ON CONNECTIONS BETWEEN GENERALIZED SOLUTIONS OF PDE'S OF THE FIRST ORDER[1]

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The paper is devoted to the connection between generalized solutions of the Cauchy problem for Hamilton–Jacobi–Bellman equation and the corresponding first order quasilinear PDE in the of case n-dimentional state space.

Keywords: Hamilton–Jacobi–Bellman equation, Quasilinear equation, Viscosity solution.

Introduction

The paper is concerned with the link between two major classes of the first order PDEs, namely Hamilton–Jacobi–Bellman equations and quasilinear PDEs. The classical solution may not exist globally for these equations. Thus both types of equations require the notion of generalized solution. For Hamilton–Jacobi equation the theory of generalized solution was developed by A.I. Subbotin [1] and M. Crandall, P.L. Lions [2], whereas the quasilinear case was studied by O.A. Oleinik, S.N. Kruzhkov [3]. The generalized solutions of Hamilton–Jacobi equations are called minimax solutions or viscosity solutions. A.I. Subbotin proved the equivalence of these notions. The generalized solutions of quasilinear PDEs are called entropy solutions. This term is due to entropy condition. The existence and uniqueness theorems for Hamilton–Jacobi PDEs can be found in [1,2], for quasilinear PDEs these theorems can be found in [3].

If we formally differentiate Hamilton–Jacobi equation with respect to phase variable, then in some cases we get the quasilinear equation. But the question about a link between these generalized solutions is opened, because the minimax/viscosity solution is only continuous function. Note that the minimax solution of Hamilton–Jacobi equation is not differentiable. Thus this differentiation should be performed with the help of methods of nonsmooth analysis.

First one-dimensional case was considered by N.N. Yanenko and B.L. Rozchdestveskii [4]. In that book a link between the Hamilton–Jacobi PDE and quasilinear equation was mentioned. The strong result is presented in [5].

In this paper we extend the results of [5] to n-dimensional case.

We assume that the Hamiltonian depends on the linear combination of impulse variables and is convex with respect to them. This type of equations describes the value function of optimal control problem with dynamics

$$\dot{x} = H_s\left(t, x, \sum_{i=1}^{n} \lambda_i s_i\right), \quad \lambda_i \in \mathbb{R}, \quad i = 1, \ldots, n,$$

where $||s|| \leq D$. The restriction D is determined by the Hamiltonian. The purpose is to minimize the functional

$$I = \sigma(x) + \int_{t_0}^{T} \langle s, H_s(t, x, s)\rangle - H(t, x, s)dt$$

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on the set of admissible controls s. It is known from the work of A.I. Subbotin, N.N. Subbotina, that if the Hamiltonian and boundary function σ are smooth enough, then the value function is locally Lipschitz continuous. Thus the value function is differentiable almost everywhere.

In the paper we prove that the generalized divergence of the minimax/viscosity solution coincides with the generalized solution of the Cauchy problem for quasilinear equation. The structure of the entropy solution of the Cauchy problem for quasilinear equation is described in the work [6] in the cases of piecewise smooth solution and the solution belonging to the class of function with locally bounded variation. This structure is valid for derivatives of the minimax/viscosity solution for Hamilton–Jacobi equation.

1. Preliminaries

1.1. Viscosity solution

Let us consider the Cauchy problem for the Hamilton–Jacobi–Bellman equation

$$\frac{\partial\varphi}{\partial t} + H\Big(t, x, \sum_{i=1}^{n} \lambda_i \frac{\partial\varphi}{\partial x_i}\Big) = 0, \quad \varphi(0,x) = \sigma(x) \tag{1.1}$$

Here $(t,x) \in \Pi_T = [0,T] \times \mathbb{R}^n$, $\lambda_i \in \mathbb{R}$. Remind the definition of Holder spaces.

D e f i n i t i o n 1. The Holder space $C^{2+\alpha}$ is the space of twice continuously differentiable functions those the second derivatives are Holder continuous of order $\alpha \in (0,1]$.

We assume that

(A1) the function $H(t,x,s) \in C^{2+\alpha}$;

(A2) the function $H(t,x,s)$ satisfies sublinear growth condition and it is convex with respect to s;

(A3) the function $\sigma(x) \in C^{2+\alpha}$ and it is bounded in $\mathbb{R}^n$.

