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SOME NOTES ABOUT THE MARTINGALE REPRESENTATION THEOREM AND THEIR APPLICATIONS

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Abstract: An important theorem in stochastic finance field is the martingale representation theorem. It is useful in the stage of making hedging strategies (such as cross hedging and replicating hedge) in the presence of different assets with different stochastic dynamics models. In the current paper, some new theoretical results about this theorem including derivation of serial correlation function of a martingale process and its conditional expectations approximation are proposed. Applications in optimal hedge ratio and financial derivative pricing are presented and sensitivity analyses are studied. Throughout theoretical results, simulation-based results are also proposed. Two real data sets are analyzed and concluding remarks are given. Finally, a conclusion section is given.

Keywords: Conditional expectation, Derivative pricing, Martingale representation theorem, Optimal hedge ratio, Sensitivity analysis, Serial correlation, Simulation, Stochastic dynamic.

1. Introduction

The martingale representation theorem states that any martingale adapted with respect to a Brownian motion can be expressed as a stochastic integral with respect to the same Brownian motion. It has many applications in construction hedging strategies for various types of assets with different stochastic dynamics, see [1]. In the current note, the time series features of this important theorem are proposed. Before going future, an important lemma is proposed. Let B_t be the standard Brownian motion on $(0,\infty)$ and $\mathfrak{F}_t$ is the sigma-field constructed by history of B_s, $s \leq t$, i.e. $\mathfrak{F}_t = \sigma\{B_s | s \leq t\}$. Hence, if $s \leq t$; then $\mathfrak{F}_s \subseteq \mathfrak{F}_t$. Indeed, $\mathfrak{F}_t$ is the augmented filtration generated by standard Brownian motion B_t. Also, assume that X, Y are the future values of two stochastic processes at some known future time T. According to the martingale representation theorem, it is necessary to assume that both of X, Y are squared integrable random variables with respect to $\mathfrak{F}_\infty$ to use this theorem for X and Y (see, [1]). These assumptions are kept fixed for all further discussions of the paper.

Lemma 1. *Sentences* (a) *and* (b) *are correct*:

(a) *The correlation* ρ_{xy} *between* X, Y, *is given as follows*

$$\rho_{xy} = \frac{\int_0^T E(u_s v_s) ds}{\sqrt{\int_0^T E(u_s^2) ds \int_0^T E(v_s^2) ds}},$$

here u_s *and* v_s *are two predictable processes used in martingale representation theorem applied to* X, Y, *respectively.*

(*b*) *Suppose that* $E(X|Y=y)=ay+b$, $E(Y|X=x)=cx+d$. *Then,*

$$\begin{cases} \mu_x = a\mu_y + b, & \mu_y = c\mu_x + d, \\ \rho_{xy}^2 = ac, & \dfrac{\sigma_x^2}{\sigma_y^2} = \dfrac{a}{c}, \end{cases}$$

where μ_A *and* σ_A^2 *are the mean and variance of* $A = X, Y$, *respectively.*

P r o o f. (*a*) The martingale representation theorem implies that (see [2]) there exist two predictable processes u_t, v_t such that

$$\begin{cases} X = E(X) + \displaystyle\int_0^T u_s dB_s, \\ Y = E(Y) + \displaystyle\int_0^T v_s dB_s. \end{cases}$$

For a review in stochastic calculus, see [7]. It is easy to see that

$$E\Big[\int_0^T u_s dB_s\Big] = E\Big[\int_0^T v_s dB_s\Big] = 0.$$

By multiplying above two equations and taking expectation, it is seen that

$$E(XY) = E(X)E(Y) + E(X)E\Big[\int_0^T v_s dB_s\Big] + E(Y)E\Big[\int_0^T u_s dB_s\Big] + E\int_0^T\int_0^T u_s v_t dB_s dB_t.$$

Notice that [7]

$$E\int_0^T\int_0^T u_s v_t dB_s dB_t = \int_0^T E(u_s v_s)\, ds.$$

Hence, it is seen that covariance between X and Y, i.e., $\sigma_{xy} = \operatorname{cov}(X,Y)$ is given by

$$\sigma_{xy} = \int_0^T E(u_s v_s)\, ds.$$

Also, using the Ito isometric lemma (see [7]), it is seen that

$$E(X^2) = (E(X))^2 + \int_0^T E(u_s^2)\, ds.$$

Therefore,

$$\sigma_x^2 = \operatorname{var} X = \int_0^T E(u_s^2)\, ds.$$

Similarly,

$$\sigma_y^2 = \int_0^T E(v_s^2)\, ds.$$

Thus, the proof is complete.

