

RESTRAINED DOUBLE MONOPHONIC NUMBER OF A GRAPH¹

A.P. Santhakumaran²

Department of Mathematics, Hindustan Institute of Technology and Science,
Chennai – 603 103, India
apskumar1953@gmail.com

K. Ganesamoorthy

Department of Mathematics, Coimbatore Institute of Technology,
Coimbatore – 641 014, India
kvgm_2005@yahoo.co.in

Abstract: For a connected graph G of order at least two, a double monophonic set S of a graph G is a *restrained double monophonic set* if either $S = V$ or the subgraph induced by $V - S$ has no isolated vertices. The minimum cardinality of a restrained double monophonic set of G is the *restrained double monophonic number* of G and is denoted by $dm_r(G)$. The restrained double monophonic number of certain classes graphs are determined. It is shown that for any integers a, b, c with $3 \leq a \leq b \leq c$, there is a connected graph G with $m(G) = a$, $m_r(G) = b$ and $dm_r(G) = c$, where $m(G)$ is the monophonic number and $m_r(G)$ is the restrained monophonic number of a graph G .

Keywords: Monophonic set, Restrained monophonic set, Restrained monophonic number, Restrained double monophonic set, Restrained double monophonic number.

AMS Subject Classification: 05C12.

Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology we refer to Harary [5]. For vertices u and v in a connected graph G , the *distance* $d(u, v)$ is the length of a shortest $u - v$ path in G . An $u - v$ path of length $d(u, v)$ is called an $u - v$ *geodesic*. It is known that d is a metric on the vertex set V of G [2]. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v . A vertex v is an *extreme vertex* if the subgraph induced by its neighbors is complete.

The *closed interval* $I[x, y]$ consists of all vertices lying on some $x - y$ geodesic of G , while for $S \subseteq V$, $I[S] = \bigcup_{x, y \in S} I[x, y]$. A set S of vertices of G is a *geodetic set* if $I[S] = V$, and the minimum cardinality of a geodetic set is the *geodetic number* $g(G)$. The geodetic number of a graph was introduced in [2, 6] and further studied in [3, 4]. It was shown in [6] that determining the geodetic number of a graph is an NP-hard problem. A geodetic set S of a graph G is a *restrained geodetic set* if the subgraph $G[V - S]$ induced by $V - S$ has no isolated vertex. The minimum cardinality of a restrained geodetic set of G is the *restrained geodetic number* of G . The restrained geodetic

¹The second author research work was supported by National Board for Higher Mathematics, INDIA (Project No. NBHM/R.P.29/2015/Fresh/157).

²Former Professor

number of a graph was introduced and studied in [1]. Let 2^V denote the set of all subsets of V . The mapping $I : V \times V \rightarrow 2^V$ defined by $I[u, v] = \{w \in V : w \text{ lies on a } u - v \text{ geodesic in } G\}$ is the *interval function* of G . One of the basic properties of I is that $u, v \in I[u, v]$ for any pair $u, v \in V$. Hence the interval function captures every pair of vertices and so the problem of double geodetic sets is trivially well-defined while it is clear that this fails in many graphs already for triplets (for example, complete graphs). This motivated us to introduce and study double geodetic sets in [7] and further double monophonic sets in [9]. Also, double monophonic parameters like the upper double monophonic number of a graph and the connected double monophonic number of a graph were studied in [10, 11]. This is the basis behind the introduction and study of the restrained double monophonic number of a graph. A set S of vertices is called a *double geodetic set* of G if for each pair of vertices x, y in G there exist vertices $u, v \in S$ such that $x, y \in I[u, v]$. The *double geodetic number* $dg(G)$ is the minimum cardinality of a double geodetic set. The double geodetic number of a graph was introduced and studied in [7].

A *chord* of a path P is an edge joining two non-adjacent vertices of P . A path P is called a *monophonic path* if it is a chordless path. A set S of vertices of G is a *monophonic set* of G if each vertex v of G lies on an $x - y$ monophonic path for some $x, y \in S$. The minimum cardinality of a monophonic set of G is the *monophonic number* of G and is denoted by $m(G)$. The monophonic number of a graph was studied and discussed in [8]. A set S of vertices of G is called a *double monophonic set* of G if for each pair of vertices x, y in G there exist vertices u, v in S such that x, y lie on a $u - v$ monophonic path. The *double monophonic number* $dm(G)$ of G is the minimum cardinality of a double monophonic set of G . The concept of double monophonic number of a graph was introduced and studied in [9].