In general case this problem has no continuously differentiable (classical) solution in the strip Π_T [1].

The generalized solution of problem (1.1) can be approximated by the solutions φ_ε of the following Cauchy problem

$$\frac{\partial\varphi_\varepsilon}{\partial t} + H\Big(t, x, \sum_{i=1}^{n} \lambda_i \frac{\partial\varphi_\varepsilon}{\partial x_i}\Big) = \varepsilon\Delta_x\varphi_\varepsilon, \quad \varphi_\varepsilon(0,x) = \sigma(x). \tag{1.2}$$

Here $\Delta_x = \sum\limits_{i=1}^{n} \dfrac{\partial^2}{\partial x_i^2}$ is the Laplace operator, $\varphi_\varepsilon \in C^{2+\alpha}$.

It is proved in the work [2] for the viscosity solution φ that

$$\varphi(t,x) = \lim_{\varepsilon\to 0} \varphi_\varepsilon(t,x) \text{ locally uniformly in } \Pi_T.$$

It is known [1], that the viscosity solution exists, it is unique and it is locally Lipschitz function in Π_T. The superdifferential of the viscosity solution is not empty at all points of Π_T.

1.2. Entropy solution

Consider the Cauchy problem

$$\frac{\partial w}{\partial t} + \sum_{i=1}^{n} \lambda_i H_{x_i}(t,x,w) = 0, \quad w(0,x) = \sum_{i=1}^{n} \lambda_i \frac{\partial\sigma(x)}{\partial x_i}. \tag{1.3}$$

Here
$$H_{x_i}(t,x,w)=\frac{\partial H}{\partial x_i}+\frac{\partial H}{\partial w}\frac{\partial w}{\partial x_i}.$$

Recall the definition of generalized (entropy) solution of problem (1.3), proposed by S.N. Kruzchkov [3].

D e f i n i t i o n 2. A Lebesgue measurable function $w:\Pi_T\to\mathbb{R}$ is called an entropy solution of problem (1.3), if for any constant $k\in\mathbb{R}$, nonnegative function $f\in C_0^\infty$, the inequality holds

$$\int\limits_{\Pi_T}\sum_{i=1}^n |w(\tau,x)-k|f_\tau+\operatorname{sign}(w(\tau,x)-k)\sum_{i=1}^n\lambda_i(H(\tau,x,k)-H(\tau,x,w))f_{x_i}dxd\tau+$$
$$\int\limits_{\Pi_T}\operatorname{sign}(w(\tau,x)-k)\sum_{i=1}^n\lambda_i H_{x_i}(\tau,x,k)f\,dxd\tau\geq 0,$$
$$w(0,x)=\sum_{i=1}^n\lambda_i\frac{\partial\sigma}{\partial x_i}(x).$$

Under assumptions (A1)–(A3) the entropy solution of problem (1.3) exists and it is unique in the class of locally bounded measurable functions [3].

Let us consider the Cauchy problem
$$\frac{\partial w^\varepsilon}{\partial t}+\sum_{i=1}^n\lambda_i H_{x_i}(t,x,w^\varepsilon)=\varepsilon\Delta_x w^\varepsilon,\quad w^\varepsilon(0,x)=\sum_{i=1}^n\lambda_i\frac{\partial\sigma(x)}{\partial x_i}. \tag{1.4}$$

It is proved [7, 8] that solution w^ε of problem (1.4) exists, it is unique and w^ε is continuously differentiable with respect to t and twice continuously differentiable with respect to x.

The classical solutions w^ε of problem (1.4) pointwise converge to the solution w of problem (1.3), as $\varepsilon\to 0$ almost everywhere in Π_T [3], [6].

D e f i n i t i o n 3. A measurable function $w:\Pi_T\to\mathbb{R}$ is piecewise smooth, if there exists not more then countable set of open domains $D_i\subset\Pi_T$ and dimension of D_i equals $n+1$, such that
$$\bigcup \mathrm{cl}D_i=\Pi_T,\ D_i\cap D_j=\varnothing \text{ for } i\neq j,$$
the function w is continuously differentiable on $\mathrm{cl}D_i$.

Here cl states for the closure.