(*b*) The proof is straightforward by using the iterated expectation law (see [2]). Therefore it is omitted. □

Example 1. Here, to give an example for assumption of part (b), a special case is considered. For a specific $t, h > 0$, let X be a martingale with respect to $\mathfrak{F}_t$, then according to the martingale representation theorem, we have

$$X = X_t = \int_0^t u_s dB_s + E(X).$$

Here, $E(X)$ is constant and independent of t. Let $Y = X_{t+h}$. For special case, suppose that u_s is a deterministic real-valued function. Then,

$$\Gamma = Y - X = \int_t^{t+h} u_s dB_s.$$

Clearly, X_t is an independent increment process and Γ has a normal distribution with zero mean and variance $\int_t^{t+h} u_s^2 ds$. Notice that $\begin{pmatrix} X \\ Y \end{pmatrix}$ is a linear combination of $\begin{pmatrix} X \\ \Gamma \end{pmatrix}$ as follows

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} X \\ X + \Gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} X \\ \Gamma \end{pmatrix},$$

and since $\begin{pmatrix} X \\ \Gamma \end{pmatrix}$ has a joint normal distribution with mean vector $\begin{pmatrix} E(X) \\ 0 \end{pmatrix}$ and covariance matrix

$$\begin{pmatrix} \sigma_X^2 & 0 \\ 0 & \sigma_\Gamma^2 \end{pmatrix},$$

therefore, $\begin{pmatrix} X \\ Y \end{pmatrix}$ has a joint distribution with mean vector $\begin{pmatrix} E(X) \\ E(X) \end{pmatrix}$ and covariance matrix

$$\begin{pmatrix} \sigma_X^2 & \sigma_X^2 \\ \sigma_X^2 & \sigma_X^2 + \sigma_\Gamma^2 \end{pmatrix}.$$

Here,

$$\sigma_X^2 = \int_0^t u_s^2 ds, \quad \sigma_\Gamma^2 = \int_t^{t+h} u_s^2 ds.$$

The correlation between X, Y is

$$\rho_{xy} = \frac{\sigma_X^2}{\sqrt{\sigma_X^2(\sigma_X^2 + \sigma_\Gamma^2)}} = \left(1 + \frac{\sigma_\Gamma^2}{\sigma_X^2}\right)^{-0.5}.$$

Thus (see [6])

$$E(X \mid Y) = E(X_t \mid X_{t+h}) = E(X) + \rho_{xy}\frac{\sigma_X}{\sigma_Y}(Y - E(X)) = E(X) + \left(1 + \frac{\sigma_\Gamma^2}{\sigma_X^2}\right)^{-1}(Y - E(X)).$$

Also, notice that $E(X_{t+h} \mid X_t) = X_t$. Hence, the parameters of a, b, c, and d of the theorem are

$$a = \left(1 + \frac{\sigma_\Gamma^2}{\sigma_X^2}\right)^{-1}, \quad b = (1-a)E(X), \quad c = 1, \quad d = 0.$$

The rest of the paper is organized as follows. In the next section the application of above Lemma 1 in deriving optimal hedge ratio is discussed. Section 3 uses the (b) of Lemma 1 to approximate the conditional mean and it is applied to financial derivative pricing. Simulation results is given throughout theoretical sections. Real data sets analysis are given in Section 4. Finally, a Conclusion section is given.

2. Optimal hedge ratio

Here, the application of above discussion in portfolio management is discussed. The cross hedging procedure is the construction of an almost riskless portfolio by using one unit of the first asset X in long position and h units of Y in short position (at the maturity) (see [4]).

