous in z. Hence, by Lemma 2, we obtain (2.13). This completes the proof of Lemma 3.□

By (2.13), the pursuer constructs their strategy based on the information about the current time t, the value $v(t)$, and the initial data z_0, ρ, σ, k.

Definition 8. *If $\rho \geq \sigma$, then the function*

$$\boldsymbol{u}_{Gr}(t, v) = v - \lambda_{Gr}(t, v)\xi_0, \quad \lambda_{Gr}(t, v) = \langle v, \xi_0 \rangle + \sqrt{\langle v, \xi_0 \rangle^2 + \delta e^{2kt}}, \tag{2.14}$$

where $\xi_0 = z_0/|z_0|$, is called the Π_{Gr}-strategy of the pursuer in the game.

Note that

$$|\mathbf{u}_{Gr}(t, v)|^2 = |v|^2 + \delta e^{2kt}. \tag{2.15}$$

2.2. Solution of the pursuit problem

Theorem 1. *If $\rho > \sigma$, then the Π_{Gr}-strategy guarantees the completion of the pursuit in the game on the time interval $[0, T_{Gr}]$, where*

$$T_{Gr} = \begin{cases} \dfrac{1}{k} \ln\left(1 + \dfrac{k|z_0|}{\rho - \sigma}\right), & k > 0, \\[3mm] \dfrac{|z_0|}{\rho - \sigma}, & k = 0. \end{cases}$$

P r o o f. Let $v(\cdot) \in \mathbb{V}$ be an arbitrary control of the evader, and let the pursuer use the Π_{Gr}-strategy. Use equations (1.5) and (2.14) to get the following initial value problem:

$$\dot{z} = \mathbf{u}_{Gr}(t, v(t)) - v(t) = -\lambda_{Gr}(t, v(t))\xi_0, \quad z(0) = z_0.$$

From this, we see that

$$z(t) = \Lambda_{Gr}(t, v(\cdot))z_0, \tag{2.16}$$

where

$$\Lambda_{Gr}(t, v(\cdot)) = 1 - \frac{1}{|z_0|}\int_0^t \lambda_{Gr}(s, v(s))ds.$$

We now study the behavior of the function $\Lambda_{Gr}(t, v(\cdot))$ with respect to t. Using the definition of the function $\lambda_{Gr}(t, v)$, we obtain

$$\Lambda_{Gr}(t, v(\cdot)) \leq 1 - \frac{1}{|z_0|}\int_0^t [\sqrt{\delta e^{2ks} + \langle v(s), \xi_0 \rangle^2} - |\langle v(s), \xi_0 \rangle|]ds.$$

The function $f(t,w) = \sqrt{\delta e^{2kt} + w^2} - w$, $w \in \mathbb{R}$, is monotonely deceasing for every $t \geq 0$. Hence, by the inequality $|\langle v(t), \xi_0 \rangle| \leq |v(t)| \leq \sigma e^{kt}$, which follows from the latter inequality in (1.6), we get

$$\Lambda_{Gr}(t, v(\cdot)) \leq 1 - \frac{1}{|z_0|} \int_0^t [\sqrt{\delta e^{2ks} + \sigma^2 e^{2ks}} - \sqrt{\sigma^2 e^{2ks}}]ds = \Phi_{Gr}(t),$$

where

$$\Phi_{Gr}(t) = \begin{cases} 1 - \dfrac{\rho - \sigma}{k|z_0|}\left(e^{kt} - 1\right), & k > 0, \\ 1 - \dfrac{(\rho - \sigma)t}{|z_0|}, & k = 0. \end{cases}$$

Clearly, the function $\Phi_{Gr}(t)$ is monotonely decreasing on $[0, T_{Gr}]$ and $\Phi_{Gr}(T_{Gr}) = 0$. Consequently, there exists a time t^*, $0 \leq t^* \leq T_{Gr}$, such that $\Lambda_{Gr}(t^*, v(\cdot)) = 0$, and hence, by (2.16), $z(t^*) = 0$.

