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GENERAL QUASILINEAR PROBLEMS INVOLVING $p(x)-$LAPLACIAN WITH ROBIN BOUNDARY CONDITION

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Abstract: This paper deals with the existence and multiplicity of solutions for a class of quasilinear problems involving $p(x)$-Laplace type equation, namely

$$\begin{cases} -\mathrm{div}\,(a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u) = \lambda f(x,u) & \text{in} \quad \Omega, \\ n\cdot a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u + b(x)|u|^{p(x)-2}u = g(x,u) & \text{on} \quad \partial\Omega. \end{cases}$$

Our technical approach is based on variational methods, especially, the mountain pass theorem and the symmetric mountain pass theorem.

Keywords: $p(x)$-Laplacian, Mountain pass theorem, Multiple solutions, Critical point theory.

1. Introduction

In this paper we study the nonlinear elliptic boundary value problem with Robin conditions

$$\begin{cases} -\mathrm{div}\,(a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u) = \lambda f(x,u) & \text{in} \quad \Omega, \\ n\cdot a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u + b(x)|u|^{p(x)-2}u = g(x,u) & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ $(N \geq 2)$, with smooth boundary, n is the outer unit normal vector on $\partial\Omega$, b is a positive continuous function defined on $\mathbb{R}^N$, $p \in C_+(\overline{\Omega})$ with

$$1 < p^- := \inf_{\overline{\Omega}} p(x) \leq p^+ := \sup_{\overline{\Omega}} p(x) < N$$

and $p(x) < p^*(x)$ where

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N-p(x)} & \text{if} \quad p(x) < N, \\ +\infty & \text{if} \quad p(x) \geq N \end{cases}$$

for any $x \in \overline{\Omega}$. It is clear that the equation in question is elliptic since it describes phenomena that do not change from moment to moment, and that the operator

$$Lu = -\mathrm{div}\,(a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u)$$

is an elliptic operator in divergence form.

Recently, the study of differential equations and variational problems involving $p(x)$-growth conditions have been extensively investigated and received much attention because they can be presented as models for many physical phenomena which arouse in the study of elastic mechanics [32], electro-rheological fluid dynamics [27] and image processing [6], electrical resistivity and polycrystal plasticity [3, 4] and continuum mechanics [2] etc, for an overview of this subject, and for more details we refer readers to [11] and [5, 10] and the references therein. The existence of nontrivial solutions to nonlinear elliptic boundary value problems has been extensively studied by many researchers [1, 7, 14, 15, 18, 21, 23, 24] and references therein.

It is known that the extension $p(x)$-Laplace operator possesses more complicated structure than the p-Laplacian. For example, it is inhomogeneous and usually it does not have the so-called first eigenvalue, since the infimum of its spectrum is zero.

However, to understand the role of the variable exponent, well, although most of the materials can be accurately modeled with the help of the classical Lebesgue and Sobolev spaces L^p and $W^{1,p}$, where p is a fixed constant, there are some nonhomogeneous materials, for which this is not adequate, e.g. the rheological fluids mentioned above, which are characterized by their ability to drastically change their mechanical properties under the influence of an exterior electromagnetic field. Thus it is necessary for the exponent p to be nonstandard, therefore, the spaces with variable exponents are required. As an introduction and a history coverage to the subject of variable exponent problems, we advice the reader to see the monograph [12] and the articles [16, 20].

Note that, the $p(x)$-Laplace operator in (1.1) is a special case of the divergence form operator $-\mathrm{div}\,(a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u)$ which appears in many nonlinear diffusion problems, in particular in the mathematical modeling of non-Newtonian fluids. When

$$a(t) = 1 + \frac{t}{\sqrt{1+t^2}}$$

we have the generalized Capillary operator (which is essential in applied fields like industrial, biomedical and pharmaceutical) initiated by W. Ni and J. Serrin [22].

Inspired by the works in [25] and [19], we study the existence and multiplicity of nontrivial solutions the problem (1.1), via the mountain pass theorem and the Rabinowitz's symmetric mountain pass theorem [26].

We assume the following conditions:

($\mathbf{A}_0$) The function $a : \mathbb{R}^+ \to \mathbb{R}$ is continuous and the mapping $\Theta : \mathbb{R}^N \to \mathbb{R}$, given by $\Theta(\xi) = A(|\xi|^{p(x)})$ is strictly convex, where A is the primitive of a, that is

$$A(t) = \int_0^t a(s)ds.$$

($\mathbf{A}_1$) There exist two constants $0 < L < K$ such that $L \leq a(t) \leq K$ for all $t \geq 0$.