Let $Z = X - hY$ be the value of portfolio at maturity T. The variance of Z is given by

$$\sigma_z^2 = \sigma_x^2 + h^2\sigma_y^2 - 2h\sigma_{xy}.$$

By minimizing σ_z^2 with respect to h, it is seen that the optimum hedge ratio h_{opt} is given by

$$h_{opt} = \frac{\sigma_{xy}}{\sigma_y^2} = \frac{\int_0^T E\left(u_s v_s\right) ds}{\int_0^T E\left(v_s^2\right) ds}.$$

Hence, the optimum value of portfolio at maturity is

$$Z = X - h_{opt}Y = E\left(X\right) - h_{opt}E\left(Y\right) + \int_0^T \left(u_s - h_{opt}v_s\right) dB_s.$$

The variance of Z at maturity is $\sigma_x^2(1-\rho_{xy}^2)$.

Next, suppose that the risk free interest rate is zero, then the value of X, Y at maturity T is the following (see [2])

$$\begin{cases} X_t = E(X|\mathfrak{F}_t) = E\left(X\right) + \int_0^t u_s dB_s, \\ Y_t = E(Y|\mathfrak{F}_t) = E\left(Y\right) + \int_0^t v_s dB_s. \end{cases}$$

Consider the self-financed portfolio $Z_t = X_t - H_tY_t$ (see [7]). Assume that $H_t = u_t/v_t$. Notice that

$$dZ = dX - HdY = \left(u - Hv\right) dB.$$

Then, $dZ = 0$. Thus, Z is constant. Indeed, $Z = E\left(X\right) - HE\left(Y\right)$. Hence,

$$X_t - H_tY_t = E\left(X\right) - H_tE\left(Y\right).$$

Hence,

$$H_t = \frac{X_t - E\left(X\right)}{Y_t - E\left(Y\right)}.$$

The following proposition summarizes the above discussion.

Proposition 1. *Sentences* (a) *and* (b) *are correct.*

(a) *Under the martingale representation, the optimum hedge ratio for cross hedging* X *by* Y*, is given by*

$$h_{opt} = \frac{\sigma_{xy}}{\sigma_y^2} = \frac{\int_0^T E\left(u_s v_s\right) ds}{\int_0^T E\left(v_s^2\right) ds}.$$

(b) *The replicating ratio for rebalancing portfolio the dynamic hedging portfolio is*

$$H_t = \frac{X_t - E\left(X\right)}{Y_t - E\left(Y\right)}.$$

P r o o f. See the above discussions. □

Next, consider the martingale representation theorem as follows

$$X_t = E(X|\mathfrak{F}_t) = E(X) + \int_0^t u_s dB_s.$$

Let $G(t) = E(u_t^2)$ and $g(t) = \log(G(t))$ and g' be its first derivative. According to Lemma 1, (a) and Ito isometric lemma, the correlation coefficient $\rho_t(h)$ between X_{t+h}, X_t is given by

$$\rho_t(ht) = \sqrt{\frac{G(t)}{G(t+h)}} = \frac{1}{\sqrt{1 + h/G(t) \times \big(G(t+h) - G(t)\big)/h}}.$$

As $h \to 0^+$, then $\rho_t(h)$ is well-approximated by $1/\sqrt{1+hg'(t)}$. The second term $g''(t)$ could be added to mentioned approximation, which is not necessary in practice.

3. Conditional mean approximation

Here, using the second part of Lemma 1, the conditional mean of $E(X_t|X_{t+h})$ is approximated and then its financial application is seen.

3.1. Approximation

Notice that one can see that X_t is a martingale with respect to filtration $\dot{F}_t$, the σ-field generated by X_s, $s \le t$. Next, assume that the conditional expectation of $E(X_t|X_{t+h})$ is well-approximated by linear combination $aX_{t+h} + b$. Then, using the Lemma 1, (b), it is seen that $\rho_t^2(h) = a$, $b = \mu(1 - \rho_t^2(h))$, where $\mu = E(X) = E(X_t)$. Therefore,

$$E(X_t \mid X_{t+h}) = \mu + \rho_t^2(h)(X_{t+h} - \mu),$$

where $\rho_t^2(h) = 1/(1 + hg'(t))$. The following proposition summarizes the above discussion.