The concept of distance in graphs is a major component in graph theory with its centrality and convexity concepts having numerous applications to real life problems. There are several interesting applications of these concepts to facility location in real life situations, routing of transport problems and communication network designs. As the paths involved in the discussion of this paper are monophonic, no intervention by hackers or enemies is possible to the respective facilities provided. Further, as monophonic paths are secured and longer than geodesic paths, it is advantageous to more customers in getting the service with protection.

The following theorems will be used in the sequel.

Theorem 1 [8]. *Each extreme vertex of a connected graph G belongs to every monophonic set of G .*

Theorem 2 [8]. *For the complete graph K_p ($p \geq 2$), $m(K_p) = p$.*

Theorem 3 [7]. *Each extreme vertex of a connected graph G belongs to every double geodetic set of G .*

Throughout this paper G denotes a connected graph with at least two vertices.

1. Restrained double monophonic number

To study the main concepts of this paper, we introduce first the restrained monophonic number of a graph and the restrained double geodetic number of a graph, and then prove some basic results and proceed.

Definition 1. *A restrained monophonic set S of a graph G is a monophonic set such that either $S = V$ or the subgraph induced by $V - S$ has no isolated vertices. The minimum cardinality of a restrained monophonic set of G is the **restrained monophonic number** of G and is denoted by $m_r(G)$.*

Example 1. For the cycle $C_5 : u, v, w, x, y, u$ of order 5, it is easily verified that $S = \{u, w\}$ is a minimum monophonic set of C_5 and so $m(C_5) = 2$. Since the subgraph induced by $V - S$ has an isolated vertex v , S is not a restrained monophonic set of C_5 . It is clear that, $S \cup \{v\}$ is a minimum restrained monophonic set of C_5 so that $m_r(C_5) = 3$. Thus the monophonic number and the restrained monophonic number of a graph are different.

It is clear that every restrained monophonic set of G is a monophonic set of G and so Theorem 1 gives the next result.

Theorem 4. *Each extreme vertex of a connected graph G belongs to every restrained monophonic set of G .*

Corollary 1. *For the complete graph $K_p(p \geq 2)$, $m_r(K_p) = p$.*

Definition 2. *A double geodetic set S of a graph G is a **restrained double geodetic set** if either $S = V$ or the subgraph induced by $V - S$ has no isolated vertices. The minimum cardinality of a restrained double geodetic set of G is the **restrained double geodetic number** of G and is denoted by $dg_r(G)$.*

Example 2. For the cycle C_4 of order 4, it is clear that any set S of two non-adjacent vertices of C_4 is a minimum double geodetic set of C_4 and so $dg(C_4) = 2$. Since the subgraph induced by $V - S$ has an isolated vertices, S is not a restrained double geodetic set of C_4 . Also, no 3-element subset of $V(C_4)$ is a restrained double geodetic set of C_4 . Thus $V(C_4)$ is the unique minimum restrained double geodetic set of C_4 and so $dg_r(C_4) = 4$. Hence the double geodetic number and the restrained double geodetic number of a graph are different.

It is clear that every restrained double geodetic set of G is a double geodetic set of G and so Theorem 3 gives the next result.

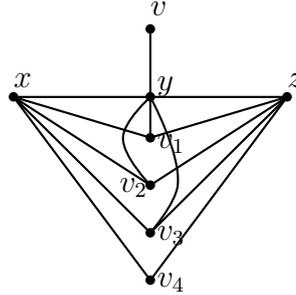
Theorem 5. *Each extreme vertex of a connected graph G belongs to every restrained double geodetic set of G .*

Now, we introduce the restrained double monophonic number of a graph and investigate.

Definition 3. *A double monophonic set S of a graph G is a **restrained double monophonic set** if either $S = V$ or the subgraph induced by $V - S$ has no isolated vertices. The minimum cardinality of a restrained double monophonic set of G is the **restrained double monophonic number** of G and is denoted by $dm_r(G)$.*

Example 3. For the graph G given in Fig. 1, it is clear that no 2-element subset of vertices of G is a monophonic set of G . Since $S = \{v, x, z\}$ is a monophonic set of G , $m(G) = 3$. It is the only minimum monophonic set of G . Since the subgraph induced by $V - S$ has the isolated vertex v_4 , it is not a restrained monophonic set of G . Now, the set $S_1 = \{v, x, z, v_4\}$ is a restrained monophonic set of G so that $m_r(G) = 4$.