We assume that $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are of Carathéodory functions, $f(x, \cdot) = g(x, \cdot) = 0$ and satisfy:

($\mathbf{F}_0$) for all $(x,t) \in \Omega \times \mathbb{R}$ $|f(x,t)| \leq f_1(x)|t|^{r(x)-1}$, such that

$$1 \leq r^- := \inf_{\overline{\Omega}} r(x) \leq r^+ := \sup_{\overline{\Omega}} r(x) < p^- := \inf_{\overline{\Omega}} p(x) \leq p^+ := \sup_{\overline{\Omega}} p(x),$$

where f_1 is nonnegative, measurable function and $f_1 \in L^{\frac{p(x)}{p(x)-r(x)}}(\Omega)$;

($\mathbf{F}_1$) for all $(x,t) \in \Omega \times \mathbb{R}$ $|f(x,t)| \geq f_2(x)|t|^{\alpha(x)-1}$,

$$1 \leq \alpha^- : \inf_{\overline{\Omega}} \alpha(x) \leq \alpha^+ := \sup_{\overline{\Omega}} \alpha(x) < r^-,$$

where $f_2 > 0$ in some nonempty open set $O \subset \Omega$;

$(\mathbf{G}_0)$ for all $(x,t) \in \partial\Omega \times \mathbb{R}$, $|g(x,t)| \leq g_1(x)|t|^{q(x)-1}$,

$$1 \leq p^+ < q^- := \inf_{\overline{\Omega}} q(x) \leq q^+ := \sup_{\overline{\Omega}} q(x), \quad q(x) < p^\partial(x),$$

where

$$p^\partial(x) = (p(x))^\partial = \begin{cases} \dfrac{(N-1)p(x)}{N-p(x)} & \text{if} \quad p(x) < N, \\ +\infty & \text{if} \quad p(x) \geq N \end{cases}$$

and there exists a positive constants C_g such that $0 \leq g_1 \leq C_g$;

$(\mathbf{G}_1)$ for all $(x,t) \in \partial\Omega \times \mathbb{R}$ $\displaystyle\lim_{t\to 0} \frac{g(x,t)t}{|t|^{p^+-1}} = 0.$

$(\mathbf{G}_2)$ there exists $\mu > p^+$ such that $\mu G(x,t) \leq g(x,t)t$ for all $(x,t) \in \partial\Omega \times \mathbb{R}$, where

$$G(x,t) = \int_0^t g(x,s)ds.$$

The main result of this paper is as follow.

Theorem 1. *Assume that* $(\mathbf{A}_0)$*–*$(\mathbf{A}_1)$*,* $(\mathbf{F}_0)$*–*$(\mathbf{F}_1)$ *and* $(\mathbf{G}_0)$*–*$(\mathbf{G}_2)$ *hold. Then there exists* $\lambda^* > 0$ *such that for every* $\lambda \in]0, \lambda^*[$*, the problem* (1.1) *admits at least one nontrivial solution. In addition, if we assume the following conditions*:

$(\mathbf{G}_3)$ *there is a nonempty open set* $U \subset \partial\Omega$ *with* $G(x,t) > 0$ *for all* $(x,t) \in U \times \mathbb{R}^+$*,*

$(\mathbf{G}_4)$ *the functions* f *and* g *are even,*

then the problem (1.1) *has infinitely many solutions for every* $\lambda > 0$*.*

The remainder of this paper is organized as follows, in Section 2 we introduce some technical results and required hypotheses in order to solve our problem, in Section 3 we state some and prove the main results of this work.

2. Preliminaries

In the sequel, let $p(x) \in C_+(\overline{\Omega})$, where

$$C_+(\overline{\Omega}) = \left\{h : h \in C(\overline{\Omega}), h(x) > 1 \text{ for any } x \in \overline{\Omega}\right\}.$$

The variable exponent Lebesgue space is defined by

$$L^{p(x)}(\Omega) = \left\{u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{p(x)}\, dx < +\infty\right\}$$

furnished with the Luxemburg norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\left\{\sigma > 0 : \int_\Omega \left|\frac{u(x)}{\sigma}\right|^{p(x)} dx \leq 1\right\}.$$

Remark 1. Variable exponent Lebesgue spaces resemble to classical Lebesgue spaces in many respects, they are separable Banach spaces and the Hölder inequality holds. The inclusions between Lebesgue spaces are also naturally generalized, that is, if $0 < \operatorname{mes}(\Omega) < \infty$ and p, q are variable exponents such that $p(x) < q(x)$ a. e. in Ω, then there exists a continuous embedding $L^{q(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$.

The variable exponent Sobolev space is defined by

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\right\}$$

equipped with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}.$$

Proposition 1 [16, 17]. *The spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable, uniformly convex, reflexive Banach spaces. The conjugate space of $L^{p(x)}(\Omega)$ is $L^{q(x)}(\Omega)$, where $q(x)$ is the conjugate function of $p(x)$, i.e.,*

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1,$$

for all $x \in \Omega$. For $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$ we have

$$\Big|\int_\Omega u(x)v(x)dx\Big| \leq \Big(\frac{1}{p^-} + \frac{1}{q^-}\Big)|u|_{p(x)}|v|_{q(x)}.$$

