URAL MATHEMATICAL JOURNAL, Vol. 5, No. 1, 2019, pp. 53–58
DOI: 10.15826/umj.2019.1.006

# JACOBI TRANSFORM OF $(\nu,\gamma,p)$-JACOBI–LIPSCHITZ FUNCTIONS IN THE SPACE $\mathrm{L}^p(\mathbb{R}^+,\Delta_{(\alpha,\beta)}(t)dt)$[1]

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**Abstract:** Using a generalized translation operator, we obtain an analog of Younis' theorem [Theorem 5.2, Younis M. S. *Fourier transforms of Dini–Lipschitz functions*, Int. J. Math. Math. Sci., 1986] for the Jacobi transform for functions from the $(\nu,\gamma,p)$-Jacobi–Lipschitz class in the space $\mathrm{L}^p(\mathbb{R}^+,\Delta_{(\alpha,\beta)}(t)dt)$.

**Keywords:** Jacobi operator, Jacobi transform, Generalized translation operator.

## 1. Introduction and preliminaries

Younis [8, Theorem 5.2] characterized the set of functions in $\mathrm{L}^2(\mathbb{R})$ satisfying the Dini–Lipschitz condition by means of an asymptotic estimate of the growth of the norm of their Fourier transforms.

**Theorem 1.** [8, Theorem 5.2] *Let $f\in\mathrm{L}^2(\mathbb{R})$. Then the following conditions are equivalent*:

(1) $\|f(.+h)-f(.)\|_{\mathrm{L}^2(\mathbb{R})}=O\left(\dfrac{h^\alpha}{(\log 1/h)^\beta}\right)$ *as* $h\to 0,\quad 0<\alpha<1,\quad \beta>0,$

(2) $\int_{|\lambda|\geq r}|\mathcal{F}(f)(\lambda)|^2d\lambda=O(r^{-2\alpha}(\log r)^{-2\beta})$ *as* $r\to+\infty,$

*where $\mathcal{F}$ stands for the Fourier transform of $f$.*

The main aim of this paper is to establish an analog of Theorem 1 for the Jacobi transform in the space $\mathrm{L}^p(\mathbb{R}^+,\Delta_{(\alpha,\beta)}(t)dt)$. For this purpose, we use a generalized translation operator which was defined by Flensted–Jensen and Koornwinder [5].

In order to confirm the basic and standard notation, we briefly overview the theory of Jacobi operators and related harmonic analysis. The main references are [1, 4, 6].

Let $\lambda\in\mathbb{C}$, $\alpha\geq\beta\geq-1/2$, and $\alpha\neq 0$. The Jacobi function $\phi_\lambda$ of order $(\alpha,\beta)$ is the unique even $\mathrm{C}^\infty$-solution of the differential equation

$$(\mathrm{D}_{\alpha,\beta}+\lambda^2+\rho^2)u=0,\quad u(0)=1,\quad u'(0)=0,$$

where $\rho=\alpha+\beta+1$, $\mathrm{D}_{\alpha,\beta}$ is the Jacobi differential operator defined as

$$\mathrm{D}_{\alpha,\beta}=\frac{d^2}{dx^2}+\left(\frac{\Delta'_{(\alpha,\beta)}(x)}{\Delta_{(\alpha,\beta)}(x)}\right)\frac{d}{dx}$$

[1]Dedicated to Professor Radouan Daher for his 61's birthday.

with

$$\Delta_{(\alpha,\beta)}(x) = (2\sinh x)^{2\alpha+1}(2\cosh x)^{2\beta+1},$$

and $\Delta'_{(\alpha,\beta)}(x)$ is the derivative of $\Delta_{(\alpha,\beta)}(x)$.

The Jacobi functions $\phi_\lambda$ can be expressed in terms of Gaussian hypergeometric functions as

$$\phi_\lambda(x) = \phi_\lambda^{(\alpha,\beta)}(x) = \mathrm{F}\left(\frac{1}{2}(\rho - i\lambda),\ \frac{1}{2}(\rho + i\lambda),\ \alpha+1,\ -\sinh^2 x\right),$$

where the Gaussian hypergeometric function is defined as

$$F(a,b,c,z) = \sum_{m=0}^{\infty} \frac{a_m b_m}{c_m m!} z^m, \quad |z| < 1,$$

with $a, b, z \in \mathbb{C}$, $c \notin -\mathbb{N}$, $a_0 = 1$, and $a_m = a(a+1)\cdots(a+m-1)$.