It is easily seen that no 4-element subset of vertices of G containing the vertex v is a double monophonic set of G . Also, it is clear that the set $S_2 = \{v, v_1, v_2, v_3, v_4\}$ is the unique minimum double monophonic set of G . Since the subgraph induced by $V - S_2$ has no isolated vertices, S_2 is a minimum restrained double monophonic set of G so that $dm_r(G) = 5$. Thus the monophonic number, the restrained monophonic number and the restrained double monophonic number of a graph are different.

Figure 1. Graph G .

Theorem 6. *Every extreme vertex of a connected graph G belongs to every restrained double monophonic set of G . In particular, if the set of all extreme vertices of G is a restrained double monophonic set, then it is the unique minimum restrained double monophonic set of G .*

P r o o f. Since every restrained double monophonic set is a monophonic set, the result follows from Theorem 1. \square

The following results are easy consequences of Theorem 6.

Result 1. *For the complete graph K_p ($p \geq 2$), $dm_r(G) = p$.*

Result 2. *For a graph G of order p with k extreme vertices, $\max\{2, k\} \leq dm_r(G) \leq p$.*

Result 3. *If T is a tree of order p with k end-vertices and $p - k \geq 2$, then $dm_r(T) = k$.*

Theorem 7. *For any graph G of order p ,*

$$2 \leq m(G) \leq m_r(G) \leq dm_r(G) \leq p, \quad m_r(G) \neq p - 1 \neq dm_r(G).$$

P r o o f. Any monophonic set needs at least two vertices and hence $m(G) \geq 2$. Since every restrained monophonic set is also a monophonic set of G , it follows that $m(G) \leq m_r(G)$. It is clear that every restrained double monophonic set of G is also a restrained monophonic set and so $m_r(G) \leq dm_r(G)$. Since the set of all vertices of G is a restrained double monophonic set of G , $dm_r(G) \leq p$. From the definitions of restrained monophonic number and the restrained double monophonic number, it is clear that $m_r(G) \neq p - 1 \neq dm_r(G)$. \square

Remark 1. The bounds in Theorem 7 are sharp. The two end-vertices of a nontrivial path P_n on n vertices is its unique minimum monophonic set so that $m(P_n) = 2$ and for the complete graph K_p ($p \geq 2$), we have $dm_r(K_p) = p$. Also, all the inequalities in Theorem 7 can be strict, for the graph G of order 8 given in Fig. 1, $m(G) = 3$, $m_r(G) = 4$ and $dm_r(G) = 5$. Thus we have $2 < m(G) < m_r(G) < dm_r(G) < p$.

Theorem 8. *For any graph G of order p , $2 \leq dm_r(G) \leq dg_r(G) \leq p$, $dm_r(G) \neq p - 1 \neq dg_r(G)$.*

P r o o f. Any restrained double monophonic set needs at least two vertices and so $dm_r(G) \geq 2$. It is clear that every restrained double geodetic set of G is also a restrained double monophonic set and so $dm_r(G) \leq dg_r(G)$. Since the set of all vertices of G is a restrained double geodetic set of G , $dg_r(G) \leq p$. From the definitions of restrained double monophonic number and the restrained

double geodetic number, it is clear that $dm_r(G) \neq p - 1 \neq dg_r(G)$. □

Remark 2. The bounds in Theorem 8 are sharp. For the path $P_n (n \geq 4)$, $dm_r(G) = dg_r(G) = 2$ and for the complete graph $K_p (p \geq 3)$, $dm_r(K_p) = dg_r(K_p) = p$. All the inequalities in Theorem 8 can be strict. For the graph G of order 7 given in Fig. 2, no 2-element subset of $V(G)$ forms a minimum restrained double monophonic set of G . The minimum restrained double monophonic sets of G are $S_1 = \{v_1, v_2, v_5\}$ and $S_2 = \{v_1, v_2, v_6\}$ so that $dm_r(G) = 3$. Also, there is no 3-element or 4-element subset of $V(G)$ forms a minimum restrained double geodetic set of G . It is easy to verify that $S_3 = \{v_1, v_2, v_4, v_5, v_6\}$ is a minimum restrained double geodetic set of G and so $dg_r(G) = 5$. Thus we have $2 < dm_r(G) < dg_r(G) < p$.