Proposition 2. *Assuming $E(X_t|X_{t+h})$ is well-approximated by a linear function of X_{t+h}, then*

$$E(X_t|X_{t+h}) = \mu + \rho_t^2(h)(X_{t+h} - \mu),$$

where $\mu = E(X) = E(X_t)$ and $\rho_t(h) = 1/\sqrt{1+hg'(t)}$. Here, $G(t) = E(u_t^2)$ and $g(t) = \log(G(t))$ and g' be its first derivative.

P r o o f. See the above discussions. □

Example 1 (*cont.*). Here, it is shown that the formula of example 1 corresponds to the approximation of Proposition 2, as $h \to 0$. Define

$$\kappa(t) = \int_0^t u_s^2 ds.$$

Thus, $\sigma_X^2 = \kappa(t)$ and $\sigma_Y^2 = \kappa(t+h) - \kappa(t) \approx h\kappa'(t)$. Hence,

$$\rho_{xy} = \frac{1}{\sqrt{1 + h\kappa'(t)/\kappa(t)}},$$

this is exactly equal to the approximation formula of Proposition 2.

Remark 1. Here some sensitivity analysis are discussed. Indeed, we have the following properties.

(*a*) As $h \to 0$, then $\rho_t(h) \to 1$ which is clear (since the correlation of each variable with its-self is one). As $h \to \infty$, then $\rho_t(h) \to 0$ which is clear since X_{t+h} and X_t are enough far from each others. Also,

$$\frac{\partial \rho}{\partial h} = -0.5 g'(t) (1 + h g'(t))^{-3/2}$$

which converges to the $-0.5 g'(t)$, as $h \to 0$. When $h \to \infty$, then $\partial\rho/\partial h$ goes to zero which is clear since the variation of $\rho_t(h)$ is too small at infinity.

(*b*) It is easy to see that

$$\frac{\partial \rho}{\partial t} = -0.5 h g''(t)(1 + h g'(t))^{-3/2}.$$

Example 2. Let $u_s = 2B_s$, then $X_t = B_t^2 - t$ is a martingale. Indeed,

$$G(t) = 4t, \quad g'(t) = \frac{1}{t}, \quad \rho_t(h) = \sqrt{\frac{t}{t+h}}.$$

For example, when $t = 0.1,\ 0.5$, the following Fig. 1 shows the behavior of

$$\rho_{0.5}(h) = \sqrt{\frac{1}{1+2h}}, \quad \rho_{0.1}(h) = \sqrt{\frac{1}{1+10h}}, \quad h \in (0,1),$$

respectively. As more $t \to 0$, then the curvature of $\rho_t(h)$ is more close to the horizontal axis. Notice that

$$\frac{\partial \rho}{\partial t} = \frac{h}{\sqrt{t}(t+h)^2} \to 0 \quad \text{as} \quad t \to \infty.$$

It is clear because as $t \to \infty$ or $t \to 0$, then $\rho_t(h) \to 1$ and its variation is too small. This is an interesting phenomena that as t gets large, then correlation B_t^2 with its future values is large for each h. For special case, when $h = t$, then $\rho_t(h) = \sqrt{2}/2$.