Next, we prove the admissibility of strategy (2.14) for all t, $t \geq 0$. Let $v(\cdot) \in \mathbb{V}$ be an arbitrary control of the evader. We obtain from (1.4) and (2.15) that

$$|\mathbf{u}_{Gr}(t, v(t))|^2 = |v(t)|^2 + \delta e^{2kt} \leq \sigma^2 + \delta e^{2kt} + 2k \int_0^t |v(s)|^2 ds$$

$$= \rho^2 + 2k \int_0^t \left(|v(s)|^2 + \delta e^{2ks}\right) ds = \rho^2 + 2k \int_0^t |\mathbf{u}_{Gr}(s, v(s))|^2 ds,$$

and this completes the proof. □

Theorem 2. *If $\rho > \sigma$, then, for any control of the pursuer, the evader's strategy $V(t) = -\sigma e^{kt}\xi_0$, $t \geq 0$, guarantees the inequality $x(t) \neq y(t)$ on the time interval $[0, T_{Gr})$.*

P r o o f. Let $0 \leq t < T_{Gr}$. Then

$$\langle x(t) - y(t), \xi_0 \rangle = |y_0 - x_0| - \int_0^t \langle v(s), \xi_0 \rangle ds + \int_0^t \langle u(s), \xi_0 \rangle ds$$

$$\geq |y_0 - x_0| + \sigma \int_0^t e^{ks} ds - \rho \int_0^t e^{ks} ds > 0.$$

Hence, $x(t) \neq y(t)$, $0 \leq t < T_{Gr}$. This completes the proof. □

Theorems 1 and 2 allows us to conclude that T_{Gr} is the optimal pursuit time, the Π_{Gr}-strategy is an optimal strategy for the pursuer, and $V(t) = -\sigma e^{kt}\xi_0$ is an optimal strategy for the evader.

2.3. Solution of the evasion problem

We now consider the game from the evader's point of view.

Theorem 3. *If $\rho \leq \sigma$, then the evasion is possible in the game.*

P r o o f. Let $\rho \leq \sigma$ and $u(\cdot) \in \mathbb{U}$. We suggest the evader to use the strategy $V(t) = -\sigma e^{kt}\xi_0$, $t \geq 0$. Obviously, $V(\cdot) \in \mathbb{V}$. Then, for any $u(t)$, we obtain

$$|z(t)| \geq |z_0 - \int_0^t V(s)ds| - \int_0^t |u(s)|ds = |z_0| + \int_0^t \sigma e^{ks}ds - \int_0^t |u(s)|ds.$$

Using the inequality $|u(s)| \leq \rho e^{kt}$, we obtain

$$|z(t)| \geq \begin{cases} |z_0| + (\sigma - \rho)(e^{kt} - 1)/k, & k > 0, \\ |z_0| + (\sigma - \rho)t, & k = 0. \end{cases}$$

This implies that $z(t) \neq 0$, $t \geq 0$. The proof of the theorem is complete. $\square$

2.4. Life-line differential game

The book of R. Isaacs [20] contains specific game problems, which are discussed in detail and proposed for further study. Among numerous examples considered in the book, the life-line differential game (Problem 9.5.1) occupies a special place as an example of a differential game with phase constraint. For the case when the controls of both the players are subject to geometric constraints, this game has been rather comprehensively studied in the works of L.A. Petrosjan [26] based on approximating measurable controls with the most efficient piecewise constant controls that realize the parallel approach strategy. About further development see [3, 5, 12, 30–34].

Here we mainly study the game with phase constraints for the evader on a given subset M of $\mathbf{R}^n$, which is called the life line (of the evader). (Note that, in the case $M = \emptyset$, we have a simple game.)

In the life-line differential game, the pursuer P aims to catch the evader E, i.e., to realize the equality $x(t) = y(t)$ for some $t > 0$, while E stays in the zone $\mathbf{R}^n \setminus M$. The aim of E is to reach the zone M before the pursuer catches him or to keep the relation $x(t) \neq y(t)$ for all t ($t \geq 0$). Note that M doesn't restrict the motion of P. Further, we assume that initial positions x_0 and y_0 are given such that $x_0 \neq y_0$ and $y_0 \notin M$.