The function $z \to F(a,b,c,z)$ is the unique solution of the differential equation

$$z(1-z)u''(z) + (c - (a+b+1)z)u'(z) - abu(z) = 0,$$

which is regular at 0 and equals 1 there.

From [7, Lemmas 3.1–3.3], we obtain the following statement.

**Lemma 1.** *The following inequalities are valid for a Jacobi function $\phi_\lambda(t)$ $(\lambda, t \in \mathbb{R}^+)$:*

(1) $|\phi_\lambda(t)| \le 1$;

(2) $|1 - \phi_\lambda(t)| \le t^2(\lambda^2 + \rho^2)$;

(3) *there is a constant $d > 0$ such that*

$$1 - \phi_\lambda(t) \ge d \quad \textit{for} \quad \lambda t \ge 1.$$

Let $\mathrm{L}^p_{\alpha,\beta}(\mathbb{R}^+) = \mathrm{L}^p(\mathbb{R}^+, \Delta_{(\alpha,\beta)}(t)dt)$, $1 \le p \le 2$, be the space of $p$-power integrable functions on $\mathbb{R}^+$ endowed with the norm

$$\|f\|_p = \left(\int_0^\infty |f(x)|^p \Delta_{(\alpha,\beta)}(x)dx\right)^{1/p} < \infty.$$

Let $\mathrm{L}^p_\mu(\mathbb{R}^+) = \mathrm{L}^p(\mathbb{R}^+, d\mu(\lambda)/2\pi)$, $1 \le p \le 2$, be the space of measurable functions $f$ on $\mathbb{R}^+$ such that

$$\|f\|_{p,\mu} = \left(\frac{1}{2\pi}\int_0^\infty |f(x)|^p d\mu(\lambda)\right)^{1/p},$$

where $d\mu(\lambda) = |c(\lambda)|^{-2}d\lambda$ and the $c$-function $c(\lambda)$ is defined as

$$c(\lambda) = \frac{2^{\rho - i\lambda}\Gamma(\alpha+1)\Gamma(i\lambda)}{\Gamma(1/2\cdot(i\lambda+\alpha+\beta+1))\Gamma(1/2\cdot(i\lambda+\alpha-\beta+1))}.$$

Now, we define the Jacobi transform

$$\widehat{f}(\lambda) = \int_0^\infty f(x)\phi_\lambda(x)\Delta_{(\alpha,\beta)}(x)dx,$$

for all functions $f$ on $\mathbb{R}^+$ and complex numbers $\lambda$ for which the right-hand side is well defined.

The Jacobi transform reduces to the Fourier transform when $\alpha = \beta = -1/2$.

We have the following inversion formula [6].

**Theorem 2.** *If* $f \in \mathrm{L}^p_{\alpha,\beta}(\mathbb{R}^+)$, *then*

$$f(x) = \frac{1}{2\pi}\int_0^\infty \widehat{f}(\lambda)\phi_\lambda(x)d\mu(\lambda).$$

From [3], we have the Hausdorff–Young inequality

$$\|\widehat{f}\|_{q,\mu} \le C_2\|f\|_p \quad \text{for all} \quad f \in \mathrm{L}^p_{\alpha,\beta}(\mathbb{R}^+),$$

where $1/p + 1/q = 1$ and $C_2$ is a positive constant.

The generalized translation operator $\mathrm{T}_h$ of a function $f \in \mathrm{L}^p_{\alpha,\beta}(\mathbb{R}^+)$ is defined as

$$\mathrm{T}_h f(x) = \int_0^\infty f(z)K(x,h,z)\Delta_{(\alpha,\beta)}(z)dz,$$

where $K$ is an explicity known kernel function such that

$$K(x,y,z) = \frac{2^{-2\rho}\Gamma(\alpha+1)(\cosh x\cosh y\cosh z)^{\alpha-\beta-1}}{\Gamma(1/2)\Gamma(\alpha+1/2)(\sinh x\sinh y\sinh z)^{2\alpha}}(1-B^2)^{\alpha-1/2}$$
$$\times F\left(\alpha+\beta,\, \alpha-\beta,\, \alpha+\frac{1}{2},\, \frac{1}{2}(1-B)\right) \quad \text{for} \quad |x-y| < z < x+y,$$

and $K(x,y,z) = 0$ elsewhere and

$$B = \frac{\cosh^2 x + \cosh^2 y + \cosh^2 z - 1}{2\cosh x\cosh y\cosh z}.$$

From [2], we have

$$\widehat{(\mathrm{T}_h f)}(\lambda) = \phi_\lambda(h)\widehat{f}(\lambda).$$

## 2. Main results

In this section, we give the main result of this paper. We need first to define the $(\nu,\gamma,p)$-Jacobi–Lipschitz class.