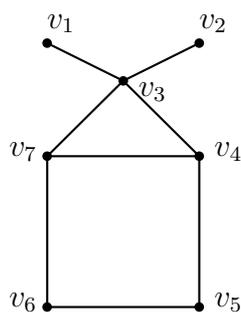


Figure 2. Graph G .

The following results are easy to prove.

Result 4. For any cycle C_p , $dm_r(C_p) = \begin{cases} p, & \text{if } p = 3, 4, \\ 3, & \text{if } p = 5, \\ 2, & \text{if } p \geq 6. \end{cases}$

Result 5. For any wheel $W_p = K_1 + C_{p-1}$, ($p \geq 4$), $dm_r(W_p) = \begin{cases} 4, & \text{if } p = 4, \\ 2, & \text{if } p \geq 5. \end{cases}$

Result 6. For the complete bipartite graph $G = K_{m,n} (2 \leq m \leq n)$,

$$dm_r(G) = \begin{cases} n + 2, & \text{if } 2 = m \leq n, \\ 4, & \text{if } 3 \leq m \leq n. \end{cases}$$

In view of Theorem 7, we have the following realization theorem.

Theorem 9. For any three integers a, b, c with $3 \leq a \leq b \leq c$, there is a connected graph G with $m(G) = a$, $m_r(G) = b$ and $dm_r(G) = c$.

P r o o f. This theorem is proved by considering four cases.

Case 1. $a = b = c$. Then, for the complete graph $G = K_a$, by Theorem 2, Corollary 1 and Result 1, $m(G) = m_r(G) = dm_r(G) = a$.

Case 2. $a = b < c$. Let G be the graph in Fig. 3 is got by adding $a - 1$ new vertices $w_1, w_2, \dots, w_{a-2}, x$ to the complete bipartite graph $K_{2,c-a+1}$ with the partite sets $U = \{u_1, u_2\}$ and $W = \{v_1, v_2, \dots, v_{c-a+1}\}$, joining each vertex $w_i (1 \leq i \leq a - 2)$ to the vertex u_1 and joining the vertex x to the vertex v_1 . By Theorems 1, 4 and 6, every monophonic set, every restrained monophonic

set and every restrained double monophonic set of G contain the set $S = \{w_1, w_2, \dots, w_{a-2}, x\}$ of all extreme vertices of G . Clearly, S is not a monophonic set of G . It is easy to verify that $S_1 = S \cup \{u_2\}$ is the unique minimum monophonic set of G and so $m(G) = a$. Since the subgraph induced by $V - S_1$ has no isolated vertices, S_1 is the unique minimum restrained monophonic set of G so that $m_r(G) = a = m(G)$. It is clear that the pair of vertices x, v_i ($i = 2, 3, \dots, c-a+1$) do not lie on any $u-v$ monophonic path, for any $u, v \in S_1$ and so S_1 is not a restrained double monophonic set of G . It is easy to verify that $S_2 = S \cup \{u_2, v_2, \dots, v_{c-a+1}\}$ is a minimum double monophonic set of G and the subgraph induced by $V - S_2$ has no isolated vertices so that $dm_r(G) = c$.

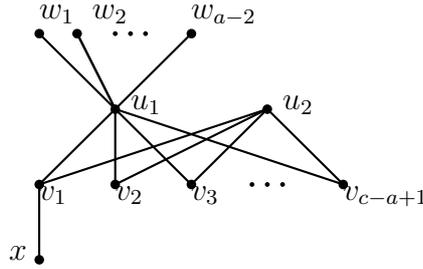


Figure 3. Graph G .

Case 3. $a < b = c$. Let G be the graph in Fig. 4 formed from the path $P_3 : v_1, v_2, v_3$ of order 3, by adding b new vertices $u_1, u_2, \dots, u_{a-2}, w_1, w_2, \dots, w_{b-a+2}$ to P_3 and joining each vertex u_i ($1 \leq i \leq a-2$) to v_2 ; and joining each vertex w_j ($1 \leq j \leq b-a+2$) to v_1 and v_3 . By Theorems 1, 4 and 6, every monophonic set, every restrained monophonic set and every restrained double monophonic set of G contain the set $S = \{u_1, u_2, \dots, u_{a-2}\}$ of all extreme vertices of G . Clearly, S is not a monophonic set of G . Also, for any $x \in V - S$, $S \cup \{x\}$ is not a monophonic set of G . It is easy to verify that $S_1 = S \cup \{v_1, v_3\}$ is a minimum monophonic set of G and so $m(G) = a$. Since the subgraph induced by $V - S_1$ has the isolated vertices $w_1, w_2, \dots, w_{b-a+2}, v_2$, S_1 is not a restrained monophonic set of G . It is clear that, every restrained monophonic set and every restrained double monophonic set of G contains $\{w_1, w_2, \dots, w_{b-a+2}\}$ and it follows that $S_2 = S \cup \{w_1, w_2, \dots, w_{b-a+2}\}$ is a minimum restrained monophonic set and a minimum restrained double monophonic set of G so that $m_r(G) = dm_r(G) = b$.