Definition 9. *A strategy* $\mathbf{u}_{Gr}(v, t)$ *of the player* P *is called* winning *on the interval* $[0, T_{Gr}]$ *in the lifeline game if, for every* $v(\cdot) \in \mathbb{V}$, *there exists some time* $t^* \in [0, T_{Gr}]$ *such that*

(1) $x(t^*) = y(t^*)$;

(2) $y(t) \notin M$ *for* $t \in [0, t^*]$.

Definition 10. *A control function* $v^*(\cdot) \in \mathbb{V}$ *of the player* E *is called* winning *in the life-line game if, for every* $u(\cdot) \in \mathbb{U}$,

(1) *there exists some time* $\overline{t}$ ($\overline{t} > 0$) *such that* $y(\overline{t}) \in M$ *and* $x(t) \neq y(t)$ *for* $t \in [0, \overline{t})$; *or*

(2) $x(t) \neq y(t)$ *for all* $t \geq 0$.

2.5. Dynamics of the attainability domain

Let conditions of Theorem 1 hold. We suppose that, at time t, $t \geq 0$, the evader E moves from a position y using the control vector

$$v(t) = \frac{w - y}{|w - y|} \sigma e^{kt}.$$

The pursuer P uses the strategy

$$\mathbf{u}_{Gr}(t, v(t)) = \frac{w - x}{|w - x|} \rho e^{kt}$$

from a position x. Then w is a point where P should meet E and

$$|w - y| = \int\limits_t^\theta |v(s)| ds, \quad |w - x| = \int\limits_t^\theta |\mathbf{u}_{Gr}(s, v(s))| ds \Rightarrow |w - x|/\rho = |w - y|/\sigma,$$

where θ is time when $x(\theta) = y(\theta) = w$. We define the attainability domain for the evader E in the following form:

$$A_{Gr}(x, y) = \big\{ w : \ |w - x| \geq (\rho/\sigma)|w - y| \big\};$$

its boundary is know as *Apollonius' sphere*. Writing the latter in the form $|w - c_{Gr}| = R_{Gr}$, one can easily find the center $c_{Gr}(x, y)$ and the radius of *Apollonius' sphere*:

$$c_{Gr}(x, y) = (\rho^2 y - \sigma^2 x)/(\rho^2 - \sigma^2),$$

$$R_{Gr}(x, y) = \rho\sigma|x - y|/|\rho^2 - \sigma^2|.$$

The pairs $(x_0, u_{Gr}(t, v(t))$ and $(y_0, v(t))$ generate the trajectories

$$x(t) = x_0 + \int\limits_0^t \mathbf{u}_{Gr}(s, v(s)) ds, \quad y(t) = y_0 + \int\limits_0^t v(s) ds,$$

respectively. Then, for every $(x(t), y(t))$, $t \in [0, \theta]$, we construct the sets

$$A_{Gr}(t) = A_{Gr}(x(t), y(t)) = \big\{ w : \ |w - x(t)| \geq (\rho/\sigma)|w - y(t)| \big\},$$

$$A_{Gr}(0) = A_{Gr}(x_0, y_0) = \big\{ w : \ |w - x_0| \geq (\rho/\sigma)|w - y_0| \big\}.$$

Theorem 4.

$$A_{Gr}(t) = x(t) + \Lambda_{Gr}(t)[A_{Gr}(0) - x_0]$$

for $t \in [0, \theta]$, where $\theta = \min\{t : z(t) = 0\}$.

P r o o f. Since $z(t) = \Lambda_{Gr}(t)z_0$, where $\Lambda_{Gr}(t) = \Lambda_{Gr}(t, v_t(\cdot))$ (see (2.16)), the relation $w \in A_{Gr}(t) - x(t)$ is equivalent to

$$|w| \geq (\rho/\sigma)\big|w + \Lambda_G(t)z_0\big|. \tag{2.17}$$

Obviously, it is sufficient to check (2.17) for $t \in [0, \theta)$ when $\Lambda_{Gr}(t) > 0$. Then (2.17) can be written as

$$|\Lambda_{Gr}^{-1}(t))w| \geq (\rho/\sigma)|\Lambda_{Gr}^{-1}(t)w + z_0|$$

or

$$\Lambda_{Gr}^{-1}(t)w \in A_{Gr}(0) - x_0.$$

The latter means that $w \in \Lambda_{Gr}(t)[A_{Gr}(0) - x_0]$. Thus, we have the equivalence