**Definition 1.** *Let* $\nu, \gamma > 0$. *A function* $f \in \mathrm{L}^p_{\alpha,\beta}(\mathbb{R}^+)$ *is said to be in the* $(\nu,\gamma,p)$*-Jacobi–Lipschitz class, denoted by* $\mathrm{Lip}(\nu,\gamma,p)$, *if*

$$\|\mathrm{T}_h f(x) - f(x)\|_p = O\left(\frac{h^\nu}{(\log 1/h)^\gamma}\right) \quad \text{as} \quad h \to 0.$$

**Theorem 3.** *Let* $f$ *belong to* $\mathrm{Lip}(\nu,\gamma,p)$. *Then*

$$\int_N^{+\infty} |\widehat{f}(\lambda)|^q d\mu(\lambda) = O(N^{-q\nu}(\log N)^{-q\gamma}) \quad \text{as} \quad N \to +\infty.$$

P r o o f. Let $f \in \mathrm{Lip}(\nu,\gamma,p)$. Then we have

$$\|\mathrm{T}_h f(x) - f(x)\|_p = O\left(\frac{h^\nu}{(\log 1/h)^\gamma}\right) \quad \text{as} \quad h \to 0.$$

Therefore,

$$\int_0^{+\infty} |1-\phi_\lambda(h)|^q|\widehat{f}(\lambda)|^q d\mu(\lambda) \le C_2^q\|\mathrm{T}_h f(x) - f(x)\|_p^q.$$

If $\lambda \in [1/h, 2/h]$, then $\lambda h \geq 1$ and inequality (3) of Lemma 1 implies that

$$1 \leq \frac{1}{d^{qk}} |1 - \phi_\lambda(h)|^{qk}.$$

Then

$$\int_{1/h}^{2/h} |\widehat{f}(\lambda)|^q d\mu(\lambda) \leq \frac{1}{d^{qk}} \int_{1/h}^{2/h} |1 - \phi_\lambda(h)|^{qk} |\widehat{f}(\lambda)|^q d\mu(\lambda)$$

$$\leq \frac{1}{d^{qk}} \int_0^{+\infty} |1 - \phi_\lambda(h)|^{qk} |\widehat{f}(\lambda)|^q d\mu(\lambda) \leq \frac{1}{d^{qk}} C_2^q \|\mathrm{T}_h f(x) - f(x)\|_p^q = O\left(\frac{h^{q\nu}}{(\log 1/h)^{q\gamma}}\right).$$

Then

$$\int_N^{2N} |\widehat{f}(\lambda)|^q d\mu(\lambda) = O\left(N^{-q\nu} (\log N)^{-q\gamma}\right) \quad \text{as} \quad N \to +\infty.$$

Thus, there exists $C_4$ such that

$$\int_N^{2N} |\widehat{f}(\lambda)|^q d\mu(\lambda) \leq C_4 N^{-q\nu} (\log N)^{-q\gamma}.$$

Furthermore, we have

$$\int_N^{+\infty} |\widehat{f}(\lambda)|^q d\mu(\lambda) = \left[\int_N^{2N} + \int_{2N}^{4N} + \int_{4N}^{8N} + ...\right] |\widehat{f}(\lambda)|^q d\mu(\lambda)$$

$$\leq C_4 N^{-q\nu} (\log N)^{-q\gamma} + C_4 (2N)^{-q\nu} (\log 2N)^{-q\gamma} + C_4 (4N)^{-q\nu} (\log 4N)^{-q\gamma} + ...$$

$$\leq C_4 N^{-q\nu} (\log N)^{-q\gamma} (1 + 2^{-q\nu} + (2^{-q\nu})^2 + (2^{-q\nu})^3 + ...$$

$$\leq C_4 C_k N^{-q\nu} (\log N)^{-q\gamma}),$$

where $C_k = (1 - 2^{-q\nu})^{-1}$ since $2^{-q\nu} < 1$.