Moreover, if $h_1, h_2, h_3 : \overline{\Omega} \to (1,\infty)$ are Lipschitz continuous functions such that

$$\frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3} = 1,$$

then for any $u \in L^{h_1(x)}(\Omega)$, $v \in L^{h_2(x)}(\Omega)$, $w \in L^{h_3(x)}(\Omega)$, the following inequality holds (see [15, Proposition 2.5])

$$\int_\Omega |uvw|dx \leq \Big(\frac{1}{h_1^-} + \frac{1}{h_2^-} + \frac{1}{h_3^-}\Big)|u|_{h_1(x)}|v|_{h_2(x)}|w|_{h_3(x)}.$$

Proposition 2 [13]. *Let $p(x)$ and $q(x)$ be measurable functions such that $p(x) \in L^\infty(\Omega)$ and $1 \leq p(x)q(x) \leq \infty$, for a.e. $x \in \Omega$. Let $u \in L^{q(x)}(\Omega)$, $u \neq 0$. Then*

$$|u|_{p(x)q(x)} \leq 1 \Rightarrow |u|^{p^+}_{p(x)q(x)} \leq ||u|^{p(x)}|_{q(x)} \leq |u|^{p^-}_{p(x)q(x)},$$
$$|u|_{p(x)q(x)} \geq 1 \Rightarrow |u|^{p^-}_{p(x)q(x)} \leq ||u|^{p(x)}|_{q(x)} \leq |u|^{p^+}_{p(x)q(x)}.$$

In particular if $p(x) = p$ is a constant, then

$$||u|^p|_{q(x)} = |u|^p_{pq(x)}.$$

Proposition 3 [16, 17]. *Assume that the boundary of Ω possesses the cone property and $p, r \in C_+(\overline{\Omega})$ such that $r(x) \leq p^*(x)$ $(r(x) < p^*(x))$ for all $x \in \overline{\Omega}$, then there is a continuous (compact) embedding*

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega),$$

Proposition 4 [9]. *For $p \in C_+(\overline{\Omega})$ and such $r \in C_+(\partial\Omega)$ that $r(x) \leq p^\partial(x)$ $(r(x) < p^\partial(x))$ for all $x \in \overline{\Omega}$, there is a continuous (compact) embedding*

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\partial\Omega).$$

Proposition 5. [8, Theorem 2.1] *For any $u \in W^{1,p(x)}(\Omega)$, let*

$$\|u\|_\partial := |u|_{L^{p(x)}(\partial\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}.$$

Then $\|u\|_\partial$ is a norm on $W^{1,p(x)}(\Omega)$ which is equivalent to

$$\|u\|_{W^{1,p(x)}(\Omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}.$$

Now, for any $u \in X := W^{1,p(x)}(\Omega)$ define

$$\|u\| := \inf \Big\{ \sigma > 0 : \int_\Omega \Big| \frac{\nabla u(x)}{\sigma} \Big|^{p(x)} dx + \int_{\partial\Omega} b(x) \Big| \frac{u(x)}{\sigma} \Big|^{p(x)} d\sigma_x \leq 1 \Big\},$$

where $b \in L^\infty(\Omega)$ and $d\sigma_x$ is the measure on the boundary $\partial\Omega$. Then by Proposition 5, $\|\cdot\|$ is also a norm on $W^{1,p(x)}(\Omega)$ which is equivalent to $\|\cdot\|_{W^{1,p(x)}(\Omega)}$ and $\|\cdot\|_\partial$, the proof of this statement can be found in [8, p. 551]. Now, we introduce the modular $\rho : X \to \mathbb{R}$ defined by

$$\rho(u) = \int_\Omega |\nabla u|^{p(x)} dx + \int_{\partial\Omega} b(x)|u(x)|^{p(x)} d\sigma_x$$

for all $u \in X$. Here, we give some relations between the norm $||\cdot||$ and the modular ρ.

Proposition 6 [16]. *For $u \in X$ we have*

(i) $\|u\| < 1 (= 1; > 1) \Leftrightarrow \rho(u) < 1 (= 1; > 1)$;

(ii) *If* $\|u\| < 1 \Rightarrow \|u\|^{p^+} \leq \rho(u) \leq \|u\|^{p^-}$;

(iii) *If* $\|u\| > 1 \Rightarrow \|u\|^{p^-} \leq \rho(u) \leq \|u\|^{p^+}$.

Proposition 7 [29]. *Suppose that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and satisfies the growth condition*

$$|f(x,t)| \leq c|t|^{\alpha(x)/\beta(x)} + h(x), \quad \text{for every } x \in \Omega,\ t \in \mathbb{R},$$

where $\alpha, \beta \in C_+(\overline{\Omega})$, $c \geq 0$ is constant and $h \in L^{\beta(x)}(\Omega)$. Then $N_f(L^{\alpha(x)}(\Omega)) \subseteq L^{\beta(x)}(\Omega)$, where $N_f(u)(x) = f(x,u(x))$. Moreover, N_f is continuous from $L^{\alpha(x)}(\Omega)$ into $L^{\beta(x)}(\Omega)$ and maps bounded set into bounded set.