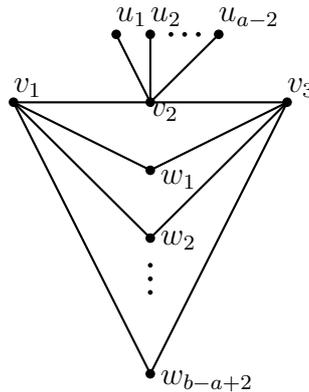


Figure 4. Graph G .

Case 4. $a < b < c$. Let G be the graph in Fig. 5 formed from the path $P_3 : x, y, z$ of order 3 by adding c new vertices $u_1, u_2, \dots, u_{a-2}, w_1, w_2, \dots, w_{b-a}, v_1, v_2, \dots, v_{c-b+2}$ to P_3 and joining each vertex v_i ($1 \leq i \leq c-b+2$) to the vertices x, y and z ; joining each vertex w_j ($1 \leq j \leq b-a$) to the

vertices x and z ; joining each vertex $u_i(1 \leq k \leq a-2)$ to the vertex y . By Theorems 1, 4 and 6, every monophonic set, every restrained monophonic set and every restrained double monophonic set of G contain the set $S = \{u_1, u_2, \dots, u_{a-2}\}$ of all extreme vertices of G . Clearly, S is not a monophonic set of G and also for any $u \in V(G) - S$, $S \cup \{u\}$ is not a monophonic set of G . It is easily verified that $S_1 = S \cup \{x, z\}$ is a minimum monophonic set of G and so $m(G) = a$. Since the subgraph induced by $V - S_1$ has the isolated vertices w_2, w_3, \dots, w_{b-a} , S_1 is not a restrained monophonic set of G . It is clear that every restrained monophonic set of G contains $\{w_2, w_3, \dots, w_{b-a}\}$. Then $S_2 = S_1 \cup \{w_1, w_2, \dots, w_{b-a}\}$ is a minimum restrained monophonic set of G and so $m_r(G) = b$.

Now, any double monophonic set of G should contain the set S . It is easily verified that the set $S' = \{u_1, u_2, \dots, u_{a-2}, w_1, w_2, \dots, w_{b-a}, v_1, v_2, \dots, v_{c-b+2}\}$ is the unique minimum double monophonic set of G . Since the subgraph induced by $V - S'$ has no isolated vertices, it follows that S' is a minimum restrained double monophonic set of G so that $dm_r(G) = c$. \square

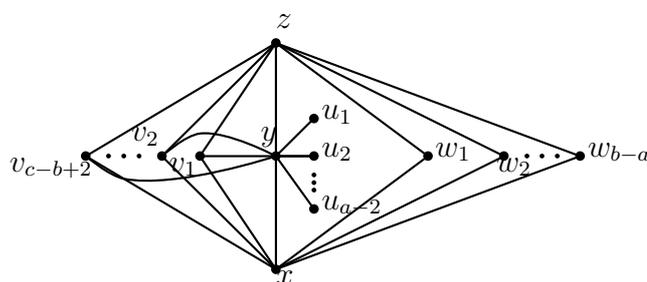


Figure 5. Graph G .

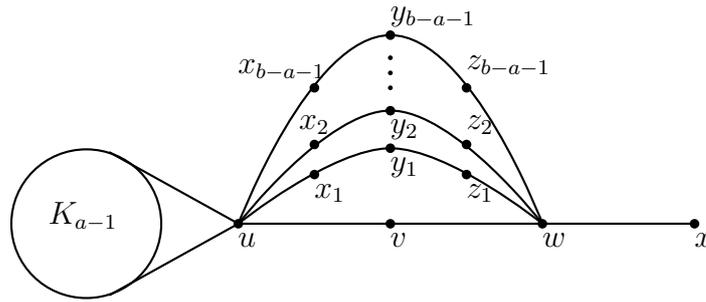
Theorem 10. For any integer $p \geq 4$ with $2 \leq k \leq p$, $k \neq p - 1$, there is a connected graph G of order p such that $dm_r(G) = k$.