This proves that

$$\int_N^{+\infty} |\widehat{f}(\lambda)|^q d\mu(\lambda) = O\left(N^{-q\nu} (\log N)^{-q\gamma}\right) \quad \text{as} \quad N \to +\infty,$$

and this completes the proof. □

**Definition 2.** *A function $f \in \mathrm{L}^p_{\alpha,\beta}(\mathbb{R}^+)$ is said to be in the $(\psi, p)$-Jacobi–Lipschitz class, denoted by* $\mathrm{Lip}(\psi, p)$, *if*

$$\|\mathrm{T}_h f(x) - f(x)\|_p = O\left(\frac{\psi(h)}{(\log 1/h)^\gamma}\right), \quad \gamma > 0, \quad \textit{as} \quad h \to 0,$$

*where*

(1) $\psi(t)$ *is a continuous increasing function on* $[0, \infty)$;

(2) $\psi(0) = 0$;

(3) $\psi(ts) \leq \psi(t)\psi(s)$ *for all* $s, t \in [0, \infty)$.

**Theorem 4.** *Let $f \in \mathrm{L}^p_{\alpha,\beta}(\mathbb{R}^+)$, $\psi$ be a fixed function satisfying the conditions of Definition 2, and let $f(x)$ belong to* $\mathrm{Lip}(\psi, p)$. *Then*

$$\int_N^{+\infty} |\widehat{f}(\lambda)|^q d\mu(\lambda) = O(\psi(N^{-q}) (\log N)^{-q\gamma}) \quad \textit{as} \quad r \to +\infty.$$

P r o o f. Let $f \in \mathrm{Lip}(\psi, p)$. Then we have

$$\|\mathrm{T}_h f(x) - f(x)\|_p = O\left(\frac{\psi(h)}{(\log 1/h)^\gamma}\right) \quad \text{as} \quad h \to 0$$

and

$$\int_0^{+\infty} |1 - \phi_\lambda(h)|^q |\widehat{f}(\lambda)|^q d\mu(\lambda) \le C_2^q \|\mathrm{T}_h f(x) - f(x)\|_p^q.$$

If $\lambda \in [1/h, 2/h]$, then $\lambda h \ge 1$ and, similarly to the proof of Theorem 3, by inequality (3) of Lemma 1, we obtain

$$1 \le \frac{1}{d^{qk}} |1 - \phi_\lambda(h)|^{qk}.$$

Then

$$\int_{1/h}^{2/h} |\widehat{f}(\lambda)|^q d\mu(\lambda) \le \frac{1}{d^{qk}} \int_{1/h}^{2/h} |1 - \phi_\lambda(h)|^{qk} |\widehat{f}(\lambda)|^q d\mu(\lambda)$$
$$\le \frac{1}{d^{qk}} C_2^q \|\mathrm{T}_h f(x) - f(x)\|_p^q = O\left(\frac{\psi(h^q)}{(\log 1/h)^{q\gamma}}\right).$$

There exists a positive constant $C_5$ such that

$$\int_N^{2N} |\widehat{f}(\lambda)|^q d\mu(\lambda) \le C_5 \frac{\psi(N^{-q})}{(\log N)^{q\gamma}}.$$

Thus,

$$\int_N^{+\infty} |\widehat{f}(\lambda)|^q d\mu(\lambda) = \left[\int_N^{2N} + \int_{2N}^{4N} + \int_{4N}^{8N} + \ldots\right] |\widehat{f}(\lambda)|^q d\mu(\lambda)$$
$$\le C_5 \frac{\psi(N^{-q})}{(\log N)^{q\gamma}} + C_5 \frac{\psi((2N)^{-q})}{(\log 2N)^{q\gamma}} + C_5 \frac{\psi((4N)^{-q})}{(\log 4N)^{q\gamma}} + \ldots$$
$$\le C_5 \frac{\psi(N^{-q})}{(\log N)^{q\gamma}} + C_5 \frac{\psi((2N)^{-q})}{(\log N)^{q\gamma}} + C_5 \frac{\psi((4N)^{-q})}{(\log N)^{q\gamma}} + \ldots$$
$$\le C_5 \frac{\psi(N^{-q})}{(\log N)^{q\gamma}} (1 + \psi(2^{-q}) + (\psi(2^{-q}))^2 + (\psi(2^{-q}))^3 + \ldots$$
$$\le C_5 K_1 \frac{\psi(N^{-q})}{(\log N)^{q\gamma}},$$

where $K_1 = (1 - \psi(2^{-q}))^{-1}$ since (1) and (3) from Definition 2 imply that $\psi(2^{-q}) < 1$.

This proves that

$$\int_N^{+\infty} |\widehat{f}(\lambda)|^q d\mu(\lambda) = O\left(\psi(N^{-q})(\log N)^{-q\gamma}\right) \quad \text{as} \quad N \longrightarrow +\infty,$$

and this completes the proof. □

## Acknowledgements

The authors would like to thank the referee for his valuable comments and suggestions.

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