As a consequence of Proposition 7, the Carathéodory function f defines an operator N_f which is called the Nemytskii operator.

Definition 1. *We say that $u \in X$ is weak solution of* (1.1) *if*

$$\int_\Omega a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u \nabla v dx + \int_{\partial\Omega} b(x)|u|^{p(x)-2}uv d\sigma_x = \lambda \int_\Omega f(x,u)v dx + \int_{\partial\Omega} g(x,u)v d\sigma_x$$

for all $v \in X$.

Now we introduce the Euler–Lagrange functional $I_\lambda : X \longrightarrow \mathbb{R}$ associated with problem (1.1) defined by

$$I_\lambda(u) = \int_\Omega \frac{1}{p(x)} A(|\nabla u|^{p(x)})dx + \int_{\partial\Omega} \frac{1}{p(x)} b(x)|u|^{p(x)} d\sigma_x - \lambda \int_\Omega F(x,u)dx - \int_{\partial\Omega} G(x,u)d\sigma_x,$$

where

$$F(x,t) := \int_0^t f(x,s)ds.$$

Furthermore, the (weak) solutions of (1.1) are precisely the critical points of the functional I_λ.

Lemma 1 [31] . *Let*

$$L(u) := \int_\Omega \frac{1}{p(x)} A(|\nabla u|^{p(x)})dx + \int_{\partial\Omega} \frac{1}{p(x)} b(x)|u|^{p(x)}d\sigma_x.$$

Then the mapping $L : X \to X^*$ is a strictly monotone, continuous bounded homeomorphism and is of type (S_+), namely assumptions $u_n \rightharpoonup u$ and $\limsup_{n\to+\infty} L(u_n)(u_n - u) \leq 0$, imply $u_n \to u$.

By Proposition 7, we can see that the functional I_λ is well defined on X and $I_\lambda \in C^1(X, \mathbb{R})$ with its Fréchet derivative is giving by

$$\begin{aligned} I'_\lambda(u) \cdot v = & \int_\Omega a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u \nabla v dx + \int_{\partial\Omega} b(x)|u|^{p(x)-2}uv d\sigma_x \\ & -\lambda \int_\Omega f(x, u)v dx - \int_{\partial\Omega} g(x, u)v d\sigma_x \end{aligned}$$

for all $u, v \in X$.

Let X be a real Banach space and let be a functional $I \in \mathcal{C}^1(X, \mathbb{R})$. We say that I satisfies the *Palais-Smale condition* on X ((PS)-condition, for short) if any sequence $(u_n) \subset X$ with $(I(u_n))$ bounded and $I'(u_n) \to 0$ as $n \to \infty$, possesses a convergent subsequence. By (PS)-*sequence* for I we understand a sequence $(u_n) \subset X$ which satisfies the conditions: $(I(u_n))$ is bounded and $I'(u_n) \to 0$ as $n \to \infty$.

The main tools used in proving Theorem 1 are the well known mountain pass theorem and its the symmetric mountain pass theorem.

Theorem 2 [26, Theorem 2.2]. *Let X be a real Banach space and let I belong to $\mathcal{C}^1(X, \mathbb{R})$ satisfying the (PS)-condition. Suppose that $I(0) = 0$ and that the following conditions hold:*

($\mathbf{I}_1$) *there exist $\rho > 0$ and $\varrho > 0$ such that $I(u) \geq \varrho$ for $\|u\| = \rho$;*

($\mathbf{I}_2$) *there exists $e \in X$ with $\|e\| > \rho$ such that $I(e) \leq 0$.*

Let

$$\Gamma = \{\gamma \in \mathcal{C}([0, 1]; X) : \gamma(0) = 0, \gamma(1) = e\}, \quad c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} I(\gamma(t)),$$

then, c is a critical value of I.

Theorem 3 [28, Theorem 2.1]. *Let X be a real Banach space and let I belong to $\mathcal{C}^1(X, \mathbb{R})$ be even, satisfies (PS)-condition and $I(0) = 0$. If $X = Y \oplus Z$ with $\dim Y < \infty$, and I satisfies*

(**I'**$_1$) *there are constants $\rho, > 0$ such that $I/_{\partial B_\rho \cap Z} \geq 0$*

(**I'**$_2$) *there a finite dimensional subspace $W \subset X$, with $\dim Y < \dim W < \infty$ and there is $M > 0$ such that $\max_{u\in W} I(u) < M$*

(**I'**$_3$) *considering $M > 0$ given by (**I'**$_2$), I satisfies $(PS)_c$ for $0 \leq c \leq M$.*

Then I possesses at least $\dim W - \dim Y$ pairs of nontrivial critical points.

3. Proof of Theorem 1

To prove Theorem 1 we recall some lemmas presented below.

Lemma 2. *Assume that* $(\mathbf{A}_1)$, $(\mathbf{F}_0)$ *and* $(\mathbf{G}_2)$ *hold. Then the functional* I_λ *satisfies the Palais–Smale condition on* X *((PS)-condition, for short) at any level* d.