The following property of solution of (1.3) is proved in [3]. If the solution of problem (1.3) is piecewise smooth in the neighborhood of the point of discontinuity, then the discontinuity surface satisfies the condition
$$\begin{gathered}|w_1-k|\cos\nu\tau+\operatorname{sign}(w_1-k)(H(\tau,x,k)-H(\tau,x,w_1))\cos(\nu,x_i)\leq\\ |w_2-k|\cos\nu\tau+\operatorname{sign}(w_2-k)(H(\tau,x,k)-H(\tau,x,w_2))\cos(\nu,x_i),\quad i=1,\dots,n\end{gathered}\tag{1.5}$$
for any $k\in\mathbb{R}$, for any $(t,x)\in\mathrm{cl}D_i\cap\mathrm{cl}D_j$, $i\neq j$.

Here ν is the normal to the discontinuity surface at the point $(t,x)\in\mathrm{cl}D_i\cap\mathrm{cl}D_j$, $w_1=\lim\limits_{(\tau,x)\to(\tau,x(\tau)+\nu)}w(\tau,x)$, $w_2=\lim\limits_{(\tau,x)\to(\tau,x(\tau)-\nu)}w(\tau,x)$.

Recall a definition from measure theory [9].

D e f i n i t i o n 4. A function w is of locally bounded variation on an open subset $A\subset\Pi_T$ if $w\in L^1_{loc}(A)$ and grad w is a ($\mathbb{R}^{n+1}$-valued) Radon measure M on A, i. e.,
$$-\int\limits_A w\ \mathrm{div}\ \eta(t,x)dxdt=\int\limits_A\eta(t,x)dM(t,x),$$
for any test function $\eta\in C_0^\infty(A)$.

Assertion 1. *If w is the entropy solution of problem* (1.3) *then* $w \in BV_{loc}(\Pi_T)$.

P r o o f. Since the function w is locally bounded measurable, $w \in L^1_{loc}(A)$. Here $A \subset \Pi_T$ is a compact set. It suffices to prove [6, 9], that

$$I = \limsup_{h \to 0} \frac{1}{h} \int\limits_A |w(t+h, x+hE_i) - w(t,x)| dx dt$$

is finite for any compact $A \subset \Pi_T$. Here vectors $E_i \subset \mathbb{R}^n,\ i = 1, \dots, n$ form the orthonormal basis in $\mathbb{R}^n$.

Remind that $w(t,x) = \lim\limits_{\varepsilon \to 0} w^\varepsilon(t,x)$ almost everywhere. Here w^ε is solution (1.4).

According to Lousin's theorem [10] there exist the sets A and A^ε such, that the measure $\mu(A \setminus A^\varepsilon) < \varepsilon$ and the function w is continuous on $(t,x) \in A^\varepsilon$.

Consider the integral I. One can get

$$I \le I_1 + I_2 + I_3,$$

where

$$I_1 = \limsup_{h \to 0} \frac{1}{h} \int\limits_A |w(t+h, x+hE_i) - w^\varepsilon(t+h, x+hE_i)| dx dt,$$

$$I_2 = \limsup_{h \to 0} \frac{1}{h} \int\limits_A |w^\varepsilon(t+h, x+hE_i) - w^\varepsilon(t,x)| dx dt,$$

$$I_3 = \limsup_{h \to 0} \frac{1}{h} \int\limits_A |w(t,x) - w^\varepsilon(t,x)| dx dt.$$

The function $w^\varepsilon \in C^{2+\alpha}$, since it belongs to BV_{loc}. Therefore, $I_2 < \infty$. If we choose the parameter $\varepsilon = h^2$, then we have the estimates

$$|w(t+h, x+hE_i) - w^\varepsilon(t+h, x+hE_i)| < \varepsilon = h^2, \quad |w(t,x) - w^\varepsilon(t,x)| < \varepsilon = h^2 \text{ for } (t,x) \in A^\varepsilon.$$