$$A_{Gr}(t) - x(t) = \left\{ w : \ |w| \geq (\rho/\sigma)|w + \Lambda_{Gr}(t)z_0| \right\} = \Lambda_{Gr}(t)[A_{Gr}(0) - x_0],$$

hence the desired result follows. $\qquad\qquad\square$

Theorem 5. *Monotony of Apollonius' sphere. The set $A_{Gr}(t)$ is monotone with respect to the inclusion for $t \in [0, \theta]$, i.e., if $0 \leq t_1 \leq t_2$, then $A_{Gr}(t_1) \supset A_{Gr}(t_2)$.*

P r o o f. By the properties (1.6) and (2.14)–(2.15), we have

$$|\mathbf{u}_{Gr}(t, v)|^2 = |v|^2 + \delta e^{2kt} \geq (\rho/\sigma)^2 |v|^2 \Rightarrow |v - \lambda_{Gr}(t, v)\xi_0| \geq (\rho/\sigma)|v|$$

or

$$\left| |z_0|v - \lambda_{Gr}(t, v)z_0 \right| \geq (\rho/\sigma)|v||z_0| \Rightarrow \left| w - \lambda_{Gr}(t, v)x_0 \right| \geq (\rho/\sigma)\left| w - \lambda_{Gr}(t, v)y_0 \right|,$$

where $w = |z_0|v + \lambda_{Gr}(t, v)y_0$. The latter relation is equivalent to

$$|z_0|v + \lambda_{Gr}(t, v)y_0 \in \lambda_{Gr}(t, v)A_{Gr}(0).$$

From this, the convexity $A_{Gr}(0)$, and the properties of the support function (see [6])

$$F(A, \psi) = \sup_{w \in A} \langle w, \psi \rangle,$$

we get

$$\langle |z_0|v, \psi \rangle - \lambda_{Gr}(t, v)F(A_{Gr}(0) - y_0, \psi) \leq 0$$

for all $\psi, |\psi| = 1$. Consequently,

$$\left\langle v - \lambda_{Gr}(t, v)\xi_0, \psi \right\rangle - \frac{1}{|z_0|}\lambda_{Gr}(t, v)F(A_{Gr}(0) - x_0, \psi) = \frac{d}{dt}F(A_{Gr}(t), \psi) \leq 0.$$

$\qquad\qquad\square$

2.6. Solution of the life-line game

In the life-line game, the pursuer P aims to catch the evader E, i.e., to realize the equality $x(t) = y(t)$ for some $t > 0$, while E stays in the zone $\mathbf{R}^n \setminus M$. The aim of E is to reach the zone M before the pursuer catches him or to keep the relation $x(t) \neq y(t)$ for all $t, t \geq 0$. Note that M doesn't restrict the motion of P.

Theorem 6. *If $\rho > \sigma$ and $M \bigcap A_{Gr}(x_0, y_0) = \emptyset$, then the Π_{Gr}-strategy is winning.*

P r o o f follows from Theorem 5. $\qquad\qquad\square$

Theorem 7. *If $\rho > \sigma$ and $M \bigcap A_{Gr}(x_0, y_0) \neq \emptyset$, then there exists a control of the evader E, which is winning.*

P r o o f. Let $w \in M \bigcap A_G(x_0, y_0)$, and let E hold the control $v^*(t) = \sigma e^{kt} \nu$, $v^*(\cdot) \in \mathbb{V}$, where $\nu = (w - y_0)/|w - y_0|$. Then the time of reaching by the evader the point w is $\bar{\theta}$, and we have

$$\int_0^{\bar{\theta}} |v^*(s)| ds = |w - y_0| \Rightarrow \varphi(\bar{\theta}) := (e^{k\bar{\theta}} - 1)/k = |w - y_0|/\sigma, \tag{2.18}$$

where $\varphi(t) = (e^{kt} - 1)/k$ increases in t. We suppose that there exists a certain control function $u^*(\cdot) \in \mathbb{U}$ of the pursuer such that $x(\bar{t}) = y(\bar{t})$ and $\bar{t} < \bar{\theta}$ or $\varphi(\bar{t}) < \varphi(\bar{\theta})$. If $z(t) = x(t) - y(t)$ and $z(0) = z_0$, then, from (1.5), we get