P r o o f. Let $d \in \mathbb{R}$ and let $(u_n) \subset X$ be (PS) sequence for I_λ, then

$$I_\lambda(u_n) \to d \quad \text{and} \quad I'_\lambda(u_n) \to 0 \quad \text{as} \quad n \to \infty. \tag{3.1}$$

First, we prove that sequence (u_n) is bounded in X. Suppose (u_n) unbounded, we may assume $\|u_n\| \to +\infty$ as $n \to \infty$.

By (2), $(\mathbf{A}_1)$, $(\mathbf{F}_0)$ and Proposition 6 we have

$$\begin{aligned} I_\lambda(u_n) &= \int_\Omega \frac{1}{p(x)} A(|\nabla u_n|^{p(x)})dx + \int_{\partial\Omega} \frac{1}{p(x)} b(x)|u_n|^{p(x)} d\sigma_x \\ &\quad -\lambda \int_\Omega F(x,u_n)dx - \int_{\partial\Omega} G(x,u_n)d\sigma_x \\ &\geq \frac{L}{p^+}\int_\Omega |\nabla u_n|^{p(x)}dx + \int_{\partial\Omega}\frac{1}{p^+} b(x)|u_n|^{p(x)}d\sigma_x - \frac{\lambda}{r^+}\int_\Omega f_1(x)|u_n|^{r(x)}dx - \int_{\partial\Omega} G(x,u_n)d\sigma_x \\ &\geq \frac{\min(L,1)}{p^+}\|u_n\|^{p^-} - \frac{\lambda}{r^+}\int_\Omega f_1(x)|u_n|^{r(x)}dx - \int_{\partial\Omega} G(x,u_n)d\sigma_x. \end{aligned} \tag{3.2}$$

From (3.2), $(\mathbf{F}_0)$ and Proposition 6 we obtain

$$\begin{aligned} \frac{1}{\mu} I'_\lambda(u_n)\cdot u_n &= \frac{1}{\mu}\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)}dx + \frac{1}{\mu}\int_{\partial\Omega} b(x)|u_n|^{p(x)}d\sigma_x \\ &\quad -\frac{\lambda}{\mu}\int_\Omega f(x,u_n)u_n dx - \frac{1}{\mu}\int_{\partial\Omega} g(x,u_n)u_n d\sigma_x \\ &\geq \frac{\min(L,1)}{\mu}\|u_n\|^{p^-} - \frac{\lambda}{\mu}\int_\Omega f_1(x)|u_n|^{r(x)}dx - \frac{1}{\mu}\int_{\partial\Omega} g(x,u_n)u_n d\sigma_x. \end{aligned} \tag{3.3}$$

Meanwhile, according to $(\mathbf{F}_0)$, Proposition 4 and Proposition 2 it yields

$$\begin{aligned} &\int_\Omega f_1(x)|u_n|^{r(x)}dx \leq \int_\Omega |f_1(x)||u_n|^{r(x)}dx \leq |f_1|_{L^{\frac{p(x)}{p(x)-r(x)}}(\Omega)} \left||u_n|^{r(x)}\right|_{\frac{p(x)}{r(x)}} \\ &\leq |f_1|_{L^{\frac{p(x)}{p(x)-r(x)}}(\Omega)} \max\left(|u_n|^{r^-}_{p(x)}, |u_n|^{r^+}_{p(x)}\right) \leq |f_1|_{L^{\frac{p(x)}{p(x)-r(x)}}(\Omega)} \max\left(C_{r^-}\|u_n\|^{r^-}, C_{r^+}\|u_n\|^{r^+}\right), \end{aligned} \tag{3.4}$$

where C_{r^-} and C_{r^+} are constants of compact embedding $X \hookrightarrow L^{p(x)}(\Omega)$. Using (3.1), (3.2), (3.3), (3.4) and (G_2) we obtain

$$\begin{aligned} d + 1 + \|u_n\| &\geq I_\lambda(u_n) - \frac{1}{\mu} I'_\lambda(u_n).u_n \geq \frac{\min(L,1)}{p^+}\|u_n\|^{p^-} - \frac{\lambda}{r^+}\int_\Omega f_1(x)|u_n|^{r(x)}dx \\ &\quad - \int_{\partial\Omega} G(x,u_n)d\sigma_x - \frac{\min(L,1)}{\mu}\|u_n\|^{p^-} - \frac{\lambda}{\mu}\int_\Omega f_1(x)|u_n|^{r(x)}dx - \frac{1}{\mu}\int_{\partial\Omega} g(x,u_n)u_n d\sigma_x \\ &\geq \min(L,1)\Big(\frac{1}{p^+} - \frac{1}{\mu}\Big)\|u_n\|^{p^-} - \Big(\frac{\lambda}{r^+} + \frac{\lambda}{\mu}\Big)\int_\Omega f_1(x)|u_n|^{r(x)}dx + \int_{\partial\Omega}\Big(\frac{1}{\mu} g(x,u_n)u_n - G(x,u_n)\Big)d\sigma_x \\ &\geq \min(L,1)\Big(\frac{1}{p^+} - \frac{1}{\mu}\Big)\|u_n\|^{p^-} - \Big(\frac{\lambda}{r^+} + \frac{\lambda}{\mu}\Big)|f_1|_{L^{\frac{p(x)}{p(x)-r(x)}}(\Omega)} C_{r^+}\|u_n\|^{r^+}, \end{aligned} \tag{3.5}$$

where d is defined in (3.1). Since $p^- \geq r^+$ (u_n) is bounded.