P r o o f. Let $G \neq K_{1,p-1}$ be any tree of order p with k end-vertices. Then clearly, $dm_r(G) = k$. \square

In view of Theorem 8, we have the following realization theorem.

Theorem 11. For every pair a, b of integers with $3 \leq a \leq b$, there is a connected graph G with $dm_r(G) = a$ and $dg_r(G) = b$.

P r o o f. For $3 \leq a = b$, by Theorem 5 and Result 1, the complete graph K_a of order a has the desired properties. So, assume that $3 \leq a < b$. Let H be the graph obtained from the complete graph K_{a-1} and the path $P_4 : u, v, w, x$ of order 4 by joining all the vertices of K_{a-1} to the vertex u of P_4 . Let G be the graph in Fig. 6 obtained from H by taking ‘ $b - a - 1$ ’ copies of the path $P_i : x_i, y_i, z_i(1 \leq i \leq b - a - 1)$ of order 3 and joining each vertex $x_i(1 \leq i \leq b - a - 1)$ in P_i and u in H ; and also joining each vertex $z_i(1 \leq i \leq b - a - 1)$ in P_i and w in H . Let $S = V(K_{a-1}) \cup \{x\}$ be the set of all extreme vertices of G . By Theorems 5 and 6, every restrained double geodetic set and every restrained double monophonic set of G contain S . Clearly, S is the unique minimum restrained double monophonic set of G and so $dm_r(G) = a$. Since the pair of vertices $v, y_i(i = 1, 2, 3, \dots, b - a - 1)$, do not lie on any geodesic joining a pair of vertices from S , S is not a restrained double geodetic set of G . Let $S' = S \cup \{v, y_1, y_2, \dots, y_{b-a-1}\}$. It is easy to verify that S' is a minimum double geodetic set of G and the subgraph induced by $V - S'$ has no isolated vertices. Thus S' is a minimum restrained double geodetic set of G and so $dg_r(G) = b$. \square

Figure 6. Graph G .

Theorem 12. If G' is a graph obtained by adding k pendant vertices to a connected graph G , then $dm_r(G) \leq dm_r(G') \leq dm_r(G) + k$.

Proof. Let G' be the connected graph obtained from G by adding k pendant vertices $v_i (1 \leq i \leq k)$ to the vertices $u_l (1 \leq l \leq k)$ where each u_l is a vertex of G and each $v_i (1 \leq i \leq k)$ is not a vertex of G . Note that u_1, u_2, \dots, u_k need not be distinct. Let S be a minimum restrained double monophonic set of G . Then $S \cup \{v_1, v_2, \dots, v_k\}$ is a restrained double monophonic set of G' and so $dm_r(G') \leq dm_r(G) + k$.

Now, we claim that $dm_r(G) \leq dm_r(G')$. Suppose that $dm_r(G) > dm_r(G')$. Let S' be a restrained double monophonic set of G' with $|S'| < dm_r(G)$. Since each $v_i (1 \leq i \leq k)$ is an extreme vertex of G' , it follows from Theorem 6 that $\{v_1, v_2, \dots, v_k\} \subseteq S'$. Let $S = (S' - \{v_1, v_2, \dots, v_k\}) \cup \{u_1, u_2, \dots, u_l\} (1 \leq l \leq k)$. Then S is a subset of $V(G)$ and $|S| \leq |S'| - k + l = |S'| - (k - l) \leq |S'| < dm_r(G)$. Now, we show that S is a restrained double monophonic set of G . Let $u, v \in V(G) - S$. Then $u, v \in V(G')$ also. Since S' is a restrained double monophonic set of G' , u and v lie on a $x - y$ monophonic path P in G' for some vertices $x, y \in S'$. If neither x nor y is $v_i (1 \leq i \leq k)$, then $x, y \in S$. If exactly one of x, y is $v_i (1 \leq i \leq k)$, say $x = v_i$, then u and v lie on a $x - y$ monophonic path in G , where $y = u_j$ and u_j is adjacent to v_j in G' where $i \neq j$. If both $x, y \in \{v_1, v_2, \dots, v_k\}$, then let $x = v_i$ and $y = v_j$ where $i \neq j$. Hence u and v lie on the $u_s - u_t$ monophonic path in G , where u_s is adjacent to v_i and u_t is adjacent to v_j in G' . Thus S is a restrained double monophonic set of G . Hence $dm_r(G) \leq |S| < dm_r(G)$, which is a contradiction. \square