Consequently $I_1, I_3 < \infty$. □

The domain Π_T is the union of three, pairwise disjoint, subsets C, J, and I with the following properties [6] :

a) C is the set of points of approximate continuity of w, i. e., with each $(\bar t, \bar x) \in C$ is associated $w_0 \in \mathbb{R}$ such that

$$\lim_{r \to 0} \frac{1}{r^{n+1}} \int\limits_{B_r(\bar t, \bar x)} |w(t,x) - w_0| dx dt = 0.$$

b) J is the set of points of approximate jump discontinuity of w, i. e., with each $(\bar t, \bar x) \subset J$ are associated N in unit sphere S^n and distinct w^- , $w^+ \in \mathbb{R}$ such that

$$\lim_{r \to 0} \frac{1}{r^{n+1}} \int\limits_{B^\pm_r(\bar t, \bar x)} |w(t,x) - w^\pm| dx dt = 0,$$

where $B^\pm_r(\bar t, \bar x)$ denote the semiballs $B_r(\bar t, \bar x) \cap \{(t,x) : \langle (t - \bar t, x - \bar x), N \rangle < 0\}$ or $B_r(\bar t, \bar x) \cap \{(t,x) : \langle (t - \bar t, x - \bar x), N \rangle > 0\}$. Moreover, J is essentially covered by the countable union of C^1 n-dimensional manifolds $\{F_i\}$ embedded in $\mathbb{R}^{n+1}$: $H^n(J \setminus \cup F_i) = 0$. Furthermore, when $(\bar t, \bar x) \in J \cap F_i$, then N is normal on F_i at (t,x).

(c) I is the set of irregular points of w; its $n{+}1$-dimensional Hausdorff measure is zero:$H^n(I) = 0$.

2. Connection between generalized solutions of problems (1.1) and (1.3)

Let us introduce a notion of generalized divergence of a nonsmooth field. This notion is similar to a notion of a divergence of a field.

D e f i n i t i o n 4. The set $\operatorname{Div} y(t, x) = \operatorname{co}\{\lim_{(t_k, x_k)\to(t, x)} \operatorname{div} y(t_k, x_k)\}$ is called generalized divergence of the field $y \in \mathbb{R}^n$ at the point (t, x). Here $\operatorname{div} y(t_k, x_k)$ is the divergence of the field y with respect to x, (t_k, x_k) are the points of differentiability of y.

The map $(t, x) \to \operatorname{Div}(\varphi(t, x)\lambda)$ is convex-valued, upper semicontinuous and locally bounded. Really, the map is convex according to definition 4. The map is upper semicontinuous, because the supergradients $\dfrac{\partial\varphi}{\partial x_i} \in D^+_x\varphi(t, x)$ and $D^+_x\varphi(t, x)$ is upper semicontinuous and locally bounded [1].

Hence we can choose a measurable selector $w : \Pi_T \to \mathbb{R}$, $w(t, x) \in \operatorname{Div}(\varphi(t, x)\lambda)$, where φ is the viscosity solution of problem (1.1) [10].

Assertion 2. *If assumptions* (A_1)–(A_3) *are true, then the solution of problem* (1.3) $w(t, x) \in \operatorname{Div}(\lambda\varphi(t, x))$, *where* φ *is the viscosity solution of problem* (1.1).

P r o o f. Since the viscosity solution φ is a Lipschitz continuous function, the map $(t, x) \to \operatorname{Div}\ \varphi(t, x)\lambda$ is single-valued almost everywhere. Let us denote by J the set of points (t, x) such, that the map $(t, x) \to \operatorname{Div}\ \varphi(t, x)\lambda$ is multivalued. The measure of J is equal zero. Consider the case $\operatorname{Div}(\varphi\lambda) < k$. Let the function f be a finite function with compact support $B \subset \Pi_T$.

$$\int_{\Pi_T} (k - \operatorname{Div}\ (\varphi(t,x)\lambda))f_\tau + \sum_{i=1}^n \lambda_i\Big(H\Big(\tau, x, \sum_{i=1}^n \lambda_i \frac{\partial\varphi}{\partial x_i}\Big) - H\Big(\tau, x, k\Big)\Big) f_{x_i} - \lambda_i H_{x_i}(\tau, x, k) f\, dxdt =$$

$$\int_{\Pi_T} kf_\tau - \sum_{i=1}^n \lambda_i (H(\tau, x, k) f_{x_i} + H_{x_i}(\tau, x, k) f) dxdt -$$

$$\int_{\Pi_T\setminus J} \sum_{i=1}^n \lambda_i \Big(\frac{\partial\varphi}{\partial x_i} f_\tau + H(\tau, x, \sum_{i=1}^n \lambda_i \frac{\partial\varphi}{\partial x_i}) f_{x_i}\Big) dxdt \quad (2.1)$$