Now, with standard arguments, we prove that any $(PS)_d$ sequence (u_n) in X has a convergent subsequence. Indeed, the space X is a Banach reflexive space then there exists $u \in X$ such that, up to subsequence still denoted by (u_n) and by the Sobolev embedding, we obtain:

- $u_n \rightharpoonup u$ in X as $n \to \infty$;
- $u_n(x) \to u(x)$ a.e. in Ω as $n \to \infty$;
- $u_n \to u$ in $L^{p(x)}(\Omega)$ as $n \to \infty$;
- $u_n \to u$ in $L^{\frac{p(x)}{p(x)-1}}(\Omega)$ as $n \to \infty$. □

Proposition 8. *If $u_n \rightharpoonup u$ in X as $n \to \infty$, then*

$$\lim_{n\to\infty} \int_\Omega f_1(x)|u_n|^{r(x)-1}(u_n - u)dx = 0, \tag{3.6}$$

and

$$\lim_{n\to\infty} \int_{\partial\Omega} g_1(x)|u_n|^{q(x)-1}(u_n - u)d\sigma_x = 0. \tag{3.7}$$

P r o o f. To demonstrate (3.6), we use Propositions 1–4 we give

$$\int_\Omega f_1(x)|u_n|^{r(x)-1}(u_n - u)dx \leq \int_\Omega |f_1(x)||u_n|^{r(x)-1}|u_n - u|dx$$
$$\leq 3C\,|f_1|_{L^{\frac{p(x)}{p(x)-r(x)}}(\Omega)} \max\left(|u_n|_{p(x)}^{r^- -1}, |u_n|_{p(x)}^{r^+ -1}\right)|u_n - u|_{p(x)},$$

where C is positive constant. By the compact embedding $X \hookrightarrow L^{p(x)}(\Omega)$ and the inequality $||u_n|_{p(x)} - |u|_{p(x)}| \leq |u_n - u|_{p(x)}$, we obtain $|u_n - u|_{p(x)} \to 0$ in $L^{p(x)}(\Omega)$ and $|u_n|_{p(x)} \to |u|_{p(x)}$.

Similar arguments establish (3.7).

Now, in virtue of (3.1) and Proposition 8, we have

$$\limsup_{n\to\infty} \int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n - \nabla u)dx + \int_{\partial\Omega} b(x)|u_n|^{p(x)-2}u_n(u_n - u)d\sigma_x$$
$$= \limsup_{n\to\infty} I'_\lambda(u_n)\cdot u_n + \limsup_{n\to\infty} \lambda \int_\Omega f(x,u_n)(u_n - u)dx + \limsup_{n\to\infty} \int_{\partial\Omega} g(x,u_n)(u_n - u)d\sigma_x = 0.$$

Finally, by Lemma 1 $u_n \to u$ in X.

To finish the proof of the Theorem 1, we check the geometrical conditions of mountain pass Theorem 2 for I_λ. Indeed

$(\mathbf{I}_1)$ since the embeddings $X \hookrightarrow L^{i(x)}(\Omega)$ $(i := p, r, q)$ and $X \hookrightarrow L^{i(x)}(\partial\Omega)$ $(i := p, q)$ is are compact, there exist positive constants C_i such that

$$|u|_{i(x)} \leq C_i\|u\|. \tag{3.8}$$

From $(\mathbf{G}_0)$–$(\mathbf{G}_1)$ it follows, for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$, such that

$$G(x,u) \leq \frac{\varepsilon}{p^+}|u|^{p^+} + C_\varepsilon|u|^{q(x)}, \quad \text{for all} \quad (x,t) \in \partial\Omega \times \mathbb{R}, \tag{3.9}$$

thus, for $u \in X$ with $\| u \| \leq 1$. By $(\mathbf{A}_1)$, (3.2), (3.4), (3.8) and (3.9), we have

$$\begin{aligned} I_\lambda(u) \geq \frac{\min(L,1)}{p^+}\|u\|^{p^+} - \frac{\lambda C_r\,|f_1|_{L^{p(x)/(p(x)-r(x))}(\Omega)}}{r^-}\|u_n\|^{r^+} - \frac{\varepsilon C_\varepsilon C_p}{p^+}\|u\|^{p^+} - C_q C_g \|u\|^{q^+} \\ \geq \|u\|^{p^+}\left[C_1 - \lambda C_2 \|u\|^{r^+-p^+} - C_3\|u\|^{q^+-p^+}\right], \end{aligned} \tag{3.10}$$

where

$$C_1 = \frac{\min(L,1)}{p^+} - \frac{\varepsilon C_\varepsilon C_p}{p^+}, \quad C_2 = \frac{C_r\,|f_1|_{L^{p(x)/(p(x)-r(x))}(\Omega)}}{r^-}, \quad C_3 = C_q C_g.$$