Also, let $h = q(t)$, for some real valued function q, and suppose that $q(t)/t$ converges to α (β) as $t \to 0$ ($t \to \infty$), then $\rho_t(h)$ tends to $1/\sqrt{1+\alpha}$ ($1/\sqrt{1+\beta}$). Fig. 1 shows the behavior of $\rho_{0.1}(h)$ and $\rho_{0.5}(h)$ which verifies the above discussion. For another example, as extension of Brownian motion, consider the Ornstein–Uhlenbeck process U_t defined by

$$dU = -\alpha U dt + \sigma dB.$$

The Ito lemma implies that $X = X_t = e^{\alpha t} U_t$ satisfies the stochastic differential equation

$$dX = \sigma e^{\alpha t} dB$$

which is martingale with respect to $\mathfrak{F}_t$. Using the Example 1, it is seen that

$$\sigma_x^2 = \int_0^t \sigma^2 e^{2\alpha s} ds = \frac{\sigma^2}{2\alpha}(e^{2\alpha(t+h)} - 1)$$

and

$$\sigma_\Gamma^2 = \frac{\sigma^2}{2\alpha}(e^{2\alpha(t+h)} - e^{2\alpha t}).$$

It is seen that

$$E(X_t | X_{t+h}) = a X_{t+h} + b$$

where

$$a = \left(1 + \frac{\sigma_\Gamma^2}{\sigma_X^2}\right)^{-1}, \quad b = (1-a)\, E(X).$$

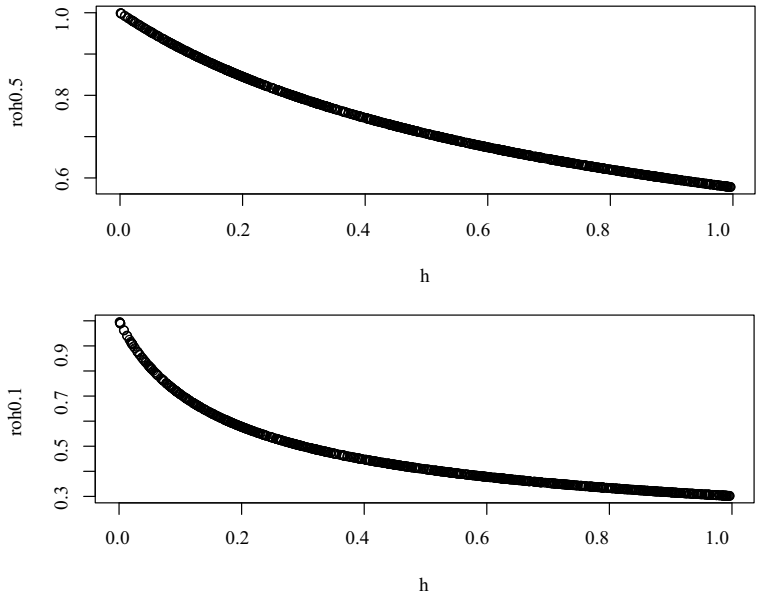

Figure 1. Plots of $\rho_{0.1}(h)$ and $\rho_{0.5}(h)$.

Equivalently,

$$E(U_t|U_{t+h}) = ae^{\alpha h}U_{t+h} + b, \quad E(U_{t+h}|U_t) = e^{-\alpha h}U_t.$$

From Dambis, Dubins–Schwarz (DDS) theorem (see [5, p. 204]), it is seen that

$$U_t = e^{-\alpha h}\tilde{B}\left(\frac{\sigma^2(e^{2\alpha t}-1)}{2\alpha}\right),$$

where $\tilde{B}$ is another Brownian motion. Again, using the results of the previous example, the same results are obtained. Using the results of Remark 1, part (b), it is seen that

$$g(t) = \log\left(\frac{\sigma^2}{2\alpha}\right) + \log(e^{2\alpha t}-1).$$

Then,

$$g''(t) = \frac{-4\alpha^2 e^{2\alpha t}}{(e^{2\alpha t}-1)^2} \to 0, \quad \text{as} \quad t \to \infty.$$

Hence,

$$\frac{\partial \rho}{\partial t} \to 0 \quad \text{as} \quad t \to \infty.$$

3.2. Pricing

In this section, the application of above approximation in pricing of financial derivative is studied. Consider the price of financial derivative f at time t which expires at maturity T $(t \leq T)$ written on a given underlying financial asset. Then,

$$f_t = e^{-r(T-t)}E_Q(f_T|\mathfrak{F}_t)$$

where r, Q, T, and $\mathfrak{F}_t$ are risk free rate, risk neutral probability measure, maturity of financial derivative and the σ-field of price time series $s_u, u \leq t$, (see [7]). Here, under the risk neutral probability measure, the dynamic of price of underlying asset is given by

$$ds = rsdt + \sigma dB$$

at which σ is volatility of price. According to the Black–Scholes formula, the price of financial derivative satisfies the partial differential equation

$$\frac{\partial f}{\partial t} + rs\frac{\partial f}{\partial s} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 f}{\partial s^2} = rf.$$