Remark 3. The bounds for $dm_r(G')$ in Theorem 12 are sharp. Consider a tree T with number of end-vertices $l \geq 3$ and at least two internal vertices. Let $S = \{v_1, v_2, \dots, v_l\}$ be the set of all end-vertices of T . Then by Result 3, $dm_r(T) = l$. If we add a pendant vertex to an end-vertex of T , then we obtain another tree T' with l end-vertices. Hence $dm_r(T) = dm_r(T')$. On the other hand, if we add k pendant vertices to a cut-vertex of T , then we obtain a tree T' with $k + l$ end-vertices. Then by Result 3, $dm_r(T') = dm_r(T) + k$.

2. Conclusions

In this paper, the concept of restrained double monophonic number of a graph is introduced and certain general properties satisfied by this parameter are studied. This parameter is determined for several standard graphs. Also, certain realisation results of this parameter are proved with regard to certain other parameters like monophonic number, restrained monophonic number and restrained double geodetic number of a graph. As a future work of this paper, new parameters like

upper restrained double monophonic number of a graph, forcing restrained double monophonic number of a graph can be developed and investigated.

Acknowledgement

The authors are thankful to the reviewers for their useful comments for the improvement of this paper.

REFERENCES

1. Abdollahzadeh Ahangar H., Samodivkin V., Sheikholeslami S.M. and Abdollah Khodkar. The Restrained Geodetic Number of a Graph. *Bull. Malays. Math. Sci. Soc.*, 2015. Vol. 38. P. 1143–1155. DOI: [10.1007/s40840-014-0068-y](https://doi.org/10.1007/s40840-014-0068-y)
2. Buckley F., Harary F. *Distance in Graphs*. Addison-Wesley, Redwood City, CA, 1990. 335 p.
3. Chartrand G., Harary F. and Zhang P. On the geodetic number of a graph. *Networks*, 2002. Vol. 39, No. 1. P. 1–6. DOI: [10.1002/net.10007](https://doi.org/10.1002/net.10007)
4. Chartrand G., Johns G.L., and Zhang P. On the detour number and geodetic number of a graph. *Ars Combin.*, 2004. Vol. 72. P. 3–15.
5. Harary F. *Graph Theory*, Addison-Wesley, 1969.
6. Harary F., Loukakis E. and Tsouros C. The geodetic number of a graph. *Math. Comput. Modelling*, 1993. Vol. 17, No. 11. P. 89–95. DOI: [10.1016/0895-7177\(93\)90259-2](https://doi.org/10.1016/0895-7177(93)90259-2)
7. Santhakumaran A. P. and Jebaraj T. Double geodetic number of a graph. *Discuss. Math. Graph Theory*, 2012, Vol. 32, No. 1. P. 109–119. DOI: [10.7151/dmgt.1589](https://doi.org/10.7151/dmgt.1589)
8. Santhakumaran A. P., Titus P. and Ganesamoorthy K. On the monophonic number of a graph. *J. Appl. Math. Inform.*, 2014. Vol. 32, No. 1–2. P. 255–266. DOI: [10.14317/jami.2014.255](https://doi.org/10.14317/jami.2014.255)
9. Santhakumaran A. P. and Venakata Raghu T. Double monophonic number of a graph. *Int. J. Comput. Appl. Math.*, 2016. Vol. 11, No. 1. P. 21–26. https://www.ripublication.com/ijcam16/ijcamv11n1_03.pdf
10. Santhakumaran A. P. and Venakata Raghu T. Upper double monophonic number of a graph. *Proyecciones*, 2018, Vol. 37, No. 2. P. 295–304. DOI: [10.4067/S0716-09172018000200295](https://doi.org/10.4067/S0716-09172018000200295)
11. Santhakumaran A. P. and Venakata Raghu T. Connected double monophonic number of a graph. *Int. J. Math. Comb.*, 2018, Special Issue 1. P. 54–60.