If $\rho = \|u\|$, we obtain

$$I_\lambda(u) \geq \rho^{p^+} \overbrace{\left[C_1 - \lambda C_2 \rho^{r^+-p^+} - C_3 \rho^{q^+-p^+}\right]}^{\psi(\rho)}. \tag{3.11}$$

A straightforward computation shows that the maximum of the function ψ is

$$\rho_m = \left(\frac{q^+(p^+-r^+)\lambda C_2}{r^+(q^+-r^+)C_3}\right).$$

Inserting this into equation (3.11), we find that the right side is zero for

$$\lambda^* := \frac{C_3}{C_2}\rho_m^{q^+-r^+} - \frac{C_1}{C_2}\rho_m^{p^+-r^+}.$$

So, there exist $\rho > 0$ and $\varrho > 0$ such that $I_\lambda(u) \geq \varrho$ for $\|u\| = \rho$, from which the demonstration of $(\mathbf{I}_1)$ is completed.

Now, put

$$h(\tau) = \tau^{-\mu} G(x, \tau t) - G(x,t) \quad \forall t \geq 1.$$

We have

$$h^{'}(t) = t^{-\mu-1}\left(g(x,t\tau)t\tau - G(x,t\tau)\right) \geq 0 \quad \forall t \geq 1$$

by $(\mathbf{G}_2)$. Hence, $h(\tau) \geq h(1)$ for all $\tau \geq 1$ that is,

$$G(x,\tau t) \geq \tau^\mu G(x,t) \quad \forall (x,t) \in \partial\Omega \times \mathbb{R}. \tag{3.12}$$

Let $u \in X$, for $t > 1$, by $(\mathbf{A}_0)$ and (3.12), we have

$$\begin{aligned} I_\lambda(tu) = \int_\Omega \frac{1}{p(x)} A(|\nabla tu|^{p(x)})dx + \int_{\partial\Omega} \frac{1}{p(x)} b(x)|tu|^{p(x)} d\sigma_x - \lambda \int_\Omega F(x,tu)dx - \int_{\partial\Omega} G(x,tu)d\sigma_x \\ \leq t^{p^+}\left(\int_\Omega \frac{1}{p(x)} A(|\nabla u|^{p(x)})dx + \int_{\partial\Omega}\frac{1}{p(x)} b(x)|u|^{p(x)} d\sigma_x\right) \\ + t^{r^+}\frac{\lambda}{r^+}\int_\Omega f_1(x)|u|^{r(x)}dx - C_4 t^\mu \int_{\partial\Omega}\left[\frac{\varepsilon}{p^+}|u|^{p^+} + C_\varepsilon |u|^{q(x)}\right] d\sigma_x. \end{aligned}$$

This shows that $I_\lambda(tu) < 0$.

Since $I_\lambda(0) = 0$, the mountain pass lemma implies the existence of a nontrivial weak solution u_1 with $I_\lambda(u_1) \geq \varrho$.

Hence problem (1.1) has at least one nontrivial weak solution in X.

To complete the proof of the Theorem 1, one must check the conditions of the Theorem 3. So we need some lemmas which we recall below.

Remark 2. [30] As the Sobolev space X is a reflexive and separable Banach space, there exist $(e_n)_{n\in\mathbb{N}^*} \subseteq X$ and $(f_n)_{n\in\mathbb{N}^*} \subseteq X^*$ such that $f_n(e_m) = \delta_{nm}$ for any $n, m \in \mathbb{N}^*$ and

$$X = \overline{\operatorname{span}\{e_n : n \in \mathbb{N}^*\}}, \quad X^* = \overline{\operatorname{span}\{f_n : n \in \mathbb{N}^*\}}^{w^*}.$$

For $k \in \mathbb{N}^*$ denote by $X_k = \operatorname{span}\{e_k\}$, $\quad Y_k = \oplus_{j=1}^{k} X_j$, $\quad Z_k = \overline{\oplus_k^{\infty} X_j}$.