Let us calculate the first integral in (2.1):

$$\int_{\Pi_T} kf_\tau - \sum_{i=1}^n \lambda_i (H(\tau, x, k) f_{x_i} + H_{x_i}(\tau, x, k) f) dxdt = \int_{\Pi_T} kf_\tau - \sum_{i=1}^n \lambda_i \frac{\partial (H(\tau, x, k) f)}{\partial x_i} dxdt = 0$$

because the function f is finite.

Integrating the second integral in (2.1) by parts several times we set, that

$$\int_{\Pi_T\setminus J} \sum_{i=1}^n \lambda_i \frac{\partial\varphi}{\partial x_i} f_\tau + \sum_{i=1}^n \lambda_i H\Big(\tau, x, \sum_{i=1}^n \lambda_i \frac{\partial\varphi}{\partial x_i}\Big) f_{x_i} dxdt = \int_{t_1}^{t_2} \sum_{i=1}^n \lambda_i \varphi f_\tau|_D dt -$$

$$\int_{\Pi_T\setminus J} \varphi \sum_{i=1}^n \lambda_i \frac{\partial f_\tau}{\partial x_i} + \sum_{i=1}^n \lambda_i H\Big(\tau, x, \sum_{i=1}^n \lambda_i \frac{\partial\varphi}{\partial x_i}\Big) f_{x_i} dxdt =$$

$$\sum_{i=1}^n \lambda_i \varphi f|_{\Pi_T\setminus J} - \int_{t_1}^{t_2} \sum_{i=1}^n \lambda_i \varphi_\tau f|_D dt - \int_D \varphi \sum_{i=1}^n \lambda_i \frac{\partial f}{\partial x_i} dx +$$

$$\int\limits_{\Pi_T\setminus J} \varphi_\tau \sum_{i=1}^{n} \lambda_i \frac{\partial f}{\partial x_i} + \sum_{i=1}^{n} \lambda_i H\Big(\tau, x, \sum_{i=1}^{n} \lambda_i \frac{\partial \varphi}{\partial x_i}\Big) f_{x_i} dxdt =$$

$$-\sum_{i=1}^{n} \lambda_i \varphi f|_{\Pi_T\setminus J} + \int\limits_{D} \sum_{i=1}^{n} \lambda_i \frac{\partial \varphi}{\partial x_i} f|_{t_1}^{t_2} dx + \int\limits_{\Pi_T\setminus J} \Big(\varphi_\tau + \lambda_i H\Big(\tau, x, \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}\Big)\Big) \sum_{i=1}^{n} \lambda_i \frac{\partial f}{\partial x_i} dxdt = 0,$$

hence φ satisfies equation (1.1) on the set $\Pi_T \setminus J$.

Similarly one can consider the case Div $\varphi\lambda > k$. □

Corollary 1. *If the viscosity solution φ of problem* (1.1) *is piecewise smooth, then the set of point $(t, x) \in \Pi_T$, where the function φ is not differentiable, has form* (1.5).

Denote by $D^+\varphi(t, x)$ the superdifferential of function φ at the point (t, x). The map $(t, x) \to D^+\varphi(t, x)$ is upper semicontinuous, concave and locally bounded. So there exists a measurable selector, which belongs to L^1_{loc}. Hence function $w \in \mathrm{Div}(\varphi\lambda) \in L^1_{loc}$. According to assertion 2 the set of definition of the viscosity solution is the union of three, pairwise disjoint, subsets C, J, and I:

a) C is the set of points, where the viscosity solution is continuously differentiable;

b) J is the set of points, where the viscosity solution has a jump. Moreover, J is essentially covered by the countable union of C^1 n-dimensional manifolds $\{F_i\}$ embedded in $\mathbb{R}^{n+1}$: $H^n(J \setminus \cup F_i) = 0$. Furthermore, when $\bar{t}, \bar{x} \in J \cap F_i$, then N is normal on F_i at (t, x).

(c) I is the set of irregular points of φ; its n-dimensional Hausdorff measure is zero:$H^n(I) = 0$.

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