$$z(\bar{t}) = z_0 + \int_0^{\bar{t}} (u^*(s) - v^*(t)) ds = 0.$$

It follows that

$$|z_0 - \int_0^{\bar{t}} v^*(t) ds| \leq \int_0^{\bar{t}} |u^*(s)| ds \leq \rho \varphi(\bar{t}) \Rightarrow (\rho^2 - \sigma^2)\varphi^2(\bar{t}) + 2\sigma \langle z_0, \nu \rangle \varphi(\bar{t}) - |z_0|^2 \geq 0.$$

Hence, we get

$$\varphi(\bar{t}) \geq (\sqrt{\sigma^2 \langle z_0, \nu \rangle^2 + |z_0|^2 (\rho^2 - \sigma^2)} - \sigma \langle z_0, \nu \rangle)/(\rho^2 - \sigma^2). \tag{2.19}$$

Since $w \in A_G(x_0, y_0)$, we have

$$|w - x_0| \geq (\rho/\sigma)|w - y_0| \Rightarrow |z_0 - (w - y_0)|^2 \geq (\rho/\sigma)^2 |w - y_0|^2 \Rightarrow$$

$$|z_0|^2 - 2\langle z_0, w - y_0 \rangle + |w - y_0|^2 \geq (\rho/\sigma)^2 |w - y_0|^2 \Rightarrow$$

$$|z_0|^2 \geq \frac{|w - y_0|^2}{\sigma^2}(\rho^2 - \sigma^2) + 2|w - y_0|\langle z_0, \nu \rangle \Rightarrow$$

$$\sigma^2 \langle z_0, \nu \rangle^2 + |z_0|^2(\rho^2 - \sigma^2) \geq \frac{|w - y_0|^2}{\sigma^2}(\rho^2 - \sigma^2)^2 + 2|w - y_0|(\rho^2 - \sigma^2)\langle z_0, \nu \rangle + \sigma^2 \langle z_0, \nu \rangle^2 \Rightarrow$$

$$\sigma^2 \langle z_0, \nu \rangle^2 + |z_0|^2(\rho^2 - \sigma^2) \geq \left[|w - y_0|(\rho^2 - \sigma^2)/\sigma + \sigma \langle z_0, \nu \rangle\right]^2 \Rightarrow$$

$$\sqrt{\sigma^2 \langle z_0, \nu \rangle^2 + |z_0|^2(\rho^2 - \sigma^2)} \geq |w - y_0|(\rho^2 - \sigma^2)/\sigma + \sigma \langle z_0, \nu \rangle \Rightarrow$$

$$\left(\sqrt{\sigma^2 \langle z_0, \nu \rangle^2 + |z_0|^2(\rho^2 - \sigma^2)} - \sigma \langle z_0, \nu \rangle\right)/(\rho^2 - \sigma^2) \geq |w - y_0|/\sigma = \varphi(\bar{\theta}).$$

Then, from (2.18)–(2.19), we get $\varphi(\bar{t}) \geq \varphi(\bar{\theta})$ or $\bar{t} \geq \bar{\theta}$, which contradict our assumption.   □

Theorem 8. *If $\sigma \geq \rho$, then there exists a control of the evader E, which is winning in the life-line game.*

P r o o f follows from Theorem 3.   □

3. Conclusion

In the present paper, we have studied a simple pursuit-evasion differential game of one pursuer and one evader. We have proposed Grönwall-type constraints on the players' controls and constructed the Π_{Gr}-strategy for the pursuer. We have shown that the Π_{Gr}-strategy is an optimal strategy for the pursuer. Also, we have constructed an optimal strategy for the evader and found the optimal pursuit time. The results obtained show that the optimal strategies U and V of the players satisfy the conditions $|U| = \rho e^{kt}$ and $|V| = \sigma e^{kt}$, respectively. For the completeness of the results, we have also studied an evasion life-line game.

There is a large scope for further investigations. For example, differential games of many players with Grönwall-type constraints on the players' controls can be studied.

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