Let $X_t = e^{-rt} f_t$. Then, using the Ito lemma, it is seen that

$$dX = e^{-rt}\frac{\partial f}{\partial s}\sigma s dB.$$

Then,

$$X = E_Q\left(e^{-rT} f_T\right) + \int_0^t e^{-ru}\frac{\partial f}{\partial s}\sigma s dB_u.$$

Thus,

$$c_t = e^{-rt}\frac{\partial f}{\partial s}\sigma s = \sigma e^{-rt} s\Delta,$$

where Δ is the Greek letter delta representing the sensitivity parameter of financial derivative with respect to variation of s. Notice that

$$G(t) = e^{-2rt}\sigma^2 E_Q(\Delta^2 s^2)$$

and

$$g(t) = -2rt + \log\left(\sigma^2\right) + l(t)$$

where $l(t) = \log\left(E_Q\left(\Delta^2 s^2\right)\right)$ and $g'(t) = -2r + l'(t)$ and

$$\rho_t(h) = \frac{1}{\sqrt{1 + h(-2r + l'(t))}}.$$

In practice, the quantity $E_Q\left(\Delta^2 s^2\right)$ is approximated using a Monte Carlo simulation. The following proposition summarizes the above discussion.

Proposition 3. *For the financial derivative with price f_t then the correlation coefficient $\rho_t(h)$ between f_{t+h}, f_t is given by*

$$\rho_t(h) = \frac{1}{\sqrt{1 + h(-2r + l'(t))}},$$

where $l(t) = \log\left(E_Q\left(\Delta^2 s^2\right)\right)$ and $g'(t) = -2r + l'(t)$.

P r o o f. The result is a direct consequence of previous discussions. □

Remark 2. Hereafter, the sensitivity analysis of $\rho_t(h)$ to its parameters σ, h is verified. For trivial derivative we have $f = s$, then $\Delta = 1$, and under the risk neutral measure Q, we have

$$ds = rsdt + \sigma s dB.$$

The solution is

$$s_t = s_0 e^{\left(r - \sigma^2/2\right)t + \sigma B_t}.$$

Thus, $E_Q\left(\Delta^2 s^2\right) = E_Q\left(s^2\right) = s_0^2 e^{2rt + \sigma^2 t}$ and $g'(t) = \sigma^2$ and $\rho_t(h) = 1/\sqrt{1 + h\sigma^2}$ independent of t. As $\sigma \to \infty$ (0), then $\rho_t(h)$ tends to the 0 (1). If $h = 1/\sigma^2$, so $\rho_t(h) = \sqrt{2}/2$.

4. Real data sets

In this section, throughout real data sets the computational aspects of above theoretical results are studied.

Example 3. In this example, the application of the formula for backward forecasting of daily stock price of *Apple co.* for period of 3 December 2019 to 2 December 2020 (including 254 observations) is studied. Backward forecasting is useful for checking the correctness of guess of traders about future price of a specified share (see [3]). According to the Proposition 2, the backward forecasting in a martingale process is given as follows:

$$E\left(X_t \mid X_{t+h}\right) = \mu + \rho_t^2\left(h\right)\left(X_{t+h} - \mu\right).$$