Lemma 3. *Assume that* $(\mathbf{A}_0)$*–*$(\mathbf{A}_1)$*,* $(\mathbf{F}_0)$*–*$(\mathbf{F}_1)$ *and* $(G_0)-(G_1)$ *hold. Then there exists* $\widetilde{\lambda} > 0$*,* $k \in \mathbb{N}$ *and* $\rho, \theta > 0$ *such that* $I_\lambda/\partial B_\rho \cap X_k \geq \theta$ *for all* $0 < \lambda < \widetilde{\lambda}$*.*

P r o o f. Similarly to (3.10), we have

$$I_\lambda(u) \geq \|u\|^{p^+}\left[C_1 - \lambda C_2 \|u\|^{r^+ - p^+}\right] - C_3\|u\|^{q^+}.$$

Taking $\rho = \|u\|$, we get

$$I_\lambda(u) \geq \rho^{p^+}\left[C_1 - \lambda C_2 \rho^{r^+ - p^+}\right] - C_3\rho^{q^+}.$$

Next, we take $\widetilde{\lambda} = C_1/C_2 \cdot \rho^{p^+ - r^+} > 0$ so that

$$I_\lambda(u) \geq \rho^{p^+}\left[C_1 - \lambda C_2 \rho^{r^+ - p^+}\right] - C_3\rho^{q^+} > 0,$$

which shows that I verifies the condition $(\mathbf{I'}_1)$ in Theorem 3. □

Finally, to show the condition $(\mathbf{I'}_2)$ in Theorem 3, we use the following lemma.

Lemma 4. *Assume that* $(\mathbf{A}_0)$*–*$(\mathbf{A}_1)$ *and* $(\mathbf{G}_2)$*–*$(\mathbf{G}_3)$ *hold. Then, given* $m \in \mathbb{N}$*, there exist a subspace* W *of* X *and a constant* $M_m > 0$*, independent of* λ*, such that* $\dim W = m$ *and* $\max_{u\in W} I_\lambda(u) < M_m$*.*

P r o o f. Let O and U be defined respectively as in $(\mathbf{F}_1)$ and in $(\mathbf{G}_3)$. We can build the space W, in the same way as in [28, Lemma 4.3]. So, we consider $v_1, \ldots\ldots, v_m$ such that $v_i \in C_O^\infty(\Omega)$, $\operatorname{supp} v_i \cap \operatorname{supp} v_j = \emptyset$, $\operatorname{supp} v_i \cap O \neq \emptyset$ and $\operatorname{supp} v_i \cap U \neq \emptyset$, where $i = 1, \ldots, m$, $j = 1, \ldots, m$, $i \neq j$.

By (2), we have

$$\begin{aligned} I_\lambda(u) &= \int_\Omega \frac{1}{p(x)} A(|\nabla u|^{p(x)})dx + \int_{\partial\Omega} \frac{1}{p(x)} b(x)|u|^{p(x)} d\sigma_x - \lambda \int_\Omega F(x,u)dx - \int_{\partial\Omega} G(x,u)d\sigma_x \\ &\leq \frac{\max(1,K)}{p^-} \max(\|u\|^{p^-}, \|u\|^{p^+}) - \lambda \int_\Omega F(x,u)dx - \int_{\partial\Omega} G(x,u)d\sigma_x, \end{aligned}$$

where K is defined in $(\mathbf{A}_0)$.

For $u \in W$, since $\operatorname{supp} u \cap O \neq \emptyset$ we get

$$I_\lambda(u) \leq \frac{\max(1,K)}{p^-} \max(\|u\|^{p^-}, \|u\|^{p^+}) - \int_{\partial\Omega} G(x,u)d\sigma_x = \widetilde{I}(u).$$

Since

$$\max_{u\in W\backslash\{0\}} I_\lambda(u) \leq \max_{u\in W\backslash\{0\}} \widetilde{I}(u) = \max_{v\in\partial B_1(0)\cap W\backslash\{0\}} \widetilde{I}(v).$$

For $t > 0$ and $u \in \partial B_1(0) \cap W\backslash\{0\}$ and ε small enough, by $(\mathbf{F}_1)$, $(\mathbf{G}_2)$–$(\mathbf{G}_3)$ and (3.9), we obtain

$$\begin{aligned} \widetilde{I}(tu) &= \frac{\max(1,K)}{p^-} \max(\|tu\|^{p^-}, \|tu\|^{p^+}) - \int_{\partial\Omega} G(x,tu)d\sigma_x \\ &\leq C_5\|tu\|^{p^-} - t^\mu \int_{\partial\Omega} \left(\frac{\varepsilon}{p^+}|u|^{p^+} + C_\varepsilon |u|^{q(x)}\right) d\sigma_x \leq C_5 t^{p^-}\|u\|^{p^-} - C_6 t^\mu \|u\|^{q^-}, \end{aligned}$$

where $C_5 = \max(1, K)/p^-$ and C_6 is the constant of embedding $X \hookrightarrow L^{q(x)}(\partial\Omega)$,

$$\lim_{t\to+\infty} \widetilde{I}(tu) \leq \lim_{t\to+\infty} \left[C_5 t^{p^-} - C_6 t^{\mu}\right]. \tag{3.13}$$

Since $\mu > p^-$, by (3.13) we get that there exist a subspace W of X and a constant $M_m > 0$, independent of λ, such that $\dim W = m$ and $\max\limits_{u\in W} I_\lambda(u) < M_m$. The proof of Lemma 4 is complete.□

According to Lemma 2, we also have that I_λ satisfies (**I'**$_3$). Since $I_\lambda(0) = 0$ and I_λ is even, we may apply Theorem 3 to conclude that I_λ has infinitely many nontrivial solutions.

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