As follows, error analyses is given to verify the accuracy of the above formula. Using the first 80 percent of data set (i.e., 202 observations, dated from 3 December 2019 to 21 September 2020), the following Ito process if fitted to the *Apple co.* stock price,

$$ds = 0.003sdt + 0.0305sdB,$$

which has solution

$$s = 64.31e^{0.00254t + 0.0305B}.$$

Here,

$$X_t = e^{0.0305B - 0.0305^2 t/2}$$

is martingale, and

$$s = 64.31X_t e^{0.003t}.$$

Hence, substituting this equation to the conditional mean approximation, we see

$$E\left(s_t \mid s_{t+h}\right) = 64.31\mu e^{0.003t}\left(1 - \rho_t^2\left(h\right)\right) + e^{-0.003t}\rho_t^2\left(h\right)s_{t+h},$$

where

$$\mu = 1, \quad u_t = \sigma e^{0.0305B - 0.0305^2 t/2}, \quad G\left(t\right) = (0.0305)^2 e^{(0.0305)^2 t},$$
$$g'(t) = (0.0305)^2, \quad \rho_t(h) = \frac{1}{\sqrt{1 + h(0.0305)^2}}.$$

Next, assuming observations 21 September 2020 to 2 December 2020 are known this is the assumption of the trader about the future, the available data (data for 3 December 2019 to 21 September 2020) are forecasted, backwardly. Here, we used the remaining 20 percent of data, as trader conjecture about future. However, in practice, he may used own dada obtained by his techniques for fundamental analysis. The following Fig. 2 gives the error obtained by different actual and backward forecast for period of 3 December 2019 to 21 September 2020. It is seen that trader guess about future is true.

Example 4. In this example, the daily stock prices of *Amazon co.* for period of 2 October 2017 to 30 September 2019 (including 502 observations) are studied. It is seen that $\sigma = 0.0199$, $r = 0.05$ per year and $s_0 = 959.19$. Consider a call option with strike price $k = 970$, with maturity $T = 1$ (12 months) and European type. The delta parameter is

$$\Delta = \Phi\left(d_1\right), \quad d_1 = \frac{\log\left(s/k\right) + \left(r + 0.5\sigma^2\right)\left(T - t\right)}{\sigma\sqrt{T - t}}.$$

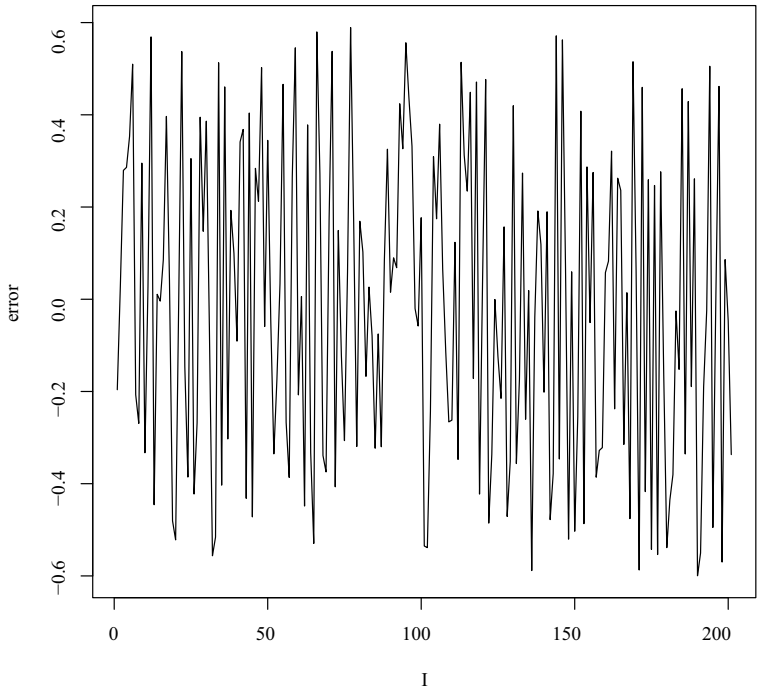

Figure 2. Time series plot of error.

Here, Φ is the normal standard distribution function. The following Fig. 3 shows the $\rho_t(0.2)$, $\rho_t(2)$ for various values of t. To simulate $l(t)$, a Monte Carlo simulation with 1000 repetitions is performed. Also, the variance reduction method is applied. It is seen that as h becomes large then, naturally, $\rho_t(h)$ becomes small. As follows, the Black-Scholes (BS) price of a call option is compared with the approximate price. Also, Δ is an important sensitivity Greek letter to obtain a riskless portfolio. Then, actual Δ is compared with its approximation. Based on these comparisons, the following table is derived. Here, min, q_i, $i = 1, 2, 3$, and max are the minimum, the first, second, third quartiles and the maximum of errors (differences between BS price and Δ, with their approximations), respectively. It is seen that the approximation works well.

Table 1. Measures of errors.

	min	q_1	q_2	q_3	max
Price	−2.59	−1.31	0.88	1.27	2.68
Δ	−0.35	−0.12	−0.01	0.17	0.25

5. Conclusion

In this paper, first, the correlation between two stochastic processes, satisfying the martingale representation theorem format, are derived. This correlation is used to obtain the optimal hedge ratio in a portfolio where two assets have the above mentioned stochastic process behaviors. Then, the results are developed to the serial correlation between a stochastic process and its lags. Then, this serial correlation is approximated. Sensitivity analyses of serial correlation to the time and lags and the parameters of underlying stochastic processes are studied and some interesting results about the relationship of process to its lags in long term (when t tends to ∞) are proposed. Using

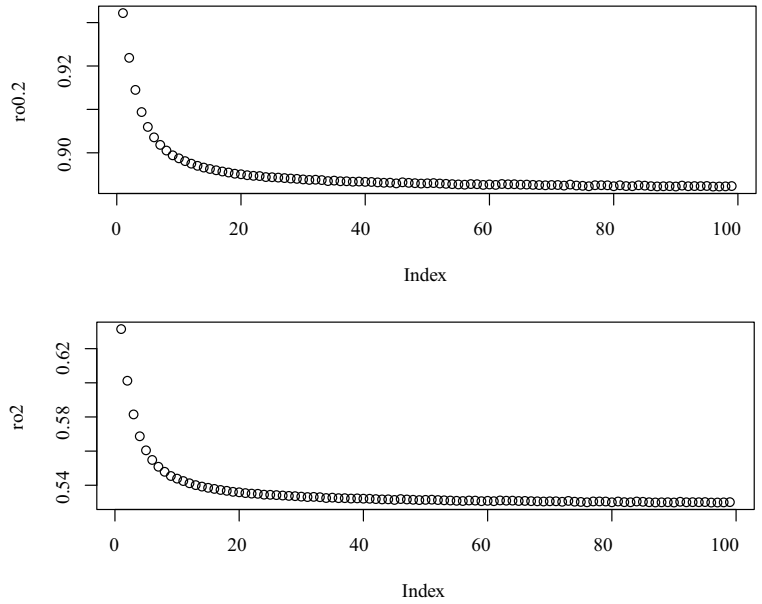

Figure 3. Plots of $\rho_t(0.2)$, $\rho_t(2)$.

the serial correlation, the backward forecast of price of financial assets such as share, equity, stocks or financial derivatives are presented. Forecasts are done using the backward conditional which is well approximated. Throughout, simulated examples and real data sets applicability of proposed methods are seen.

REFERENCES

1. Baxter M. *Financial Calculus: an Introduction to Derivative Pricing*. 1st ed. UK: Cambridge University Press, 1996. 233 p.
2. Duffie D. *Dynamic Asset Pricing Theory*. 1st ed. USA: Princeton University Press, 1992. 472 p.
3. Gann W. D. *Method for Forecasting the Stock Market*. USA: Create Space Independent Publishing Platform, 2012. 28 p.
4. Hull J. *Options, Futures and Other Derivatives*. 1st ed. USA: Prentice-Hall, 1993. 496 p.
5. Klebaner F. *Introduction to Stochastic Calculus with Applications*. 2nd ed. UK: Imperial College, 2005. 430 p.
6. Ross S. *A First Course in Probability*. 8th Ed. USA: Prentice Hall, 2010. 552 p.
7. Wilmott P. *Option Pricing: Mathematical Models and Computation*. UK: Oxford Financial, 1993. 457 p.